Section 5.11 Stiff Differential Equations

Stability on unbounded intervals
Motivation

Example 1. The initial-value problem \( y' = -30y, \quad 0 \leq t < \infty, \quad y(0) = 1 \) has exact solution \( y(t) = e^{-30t} \). Use Euler’s method to solve with step size \( h = 1 \).

Solution: Euler’s method

\[
\begin{align*}
  w_1 &= (1 - 30h)w_0 = (1 - 30h) = -29 \\
  w_2 &= (1 - 30h)w_1 = (1 - 30h)^2 = (-29)^2 \\
  &\quad \vdots \\
  w_{i+1} &= (1 - 30h)w_i = (1 - 30h)^{i+1}
\end{align*}
\]

If \( h > \frac{1}{15} \), then \( |1 - 30h| > 1 \), and \( (1 - 30h)^{i+1} \) grows geometrically, in contrast to the true solution which decays to zero.

Facts:

1) A stiff differential equation is numerically unstable unless the step size is extremely small.

2) Stiff differential equations are characterized as those whose exact solution has a term of the form \( e^{-ct} \), where \( c \) is a large positive constant.
Absolute Stability

Definition (The test equation)

\[ y' = \lambda y, \quad 0 \leq t < \infty, \quad y(0) = \alpha, \quad \text{where } \lambda < 0. \]

The test equation has exact solution \( y(t) = \alpha e^{\lambda t} \), which decays to zero as \( t \to \infty \).

Definition (Absolute Stability). Let \( \{w_i\}_{i=0}^{\infty} \) be the sequence generated by a difference method applied to the test equation with a fixed step size \( h \).

a) If the limit \( \lim_{n \to \infty} w_n = 0 \), then the method with step-size \( h \) is absolute stable.

b) If the limit \( \lim_{n \to \infty} w_n \neq 0 \), then the method with step-size \( h \) is not stable.
Example 2. The initial-value problem \( y' = -30y \), \( 0 \leq t < \infty \), \( y(0) = 1 \) has exact solution \( y(t) = e^{-30t} \).

Under what condition on the step size \( h \) ensures the stability of Euler method for this problem.
One-step methods for solving test equation

• The test equation:
  \[ y' = \lambda y, \quad 0 \leq t < \infty, \quad y(0) = \alpha, \quad \text{where } \lambda < 0. \]

• Any one-step method for the test equation can be written as
  \[ w_{j+1} = Q(h\lambda)w_j, \quad j \geq 0, \]
  where \( Q(h\lambda) \) is a function of \( h\lambda \).

Example 3.

a) Find \( Q(h\lambda) \) for Euler method

b) Find \( Q(h\lambda) \) for Taylor method of order \( n \)

c) Find \( Q(h\lambda) \) for RK4.
Region of Stability

Definition 5.25 The region \( R \) of absolute stability for a one-step method is \( R = \{ h\lambda \in C \mid |Q(h\lambda)| < 1 \} \).

Example 3.

a) Draw the stability region of the Euler method.

b) Draw the stability region of the following backward Euler method:

\[
    w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})
\]

Definition A numerical method is said to be A-stable if its region \( R \) of absolute stability contains the entire left half-plane.
The A-stable (implicit) **backward Euler method**.

\[ w_0 = \alpha \]

\[ w_{j+1} = w_j + hf(t_{j+1}, w_{j+1}), \quad \text{for} \quad j \geq 0. \]

- Backward Euler method has \( Q(h\lambda) = \frac{1}{1-h\lambda} \).
- Stability implies \( |\frac{1}{1-h\lambda}| < 1 \).
Example 4. Show the **implicit Trapezoidal method** is A-stable.

\[ w_0 = \alpha \]

\[ w_{j+1} = w_j + \frac{h}{2} \left[ f(t_j, w_j) + f(t_{j+1}, w_{j+1}) \right], \quad \text{for} \quad j \geq 0. \]
Multistep Method for solving test equation

Apply a multistep method to the test equation:
\[ w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} \\
+ h\lambda [b_m w_{i+1} + b_{m-1}w_i + \cdots + b_0w_{i+1-m}], \]

This leads to:
\[ (1 - h\lambda b_m)w_{i+1} - (a_{m-1} + h\lambda b_{m-1})w_i - \cdots - (a_0 + h\lambda b_0)w_{i+1-m} = 0 \]

Define the associated **characteristic polynomial** to this difference equation
\[ Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0). \]
Let \( \beta_1, \beta_2, \ldots, \beta_m \) be the zeros of the characteristic polynomial to the difference equation. Then \( c_1, c_2, \ldots, c_m \) exist with

\[
w_i = \sum_{k=1}^{m} c_k (\beta_k)^i , \quad \text{for} \quad i = 0, \ldots, N
\]

and \( |\beta_k| < 1 \) is required for stability for multistep method.
Region of Stability

**Definition 5.25** The region $R$ of absolute stability for a multistep method, it is

$$R = \{ h\lambda \in C \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda) \}.$$ 

**Remark:** The only $A$-stable multistep method is the implicit Trapezoidal method (also known as the Crank-Nicolson method):

$$w_{j+1} = w_j + \frac{h}{2} [f(t_j, w_j) + f(t_{j+1}, w_{j+1})]$$