lecture 3

Optics Course

Beam Optics

~ the spherical waves introduced in the previous lecture are a good way of thinking about scattering from a point dipole

~ not so great for a focussed laser beam...

Fresnel approximation for spherical wave

$$E(r) \simeq \frac{\mathcal{E}_0}{z} \exp\left(-ik\frac{x^2+y^2}{2z}\right) e^{-ikz}$$

This is appropriate for large z, where the amplitude

$$A(\vec{r}) = \frac{\mathcal{E}_0}{z} e^{-ik\frac{x^2+y^2}{2z}}$$ is varying slowly w/ z

~ waves of the form

$$\mathcal{E}(\vec{r}) = A(\vec{r})\, e^{-ikz}$$

where $A(\vec{r})$ varies slowly w/ z are called "Paraxial waves"

~ they obey the "paraxial Helmholtz" eqn.

Helmholtz Equation: $\nabla^2 \mathcal{E}(\vec{r}) + k^2 \mathcal{E}(\vec{r}) = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

~ consider $\displaystyle \frac{\partial^2}{\partial z^2} \mathcal{E}(r) = \frac{\partial^2}{\partial z^2}\left(A(r)\, e^{-ikz} \right)$

$$= \frac{\partial}{\partial z}\left(\frac{\partial A}{\partial z} e^{-ikz} - ikAe^{-ikz} \right)$$

$$= \frac{\partial^2 A}{\partial z^2} e^{-ikz} - 2ik \frac{\partial A}{\partial z} e^{-ikz} - k^2 A e^{-ikz}$$

~ if A is slowly varying w.r.t z, then we can neglect the $\partial^2 A/\partial z^2$ term

Helmholtz Eqn becomes:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A(\vec{r}) e^{-ikz} - 2ik \frac{\partial A}{\partial z} e^{-ikz} - k^2 A e^{-ikz} + k^2 A e^{-ikz} = 0$$

$$\Rightarrow \boxed{\nabla_T^2 A(\vec{r}) - 2ik \frac{\partial A(\vec{r})}{\partial z} = 0}$$

where we have defined $\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

"transverse Laplacian"

Fresnel approx to spherical wave:

$$A(\vec{r}) = \frac{E_0}{z} \exp\left(-ik \frac{x^2+y^2}{2z}\right)$$

~ the amplitude $A(\vec{r}) = \frac{\varepsilon_0}{z} \exp\left(-ik \frac{x^2+y^2}{2z}\right)$

solves the paraxial Helmholtz Eqn.

~ prove using Mathematica...

~ define function F[x, y, z]

$\frac{\partial F}{\partial x} \equiv$ D[F[x, y, z], x]

$\frac{\partial^2 F}{\partial x^2} \equiv$ D[D[F[x, y, z], x], x]

In[1]:= $F[x\_, y\_, z\_] := \frac{B}{z} \mathrm{Exp}\left[-\frac{i k}{2 z}(x^2+y^2)\right]$

D[D[F[x, y, z], x], x] + D[D[F[x, y, z], y], y] - i 2 k D[F[x, y, z], z]

Out[2]= $-2 i k \left(\frac{i B e^{-\frac{i k (x^2+y^2)}{2 z}} k (x^2+y^2)}{2 z^3} - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}}}{z^2}\right) - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}} k^2 x^2}{z^3} - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}} k^2 y^2}{z^3} - \frac{2 i B e^{-\frac{i k (x^2+y^2)}{2 z}} k}{z^2}$

In[3]:= FullSimplify$\left[-2 i k \left(\frac{i B e^{-\frac{i k (x^2+y^2)}{2 z}} k (x^2+y^2)}{2 z^3} - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}}}{z^2}\right) - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}} k^2 x^2}{z^3} - \frac{B e^{-\frac{i k (x^2+y^2)}{2 z}} k^2 y^2}{z^3} - \frac{2 i B e^{-\frac{i k (x^2+y^2)}{2 z}} k}{z^2}\right]$

Out[3]= 0

~ more generally, waves of the form

$$A(\vec{r}) = \frac{\mathcal{E}_0}{q(z)} \exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)$$

w/ $q(z) = z + i z_0$

solve the paraxial H.E.

~ this equation defines the "Gaussian Beam"

where z_0 = "Rayleigh Range"

~ more common form:

$$E(\vec{r}) = \mathcal{E}_0 \frac{w_0}{w(z)} \exp\left(-\frac{\rho^2}{w(z)^2}\right) e^{-ikz - ik\frac{\rho^2}{2R(z)} + i\zeta(z)}$$

important parameters: $w_0 = (\lambda z_0/\pi)^{1/2}$

$w(z) = w_0\left(1 + (z/z_0)^2\right)^{1/2}$

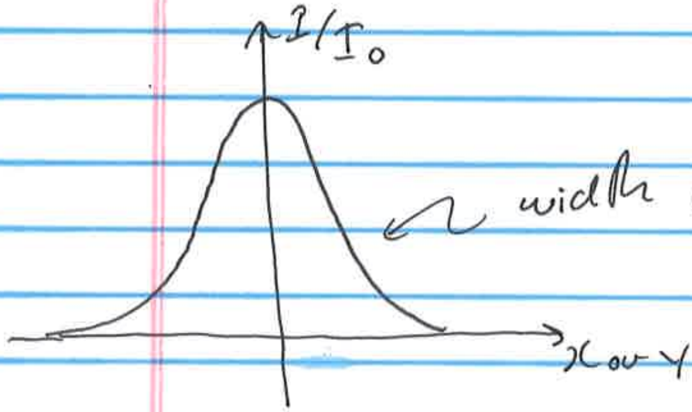
and $\rho = x^2 + y^2$

Properties of Gaussian Beam

intensity $I(\vec{r}) = |E(\vec{r})|^2$ is given by

$$I(\rho, z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2\rho^2}{w^2(z)}\right]$$

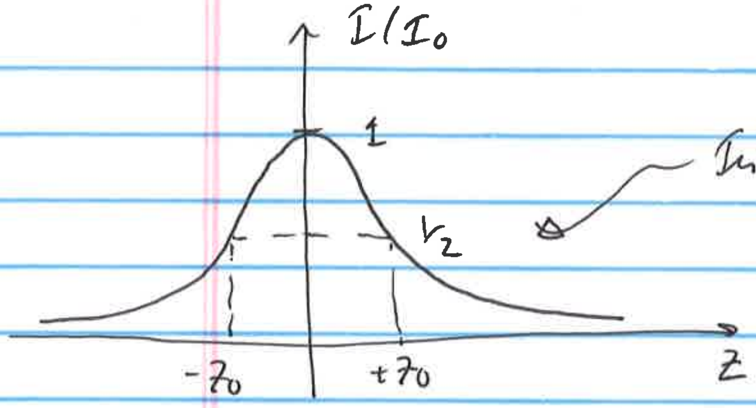
for any specific value of z the intensity distⁿ in (x, y) is a Gaussian fⁿ.



width changes w/ z

~ look at the intensity @ the center ($\rho = 0$)

$$I(0, z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 = \frac{I_0}{1 + (z/z_0)^2}$$



Intensity drops to $I_0/2$
at $z = \pm z_0$

~ more useful to describe beams through power

$$P = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, I(\rho, z) = \int_0^{\infty} I(\rho, z) \, 2\pi \rho \, d\rho$$

$$= I_0 \left(\pi w_0^2 / 2 \right)$$ independent of z

Note: $I_0 = 2P / \pi w_0^2$, thus, in terms of P

$$I(\rho, z) = \frac{2P}{\pi w^2(z)} \exp\left(-\frac{2\rho^2}{w^2(z)} \right)$$

at $z = 0$

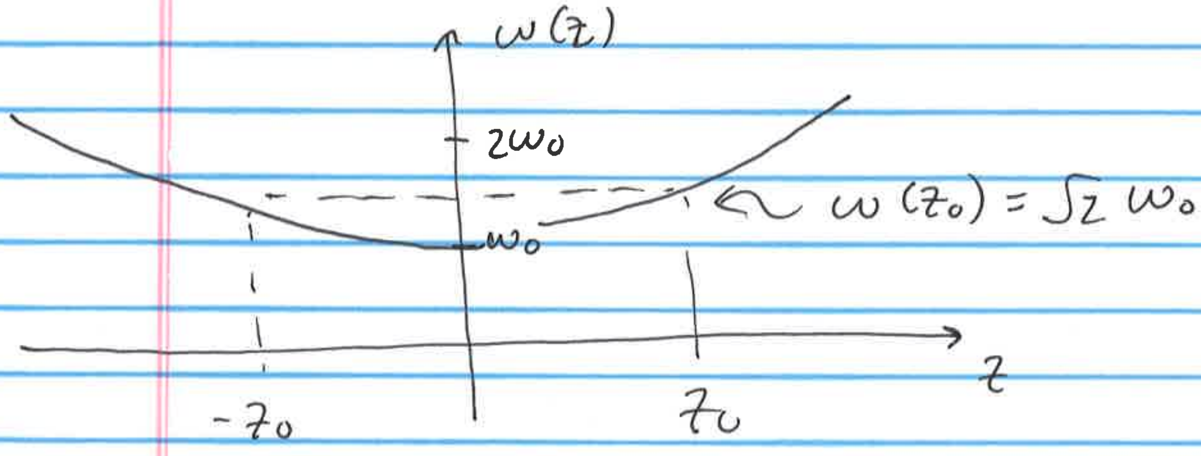
$$I(\rho, 0) = \frac{2P}{\pi w_0^2} \exp\left(-\frac{2\rho^2}{w_0^2} \right)$$

~ $w(z)$ characterizes the size of the beam

~ about 86% of the power of the beam is contained with a circle w/ radius $\rho = w(z)$

~ minimum beam waist is at $z = 0$

~ plot $w(z) = w_0\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2}$



~ note at large values of z $(z \gg z_0)$

$$w(z) \simeq \frac{w_0 z}{z_0}$$

$\Rightarrow$ beam radius (size) increases linearly w/ z

~ the divergence of the beam, how quickly it expands, is defined as

$$\theta_0 = w_0 / z_0$$

note, $w_0 = \left(\frac{\lambda z_0}{\pi}\right)^{1/2} \Rightarrow z_0 = \pi w_0^2 / \lambda$

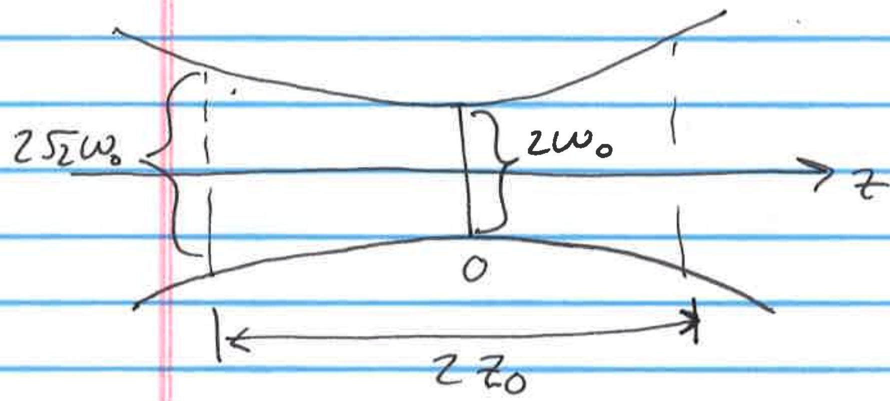
$$\Rightarrow \theta_0 = \lambda / \pi w_0$$

~ this brings out an important ~~the~~ point about focussed beams

~ tightly focussed beams (small w_0) with long wavelengths diverge rapidly

~ conversely, to achieve a tight focus (small w_0) you need to create a strongly converging beam

~ look at the focus of a Gaussian Beam



↜ think of the beam as being focussed over a range of $2z_0$

i.e. $2\times$ Rayleigh range

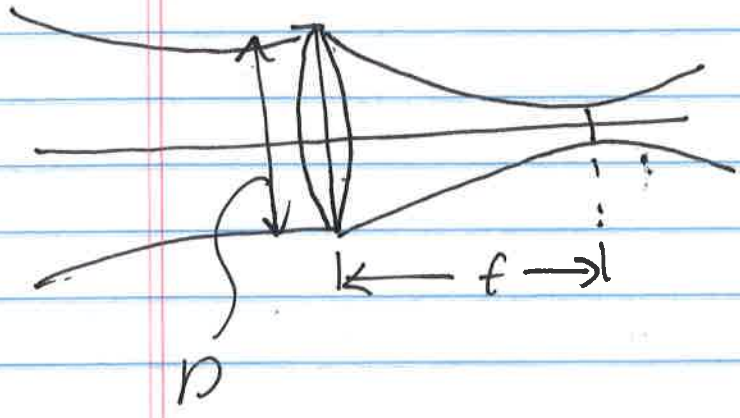
i.e. beam is focussed over

$$2z_0 = \frac{2\pi w_0^2}{\lambda}$$

↜ if w_0 is small
$\Rightarrow$ beam is only focussed over a short range

~ lets look at (approximately) the effect of a lens on a Gaussian Beam

~ assume that the focal length is $\gg 2z_0$ (always the case...)



$$w(z) = w_0\left(1+\left(z/z_0\right)^2\right)^{\frac{1}{2}}$$

at $z = f$ $\quad w(f) = D/2 = w_0\left(1+\left(f/z_0\right)^2\right)^{1/2}$

for $f \gg z_0$ $\quad D/2 \simeq w_0 f/z_0$

now $w_0^2 = \dfrac{\lambda z_0}{\pi}$ $\quad \Rightarrow \quad D/2 \simeq \dfrac{w_0 f}{(\pi w_0^2/\lambda)}$

$$D/2 \simeq \frac{\lambda f}{\pi w_0}$$

~ rearrange for w_0: $\quad w_0 \simeq \dfrac{2\lambda f}{\pi D}$

$\Rightarrow$ at the focus the "spot size"

$$2w_0 \simeq \frac{4\lambda}{\pi} F_{\#} \qquad F_{\#} = f/D$$

F# related to the "numerical aperture"

by $F\# \simeq 1/2NA$

$$\Rightarrow 2w_0 \simeq \frac{2\lambda}{\pi} \times 1/NA$$

~ best objectives we can get have $NA = 1.4$

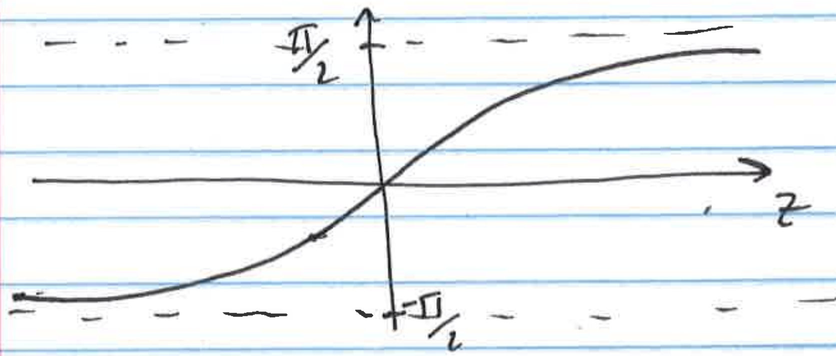
$$\Rightarrow 2w_0 \simeq \frac{2\lambda}{1.4\pi} = \lambda/2.2$$

~ this is close to the common definition of the resolving power of a microscope, which is $\lambda/2NA$

~ the last point to look at for a Gaussian Beam is how the phase changes through the focus

~ on axis ($\rho = 0$) the phase is given by the $e^{i\zeta(z)}$ term where

$$\zeta(z) = \tan^{-1} z/z_0$$

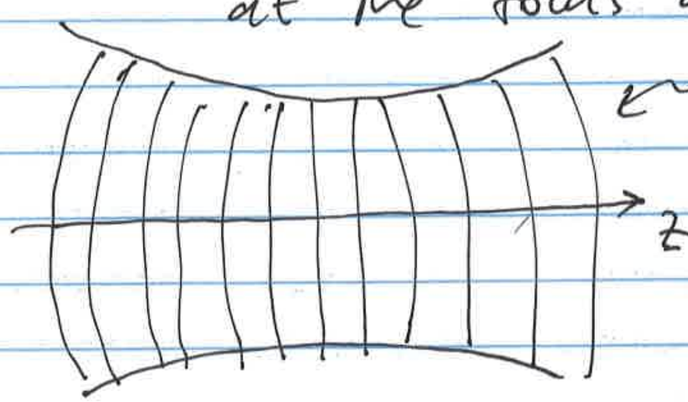


phase changes sign through the focus

~ this is called the "Gouy Effect"

~ it arises from an unusual property of Gaussian beams

go from a paraboloidal wave at large z to a plane wave at the focus and back again



curved wavefronts at large z