Focussing & the Diffraction Limit

Ray Optics

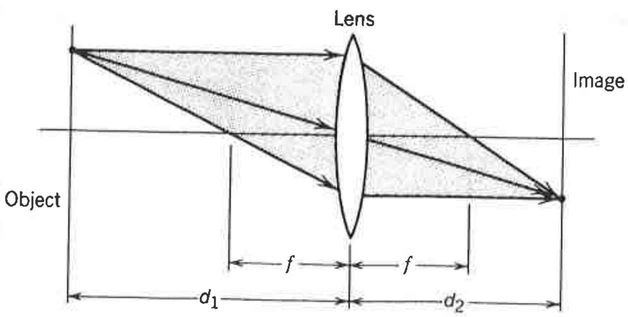


Figure 4.4-1 Rays in a focused imaging system.

in focus when

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$$

~ define focussing error $\epsilon = \frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{f}$

~ note, for object a long way away

$$d_1 \to \infty \Rightarrow d_2 = f$$

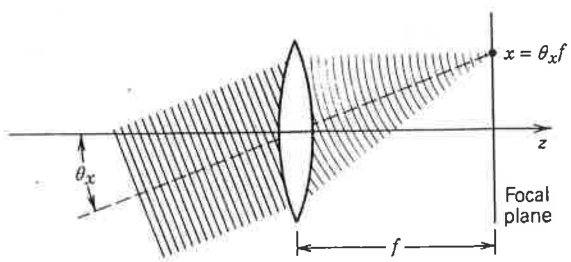


Figure 4.2-2 Focusing of a plane wave into a point. A direction (θ_x, θ_y) is mapped into a point $(x, y) = (\theta_x f, \theta_y f)$.

Wave Optics

~ off axis rays focussed at different points

~ position at the focal plane

$$x = \theta_x f = \frac{x_0}{d_1} f = \frac{k_x}{k}$$

~ lets look at an imaging system with one lens

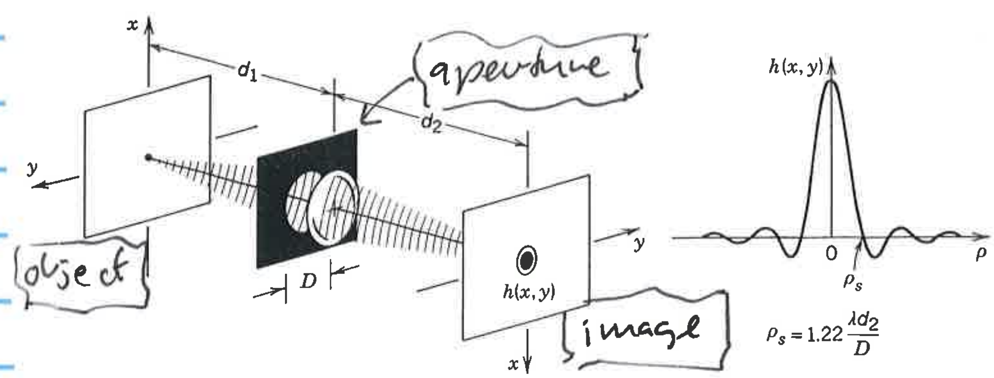


Figure 4.4-8 Impulse-response function of an imaging system with a circular aperture.

point on the object plane produces a spherical wave

@ the aperture we have

$$\left.\begin{aligned} a(x,y) &= h_1 e^{-ik(x^2+y^2)/2d_1} \\ h_1 &= \frac{i}{\lambda d_1} e^{-ikd_1} \end{aligned}\right\} \text{Fresnel approximation}$$

~ just after the aperture

$$u(x,y) = h_1 e^{-ik(x^2+y^2)/2d_1} \underbrace{p(x,y)}_{\text{function that describes the aperture}}$$

~ this field is multiplied by the transfer function of the lens

$$u(x,y) = h_1 e^{-ik\frac{x^2+y^2}{2d_1}} p(x,y) e^{ik\frac{x^2+y^2}{2f}}$$

~ the field then propagates to the image plane at d_2

~ find the field at the image plane by convolating $u(x,y)$ w/ the free space impulse-response function

$$h(x,y) = h_1 h_2 \iint e^{-ik\frac{x'^2+y'^2}{2d_1}} p(x',y') e^{ik\frac{x'^2+y'^2}{2f}}$$

$$\times e^{-i\pi/\lambda d_2 \left((x-x')^2+(y-y')^2\right)} dx'dy'$$

$$h_2 = \frac{i}{\lambda d_2} e^{-ikd_2}$$

~ we can now use the Fraunhofer approximation ← ignore x^2, y^2 terms in the exponentials ($x^2, y^2 \ll \lambda d$)

$$h(x,y) = h_1 h_2 \iint e^{-\frac{ik}{2}\left(\frac{1}{d_1}+\frac{1}{d_2}-\frac{1}{f}\right)(x'^2+y'^2)} \times p(x',y') \times e^{i2\pi\left(\frac{x}{\lambda d_2}x' + \frac{y}{\lambda d_2}y'\right)} \, dx'dy'$$

$$= h_1 h_2 \iint e^{-\frac{ik}{2}\epsilon(x'^2+y'^2)} p(x',y') \, e^{i2\pi(\nu_x x' + \nu_y y')} \, dx'dy'$$

where $\nu_x = x/\lambda d_2$, $\nu_y = y/\lambda d_2$ ~ as before w/ diff.^{n}

and $\epsilon = 1/d_1 + 1/d_2 - 1/f$ ← focussing error

~ when the system is "in focus"

$$1/d_1 + 1/d_2 = 1/f \Rightarrow \epsilon = 0$$

$$\Rightarrow h(x,y) = h_1 h_2 \iint p(x',y')\, e^{i2\pi(\nu_x x' + \nu_y y')}\, dx'dy'$$

field at the image plane is the F.T. of the aperture function $p(x,y)$

write as: $h(x,y) = h_1 h_2 P\left(\frac{x}{\lambda d_2}, \frac{y}{\lambda d_2}\right)$

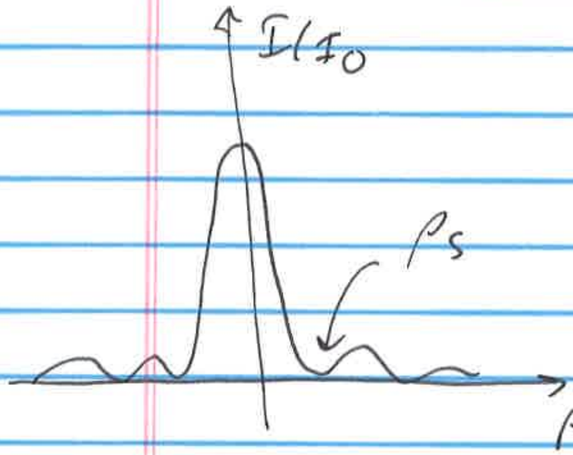
this is the same as the diffraction problem.

eg circular aperture

$$h(x,y) = h_1 h_2 \frac{D}{2} \frac{J_1(\pi D \rho/\lambda d)}{\rho/\lambda d}$$

$\Rightarrow$ pt. spot at the object plane appears as an "Airy disk" at the image plane

~ characterize size by the radius where the Airy disk has its 1st zero



1st zero occurs at

$$\frac{\pi D \rho_s}{\lambda d_2} = 3.832$$

$$\Rightarrow \rho_s = \frac{1.22\,\lambda d_2}{D}$$

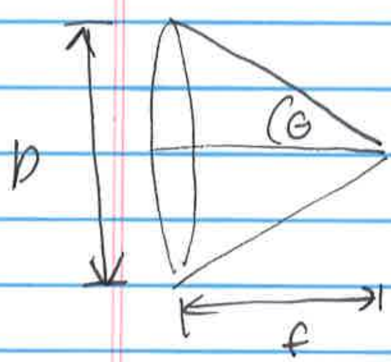
~ if the object plane is far away ($d_1 \to \infty$)

then $d_2 \simeq f$

$$\Rightarrow \rho_s \simeq \frac{1.22\,\lambda f}{D} = 1.22\,\lambda F_{\#}$$

F-number of the lens

$F_{\#} = f/D$



Numerical Aperture defined as $NA = n\sin\theta$

$$= n\sin\left(\arctan\left(D/2f\right)\right)$$

$$\simeq \frac{nD}{2f} = \frac{n}{2F_{\#}}$$

$\Rightarrow NA = 1/2F_{\#}$ for air $(n=1)$

$\Rightarrow$ critical radius $\rho_s = \dfrac{1.22\lambda}{2NA} = \lambda/1.6NA$

This is the "diffraction limit"

<u>Note</u> ~ the "diffraction limit" comes from diffraction through the optical system i.e. it is from the limited NA of the lens

~ high spatial resolution is achieved by collecting light over a large angle $\Leftrightarrow$ high spatial frequencies and $\therefore$ fine distance resolution

$\nu_x \simeq \theta_x/\lambda$

- our ability to resolve 2 objects, in either a telescope or a microscope, is determined by the diffraction limit

~ for example, imaging waves originating from 2 pts at the object plane at $x=0$ and $x=x_0$

~ waves are focussed at different points on the image plane according to the angle that the waves make with the lens

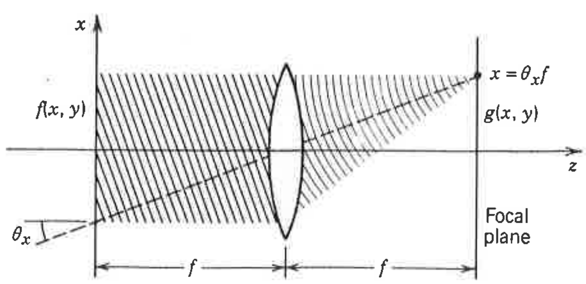


Figure 4.2-4 Fourier transform system. The Fourier component of $f(x, y)$ with spatial frequencies ν_x and ν_y generates a plane wave at angles $\theta_x = \lambda\nu_x$ and $\theta_y = \lambda\nu_y$ and is focused by the lens to the point $(x, y) = (f\theta_x, f\theta_y) = (\lambda f\nu_x, \lambda f\nu_y)$ so that $g(x, y)$ is proportional to the Fourier transform $F(x/\lambda f, y/\lambda f)$.

$$\theta_x = \frac{x_0}{d_1}$$

~ lets be more rigorous

~ for the pt. @ (x_0, y_0) in the object plane, light has the form

$$u(x,y) = h_1\, e^{ik((x-x_0)^2 + (y-y_0)^2)/2d_1}$$

~ following the same steps as we did at the start of the lecture we get

$$h(x,y) = h_1 h_2 \iint e^{-ik((x'-x_0)^2 + (y'-y_0)^2)/2d_1} \times p(x',y')\, e^{ik(x'^2+y'^2)/2f} \times e^{-ik((x-x')^2 + (y-y')^2)/2d_2}\, dx'dy'$$

~ if we ignore the x^2, y^2, x_0^2, y_0^2 terms (Fresnel approximation)

$$h(x,y) = h_1 h_2 \iint e^{-\frac{ik}{2}(1/d_1 + 1/d_2 - 1/f)(x'^2+y'^2)} \times p(x',y') \times e^{i2\pi\left(\frac{x_0}{\lambda d_1}x' + \frac{y_0}{\lambda d_1}y' + \frac{x}{\lambda d_2}x' + \frac{y}{\lambda d_2}y'\right)}\, dx'dy'$$

again, assume we are in focus $\Rightarrow \epsilon = 1/d_1 + 1/d_2 - 1/f = 0$

$$\Rightarrow h(x,y) = h_1 h_2 \iint p(x',y')\, e^{i2\pi\left[\left(\frac{x_0}{\lambda d_1} + \frac{x}{\lambda d_2}\right)x' + (\;)y'\right]} dx'dy'$$

$$= h_1 h_2 \tilde{P}\left[\frac{x_0}{\lambda d_1} + \frac{x}{\lambda d_2}, \frac{y_0}{\lambda d_1} + \frac{y}{\lambda d_2}\right]$$

F.T. of $p(x,y)$ evaluated

at $\nu_x = \frac{x_0}{\lambda d_1} + \frac{x}{\lambda d_2}$; $\nu_y = \frac{y_0}{\lambda d_1} + \frac{y}{\lambda d_2}$

write $\nu_x = \frac{1}{\lambda d_2}\left(x + \frac{d_2}{d_1} x_0\right)$

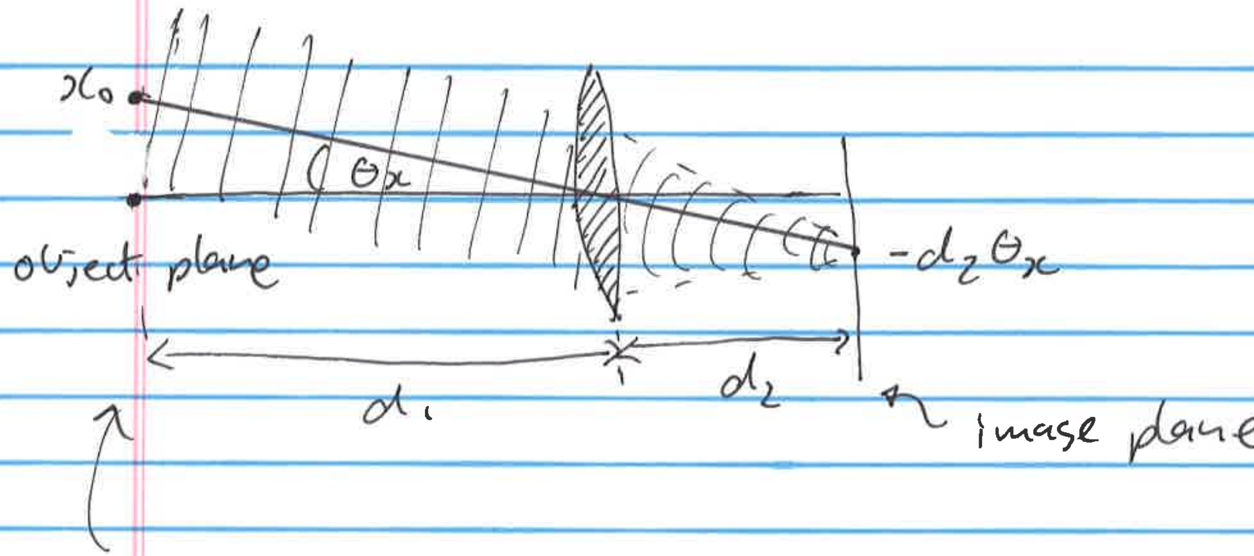
$$= \frac{1}{\lambda d_2}\left(x + d_2 \theta_x\right)$$

likewise $\nu_y = \frac{1}{\lambda d_2}\left(y + d_2 \theta_y\right)$

$\Rightarrow$ pt at (x_0, y_0) in the object plane

produces an Airy disk at

$(x,y) = (-d_2\theta_x, -d_2\theta_y)$ in the image plane



whether light from an object at (x_0, y_0) can be resolved from light from an object at $(0,0)$ depends on the size of the Airy disk

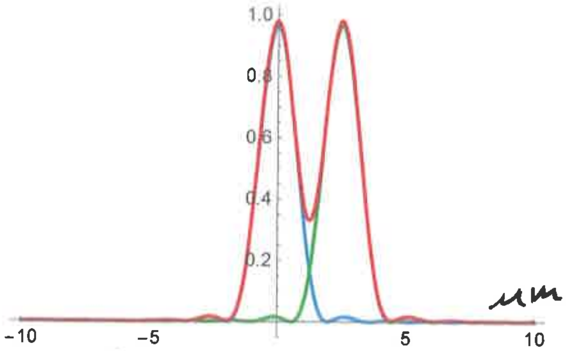
Example (resolution for a telescope)

1" ϕ lens w/ 2" focal length.

$\Rightarrow NA = 0.25 \quad f \simeq 50\text{ mm} = 5\times10^4\ \mu\text{m}$

~ assume $\lambda = 0.8\ \mu\text{m}$

plot $I = \left(\dfrac{J_1(\pi D\rho/\lambda d_2)}{\rho}\right)^2 = \left(\dfrac{J_1(2\pi NA\rho/\lambda)}{\rho}\right)^2$



$\theta_y = 0$

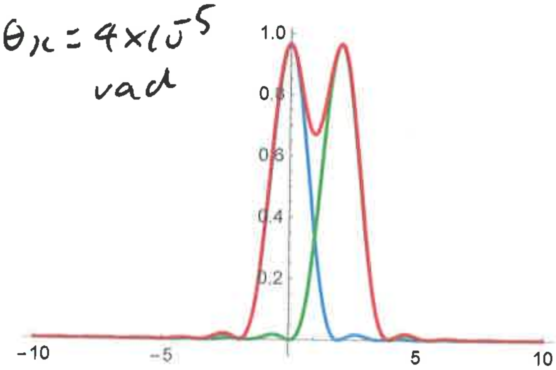
$\theta_x = 5\times10^{-5}$ rad

~ peak @

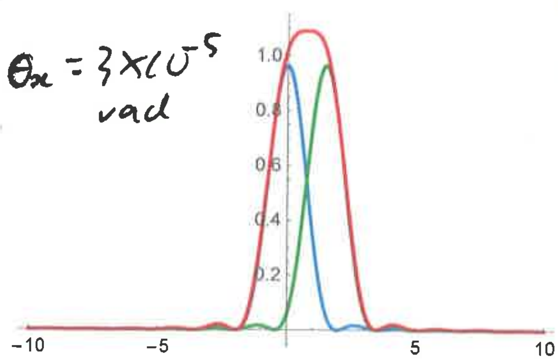
$x = f\theta_x = 2.5$ um.

~ assume $d_1 = 1$ km $\Rightarrow$ $x_0 = d_1\theta_x = 5$ cm

$\theta_x = 4\times10^{-5}$ rad



$\theta_x = 3\times10^{-5}$ rad



$x_0 = 4$ cm

$\Rightarrow$ just resolved

$x_0 = 3$ cm

~ not resolved

Rayleigh Criterion: resolve 2 objects when the center of the Airy disk of one occurs at the 1st zero of the other

1st zero occurs when $\frac{2\pi NA x}{\lambda} = 3.832$

$$\Rightarrow x = \frac{0.61\lambda}{NA}$$

$$\theta_x = \frac{x}{f} = \frac{x_0}{d_1} \Rightarrow x_0 = \frac{0.61\lambda}{NA} \times \frac{d_1}{f}$$

$= 3.9$ cm for our example

Microscope: more productive to think of resolution in terms of the spot size

Airy disk: spot size $= \frac{2 \times 0.61\lambda}{NA} = \frac{1.22\lambda}{NA}$

Gaussian beam: spot size $= 2w_0 = \frac{2\lambda}{\pi NA} = \frac{0.64\lambda}{NA}$