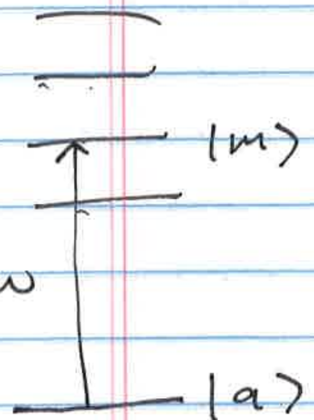
Optics & Spectroscopy — Lecture 15



Prob^y of being in $|m\rangle$

$$|c_m|^2 = \frac{I_0}{4}\frac{|\hat{\epsilon}\cdot\mu|^2}{\hbar^2}\left|\frac{e^{-i(\omega-\omega_{ma})t}-1}{-i(\omega-\omega_{ma})}\right|^2$$

for $E(t) \sim \frac{E_0}{2}\left(e^{-i\omega t}+e^{i\omega t}\right)$

~ multiply out $|c_m|^2$

$$|c_m|^2 = (\quad)\left\{\frac{e^{-i(\omega-\omega_{ma})t}-1}{-i(\omega-\omega_{ma})}\right\}\left\{\frac{e^{i(\omega-\omega_{ma})t}-1}{i(\omega-\omega_{ma})}\right\}$$

$$= (\quad)\frac{1}{(\omega-\omega_{ma})^2}\left(2-2\cos(\omega-\omega_{ma})t\right)$$

$$= \frac{I_0|\hat{\epsilon}\cdot\mu|^2}{\hbar^2}\frac{\sin^2(\omega-\omega_{ma})t/2}{(\omega-\omega_{ma})^2}$$

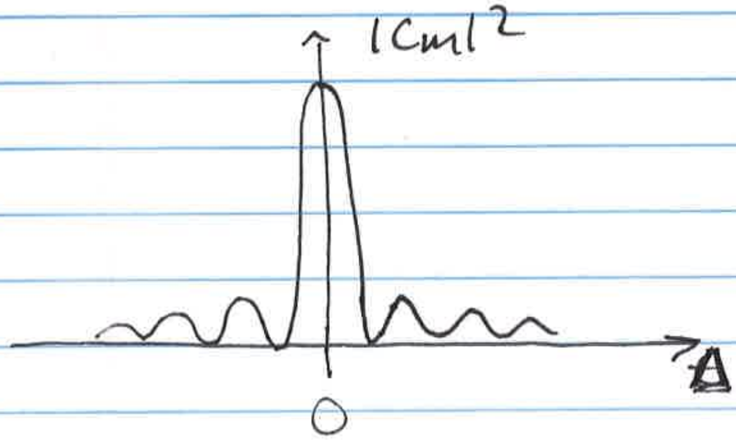
~ write $\omega - \omega_{ma} = \Delta \leftarrow$ "detuning"

$$|C_m|^2 = \frac{|I_0 l|^2}{\hbar^2} \frac{\sin^2 \Delta t/2}{\Delta^2}$$

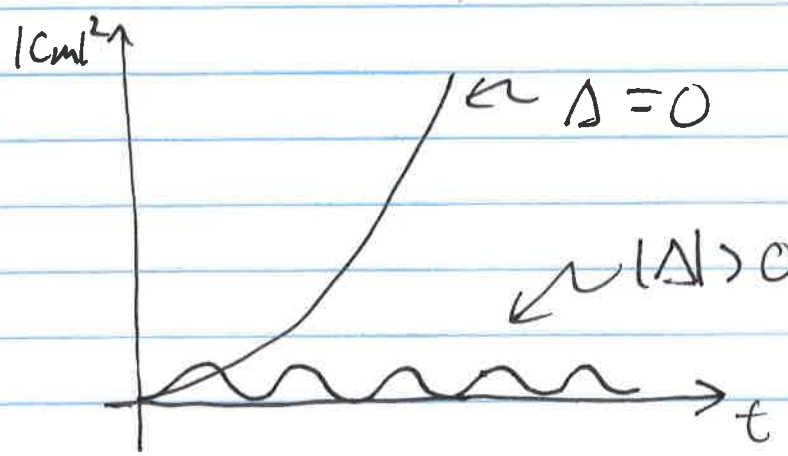
Note: $\Delta \to 0 \quad \sin^2 \Delta t/2 \to (\Delta t/2)^2$

$$\Rightarrow |C_m|^2 = \frac{|I_0 l|^2}{\hbar^2} \times t^2/4$$

Plot $|C_m|^2$ as a f^n of Δ, fixed t



Plot $|C_m|^2$ as a f^n of t, fixed Δ



~ go back to $|C_m|^2$ versus Δ

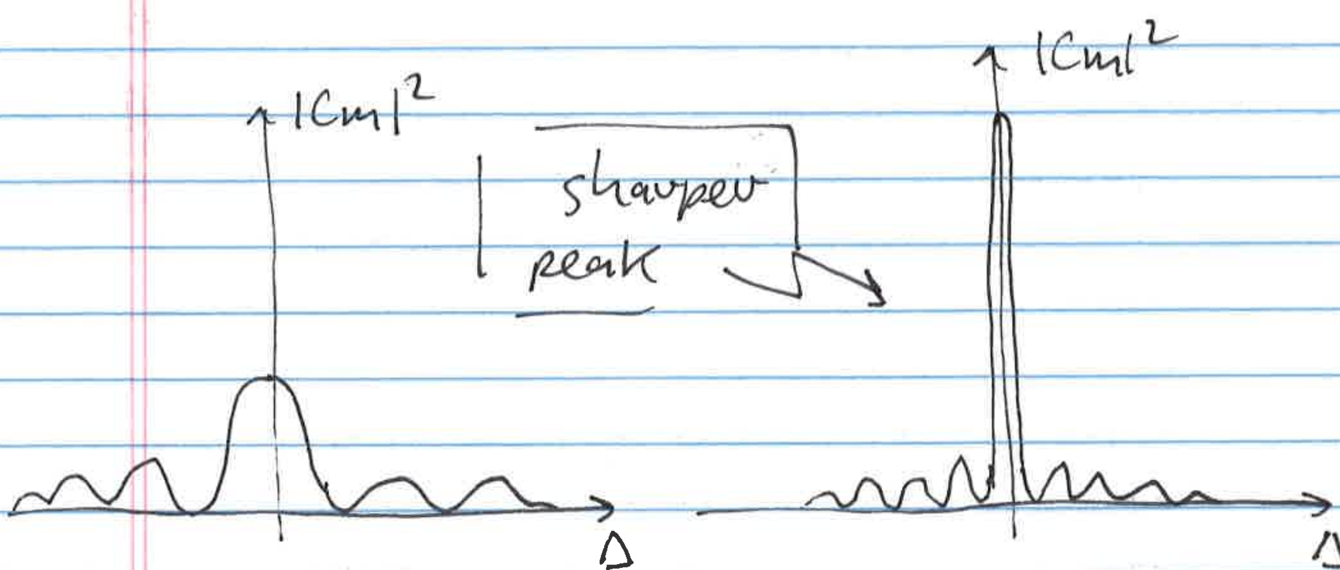
~ what happens when $t \uparrow$?

longer observation times

$\Delta \to 0$ $\quad \frac{\sin^2 \Delta t/2}{\Delta^2} \to t^2/4$ $\quad$ ~ ampl. $\uparrow$

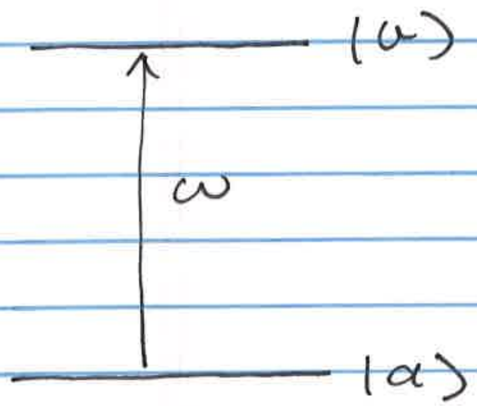
width? $\quad \sin^2 \Delta t/2 = 0$ when $\frac{\Delta t}{2} = \pm \pi$

$\Rightarrow \Delta = \pm 2\pi/t$



small Δ $\qquad$ large Δ

Exact Solution for 2-level System



$$H = H_0 + \mu_{ab} E_0 \cos\omega t$$

$$= H_0 + \frac{\mu_{ab} E_0}{2}\left(e^{-i\omega t} + e^{i\omega t}\right)$$

$$|\psi(t)\rangle = a(t)\, e^{-i\omega_a t}\,|a\rangle + b(t)\, e^{-i\omega_b t}\,|b\rangle$$

time-dependent coefficients

General Result:

$$\frac{dc_m}{dt} = \frac{-i}{\hbar} \sum_n c_n(t)\, e^{-i\omega_{nm} t}\, V_{mn}(t)$$

$$= \frac{-i E_0}{2\hbar} \sum_n c_n(t)\, e^{-i\omega_{nm} t} \times \left(e^{-i\omega t} + e^{i\omega t}\right) \mu_{mn}$$

For 2 levels, this becomes

$$\frac{da(t)}{dt} = \frac{-i}{\hbar} \frac{\mu_{ab} E_0}{2} \left(e^{-i\omega t} + e^{i\omega t}\right) e^{-i\omega_{ba} t} \times b(t)$$

& $$\frac{db(t)}{dt} = \frac{-i}{\hbar} \frac{\mu_{ba} E_0}{2} \left(e^{-i\omega t} + e^{i\omega t}\right) e^{-i\omega_{ab} t} \times a(t)$$

~ use the "Rotating Wave" approximation

$$\frac{da}{dt} = -\frac{i}{\hbar} \frac{\mu_{ab} E_0}{2} e^{i(\omega - \omega_{ba})t} b(t)$$

$$\frac{db}{dt} = -\frac{i}{\hbar} \frac{\mu_{ba} E_0}{2} e^{-i(\omega - \omega_{ba})t} a(t)$$

~ 2 coupled, linear differential equations write as $$\frac{da}{dt} = -\frac{i}{\hbar}\frac{\mu E_0}{2} e^{i\Delta t} b(t)$$

$$\frac{db}{dt} = -\frac{i}{\hbar}\frac{\mu E_0}{2} e^{-i\Delta t} a(t)$$

$$\Delta = \omega - \omega_{ba} \; ; \quad \mu = \mu_{ab} = \mu_{ba}$$

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on resonance: $\Delta = 0$

$$\frac{da}{dt} = -i \frac{\mu E_0}{2\hbar} b(t)$$

$$\frac{db}{dt} = -i \frac{\mu E_0}{2\hbar} a(t)$$

write $\frac{\mu E_0}{\hbar} = K$

$$\Rightarrow b(t) = i \frac{2}{K}\frac{da}{dt}$$

$$\Rightarrow i \frac{2}{\Omega}\frac{d^2 a}{dt^2} = -i \frac{K}{2} a \quad \text{or} \quad \frac{d^2 a}{dt^2} = -\frac{K^2}{4} a$$

Try $\quad a(t) = A e^{iKt/2} + B e^{-iKt/2}$

$$\Rightarrow \quad b(t) = \frac{i2}{K}\left(\frac{iK}{2} A e^{iKt/2} - \frac{iK}{2} B e^{-iKt/2}\right)$$

$$= -A e^{iKt/2} + B e^{-iKt/2}$$

$t = 0$: $\quad a(0) = 1\,, \quad b(0) = 0$

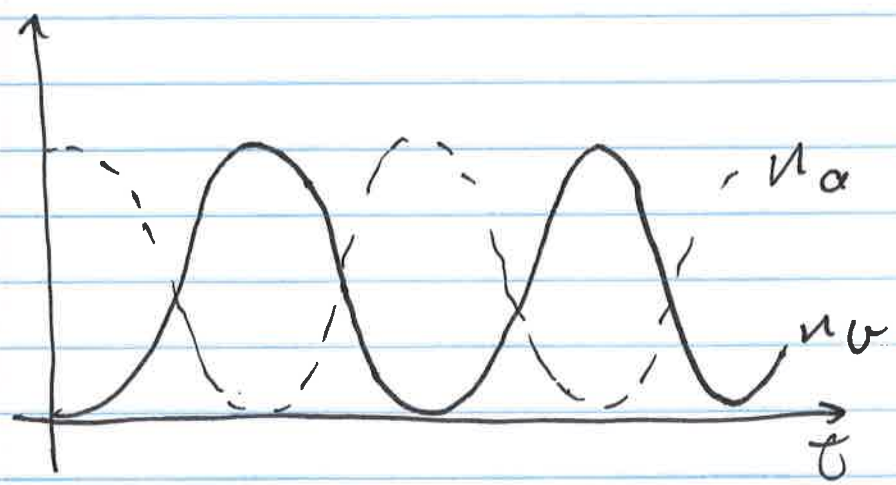
$$\Rightarrow \quad A + B = 1 \quad \text{and} \quad -A + B = 0$$

$$\Rightarrow \quad A = B = 1/2$$

$\leadsto \quad a(t) = \cos Kt/2\,, \qquad b(t) = -i \sin Kt/2$

populations: $\quad n_a = |a(t)|^2 = \cos^2 Kt/2$

$\qquad\qquad\qquad n_b = |b(t)|^2 = \sin^2 Kt/2$



population oscillates b/w $|a\rangle$ and $|b\rangle$

period $= T = 2\pi/K = \frac{2\pi\hbar}{\mu E_0}$

"Rabi Frequency" $\omega = \mu E_0/\hbar$

~ how fast we cycle population b/w $|a\rangle$ and $|b\rangle$ depends on the transition dipole (μ) & the electric field strength (E_0)

Not on resonance?

$$\frac{da}{dt} = -i\frac{K}{2} e^{i\Delta t} b(t) \quad - (1)$$

$$\frac{db}{dt} = -i\frac{K}{2} e^{-i\Delta t} a(t) \quad - (2)$$

from (2) $a(t) = \frac{i2}{K} e^{i\Delta t} \frac{db}{dt}$

~ substitute into (1)

$$\frac{i2}{\kappa} i\Delta e^{i\Delta t} \frac{db}{dt} + \frac{i2}{\kappa} e^{i\Delta t} \frac{d^2b}{dt^2} = -i\frac{\kappa}{2} e^{i\Delta t} b$$

$$\Rightarrow \frac{d^2b}{dt^2} + i\Delta \frac{db}{dt} + \frac{\kappa^2}{4} b = 0$$

~ solve this w/ the initial conditions that $b(0) = 0$ and $\left.\frac{db}{dt}\right|_{t=0} = -i\frac{\kappa}{2}$

Use Mathematica:

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In[6]:= DSolve[{b''[x] + i Δ b'[x] + κ²/4 b[x] == 0, b'[0] == -i κ/2, b[0] == 0}, b, x]
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$$\text{Out[6]= } \left\{\left\{b \to \text{Function}\left[\{x\}, \frac{i\left(e^{\frac{1}{2}x\left(-i\Delta-\sqrt{-\Delta^2-\kappa^2}\right)} - e^{\frac{1}{2}x\left(-i\Delta+\sqrt{-\Delta^2-\kappa^2}\right)}\right)\kappa}{2\sqrt{-\Delta^2-\kappa^2}}\right]\right\}\right\}$$

$$\Rightarrow b(t) = \frac{\kappa}{2\sqrt{\Delta^2+\kappa^2}} e^{-i\Delta t/2} \left\{ e^{-i\sqrt{\Delta^2+\kappa^2}\,t/2} - e^{i\sqrt{\Delta^2+\kappa^2}\,t/2} \right\}$$

~ rewrite w/ $\Omega = \sqrt{\Delta^2 + K^2}$

$$\Rightarrow b(t) = \frac{K}{2\Omega} e^{-i\Delta t/2} \left(e^{-i\Omega t/2} - e^{i\Omega t/2} \right)$$

$$= \frac{-iK}{\Omega} e^{-i\Delta t/2} \sin \Omega t/2$$

$$a(t) = \frac{i2}{K} e^{i\Delta t} \frac{db}{dt}$$

$$= \frac{i2}{K} e^{i\Delta t} \left[\frac{-iK}{\Omega} \left\{ \frac{-i\Delta}{2} e^{-i\Delta t/2} \sin \Omega t/2 + \frac{\Omega}{2} e^{-i\Delta t/2} \cos \Omega t/2 \right\} \right]$$

$$= \frac{2}{\Omega} e^{i\Delta t/2} \left\{ \frac{-i\Delta}{2} \sin \Omega t/2 + \frac{\Omega}{2} \cos \Omega t/2 \right\}$$

Populations: $n_b = |b(t)|^2 = \frac{K^2}{\Omega^2} \sin^2 \Omega t/2$

$$n_a = |a(t)|^2 = \frac{4}{\Omega^2}\left(\frac{\Omega}{2}\cos\frac{\Omega t}{2} - \frac{i\Delta}{2}\sin\frac{\Omega t}{2}\right)$$

$$\times\left(\frac{\Omega}{2}\cos\frac{\Omega t}{2} + \frac{i\Delta}{2}\sin\frac{\Omega t}{2}\right)$$

$$= \frac{4}{\Omega^2}\left(\frac{\Omega^2}{4}\cos^2\frac{\Omega t}{2} + \frac{\Delta^2}{4}\sin^2\frac{\Omega t}{2}\right)$$

$$= \cos^2\frac{\Omega t}{2} + \frac{\Delta^2}{\Omega^2}\sin^2\frac{\Omega t}{2}$$

$$= \frac{\Delta^2}{\Omega^2} + \left(1 - \frac{\Delta^2}{\Omega^2}\right)\cos^2\frac{\Omega t}{2}$$

$$= \frac{\Delta^2}{\Omega^2} + \frac{(\Omega^2 - \Delta^2)}{\Omega^2}\cos^2\Omega t/2$$

$$= \frac{\Delta^2}{\Omega^2} + \frac{K^2}{\Omega^2}\cos^2\Omega t/2$$

Note: $$n_a + n_b = \frac{\Delta^2}{\Omega^2} + \frac{K^2}{\Omega^2}\cos^2\frac{\Omega t}{2} + \frac{K^2}{\Omega^2}\sin^2\frac{\Omega t}{2}$$

$$= \frac{\Delta^2}{\Omega^2} + \frac{K^2}{\Omega^2} = 1$$