2.2 Fixed-Point Iteration

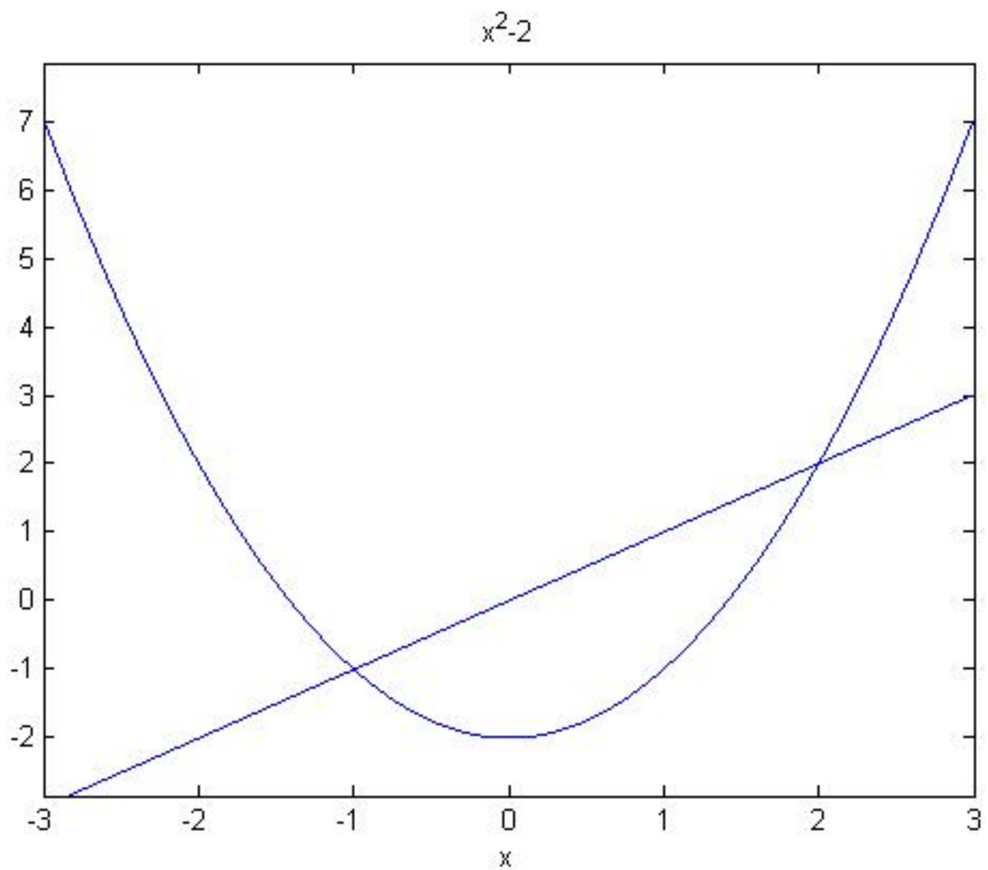
Definition 2.2. The number p is a **fixed point** for a given function $g(x)$ if $g(p) = p$.

Geometric interpretation of **fixed point.**

- Consider the graph of function $g(x)$, and the graph of equation $y = x$.
- If they intersect, what are the coordinates of the intersection point?

Example 1.

Determine the fixed points of the function $g(x) = x^2 - 2$.



Connection between fixed-point problem and root-finding problem

1. Given a root-finding problem, i.e., to solve $f(x) = 0$. Suppose a root is p, so that $f(p) = 0$.

   There are many ways to define $g(x)$ with fixed-point at p.

   For example, $g(x) = x - f(x)$,

   $$g(x) = x + 3f(x),$$

   $$\ldots$$

2. If $g(x)$ has a fixed-point at p, then $f(x)$ defined by $f(x) = x - g(x)$ has a zero at p.

Sufficient conditions for existence and uniqueness of a fix point

Theorem 2.3. Existence and Uniqueness Theorem

a. If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one **fixed-point** in $[a,b]$

b. If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with
$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b),$$
then there is **exactly one fixed-point** in $[a,b]$.

Note:

1. $g \in C[a,b] \rightarrow$ g is continuous in $[a,b]$

2. $g(x) \in [a,b] \rightarrow$ range of g is in $[a,b]$

Example 2. Show $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on [-1, 1].

Remark: The conditions in Theorem 2.3 are sufficient to guarantee existence and uniqueness of a fixed point, but are not necessary. [See, e.g., *Example 3* in textbook page 58]

Fixed-Point Iteration Algorithm

- Choose an initial approximation p_0, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$.
- If the sequence converges to p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

Remark:

a) $g(x)$ might not have a fixed point.

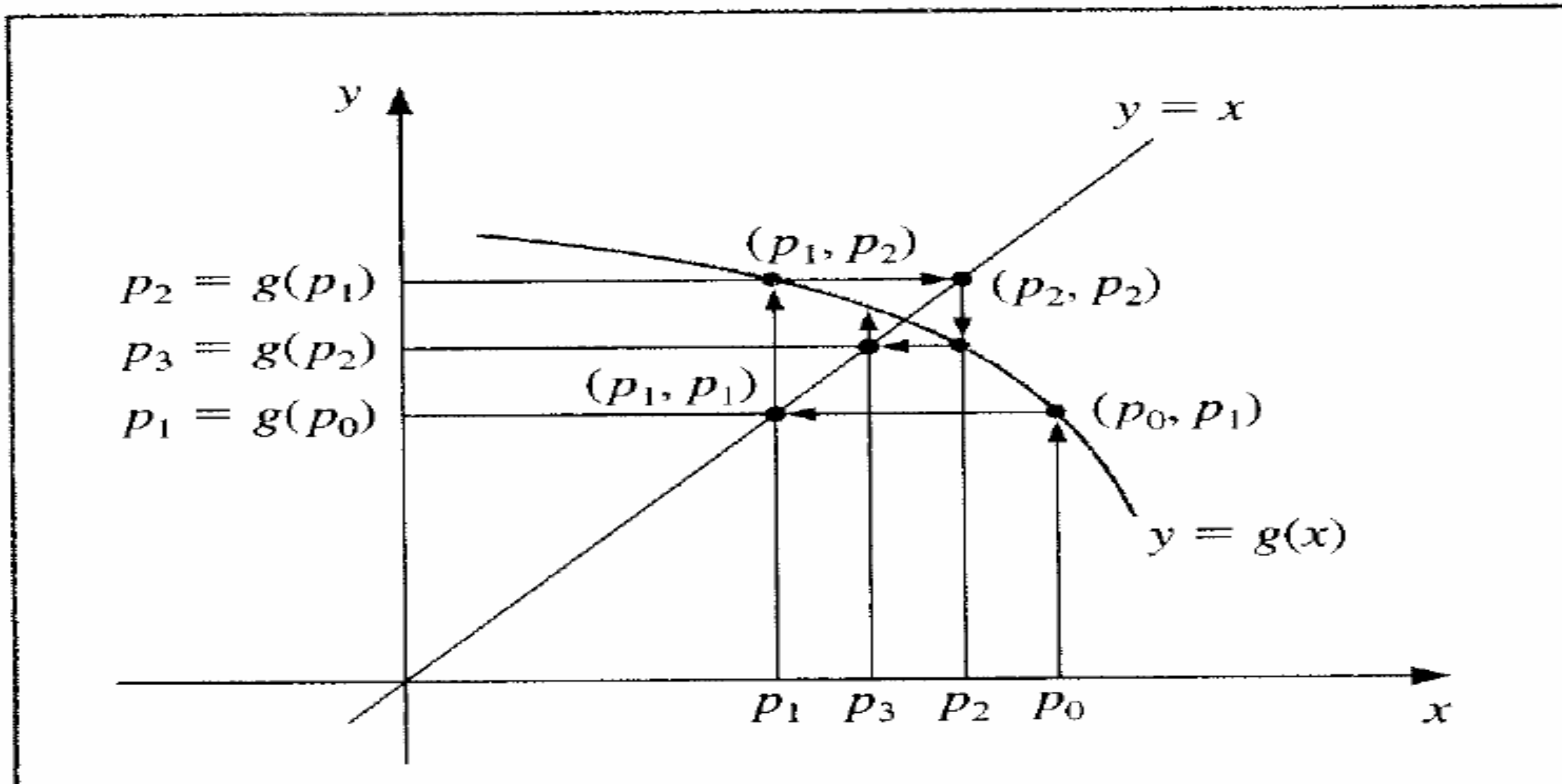
b) Even if $g(x)$ has a fixed point, the sequence $\{p_n\}_{n=0}^{\infty}$ generated by fixed-point iteration might not converge. (bad initial approximation p_0)

Example 3. Let $f(x) = x^4 + 3\,x^2 - 2$.

a) Verify that the fixed-point of $g(x) = \sqrt[4]{2 - 3x^2}$ is a solution of the equation $f(x) = 0$.

b) Apply the fixed-point iteration algorithm to $g(x)$ with initial approximation $p_0 = 0$ to compute the iterations p_1, p_2.

Geometric illustration: fixed-pt



(a)

Algorithm 2.2

INPUT **p0**; tolerance **TOL**; maximum number of iteration **N0**.

OUTPUT solution **p** or message of failure

STEP1 Set i = 1. // init. counter

STEP2 While i ≤ N0 do Steps 3-6

STEP3 Set **p** = g(**p0**).

STEP4 If |**p-p0**| < **TOL** then

OUTPUT(p); // successfully found the solution

STOP.

STEP5 Set i = i + 1.

STEP6 Set **p0** = **p**. // update **p0**

STEP7 OUTPUT("The method failed after **N0** iterations");

STOP.

Example 4 Equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1,2]. Use algebraic manipulation to obtain fixed-point iteration function g to solve this root-finding problem.

Then implement the algorithm in MATLAB to see whether it converges. (interactive matlab session)

MATLAB snippet for ALG 2.2

(see complete code in course webpage)

```matlab
% ------------------------- Inputs ---------------------
g = @(x) sqrt(10/(x+4));
p0 = 1.5;
TOL = 1e-10; NI = 100;
% -------------------------------------------------------

% STEP 1
i = 1;
converge = false; % convergence flag
% STEP 2
while i<=NI
    % STEP 3: compute p(i)
    p = g(p0);
    err = abs(p-p0);
    % STEP 4: check if meets the stopping criteria
    if (err< TOL)
        converge = true; break
    else
        i = i+1; % STEP 5
        p0 = p;  % STEP 6: update p0
    end
end

if converge
    fprintf('\n\nApproximate solution P = %.8f\n',p)
    fprintf('Number of iterations = %3i\n',i)
    fprintf('Tolerance = %.3e  |p-pold| = %.3e\n',TOL, err)
else
    fprintf('\n\nInteration number = %3i\n',NI)
    fprintf(' gave approximation  %.8f\n',p)
    fprintf('|p-pold| = %.3e not within tolerance %.3e\n',err, TOL)
end
```

Convergence

Fixed-Point Theorem 2.4

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a,b)$$

Then, for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1})$$

converges to the unique fixed point p in $[a,b]$.

Corollary 2.5

If g satisfies the above hypotheses, then bounds for the error involved using p_n to approximating p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0|$$