2.5 Accelerating Convergence

Aitken's Δ^2 Method

- **Assume** $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p.
- Further assume $\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p}$ when n is large
- Solving for p yields:

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

A little algebraic manipulation gives:

$$p \approx p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

- **Define** $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

Remark: The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster.

Definition 2.13

The **forward difference** Δp_n is defined by

$\Delta p_n = p_{n+1} - p_n$. High powers of Δ are defined recursively by $\Delta^k p_n = \Delta\left(\Delta^{k-1} p_n\right)$.

Remark: $\widehat{p_n}$ can also be rewritten as

$$\widehat{p_n} = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

Theorem 2.14:

Suppose that $\{p_n\}_{n=0}^{\infty}$ converges linearly to the limit p and that $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n - p} < 1$. Then the Aitken's Δ^2 sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that $\lim_{n\to\infty} \frac{\widehat{p_n}-p}{p_n - p} = 0$.

Example. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ generated by the fixed point iteration $p_{n+1} = \cos(p_n)\,, p_0 = 0.$

iteration	p_n	$\widehat{p_n}$
0	0.000000000000000	**0** .685073357326045
1	1.000000000000000	**0.7** 28010361467617
2	**0** .540302305868140	**0.73** 3665164585231
3	**0** .857553215846393	**0.73** 6906294340474
4	**0** .654289790497779	**0.73** 8050421371664
5	**0.7** 93480358742566	**0.73** 8636096881655
6	**0.7** 01368773622757	**0.73** 8876582817136
7	**0.7** 63959682900654	**0.73** 8992243027034
8	**0.7** 22102425026708	**0.7390** 42511328159
9	**0.7** 50417761763761	**0.7390** 65949599941
10	**0.73** 1404042422510	**0.7390** 76383318956
11	**0.7** 44237354900557	**0.73908** 1177259563*
12	**0.73** 5604740436347	**0.73908** 3333909684*

Example 1. Generate the first two terms of the sequence $\{\widehat{p_n}\}_{n=1}^{\infty}$ using Aitken's Δ^2 method if

$$p_n = \frac{1}{n}, n \geq 1.$$

Steffensen's Method

- Steffensen's Method combines fixed-point iteration and the Aitken's Δ^2 method:

Step 0. Suppose we have a fixed point iteration:

$$p_0, \qquad \mathrm{p}_1 = \mathrm{g}(\mathrm{p}_0), \qquad \mathrm{p}_2 = \mathrm{g}(\mathrm{p}_1)$$

Once we have we have p_0, p_1 and p_2, we can compute

$$p_0^{(1)} = p_0 - \frac{(p_1 - p_0)^2}{(p_2 - 2p_1 + p_0)}$$

Step 1. Then we "restart" the fixed point iteration with

$$p_1^{(1)} = g\left(p_0^{(1)}\right), \quad p_2^{(1)} = g\left(p_1^{(1)}\right)$$

and compute:

$$p_0^{(2)} = p_0^{(1)} - \frac{\left(p_1^{(1)} - p_0^{(1)}\right)^2}{\left(p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}\right)}.$$

Step 2. We "restart" the fixed point iteration with

$$p_1^{(2)} = g\left(p_0^{(2)}\right), \quad p_2^{(2)} = g\left(p_1^{(2)}\right)$$

and compute:

$$p_0^{(3)} = p_0^{(2)} - \frac{\left(p_1^{(2)} - p_0^{(2)}\right)^2}{\left(p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}\right)}.$$

Example 2. Use Steffensen's method for solving:

$$f(x) = x^3 + 4x^2 - 10 = 0.$$

The base fixed point iteration takes $g(x) = \sqrt{\frac{10}{x+4}}$,

and $p_0 = 1.5$.

Remark: Steffensen's method gives quadratic convergence. (see **Theorem 2.15** in page 89)