Section 5.10 Zero-Stability

Consistency and Convergence: one-step methods

Definition 5.18 A one-step difference method with local truncation error $\tau_i(h)$ is said to be ***consistent*** if

$$\lim_{h \to 0} \max_{1 \le i \le N} |\tau_i(h)| = 0$$

Remark: A method is consistent implies that the difference equation approaches the differential equation as $h \to 0$.

Definition 5.19 A one-step difference method is said to be ***convergent*** if

$$\lim_{h \to 0} \max_{1 \le i \le N} |w_i - y(t_i)| = 0$$

where $y(t_i)$ is the exact solution and w_i is the approximate solution.

Example 1. Consider to solve $y' = f(t,y), \quad a \le t \le b, \;\; y(a) = \alpha$. Let $|y''(t)| \le M$, an $f(t,y)$ be continuous and satisfy a Lipschitz condition with Lipschitz constant L. Show that Euler's method is consistent and convergent.

Solution:

$$|\tau_{i+1}(h)| = |\tfrac{h}{2} y''(\xi_i)| \le \tfrac{h}{2} M$$

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| \le \lim_{h\to 0} \frac{h}{2} M = 0$$

Thus Euler's method is consistent.

By **Theorem 5.9**,

$$\max_{1\le i\le N} |w_i - y(t_i)| \le \frac{Mh}{2L}[e^{L(b-a)} - 1]$$

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| \le \lim_{h\to 0} \frac{Mh}{2L}\left[e^{L(b-a)} - 1\right] = 0$$

Thus Euler's method is convergent.

The rate of convergence of Euler's method is $O(h)$.

- **Stability:** small changes in the initial conditions produce correspondingly small changes in the subsequent approximations.

- The concept of **Zero-stability** guarantee that, in a *fixed bounded interval* $[a, b]$, small perturbations of data yield bounded perturbations of the numerical solution when $h \to 0$.

- The one-step method is **zero-stable** if there is a constant K and a step size $h_0 > 0$ such that the difference between two solutions w_i and $\widetilde{w}_i$ with initial values α and $\tilde{\alpha}$ respectively, satisfies $|w_i - \widetilde{w}_i| < K|\alpha - \tilde{\alpha}|$ whenever $h < h_0$ and $0 \leq i \leq N = \frac{b-a}{h}$.

Theorem 5.20 Suppose the IVP $y' = f(t, y), \quad a \le t \le b, \; y(a) = \alpha$ is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + h\phi(t_i, w_i, h) \qquad \text{where } i = 0, 1, \dots N-1.$$

Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in w with constant L on D,

$D = \{(t, w, h) |\; a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}$. Then:

a) The method is **zero-*stable***;

b) The method is ***convergent*** if and only if it is ***consistent***, which is equivalent to $\phi(t, y, 0) = f(t, y), \qquad \text{for all } a \le t \le b.$

c) If a function τ exists s.t. $|\tau_i(h)| \le \tau(h)$ when $0 \le h \le h_0$, then

$$|w_i - y(t_i)| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

Example 2. Show modified Euler method

$$w_{i+1} = w_i + \frac{h}{2}\big(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\big)$$

is stable and convergent. Suppose $f(t, y)$ satisfied a Lipschitz condition on $\{(t, w)\ |a \le t \le b, \text{and} -\infty < w < \infty\}$ for y variable with Lipschitz constant L, $f(t, y)$ is also continuous.

Consistency and Convergence: multistep methods

Definition. A m-step multistep method is **consistent** if $\lim_{h\to 0} |\tau_i(h)| = 0$, for all $i = m, m+1, \dots, N$ and $\lim_{h\to 0} |\alpha_i - y(t_i)| = 0$, for all $i = 1,2, \dots, m-1$. $\{\alpha_i\}$ are the starting values computed by some one-step method.

Definition. A m-step multistep method is **convergent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0$$

Zero-Stability

Definition. Consider to solve the IVP: $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$. by an m-step multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

The **characteristic polynomial** of the method is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

Definition 5.22 Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of the **characteristic equation**

$$P(\lambda) = 0.$$

If $|\lambda_i| \le 1$ and all roots with absolute value 1 are simple roots, then the difference equation is said to satisfy the **root condition**.

Zero-Stability

Definition 5.23

i. Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly zero-stable**.

ii. Methods that satisfy the root condition and have more than one distinct roots with magnitude one are called **weakly zero-stable**.

iii. Methods that do not satisfy the root condition are called **unstable**.

The fundamental theorem of Numerical Analysis

Theorem 5.24 (Lax-Richtmyer).
Any consistent method is convergent if and only if it is zero-stable.

OR, *consistency plus zero-stability implies convergence.*

Remark: A one-step methods is consistent if and only if it is convergent. [see **Thm 5.20**]

Example 3. Show AB2, AB4, AM2, AM3 methods are strongly zero-stable.

Example 4. Investigate the stability of the difference method

$$w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1})).$$

[See what happens if you implement this method!!!!]