

You are encouraged to work with other students on this assignment but you are expected to write and work on your own answers. You don't need to provide the name of students you worked with.

You are expected to submit

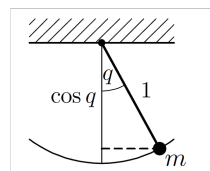
- a zipped file containing all your work online on Sakai so the grader can reproduce your results using only your online submission. **Due on Sakai, Wednesday, Dec. 11.**
- a print-out copy of your obtained results (don't print out the *.dat files) for each problem. **Due in class, Wednesday, Dec. 11.**

Problem 1 : Mathematical pendulum

The following ODE system model a mathematical pendulum (mass $m = 1$, massless rod of length $l = 1$, gravitational acceleration $g = 1$), see figure below,

$$\frac{d}{dt}p(t) = -\sin(q),$$

$$\frac{d}{dt}q(t) = p.$$



Here q is the angle between the string position to the string position at rest (radians), and p is the angular velocity. For this system, the *Hamiltonian* $H(p, q) = \frac{1}{2}p^2 - \cos(q)$ is conserved in the sense that it does not change over time, because:

$$\frac{d}{dt}H(p, q) = \frac{\partial}{\partial p}H(p, q)\frac{d}{dt}p + \frac{\partial}{\partial q}H(p, q)\frac{d}{dt}q = p(-\sin(q)) + \sin(q)p = 0.$$

In the following, you are asked to implement numerical methods for the above ODE system for $t \in [0, 20]$. (Final time is $t_{end} = 20$) Take initial data as $p(0) = 0$, $q(0) = \pi/3$.

- Implement Euler's method for the above ODE system with step size $h = 0.1$. Save the computed values as a matrix of size $2 \times (N + 1)$, whose first row contains to your computed p -values, and the second row contains to your computed q -values, in **A11.dat**.

Plot (a) time against your computed q -values, i.e., the x -axis is time steps t_0, t_1, \dots, t_N , the y -axis is the corresponding q -values q_0, q_1, \dots, q_N . (b) your computed p -values (x -axis) against your computed q -values (y -values). This plot is usually called the *phase plane*.

- Repeat (i) using the following *symplectic Euler's method*:

$$(1a) \quad p_{n+1} = p_n - h \sin(q_n),$$

$$(1b) \quad q_{n+1} = q_n + h p_{n+1}.$$

with $p_0 = 0$, $q_0 = \pi/3$. Save data as **A12.dat**.

Note: The only difference for the original Euler and symplectic Euler's method lies on the update for q . For the Euler's method, $q_{n+1} = q_n + h p_n$, while for symplectic Euler's method, q_{n+1} is obtained via (1b). You will see from your plots that the symplectic Euler method is much more stable than the Euler method.

- Repeat (i) using the Runge-Kutta method of order 4, save data as **A13.dat**.

For this problem, please (1) include all your code and generated .dat files in the online submission, and (2) report the plots (6 figures in total) in your print-out submission.

Problem 2 : SIR Model with real data

This example was given by Murray¹, based on data compiled by the British Communicable Disease Surveillance Centre (British Medical Journal, March 4 1978, p. 587). The event was a flu epidemic in a boys boarding school in the north of England. There were 763 resident boys, including one initial infective. The data (given in graphical form only) is for residents confined to bed, and, following Murray, we will assume those to be the infectives. The data for the two-week epidemic are given in the table below, constructed by reading values from the graph in the original publication.

day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
number	1	3	7	25	72	222	282	256	233	189	123	70	25	11	4

Table 1: Individuals confined to bed (infectives).

The following SIR model is used to fit the data:

$$\begin{cases} S'(t) = -\beta SI, \\ I'(t) = \beta SI - \gamma I, \\ R'(t) = \gamma I. \end{cases}$$

Initial condition is taken to be $S_0 = 762, I_0 = 1, R_0 = 0$, and, according to Murray, the parameters are taken to be $\beta = 2.18 \times 10^{-3}, \gamma = 0.441$.

- (i) Use Adams 4th order predictor-corrector method to solve the above SIR model. Compute the first three starting values using RK4 method. Take final time to be $t_{\text{end}} = 14$, and take time step size $h = 0.2$. Save the computed values as a matrix of size $3 \times (N + 1)$, whose first row contains to your computed S-values, the second row contains to your computed I-values, and the last row contains your computed R-values, in `A21.dat`.

Then

- plot time against your computed S-values, i.e., the x-axis is time steps t_0, t_1, \dots, t_N , the y-axis is the corresponding S-values S_0, S_1, \dots, S_N .
 - plot time against your computed I-values. On the same figure, plot the scattered data in Table 1 above. (you will see the curve fits the data quite well as the model predicts the right general features)
 - plot time against the sum of your computed S, I, R-values, i.e., the y-axis is the corresponding sum values $S_0 + I_0 + R_0, S_1 + I_1 + R_1, \dots, S_N + I_N + R_N$. (you will see the sum, which is the total number of residents, does not change over time).
- (ii) Extend the final time of your code in part (i) to be $t_{\text{end}} = 30$, and use your obtained result to
- predict the total number of residents who do not get the flu, which is nothing but the S-value at your final time step.
 - predict the peak number of infectives, which is nothing but the maximum number from all your computed I-values.
 - plot the S – I phase plane, i.e. x-axis is the S-values, and y-axis is the I-values. Set the axis to be $[0, 800] \times [0, 800]$. What can you say about the relation between S and I?

For this problem, please (1) include all your code and generated .dat files in the online submission, and (2) report the plots in part (i), and the answer and plot in part (ii) in your print-out submission.

¹Murray, J.D., “Mathematical Biology.” Springer-Verlag, 1989.

Optional Problem: Lorenz system

This problem is optional, and will not be graded towards your score. There is no extra credit if you do this problem, but I strongly encourage you to do this interesting problem.

Apply RK4 method to solve the following 3-component ode system (Lorenz system):

$$(2) \quad \begin{cases} \frac{dx}{dt} = \sigma(y - x), \\ \frac{dy}{dt} = x(\rho - z) - y, \\ \frac{dz}{dt} = xy - \beta z, \end{cases}$$

for $0 < t < 100$ with initial condition

$$x(0) = -6, \quad y(0) = 8, \quad z(0) = 27.$$

Here the parameters $\sigma = 10, \rho = 28, \beta = 8/3$. And x, y, z are functions of time t .

- (i) Run the simulation using RK4 method with the time step size taken to be (a) $h = 0.01$ (b) $h = 0.001$.
 - (a) Plot on the same figure the $x - z$ phase plane for your two obtained solutions (you will see a butterfly-like figure (which is called the Lorenz's attractor)).
 - (b) Plot on the same figure the numerical solution x against time t . What can you say about these plots.
- (ii) Now perturb the initial data $x(0) = -6 + 10^{-8}$. And run your RK4 method for this perturbed data, take time step size to be $h = 0.001$. Plot on the same figure solution x against time for both RK4 methods using the unperturbed initial data $x(0) = -6, y(0) = 8, z(0) = 27$, and the perturbed initial data $x(0) = -6 + 10^{-7}, y(0) = 8, z(0) = 27$. What do you see? Are your obtained results similar to each other?

For this problem, you may submit (1) your code in the online submission, and (2) the plots and comments in your print-out submission. **Again, the problem will not be graded towards your score, and it's OK to skip the submission of this problem.**