Section 4.7 Gaussian Quadrature

Motivation

When approximate $\int_a^b f(x)dx$, nodes $x_0, x_1, \cdots, x_n$ in $[a, b]$ do not need to be equally spaced. This can lead to the greatest degree of precision (accuracy).

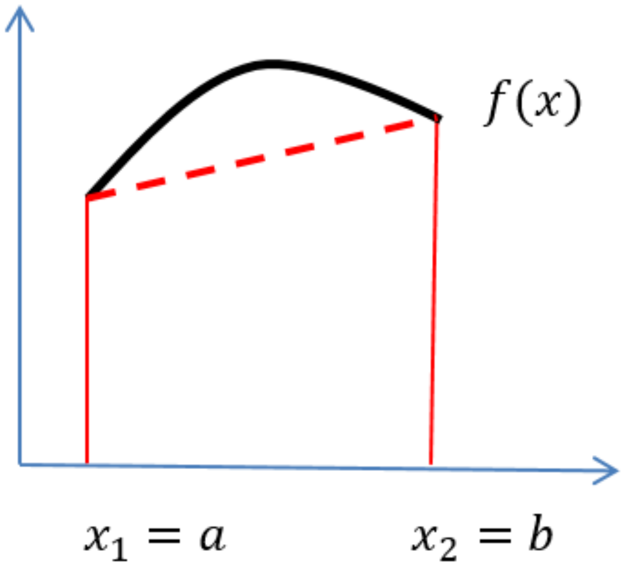


Figure 1. Trapezoidal rule

D.O.P.: 1

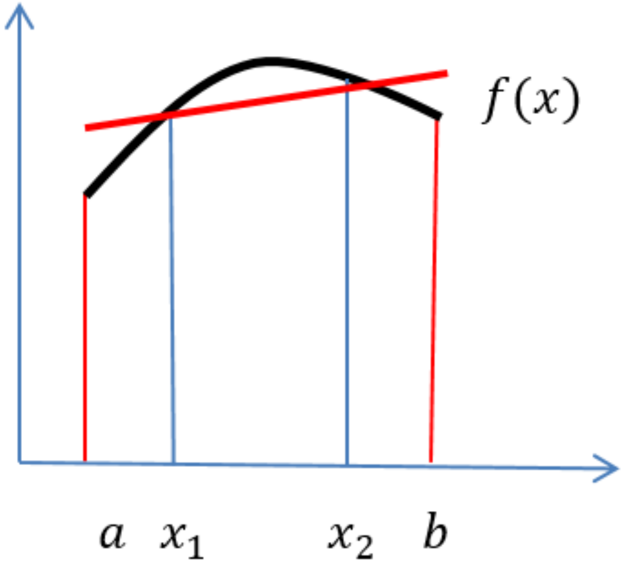


Figure 2. Gaussian quadrature

D.O.P.: 3

Deriving Gaussian Quadrature by Naïve approach

Consider $\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$. Here $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$ are $2n$ parameters. We therefore determine a class of polynomials of degree at most $2n - 1$ for which the quadrature formulas have the degree of precision less than or equal to $2n - 1$.

(The nodes and weights in Gaussian Quadrature are chosen in an optimal manner.)

Example Consider $n = 2$ and $[a, b] = [-1,1]$. We want to determine x_1, x_2, c_1 and c_2 so that quadrature formula $\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$ has degree of precision 3.

Solution: Let $f(x) = 1$. $c_1 + c_2 = \int_{-1}^{1} 1dx = 2$ **(Eq. 1)**

Let $f(x) = x$. $c_1 x_1 + c_2 x_2 = \int_{-1}^{1} xdx = 0$ **(Eq. 2)**

Let $f(x) = x^2$. $c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$ **(Eq. 3)**

Let $f(x) = x^3$. $c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^{1} x^3 dx = 1$ **(Eq. 4)**

Use (nonlinear) equations **(1)-(4)** to solve for x_1, x_2, c_1 and c_2. We obtain:

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

Remark: Quadrature formula $\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ has degree of precision 3; while Trapezoidal rule has degree of precision 1.

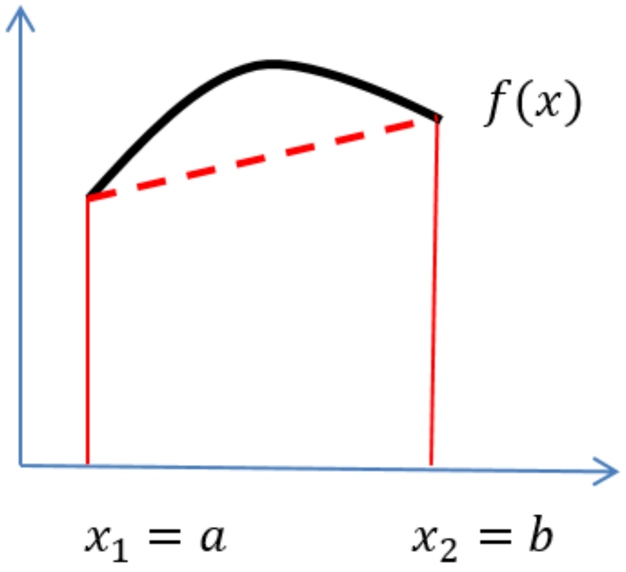


Figure 1. Trapezoidal rule

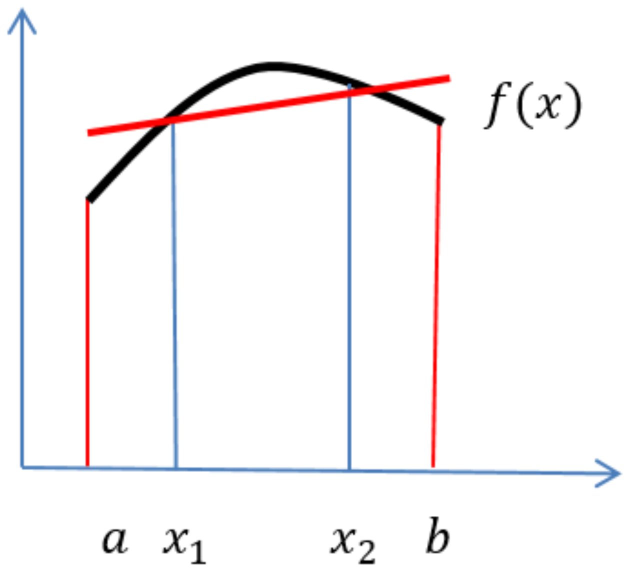


Figure 2. Gaussian quadrature

Deriving Gaussian Quadrature by Legendre Polynomial

Definition. Legendre Polynomials

Legendre polynomials $P_n(x)$, defined on $[-1,1]$ satisfy:

1) For each n, $P_n(x)$ is a polynomial of degree n, and $P_n(1) = 1.$

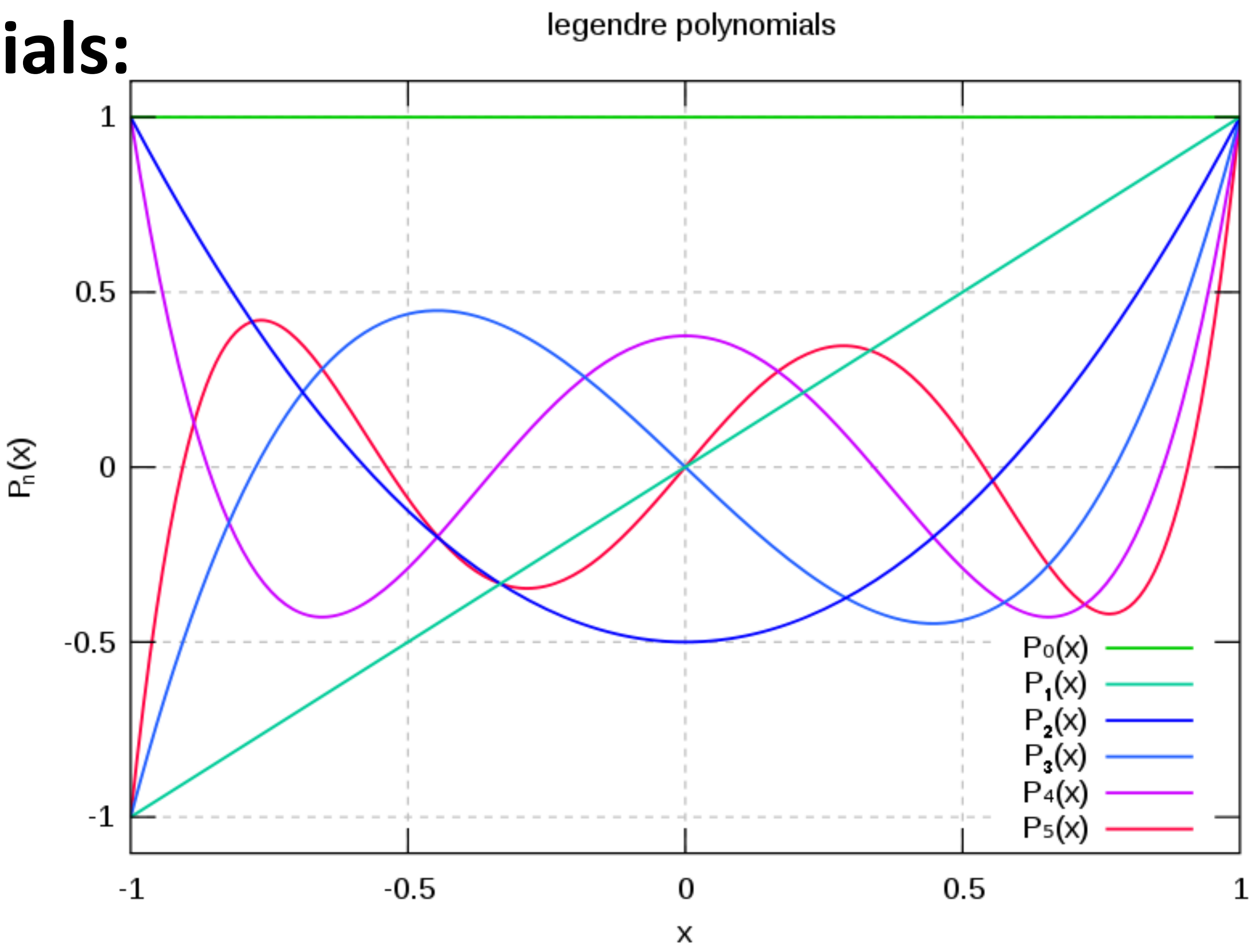
2) $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n

Remark:

- In textbook, condition $P_n(1) = 1$ in Property **1)** is replaced by requiring leading coefficient of $P_n(x)$ is 1, *which is not standard*.
- Property **2)** is usually referred to as $P(x)$ and $P_n(x)$ are orthogonal.

The first five Legendre Polynomials:

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$



Bonnet's recursion formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), n \geq 1$$

Theorem of Gaussian Quadrature by Legendre Polynomial

Theorem 4.7 Suppose that $x_1, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1,2, \cdots n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1;\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x) dx = \sum_{i=1}^{n} c_i P(x_i)$$

- Remark: $\prod_{\substack{j=1;\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$ is a (n-1)th degree Lagrange basis polynomial using nodes $x_1, \cdots, x_n$.

Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

n	Abscissae (x_i)	Weights (c_i)	Degree of Precision
2	$\sqrt{3}/3$	1.0	3
	$-\sqrt{3}/3$	1.0	
3	0. 7745966692	0.5555555556	5
	0.0	0.8888888889	
	-0.7745966692	0.5555555556	

More in **Table 4.12** of the textbook

Example 1. Approximate $\int_{-1}^{1} e^x dx$ using Gaussian quadrature with n = 3.

Gaussian Quadrature on Arbitrary Intervals

Use substitution or transformation to transform $\int_a^b f(x)dx$ into an integral defined over $[-1,1]$.

Let $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$, with $t \in [-1, 1]$

Then

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\right)\left(\frac{b-a}{2}\right)dt$$

Example 2. Approximate $\int_1^3 \cos x \, dx$ using Gaussian quadrature with $n = 2$.