Section 5.1 Elementary Theory of Initial-Value Problems

- Ordinary differential equations (ODE) describe the change of some variables with respect to another:

$$\frac{dy}{dt} = f(t, y), \qquad \text{for } a \le t \le b$$

$$y(a) = \alpha$$

Definition 5.1. A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset R^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a **Lipschitz constant** for f.

Definition 5.2 A set $D \subset R^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belongs to D and $\lambda \in [0,1]$, the point $((1-\lambda)t_1 + \lambda t_2,$ $(1-\lambda)y_1 + \lambda y_2)$ also belongs to D.

Remark:

1. Convex means that line segment connecting (t_1, y_1) and (t_2, y_2) is in D whenever (t_1, y_1) and (t_2, y_2) belongs to D.

2. The set $D = \{(x, y)\ \ a \le t \le b\ and - \infty \le y \le \infty\}$ is convex.

Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D \subset R^2$. If a constant $L > 0$ exists with $|\frac{\partial f}{\partial y}(t, y)| \leq L$ for all $(t, y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem 5.4 (existence & uniqueness) Suppose that $D = \{(x, y) \ a \leq t \leq b \ and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem (IVP)

$$y' = f(t, y), \qquad a \leq t \leq b, \qquad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Well-posedness

Definition 5.5 The initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is said to be **a well-posed problem** if:

1. There exists a unique solution $y(t)$.
2. Small perturbations in the statement of the problem

$$f(t, y) \dashrightarrow f(t, y) + \delta(t), \qquad \alpha \dashrightarrow \alpha + \delta_0$$

introduce correspondingly small changes in the solution

$$y(t) \dashrightarrow y(t) + \epsilon(t)$$

Why well-posedness? Numerical methods always solve perturbed problem because of, e.g., round-off errors.

Well-posedness

Theorem 5.6 Suppose $D = \{(x, y)\ \ a \ \leq t \leq b\ and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y, then IVP

$$y' = f(t, y), \quad\quad a \ \leq t \ \leq b, \ \ y(a) = \ \beta,$$

is well-posed.

Example 1. Show the IVP $y' = y - t^2 + 1, \ \ 0 \ \leq t \ \leq 2, \ \ y(0) = \ 0.5$ is well-posed on $D = \{(x, y)\ \ 0 \ \leq t \leq 2\ and - \infty < y < \infty\}$

Section 5.2 Euler’s method

Some problems modeled by differential equations

1) Epidemics (spread of a disease in population)

Population in three categories: Susceptible $(S(t))$, Infectious $(I(t))$, Recovered $(R(t))$.

$$\begin{cases} \dfrac{dS(t)}{dt} = -\beta S(t)I(t) \\ \dfrac{dI(t)}{dt} = \beta\, S(t)I(t) - \gamma\, I(t) \\ \dfrac{dR(t)}{dt} = \gamma\, I(t) \end{cases}$$

2) Income and wealth distribution (with wealth a and income z)

$$\max E_0 \int_0^\infty e^{-\rho t} u(c_t)dt \quad s.t.$$

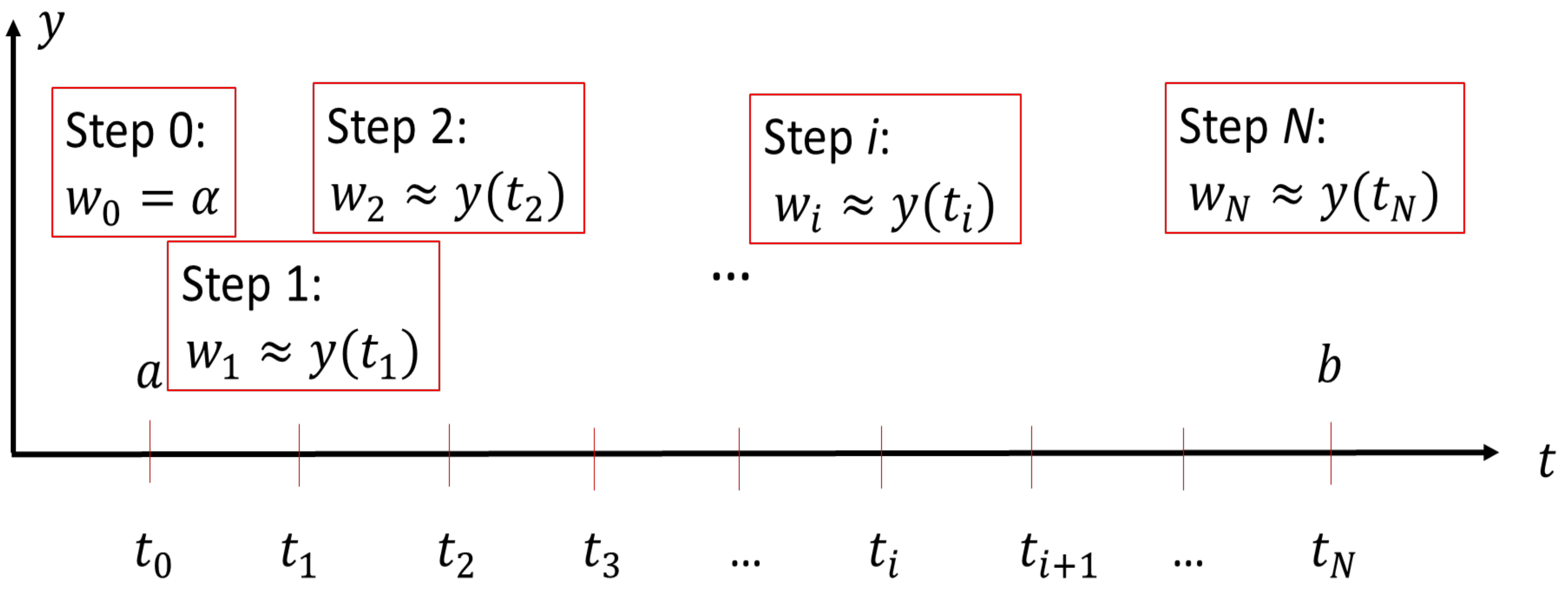
$$da_t = (z_t + r(t)a_t - c_t)dt$$

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$$

$$a_t \geq a.$$

Summary of Problems to Be Solved

- Consider to solve $\begin{cases} \dfrac{dy}{dt} = f(t, y) & \qquad a \le t \le b \\ y(a) = \alpha \end{cases}$
- Choose integer N. Let $h = \frac{b-a}{N}$, and $t_i = a + ih$ with $i = 0,1, \ldots, N$.
  h is called the step size, t_i are called mesh points.
- **We want to compute approximate solutions** $w_0, w_1, w_2, \ldots, w_i, w_{i+1}, \ldots w_N$ step by step



Deriving Euler's method by Taylor's Theorem

1) For each $i = 0,1, \ldots, N$,

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i).$$

Since $\frac{dy}{dt} = f(t, y)$, $y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$

2) Drop $\frac{h^2}{2} y''(\xi_i)$, and let $w_0 = \alpha, w_i \approx y(t_i)$, we obtain:

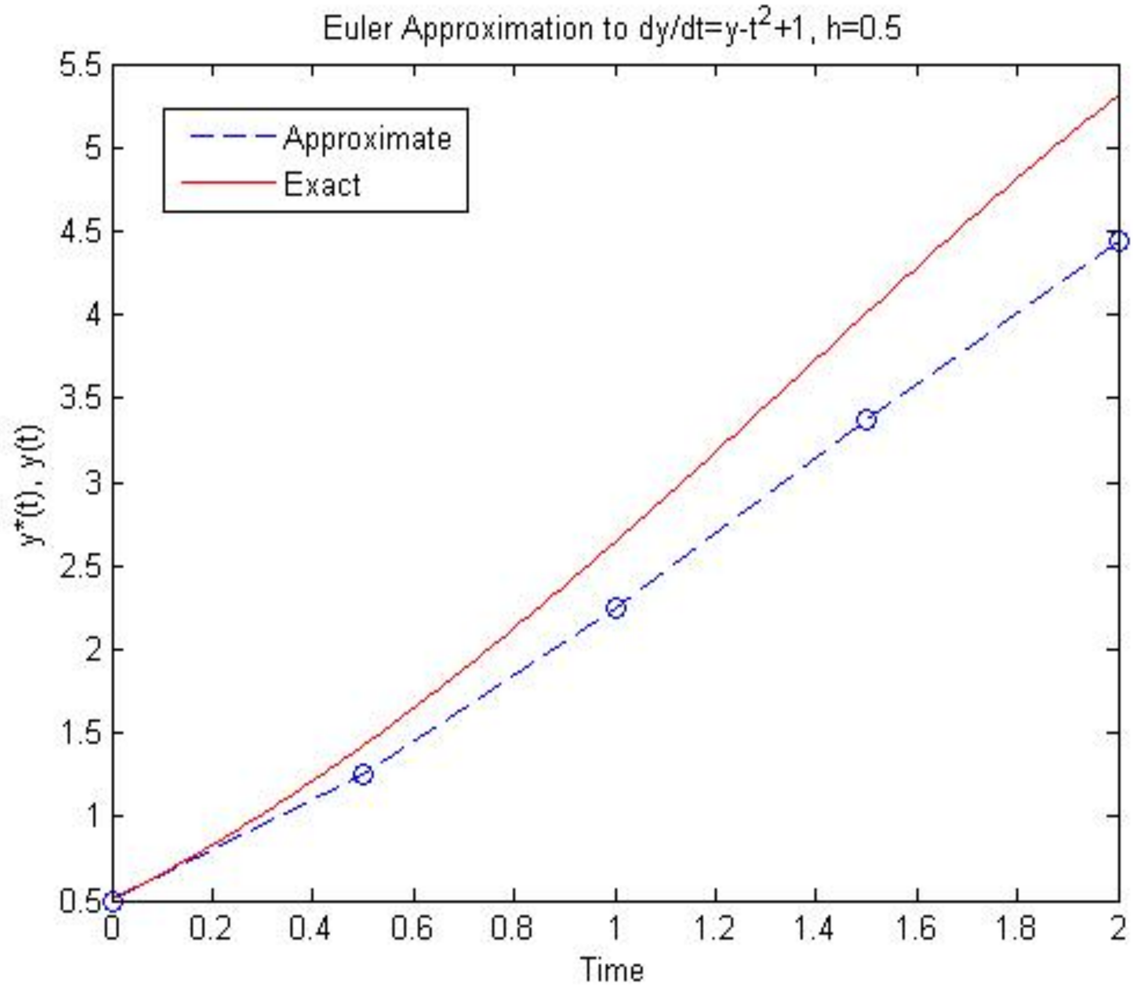
Euler's method:

$w_0 = \alpha$

$w_{i+1} = w_i + hf(t_i, w_i), \quad$ for each $i = 0,1, \ldots, N-1.$

Geometric Interpretation of Euler's Method

$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$ implies $f(t_i, w_i)$ is an approximation to slope of $y(t)$ at t_i.



Example 1. Solve $y' = y - t^2 + 1, \quad 0 \le t \le 2, \ \ y(0) = 0.5$ numerically using Euler's method with time step size $h = 0.5$.

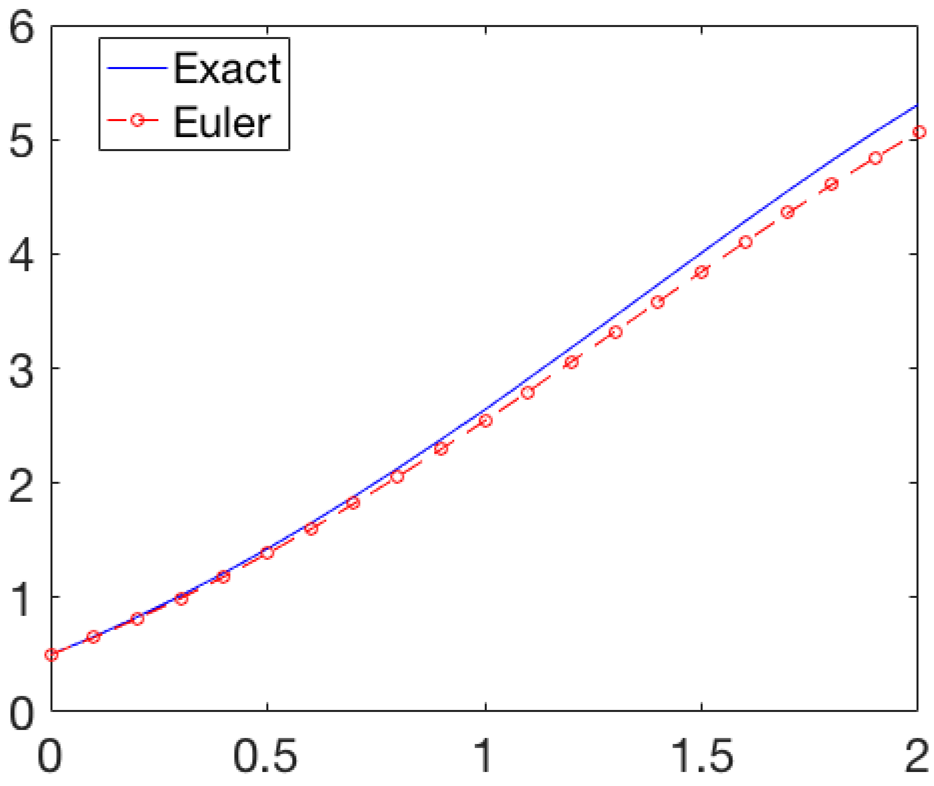
Exact value: $y(t) = (t+1)^2 - 0.5\, e^t$.

(MATLAB) Implement Euler's method for Example 1 using $h = 0.1$.

```matlab
% inputs
f = @(t,y) y - t.^2 +1;
tend = 2;
y0 = 0.5;
h = 0.1;
yex = @(t) (t+1).^2-0.5*exp(t); % exact solution

% Euler's method
tGrid = [0:h:tend];
N = length(tGrid)-1;
wGrid = zeros(1,N+1);
wGrid(1) = y0; % initial data
for i = 1:N
    ti = tGrid(i); wi = wGrid(i);
    wGrid(i+1) = wi + h*f(ti, wi); % Euler update
end

plot(tGrid, yex(tGrid),'b', tGrid, wGrid, 'ro--')
legend('Exact', 'Euler','Location','Best')
set(gca, 'FontSize',24)
shg
```



Error Bound of Euler's Method

Theorem 5.9 Suppose $D = \{(x, y)\ \ a \ \leq t \leq b \; and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L and that a constant M exists with

$$|y''(t)| \leq M, \qquad \text{for all } t \ \in [a, b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y), \qquad a \ \leq t \ \leq b, \qquad y(a) = \ \beta,$$

and $w_0, w_1, \cdots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Example 2 The solution to the IVP $y' = y - t^2 + 1, 0 \le t \le 2, \quad y(0) = 0.5$ was approximated by Euler's method with $h = 0.2$. Find the bound for approximation. Compare the actual error at each step to the error bound.

Table 5.2

t_i	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Actual Error	0.02930	0.06209	0.09854	0.13875	0.18268	0.23013	0.28063	0.33336	0.38702	0.43969
Error Bound	0.03752	0.08334	0.13931	0.20767	0.29117	0.39315	0.51771	0.66985	0.85568	1.08264