Section 5.11 Stiff Differential Equations

Stability on unbounded intervals

Motivation

Example 1. The initial-value problem $y' = -30y, \quad 0 \le t < \infty, \; y(0) = 1$ has exact solution $y(t) = e^{-30t}$. Use Euler's method to solve with step size $h = 1$.

Solution: Euler's method

$$w_1 = (1 - 30h)w_0 = (1 - 30h) = -29$$

$$w_2 = (1 - 30h)w_1 = (1 - 30h)^2 = (-29)^2$$

$$\dots$$

$$w_{i+1} = (1 - 30h)w_i = (1 - 30h)^{i+1}$$

If $h > \frac{1}{15}$, then $|1 - 30h| > 1$, and $(1 - 30h)^{i+1}$ grows geometrically, in contrast to the true solution which decays to zero.

Facts:

1) A stiff differential equation is numerically unstable unless the step size is extremely small.

2) Stiff differential equations are characterized as those whose exact solution has a term of the form e^{-ct}, where c is a large positive constant.

Absolute Stability

Definition (The *test equation*)

$$y' = \lambda y, \qquad 0 \le t < \infty, \qquad y(0) = \alpha, \qquad \text{where } \lambda < 0.$$

The test equation has exact solution $y(t) = \alpha e^{\lambda t}$, which decays to zero $as\ t \to \infty$.

Definition(Absolute Stability). Let $\{w_i\}_{i=0}^{\infty}$ be the sequence generated by a difference method applied to the *test equation* with a fixed step size h.

a) If the limit $\lim_{n\to\infty} w_n = 0$, then the method with step-size h is absolute stable.

b) If the limit $\lim_{n\to\infty} w_n \neq 0$, then the method with step-size h is not stable.

Example 2. The initial-value problem $y' = -30y, \quad 0 \le t < \infty$, $y(0) = 1$ has exact solution $y(t) = e^{-30t}$.

Under what condition on the step size h ensures the stability of Euler method for this problem.

One-step methods for solving test equation

- The test equation:

$$y' = \lambda y, \qquad 0 \le t < \infty, \qquad y(0) = \alpha, \qquad \text{where } \lambda < 0.$$

- Any one-step method for the test equation can be written as

$$w_{j+1} = Q(h\lambda) w_j, \quad j \ge 0,$$

where $Q(h\lambda)$ is a function of $h\lambda$.

Example 3.

a) Find $Q(h\lambda)$ for Euler method
b) Find $Q(h\lambda)$ for Taylor method of order n
c) Find $Q(h\lambda)$ for RK4.

Region of Stability

Definition 5.25 The **region $\boldsymbol{R}$ of absolute stability** for a one-step method is $R = \{h\lambda \in C \mid |Q(h\lambda)| < 1\}$.

Example 3.

a) Draw the stability region of the Euler method.

b) Draw the stability region of the following backward Euler method:

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$$

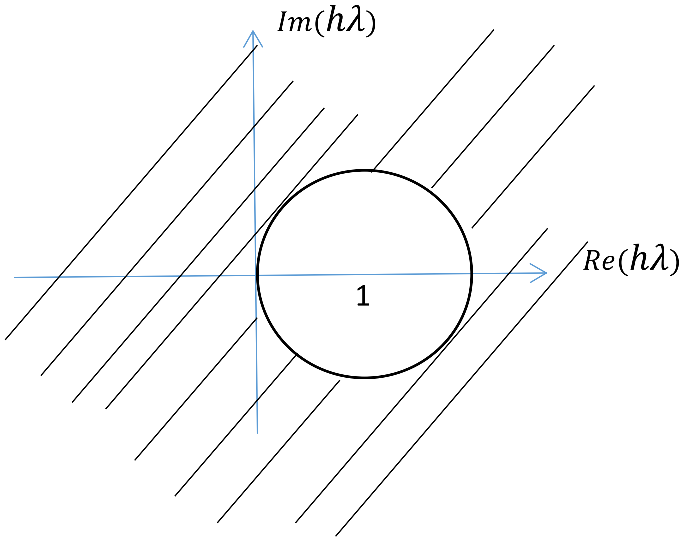
Definition A numerical method is said to be A-stable if its region R of absolute stability contains the entire left half-plane.

The A-stable (implicit) **backward Euler method**.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + hf\left(t_{j+1}, w_{j+1}\right), \qquad \text{for} \quad j \geq 0.$$

- Backward Euler method has $Q(h\lambda) = \frac{1}{1-h\lambda}$.
- Stability implies $|\frac{1}{1-h\lambda}| < 1$.



Example 4. Show the **implicit Trapezoidal method** is A-stable.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + \frac{h}{2}\left[f\left(t_j, w_j\right) + f\left(t_{j+1}, w_{j+1}\right)\right], \qquad \text{for} \quad j \geq 0.$$

Multistep Method for solving test equation

Apply a multistep method to the test equation:

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$+h\lambda[b_m w_{i+1} + b_{m-1}w_i + \cdots + b_0 w_{i+1-m}],$$

This leads to:

$$(1 - h\lambda b_m)w_{i+1} - (a_{m-1} + h\lambda b_{m-1})w_i - \cdots - (a_0 + h\lambda b_0)w_{i+1-m} = 0$$

Define the associated **characteristic polynomial** to this difference equation

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} + h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 + h\lambda b_0).$$

Let $\beta_1, \beta_2, \dots, \beta_m$ be the zeros of the **characteristic polynomial** to the difference equation. Then $c_1, c_2, \dots, c_m$ exist with

$$w_i = \sum_{k=1}^{m} c_k (\beta_k)^i, \qquad \text{for} \quad i = 0, \dots N$$

and $|\beta_k| < 1$ is required for stability for multistep method.

Region of Stability

Definition 5.25 The **region $\boldsymbol{R}$ of absolute stability** for a multistep method,

it is $R = \{h\lambda \in C \mid |\beta_k| < 1, \text{ for all zeros } \beta_k\ of\ Q(z, h\lambda)\}$.

Remark: The only A-stable multistep method is the implicit Trapezoidal method (also known as the Crank-Nicolson method):

$$w_{j+1} = w_j + \frac{h}{2}\left[f(t_j, w_j) + f(t_{j+1}, w_{j+1})\right]$$