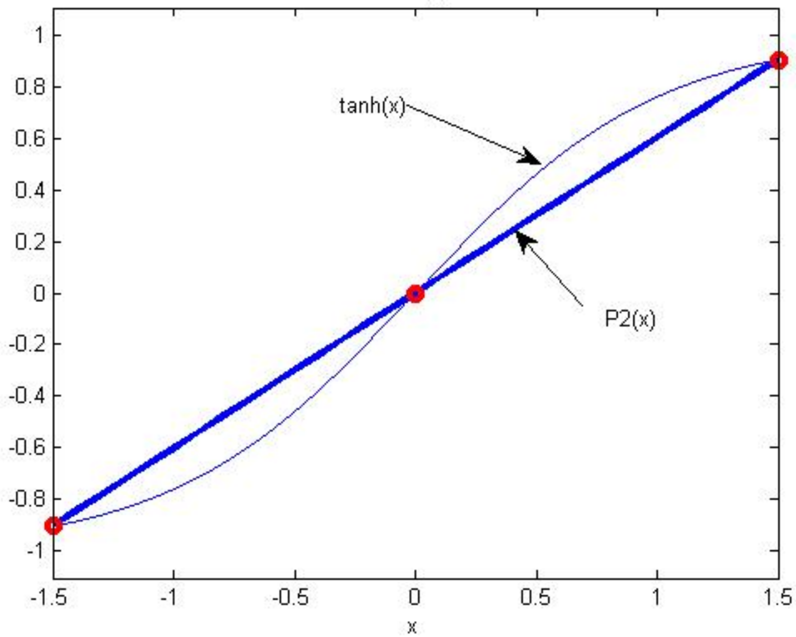
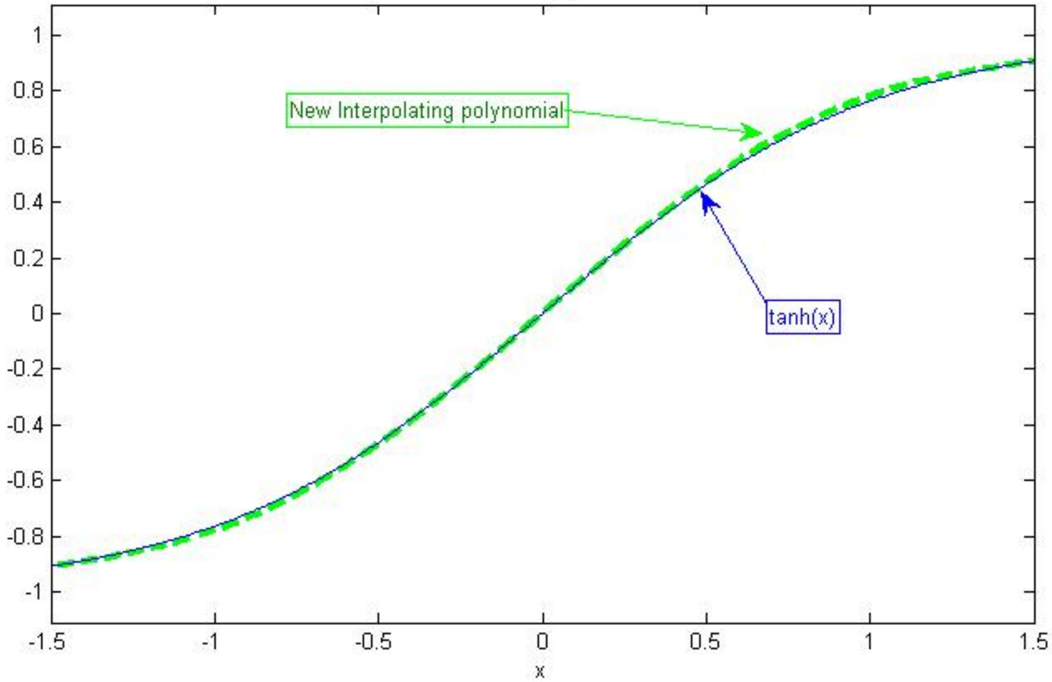
3.4 Hermite Interpolation

Illustration. Consider to interpolate $\tanh(x)$ using Lagrange polynomial and nodes $x_0 = -1.5, x_1 = 0, x_2 = 1.5$.



Now interpolate $\tanh(x)$ using nodes $x_0 = -1.5, x_1 = 0, x_2 = 1.5$. Moreover, Let 1st derivative of interpolating polynomial agree with derivative of $\tanh(x)$ at these nodes.



Remark: This is called Hermite interpolating polynomial.

Hermite Polynomial

Definition. Suppose $f \in C^1[a, b]$. Let $x_0, \dots, x_n$ be distinct numbers in $[a, b]$, the Hermite polynomial $P(x)$ approximating f is that:

1. $P(x_i) = f(x_i)$, for $i = 0, \dots, n$
2. $\frac{dP(x_i)}{dx} = \frac{df(x_i)}{dx}$, for $i = 0, \dots, n$

Remark:

(a) $P(x)$ and $f(x)$ agree not only function values but also 1st derivative values at x_i, $i = 0, \dots, n$.

(b) The degree of $P(x)$ is at most $2n + 1$.

3rd Degree Hermite Polynomial Formula

- Given distinct x_0, x_1 and values of f and f' at these numbers.

$$
\begin{aligned}
&H_3(x) \\
&= \left(1 + 2\,\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f(x_0) \\
&+ \left(1 + 2\,\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f(x_1) \\
&+ (x - x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f'(x_0) \\
&+ (x - x_1)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f'(x_1)
\end{aligned}
$$

Theorem 3.9 If $f \in C^1[a,b]$ and $x_0, \ldots, x_n \in [a,b]$ distinct numbers, the Hermite polynomial of degree at most $2n+1$ is:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

Where

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L_{n,j}^2(x)$$
$$\hat{H}_{n,j}(x) = (x - x_j) L_{n,j}^2(x)$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some $\xi(x) \in (a,b)$.

Recall: $L_{n,j}(x)$ is the jth Lagrange basis polynomial of degree n.

Remark: (proof of 2, 3 on the next slide)

1. $H_{n,j}(x), \widehat{H}_{n,j}(x)$ are the (degree $2n+1$) Hermite basis functions.

2. $H_{n,j}(x)$ satisfies the following property:

   a) $H_{n,j}(x_j) = 1, \quad H_{n,j}(x_i) = 0, \forall\, i \neq j$

   b) $H'_{n,j}(x_j) = 0, \quad H'_{n,j}(x_i) = 0, \forall\, i \neq j$

3. $\widehat{H}_{n,j}(x)$ satisfies the following property:

   a) $\widehat{H}_{n,j}(x_j) = 0, \quad \widehat{H}_{n,j}(x_i) = 0, \forall\, i \neq j$

   b) $\widehat{H}'_{n,j}(x_j) = 1, \quad \widehat{H}'_{n,j}(x_i) = 0, \forall\, i \neq j$

$$\Longrightarrow H_{2n+1}(x_j) = f(x_j), \;\; H'_{2n+1}(x_j) = f'(x_j).$$

1. When $i \neq j$: $H_{n,j}(x_i) = 0; \widehat{H}_{n,j}(x_i) = 0.$

2. When $i = j$:

$$H_{n,j}(x_j) = [1 - 2(x_j - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x_j) = 1$$

$$\widehat{H}_{n,j}(x_j) = (x_j - x_j)L^2_{n,j}(x_j) = 0$$

3. $H'_{n,j}(x) = L_{n,j}(x)\left[-2L'_{n,j}(x_j)L_{n,j}(x) + \left(1 - 2(x - x_j)L'_{n,j}(x_j)\right) 2L'_{n,j}(x)\right]$

$\Longrightarrow$ When $i \neq j$: $H'_{n,j}(x_i) = 0$; When $i = j$: $H'_{n,j}(x_j) = 0.$

4. $\widehat{H}'_{n,j}(x) = L^2_{n,j}(x) + 2(x - x_j)L_{n,j}(x)L'_{n,j}(x)$

$\Longrightarrow$ When $i \neq j$: $\widehat{H}'_{n,j}(x_i) = 0$; When $i = j$: $\widehat{H}'_{n,j}(x_j) = 1.$

Example 1. Use Hermite polynomial that agrees with the data in the table to find an approximation of $f(0.5)$

k	x_k	$f(x_k)$	$f'(x_k)$
0	0	1	-1
1	1	$\frac{1}{2}$	$-\frac{1}{4}$
2	2	$\frac{1}{3}$	$-\frac{1}{9}$

Hermite Polynomial by Divided Differences

Suppose $x_0, \dots, x_n$ and f, f' are given at these numbers.

- Define $z_0, \dots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \qquad \text{for } i = 0, \dots, n$$

- Construct divided difference table, but use

$$f'(x_0), f'(x_1), .., f'(x_n)$$

  to set the following undefined divided difference:

$$f[z_0, z_1], f[z_2, z_3], \dots, f[z_{2n}, z_{2n+1}].$$

  Namely, $f[z_0, z_1] = f'(x_0), f[z_2, z_3] = f'(x_1), \dots$

$$f[z_{2n}, z_{2n+1}] = f'(x_n).$$

- Next, proceed consecutive to find

$$f[z_0], f[z_0, z_1], f[z_0, z_1, z_2], \dots, f[z_0, z_1, \dots, z_{2n+1}]$$

- Finally, the Hermite polynomial is

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0) \dots (x - z_{k-1})$$

Table 3.16

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Example 2. Use divided difference method to determine the Hermite polynomial that agrees with the data in the table to find an approximation of $f(0.5)$

k	x_k	$f(x_k)$	$f'(x_k)$
0	0	1	-1
1	1	$\frac{1}{2}$	$-\frac{1}{4}$
2	2	$\frac{1}{3}$	$-\frac{1}{9}$

$\mathbf{z}$	$f(z)$	$first\ diff.$	$second\ diff.$	$thrid\ diff.$
$\mathbf{z_0} =$	$f[z_0] =$	$f[z_0, z_1] =$	$f[z_0, z_1, z_2] =$	$f[z_0, z_1, z_2, z_3] =$
$\mathbf{z_1} =$	$f[z_1] =$	$f[z_1, z_2] =$	$f[z_1, z_2, z_3] =$	
$\mathbf{z_2} =$	$f[z_2] =$	$f[z_2, z_3] =$		
$\mathbf{z_3} =$	$f[z_3] =$			

Example 3. The cubic Hermite polynomial

$$H_3(x) = -1 + 2x - \frac{3}{2}x^2(x-2)$$

interpolates two points on the graph of function $f(x)$ Hermite polynomial $H_3(x)$ is constructed based on the divided difference table below.

a) Complete the table.

b) Find $f'(0), f'(2)$.

$\mathbf{z}$	$\boldsymbol{f(z)}$	$\boldsymbol{first\ diff.}$	$\boldsymbol{second\ diff.}$	$\boldsymbol{thrid\ diff.}$
$\mathbf{z_0} =$	$f[z_0] =$	$f[z_0, z_1] =$	$f[z_0, z_1, z_2] =$	$f[z_0, z_1, z_2, z_3] =$
$\mathbf{z_1} =$	$f[z_1] =$	$f[z_1, z_2] =$	$f[z_1, z_2, z_3] =$	
$\mathbf{z_2} =$	$f[z_2] =$	$f[z_2, z_3] =$		
$\mathbf{z_3} =$	$f[z_3] =$			