3.3 Divided Differences

Divided difference form of Lagrange Polynomial

- Let $P_n(x)$ be the nth degree Lagrange interpolating polynomial that agrees with $f(x)$ at the points $\{x_0, x_1, \dots, x_n\}$, i.e.,

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x).$$

- We express $P_n(x)$ in the following form (divided difference):

$$P_n(x) = a_0 + a_1(x - x_0) + \\ a_2(x - x_0)(x - x_1) + \\ a_3(x - x_0)(x - x_1)(x - x_2) + \\ \dots + a_n(x - x_0)(x - x_1)(x - x_2)\dots(x - x_{n-1})$$

- How to find constants $a_0, \dots, a_n$?

Finding constants $a_0, \ldots, a_n$

Given interpolating polynomial $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})$

- At x_0: $a_0 = P_n(x_0) = f(x_0)$
- At x_1: $f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1)$

$$\Longrightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- At x_2: $f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = P_n(x_2) = f(x_2)$

$$\Longrightarrow a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

- …

Newton's Divided Difference

- Zeroth divided difference:

$$f[x_i] = f(x_i).$$

- First divided difference:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

- Second divided difference:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

- Third divided difference:

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}.$$

- Kth divided difference:

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Finding constants $a_0,\ldots, a_n$-**revisited**

Given interpolating polynomial $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})$

- $a_0 = f(x_0) = f[x_0]$
- $a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f[x_1]-f[x_0]}{x_1-x_0} = f[x_0, x_1].$
- $a_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = f[x_0, x_1, x_2].$
- $a_3 = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0} = f[x_0, x_1, x_2, x_3].$
- $a_k = f[x_0, x_1, \ldots, x_k].$

Interpolating Polynomial Using Newton's (forward) Divided Difference Formula

$$P_n(x)$$
$$= f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$
$$+ f[x_0, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Or

$$P_n(x)$$
$$= f[x_0] + \sum_{k=1}^{n} [f[x_0, \dots, x_k](x - x_0) \dots (x - x_{k-1})]$$

Remark: $a_k = f[x_0, x_1, \dots, x_k]$ for $k = 0, \dots, n$

Table for Computing (Table 3.9)

x	f(x)	1st Div. Diff.	2nd Div. Diff.	
x_0	$f[x_0]$			
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	...
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$		
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3, x_4] = \frac{f[x_4]-f[x_3]}{x_4-x_3}$		
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4, x_5] = \frac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Example 1. Construct the interpolating polynomial of degree three for the following data using Newton' forward divided difference formula:

$$f(0) = 6, f(1) = 3, f(2) = 2, f(3) = 1.5.$$

Theorem 3.6 Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \dots, x_n$ are distinct numbers in $[a,b]$. Then $\exists \xi \in (a,b)$ with $f[x_0, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$.

Remark: When $n = 1$, it's just the Mean Value Theorem.

Algorithm 3.2: Newton's Divided Differences

Input: $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$

Output: Divided differences $F_{0,0}, \ldots, F_{n,n}$

//comment: $P_n(x) = F_{0,0} + \sum_{i=1}^{n}[F_{i,i}(x - x_0) \ldots (x - x_{i-1})]$

Step 1: For $i = 0, \ldots, n$

set $F_{i,0} = f(x_i)$

Step 2: For $i = 1, \ldots, n$

For $j = 1, \ldots, i$

set $F_{i,j} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$

End

End

Output($F_{0,0}, \ldots F_{i,i}, \ldots, F_{n,n}$)

STOP.

D.D. coef. (function .m file)

divided_diff.m

```
function F = divided_diff(x, y)

% initialization
n = length(x)-1; % # of points -1
Q = zeros(n+1, n+1); % a matrix store all the divided diff. coef. Fij
F = zeros(n+1,1); % a vector store Fii

% STEP 1
for i = 0:n
    Q(i+1, 1) = y(i+1); % MATLAB used 1-based indexing
end

% STEP 2
for i = 1:n
    for j = 1:i
        Q(i+1,j+1) = (Q(i+1,j)-Q(i,j))/(x(i+1)-x(i-j+1));
    end
end

% STEP 3: output
for i = 0:n
    F(i+1) = Q(i+1,i+1);
end

return
```

Example 2 (MATLAB). Apply Newton's divided difference formula to implement the 4th Lagrange interpolating polynomial for

$$f(x) = \frac{1}{1+25\,x^2}$$

on the interval $[-1, 1]$ using 5 uniform nodes $x_0 = -1, x_1 = -0.5, x_2 = 0, x_3 = 0.5, x_4 = 1$. Plot the function. On the same figure, plot the original function $f(x)$ and the interpolation nodes.

Newton formula (script .m file)

lagrange_divided_diff.m

```
% function and interpolation nodes
f = @(x) 1./(1+25*x.*x);
n = 4;
xNodes = linspace(-1,1,n+1);
yNodes = f(xNodes);

% the points to be evaluated at
m = 1001;
xGrid = linspace(-1,1,m);
pGrid = zeros(size(xGrid)); % the interpolation values

% STEP 1: the divided differences coefficients
F = divided_diff(xNodes,yNodes);

% STEP 2: loop over points
for k = 1:m
    xk = xGrid(k); % the point to be evaluated at
    % nested arithmic
    tmp = F(n+1);
    for i=0:n-1
        tmp = tmp*(xk-xNodes(n-i))+F(n-i);
    end
    pGrid(k) = tmp;
end

plot(xGrid, f(xGrid),'b', xGrid, pGrid, 'r', xNodes, yNodes, 'ko',...
    'LineWidth', 4)
legend('F(x)', 'P(x)','Nodes')
set(gca, 'FontSize',24)
```

The figure (run ≫ lagrange_divided_diff on command window)