Section 5.3 High-Order Taylor Methods

Local Truncation Error

Consider to solve $\begin{cases} \frac{dy}{dt} = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$

Definition 5.11: The difference method

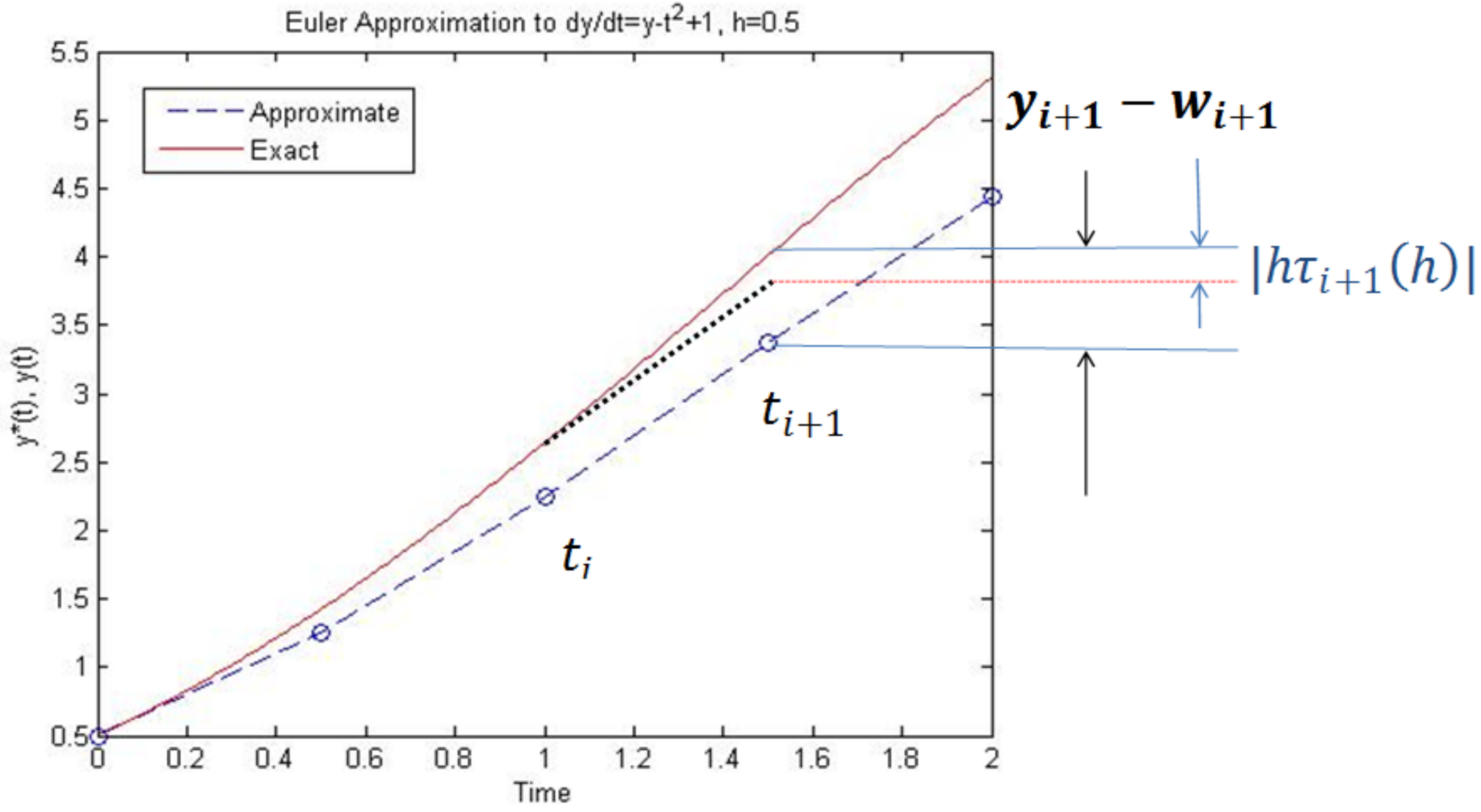
$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\phi(t_i, w_i) \quad \text{for each } i = 0, 1, 2, \cdots, N-1 \end{cases}$$

$with$ step size $h = \frac{b-a}{N}$ has **Local Truncation Error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - \left(y_i + h\phi(t_i, y_i)\right)}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) \quad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

- Remark: $y_i \coloneqq y(t_i)$ and $y_{i+1} \coloneqq y(t_{i+1})$.

Geometric View of Local Truncation Error



Example 1. Analyze the local truncation error of Euler's method for solving $y' = f(t, y), \quad a \le t \le b, \;\; y(a) = \alpha$. Assume $|y''(t)| < M$ with $M > 0$ constant.

Example 2. Consider the IVP

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha.$$

Compute $y'', y^{(3)}$ by using $f(t, y)$ and its derivatives.

Derivation of Higher-Order Taylor Methods

- Consider $\begin{cases} \frac{dy}{dt} = f(t,y), \;\; a \le t \le b \\ \quad y(a) = \alpha \end{cases}$ with step size $h = \frac{b-a}{N}$, $t_{i+1} = a + ih$.
- Expand $y(t)$ in the *n*th Taylor polynomial about t_i, and evaluate $y(t)$ at t_{i+1}

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots$$

$$+ \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

$$y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2}f'\big(t_i, y(t_i)\big) + \cdots$$

$$+ \frac{h^n}{n!}f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)), \qquad \text{for } \xi_i \in (t_i, t_{i+1})$$

- **Denote**

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

- **Taylor method of order *n***

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i) \qquad for\ each\ \ i = 0, 1, 2, \cdots, N-1.$$

Remark: Euler's method is the Taylor method of order one.

Taylor method of order 2

$$w_{i+1} = w_i + hT^{(2)}(t_i, w_i), \text{ where}$$

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$

$$= f(t_i, w_i) + \frac{h}{2}\left(\frac{\partial f}{\partial t}(t_i, w_i) + \frac{\partial f}{\partial y}(t_i, w_i) f(t_i, w_i)\right)$$

Remark:

$$f'(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)\frac{dy}{dt}(t) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) f(t, y)$$

Taylor method of order 3

- $f'(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\frac{dy}{dt}(t) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)f(t,y)$
- $f''(t,y) = \left[\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)f(t,y)\right]'$

$$= \frac{\partial^2 f}{\partial t^2}(t,y) + 2\frac{\partial^2 f}{\partial t \partial y}(t,y)f(t,y) + \frac{\partial^2 f}{\partial y^2}(t,y)[f(t,y)]^2 + \frac{\partial f}{\partial y}(t,y)f'(t,y)$$

$$w_{i+1} = w_i + hT^{(3)}(t_i, w_i), \text{ where}$$

$$T^{(3)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i)$$

Remark: explicit form of higher-order (>3) Taylor method is significantly more involved to derive.

Example 3. Use Taylor method of orders (a) two and (b) three with $N = 2$ to the IVP:

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \qquad y(0) = 0.5.$$

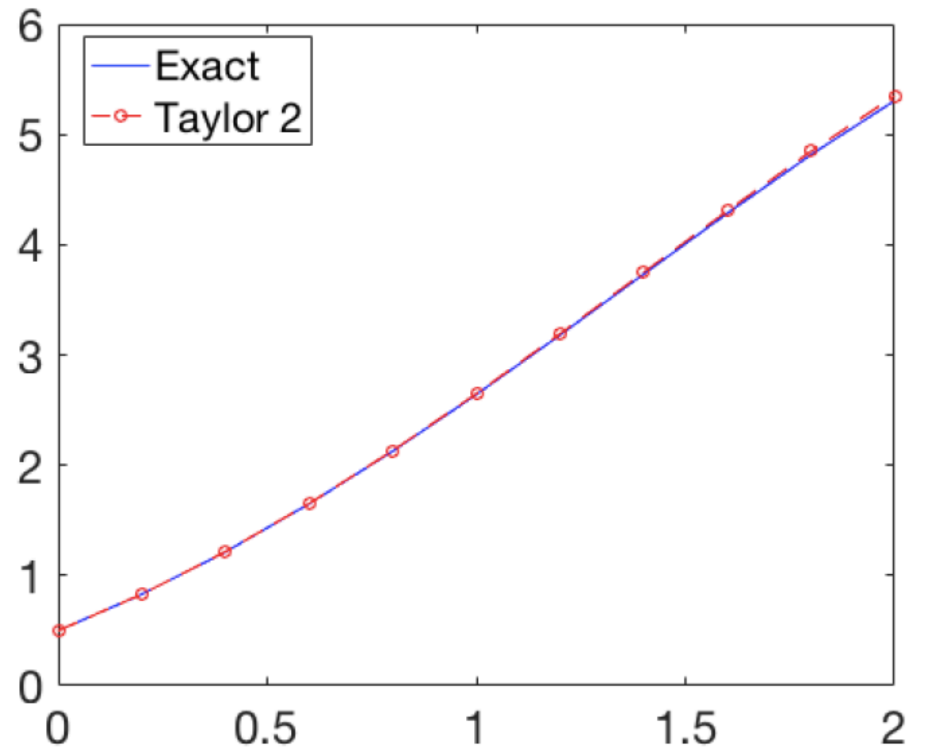
(MATLAB) Implement both methods for Example 3 using $h = 0.2$. Record the maximum error.

taylor2.m

```
% inputs
f = @(t,y) y - t.^2 +1;
dtdf = @(t,y) -2*t;
dydf = @(t,y) 1;

tend = 2;
y0 = 0.5;
h = 0.2;
yex = @(t) (t+1).^2-0.5*exp(t); % exact solution

% Taylor method order 2
tGrid = [0:h:tend];
N = length(tGrid)-1;
wGrid = zeros(1,N+1);
wGrid(1) = y0; % initial data
for i = 1:N
    % DATA at step i
    ti = tGrid(i); wi = wGrid(i);
    fi = f(ti,wi); % function evaluation
    dtdfi = dtdf(ti,wi); % first derivatives
    dydfi = dydf(ti,wi);

    DtDfi = dtdfi+dydfi*fi;        % f'

    t2 = fi+h/2*DtDfi;
    wGrid(i+1) = wi + h*t2; % update
end
```

```
% print the max err
err = max(abs(yex(tGrid)-wGrid));
fprintf('\n max err: %.3e\n', err)

plot(tGrid, yex(tGrid),'b', tGrid, wGrid, 'ro--')
legend('Exact', 'Taylor 2','Location','Best')
set(gca, 'FontSize',24)
shg
```

max err: 4.221e-02

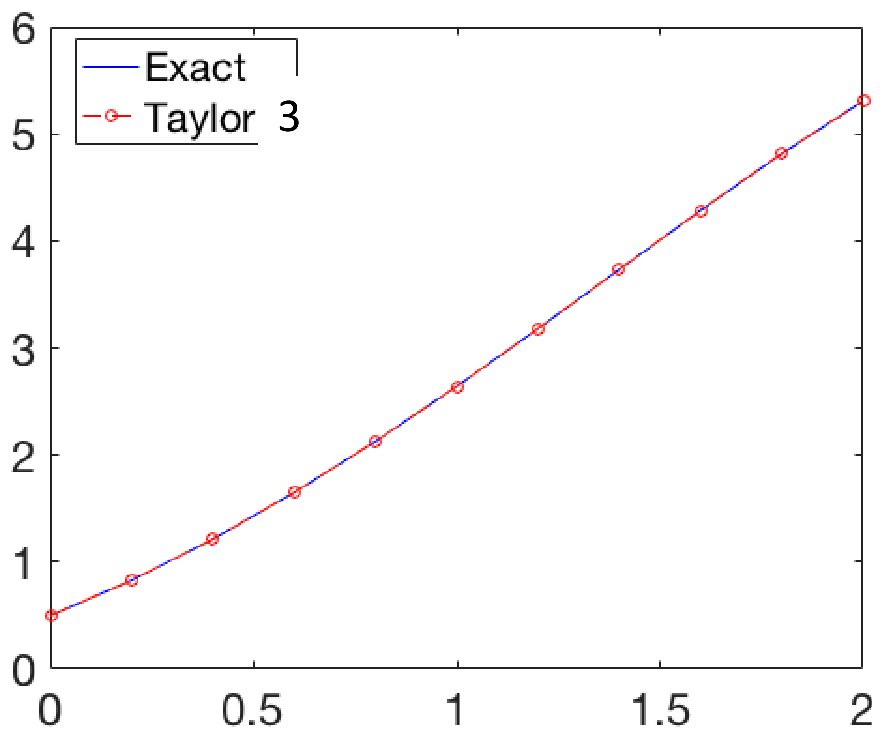


taylor3.m

```
% inputs
f = @(t,y) y - t.^2 +1;
dtdf = @(t,y) -2*t;
dydf = @(t,y) 1;
dttdf = @(t,y) -2;
dtydf = @(t,y) 0;
dyydf = @(t,y) 0;

tend = 2;
y0 = 0.5;
h = 0.2;
yex = @(t) (t+1).^2-0.5*exp(t); % exact solution

% Taylor method order 3
tGrid = [0:h:tend];
N = length(tGrid)-1;
wGrid = zeros(1,N+1);
wGrid(1) = y0; % initial data
for i = 1:N
    % DATA at step i
    ti = tGrid(i); wi = wGrid(i);
    fi = f(ti,wi); % function evaluation
    dtdfi = dtdf(ti,wi); % first derivatives
    dydfi = dydf(ti,wi);
    dttdfi = dttdf(ti,wi);% second derivatives
    dtydfi = dtydf(ti,wi);
    dyydfi = dyydf(ti,wi);

    DtDfi = dtdfi+dydfi*fi;         % f'
    DttDfi = dttdfi+2*dtydfi*fi...
        +dyydfi*fi^2+dydfi*DtDfi;   % f''

    t3 = fi+h/2*DtDfi+h^2/6*DttDfi;
    wGrid(i+1) = wi + h*t3; % update
end
```

```
% print the max err
err = max(abs(yex(tGrid)-wGrid));
fprintf('\n max err: %.3e\n', err)

plot(tGrid, yex(tGrid),'b', tGrid, wGrid, 'ro--')
legend('Exact', 'Taylor 3' ,'Location','Best')
set(gca, 'FontSize',24)
shg
```

max err: 2.099e-03



Error Analysis of Taylor Method

Theorem 5.12 If Taylor method of order *n* is used to approximate the solution to the IVP

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha$$

with step size *h* and if $y \in C^{n+1}[a, b]$, then the **local truncation error** is $O(h^n)$.

- **Remark** for Theorem 5.12:

Since $y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} f'\big(t_i, y(t_i)\big) + \cdots + \frac{h^n}{n!} f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$.

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}\big(\xi_i, y(\xi_i)\big).$$

Assume $y^{(n+1)}(t) = f^{(n)}\big(t, y(t)\big)$ is bounded by $\left|y^{(n+1)}(t)\right| \le M$.

Thus $|\tau_{i+1}(h)| \le \frac{h^n}{(n+1)!} M$.