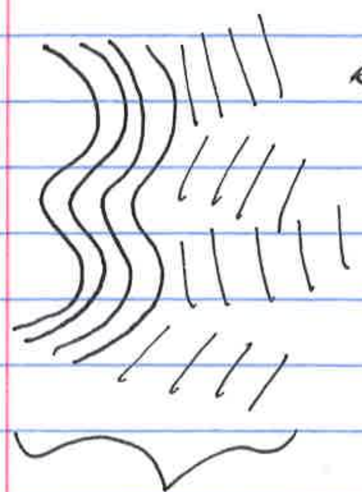
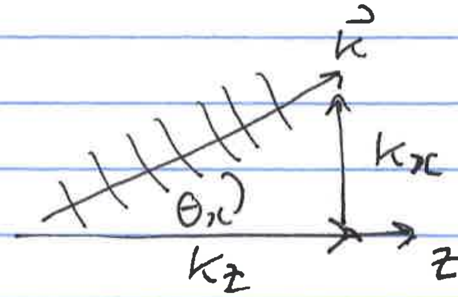
Optical Fourier Transform

$u(x, y, 0) = f(x, y)$





$$\theta_x = \sin^{-1} k_x/k$$

sum of harmonic waves $\left(e^{-i2\pi(\nu_x x + \nu_y y)} \right)$
w/ different spatial frequencies

~ waves travel at angles

$$\theta_x = \sin^{-1} k_x/k \simeq k_x/k = \lambda \nu_x$$

$$\theta_y = \sin^{-1} k_y/k \simeq k_y/k = \lambda \nu_y$$

~ amplitudes of diff$^{\underline{t}}$ spatial frequencies
are given by $F(\nu_x, \nu_y)$

$$\Rightarrow f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu_x, \nu_y)\, e^{-i2\pi(\nu_x x + \nu_y y)}\, d\nu_x d\nu_y$$

F.T. pair

~ find $F(\nu_x, \nu_y)$ if we know $f(x,y)$ and vice-versa

~ consider how the waves propagate

$$g(x,y) = h_0 \iint_{-\infty}^{\infty} f(x',y')\, e^{-\frac{i\pi}{\lambda d}\left((x-x')^2+(y-y')^2\right)} dx'dy'$$

field @ distance d from the input plane

where $h_0 = \frac{i}{\lambda d} e^{-ikd}$

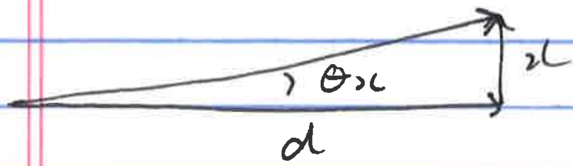
Fresnel approximation

"Fraunhofer" approximation:

$$\frac{\pi}{\lambda d}\left((x-x')^2 + (y-y')^2\right) = \frac{\pi}{\lambda d}\left(x^2+y^2+x'^2+y'^2 - 2(xx'+yy')\right)$$

~ for small angles (paraxial rays) we can ignore x'^2 and y'^2 terms

$$g(x,y) = h_0\, e^{-\frac{i\pi}{\lambda d}(x^2+y^2)} \times \iint f(x',y')\, e^{i2\pi\left(\frac{x}{\lambda d}x' + \frac{y}{\lambda d}y'\right)} dx'\,dy'$$



$$\theta_x = \frac{x}{d} = \lambda \nu_x \quad \Rightarrow \quad \nu_x = \frac{x}{\lambda d}$$

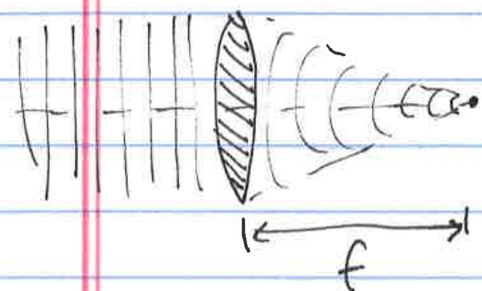
$$g(x,y) = h_0\, e^{-\frac{i\pi}{\lambda d}(x^2+y^2)} \times \iint_{-\infty}^{\infty} f(x',y')\, e^{i2\pi(\nu_x x' + \nu_y y')}\, dx'dy'$$

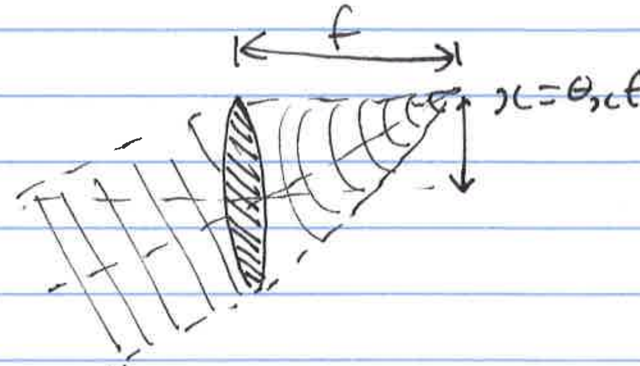
$$\boxed{g(x,y) \simeq h_0\, F(\nu_x, \nu_y)}$$ ← dropped the $e^{-\frac{i\pi}{\lambda d}(x^2+y^2)}$ ~ which is not impt. for paraxial rays

⇒ field pattern in the far field is the Fourier Transform of the field at $z=0$ $(f(x,y))$

~ we can separate the different spatial frequencies by letting the fields propagate over large distances, or by using a lens

Basic Idea:





off-axis ray

~ focussed at pt

$x = \theta_x f = \nu_x \lambda f$

~ to be more rigorous we need to determine the transfer function of a lens

~ consider the function $e^{i\pi x^2/\lambda f}$

this creates a phase shift in the beam $\phi(x) = \frac{-x^2}{2\lambda f}$

$$\left(e^{i\pi x^2/\lambda f} \equiv e^{-i2\pi\phi(x)} \right)$$

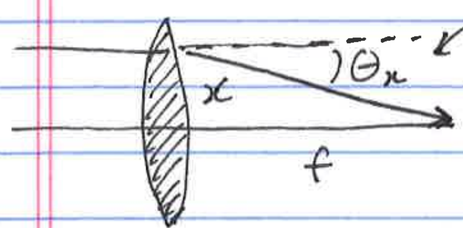
Now, ... $e^{-i2\pi\phi(x)} = e^{-i2\pi\left(\phi_0 + (x - x_0)\frac{\partial\phi}{\partial x} + \dots\right)}$

$$= e^{-i2\pi\left(\phi_0 - x_0\frac{\partial\phi}{\partial x}\right)} e^{-i2\pi x\partial\phi/\partial x}$$

~ so, a phase function $\phi(x)$ that varies in space introduces a spatial frequency

$$\nu_x = \frac{\partial \phi}{\partial x}$$ ← this depends on x in our case

for $\phi(x) = \frac{-x^2}{2\lambda f}$ $\quad \frac{\partial \phi}{\partial x} = -x/\lambda f$



← $\theta_x = \lambda \nu_x = -x/f$

$\Rightarrow e^{i\pi x^2/\lambda f}$ describes a cylindrical lens w/ a focal length f

Likewise, the transfer function for a spherical lens is given by

$$e^{i\pi(x^2+y^2)/\lambda f}$$

~ now, we consider the effect of a spherical lens on a plane wave

$$f(x,y) = A e^{-i2\pi(\nu_x x + \nu_y y)}$$

$$g(x,y) = e^{i\pi(x^2+y^2)/\lambda f} \; A e^{-i2\pi(\nu_x x + \nu_y y)}$$

$$= A e^{i\pi\left(\frac{x^2}{\lambda f} - 2\nu_x x + \dots\right)}$$

↑ y terms (same)

$$\frac{x^2}{\lambda f} - 2\nu_x x = \frac{(x^2 - 2\lambda f \nu_x x)}{\lambda f}$$

$$= \frac{(x - \lambda f \nu_x)^2 - (\lambda f \nu_x)^2}{\lambda f}$$

write $x_0 = \lambda f \nu_x$; $y_0 = \lambda f \nu_y$

$$\Rightarrow g(x,y) = A e^{\frac{i\pi}{\lambda f}\left((x-x_0)^2 - x_0^2 + (y-y_0)^2 - y_0^2\right)}$$

$$= A e^{-\frac{i\pi}{\lambda f}(x_0^2 + y_0^2)} \times \underbrace{e^{\frac{i\pi}{\lambda f}\left((x-x_0)^2 + (y-y_0)^2\right)}}_{\text{Paraboloid wave that focusses at } (x_0, y_0, f)}$$

$$(x_0, y_0, f) = (\lambda \nu_x f, \lambda \nu_y f, f)$$

$$= \left(\frac{k_x}{k} f, \frac{k_y}{k} f, f\right)$$

⇒ lens creates a spatial Fourier transform at its focus

Example: spatial filtering.

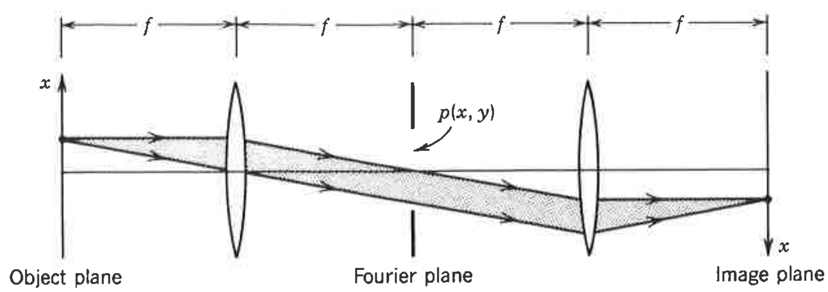
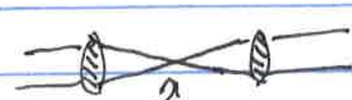


Figure 4.4-3 The 4–f imaging system. If an inverted coordinate system is used in the image plane, the magnification is unity.

"4-f" lens system

for laser beams this looks like



focus @ f

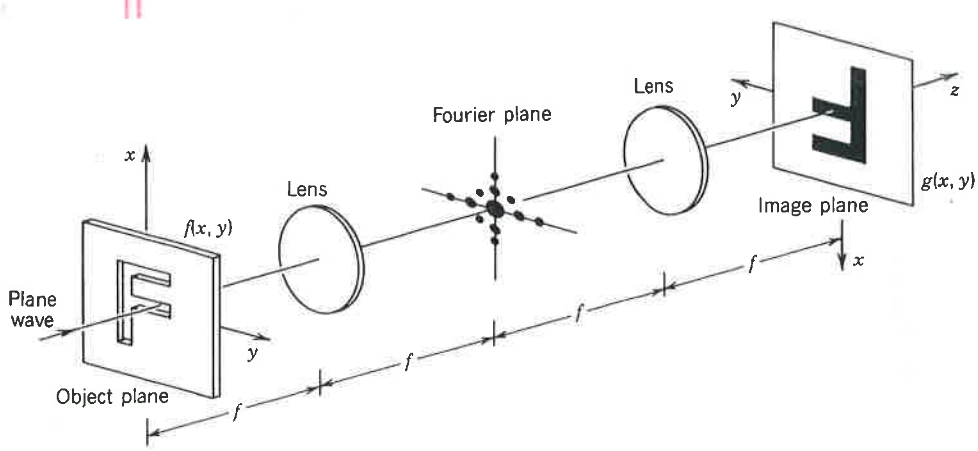
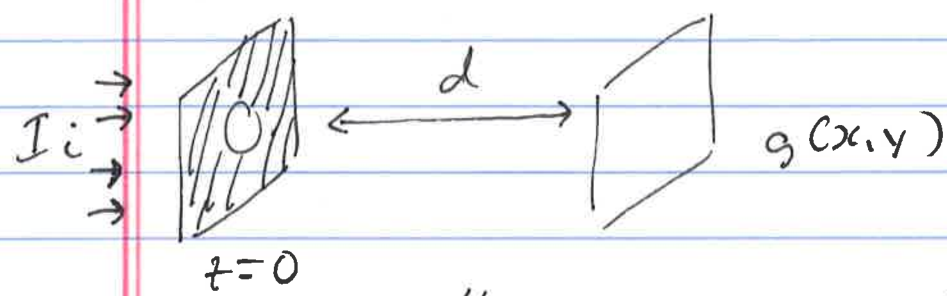


Figure 4.4-4 The 4–f system performs a Fourier transform followed by an inverse Fourier transform, so that the image is a perfect replica of the object.

remove different spatial frequency components by placing a "block" at the Fourier plane

Diffraction:

~ imagine we have an aperture @ $z=0$



$$f(x,y) = I_i^{1/2}\, p(x,y)$$ ~ $p(x,y)$ describes the aperture

In the Fraunhofer limit

$$g(x,y) = I_i^{1/2}\, h_0\, P\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$ ← F.T. of $p(x,y)$ evaluated at $\nu_x = \frac{x}{\lambda d}$, $\nu_y = \frac{y}{\lambda d}$

$$h_0 = \frac{i}{\lambda d} e^{-ikd}$$

Intensity pattern

$$I(x,y) = |g(x,y)|^2 = \frac{I_i}{(\lambda d)^2} \left| P\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)\right|^2$$

$\Rightarrow$ to calculate the intensity pattern in the far field, we just need to know the function that describes the aperture & calculate

$$P(\nu_x, \nu_y) = \iint p(x,y)\, e^{i2\pi(\nu_x x + \nu_y y)}\, dx\, dy$$

this is evaluated at $\nu_x = x/\lambda d$, $\nu_y = y/\lambda d$

Example: Square aperture with side length "D"

$$P(\nu_x, \nu_y) = \int_{-D/2}^{D/2}\int_{-D/2}^{D/2} dx\, dy\, e^{i2\pi(\nu_x x + \nu_y y)}$$

$$= \frac{\sin(D\pi\nu_x)}{\pi\nu_x}\,\frac{\sin(D\pi\nu_y)}{\pi\nu_y}$$

$$\Rightarrow I(x,y) = \frac{I}{(\lambda d)^2}\,\frac{\sin^2(D\pi x/\lambda d)}{(\pi x/\lambda d)^2} \times \frac{\sin^2(D\pi y/\lambda d)}{(\pi y/\lambda d)^2}$$

~ rewrite $I(x,y)$ using the definition

$$\frac{\sin \pi x}{\pi x} = \operatorname{sinc} x$$

$$I(x,y) = \frac{I\, D^4}{(\lambda d)^2} \, \frac{\sin^2(\pi D x/\lambda d)}{(\pi D x/\lambda d)^2} \, \frac{\sin^2(\pi D y/\lambda d)}{(\pi D y/\lambda d)^2}$$

$$= I \left(\frac{D^2}{\lambda d}\right)^2 \operatorname{sinc}^2(\pi D x/\lambda d)\, \operatorname{sinc}^2(\pi D y/\lambda d)$$

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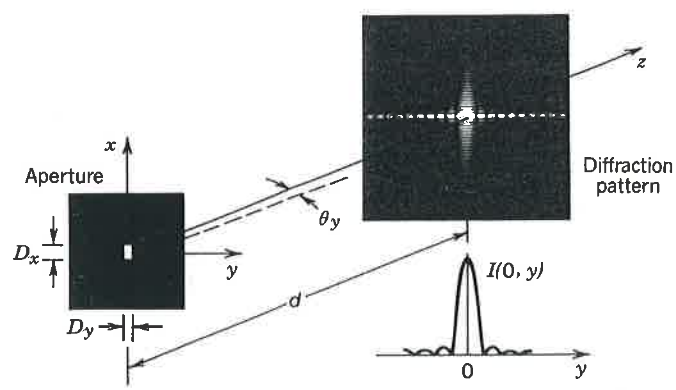


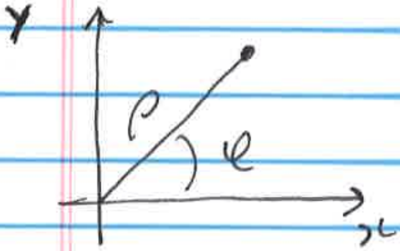
Figure 4.3-3 Fraunhofer diffraction from a rectangular aperture. The central lobe of the pattern has half-angular widths $\theta_x = \lambda/D_x$ and $\theta_y = \lambda/D_y$.

~ minimum in Int. occurs when

$$\frac{\pi D x}{\lambda d} = n\pi \implies x = n\frac{\lambda d}{D}$$

Circular aperture

~ easiest to do in polar co-ordinates



$$\iint_{-\infty}^{\infty} dx\,dy \to \int_0^{\infty}\int_0^{2\pi} \rho\, d\rho\, d\varphi$$

$x = \rho\cos\varphi$; $y = \rho\sin\varphi$

~ for a spherical hole @ $z=0$

$$P(\nu_x,\nu_y) = \int_0^{D/2}\int_0^{2\pi} e^{i2\pi(\nu_x\rho\cos\varphi + \nu_y\rho\sin\varphi)}\,\rho\, d\rho\, d\varphi$$

do the integral over φ first in Mathematica

$$\int_0^{2\pi} \rho\, e^{i2\pi(\nu_x\rho\cos\varphi + \nu_y\rho\sin\varphi)}\, d\varphi$$

$$= 2\pi\rho\, J_0\left(2\pi\rho\sqrt{\nu_x^2+\nu_y^2}\right)$$ ← Bessel Function of the first kind

(BesselJ[0, —])

$$\int_0^{D/2} 2\pi\rho \, J_0(2\pi\rho\sqrt{(\,)})\, d\rho$$

$$= \left(\frac{D}{2}\right) \frac{J_1\left(\pi D\sqrt{\nu_x^2+\nu_y^2}\right)}{\sqrt{\nu_x^2+\nu_y^2}} = P[\nu_x, \nu_y]$$

Now, $I(x,y) = \dfrac{I_i}{(\lambda d)^2}\left|P\left[\frac{x}{\lambda d}, \frac{y}{\lambda d}\right]\right|^2$

where $\sqrt{\nu_x^2+\nu_y^2} = \frac{1}{\lambda d}\sqrt{x^2+y^2} = \rho/\lambda d$

$$\Rightarrow I(x,y) = \frac{I_i}{(\lambda d)^2}(D/2)^2 \frac{J_1(\pi D\rho/\lambda d)^2}{(\rho/\lambda d)^2}$$

$$= \frac{I_i D^2}{4}\frac{J_1(\pi D\rho/\lambda d)^2}{\rho^2}$$

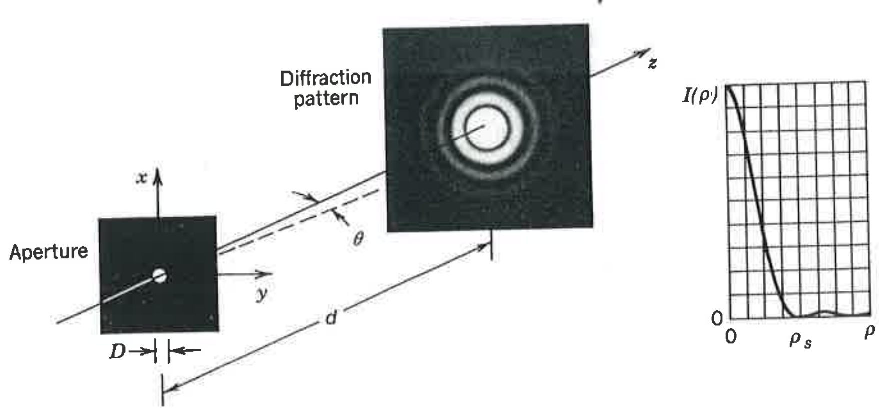
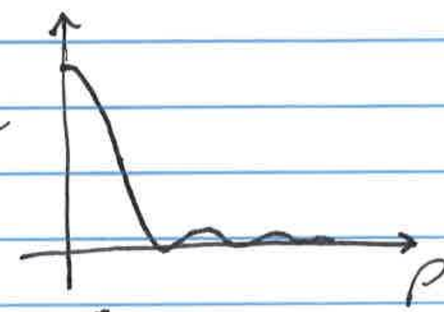


Figure 4.3-4 The Fraunhofer diffraction pattern from a circular aperture produces the Airy pattern with the radius of the central disk subtending an angle $\theta = 1.22\lambda/D$.

~ pattern produced by diffraction from a circular aperture is called an "Airy Pattern"

$$\frac{I_i D^2}{4} \frac{\left(J_1(\pi D \rho/(\lambda d))\right)^2}{\rho^2}$$ ~



most of the intensity is in the "1st lobe"

~ 1st minimum occurs at $\frac{\pi D \rho}{\lambda d} = 3.8317$

$\Rightarrow \rho/d = \frac{1.22\lambda}{D}$ ← light diverges w/ an angle

$$\theta = \rho/d = \frac{1.22\lambda}{D}$$

$\Rightarrow$ larger diffraction (larger θ) for large λ or small D