The Stabilizing Effect of the Temperature on Buoyancy-Driven Fluids

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ABSTRACT. The Boussinesq system for buoyancy-driven fluids couples the momentum equation forced by the buoyancy with the convection-diffusion equation for the temperature. One fundamental issue on the Boussinesq system is the stability problem on perturbations near the hydrostatic balance. This problem can be extremely difficult when the system lacks full dissipation. This paper solves the stability problem for a two-dimensional Boussinesq system with only vertical dissipation and horizontal thermal diffusion. We establish the stability for the nonlinear system and derive precise large-time behavior for the linearized system. The results presented in this paper reveal a remarkable phenomenon for buoyancy-driven fluids. That is, the temperature actually smooths and stabilizes the fluids. If the temperature were not present, the fluid is governed by the 2D Navier-Stokes with only vertical dissipation, and its stability remains open. It is the coupling and interaction between the temperature and the velocity in the Boussinesq system that makes the stability problem studied here possible. Mathematically, the system can be reduced to degenerate and damped wave equations that fuel the stabilization.

1. INTRODUCTION

This paper intends to reveal and rigorously prove the fact that the temperature can actually have a stabilizing effect on the buoyancy-driven fluids. As we know, buoyancy-driven flows such as geophysical fluids and various Rayleigh-Bénard convection are modeled by the Boussinesq equations. Our study is based on the following special two-dimensional (2D) Boussinesq system with partial dissipation:

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu\, \partial_{22} U + \Theta \mathbf{e}_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \Theta + U \cdot \nabla \Theta = \eta\, \partial_{11} \Theta, \\ \nabla \cdot U = 0, \end{cases} \tag{1.1}$$

where U denotes the fluid velocity, P the pressure, Θ the temperature, $\nu > 0$ the kinematic viscosity, and η the thermal diffusivity. Here, $\mathbf{e}_2$ is the unit vector in the vertical direction. The dissipation in the velocity equation is anisotropic and is only in the vertical direction. The partial differential equations (PDEs) with only degenerate dissipation are relevant in certain physical regimes, and one of the most notable examples is Prandtl's equation. Another reason for including only partial dissipation in the velocity equation is to help better reveal the smoothing and stabilization effect of the temperature. More precise explanation will be presented later.

The Boussinesq equations for buoyancy-driven fluids are widely used in the modeling and study of atmospheric and oceanographic flows and the Rayleigh-Bénard convection (see, e.g., [14, 22, 42, 45]). The Boussinesq equations are also mathematically important. The 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [43]. Fundamental issues on the Boussinesq systems such as the global well-posedness problem have attracted a lot of interests recently, especially when the systems involve only partial dissipation or no dissipation at all (see, e.g., [1–5, 7, 9–13, 15, 17–19, 24, 26–35, 37–41, 44, 46, 49, 54, 55, 57–64]). The study on the stability of several steady states to the Boussinesq system has recently gained momentum due to their physical applications. More details will be described in the later part of the introduction.

The main purpose of this paper is to understand the stability and large-time behavior of perturbations near the hydrostatic equilibrium $(U_{\text{he}}, \Theta_{\text{he}})$ with

$$U_{\text{he}} = 0, \quad \Theta_{\text{he}} = x_2.$$

For the static velocity U_{he}, the momentum equation is satisfied when the pressure gradient is balanced by the buoyancy force, namely,

$$-\nabla P_{\text{he}} + \Theta_{\text{he}} \mathbf{e}_2 = 0 \quad \text{or} \quad P_{\text{he}} = \frac{1}{2} x_2^2.$$

Here, $(U_{\text{he}}, P_{\text{he}}, \Theta_{\text{he}})$ is a very special steady solution with great physical significance. In fact, our atmosphere is mostly in hydrostatic equilibrium with the upward pressure gradient force balanced by the buoyancy due to the gravity.

To understand the desired stability, we write the equation of the perturbation denoted by (u, p, θ), where

$$u = U - U_{\text{he}}, \quad p = P - P_{\text{he}} \quad \text{and} \quad \theta = \Theta - \Theta_{\text{he}}.$$

It follows easily from (1.1) that the perturbation (u, p, θ) satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{22} u + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{1.2}$$

The only difference between (1.1) and (1.2) is an extra term u_2 (the vertical component of u) in (1.2), which plays a very important role in balancing the energy. In order to assess the stability, we need to establish that the solution (u, b) of (1.2) corresponding to any sufficiently small initial perturbation (u_0, b_0) (measured in the Sobolev norm $H^2(\mathbb{R}^2)$) remains small for all time. This does not appear to be an easy task when there is only vertical velocity dissipation and horizontal thermal diffusion.

The lack of horizontal dissipation makes it hard to control the growth of the vorticity $\omega = \nabla \times u$, which satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega + \partial_1 \theta, \quad x \in \mathbb{R}^2,\ t > 0. \tag{1.3}$$

We can obtain a uniform bound on the L^2-norm of the vorticity ω itself, but it does not appear possible to control the L^2-norm of the gradient of the vorticity, $\nabla \omega$. If θ were identically zero, (1.3) becomes the 2D Navier-Stokes equation with degenerate dissipation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega, \qquad x \in \mathbb{R}^2,\ t > 0. \tag{1.4}$$

One can note that (1.4) always has a unique global solution ω for any initial data $\omega_0 \in H^1(\mathbb{R}^2)$, but the issue of whether $\|\nabla \omega(t)\|_{L^2}$ for the solution ω of (1.4) grows as a function of t remains an open problem. When $\nu = 0$, (1.4) becomes the 2D Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \qquad x \in \mathbb{R}^2,\ t > 0.$$

As demonstrated in several beautiful works (see, e.g., [21, 36, 66]), $\nabla \omega(t)$ can grow even double exponentially in time. In contrast, solutions to the 2D Navier-Stokes equations with full dissipation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega, \quad x \in \mathbb{R}^2,\ t > 0$$

have been shown to always decay in time (see, e.g., [47, 48]). The lack of the horizontal dissipation in (1.4) prevents us from mimicking the approach designed for the fully dissipative Navier-Stokes equations. In fact, when we estimate $\|\nabla\omega(t)\|_{L^2}$, the issue is how to proceed from the energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t)\|_{L^2}^2 + \nu\|\partial_2\nabla\omega(t)\|_{L^2}^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x.$$

It appears impossible to control the term on the right. To make use of the anisotropic dissipation, we can further decompose the nonlinearity into four component terms:

$$\begin{aligned}\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x &= \int \partial_1 u_1(\partial_1\omega)^2\,\mathrm{d}x + \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,\mathrm{d}x \\ &\quad + \int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,\mathrm{d}x + \int \partial_2 u_2(\partial_2\omega)^2\,\mathrm{d}x.\end{aligned} \tag{1.5}$$

Yet, the first two terms in (1.5) do not appear to admit suitable bounds because of the lack of control on the horizontal derivatives in the dissipation. Whether $\|\nabla\omega(t)\|_{L^2}$ for the solution ω of (1.4) grows in time remains an open problem.

When we deal with the stability problem on (1.2), we encounter exactly the same term in (1.5). How would it be possible to deal with the same difficulty when we now have a more complex system like (1.2)? It is the smoothing and stabilization effect of the temperature through the coupling and interaction that makes the stability problem on (1.2) possible. We give a quick explanation on this mechanism. Since the linear portion of the nonlinear system in (1.2) plays a crucial role in the stability properties, we first eliminate the pressure term in (1.2) to separate the linear terms from the nonlinear ones. Applying the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation yields

$$\partial_t u = \nu\,\partial_{22}u + \mathbb{P}(\theta\mathbf{e}_2) - \mathbb{P}(u\cdot\nabla u). \tag{1.6}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(\theta\mathbf{e}_2) = \theta\mathbf{e}_2 - \nabla\Delta^{-1}\nabla\cdot(\theta\mathbf{e}_2) = \begin{bmatrix} -\partial_1\,\partial_2\Delta^{-1}\theta \\ \theta - \partial_2^2\Delta^{-1}\theta \end{bmatrix}. \tag{1.7}$$

Inserting (1.7) in (1.6) and writing (1.6) in terms of its component equations gives

$$\begin{cases} \partial_t u_1 = \nu\,\partial_{22}u_1 - \partial_1\,\partial_2\Delta^{-1}\theta + N_1, \\ \partial_t u_2 = \nu\,\partial_{22}u_2 + \partial_1\,\partial_1\Delta^{-1}\theta + N_2, \end{cases} \tag{1.8}$$

where N_1 and N_2 are the nonlinear terms

$$\begin{aligned} N_1 &= -(u\cdot\nabla u_1 - \partial_1\Delta^{-1}\nabla\cdot(u\cdot\nabla u)), \\ N_2 &= -(u\cdot\nabla u_2 - \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)). \end{aligned}$$

By differentiating, the first equation of (1.8) in t yields

$$\partial_{tt}u_1 = \nu\,\partial_{22}\,\partial_t u_1 - \partial_1\,\partial_2\Delta^{-1}\partial_t\theta + \partial_t N_1.$$

Replacing $\partial_t\theta$ by the equation of θ, namely $\partial_t\theta = \eta\,\partial_{11}\theta - u_2 - u\cdot\nabla\theta$, gives

$$\partial_{tt}u_1 = \nu\,\partial_{22}\,\partial_t u_1 + \partial_1\,\partial_2\Delta^{-1}u_2 - \eta\,\partial_{11}\,\partial_1\,\partial_2\Delta^{-1}\theta + \partial_1\,\partial_2\Delta^{-1}(u\cdot\nabla\theta) + \partial_t N_1.$$

By further replacing $\partial_1\,\partial_2\Delta^{-1}\theta$ by the first equation of (1.8), namely,

$$-\partial_1\,\partial_2\Delta^{-1}\theta = \partial_t u_1 - \nu\,\partial_{22}u_1 - N_1,$$

we obtain

$$\begin{aligned}\partial_{tt}u_1 = {}&\nu\,\partial_{22}\,\partial_t u_1 + \partial_1\,\partial_2\Delta^{-1}u_2 + \eta\,\partial_{11}(\partial_t u_1 - \nu\,\partial_{22}u_1 - N_1)\\ &+ \partial_1\,\partial_2\Delta^{-1}(u\cdot\nabla\theta) + \partial_t N_1,\end{aligned}$$

which leads to, because of the divergence-free condition $\partial_2 u_2 = -\partial_1 u_1$,

$$\partial_{tt}u_1 - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t u_1 + \nu\eta\,\partial_{11}\,\partial_{22}u_1 + \partial_{11}\Delta^{-1}u_1 = N_3. \tag{1.9}$$

Here, N_3 contains the nonlinear terms

$$N_3 = (\partial_t - \eta\,\partial_{11})N_1 + \partial_1\,\partial_2\Delta^{-1}(u\cdot\nabla\theta).$$

Through a similar process, u_2 and θ can be shown to satisfy

$$\begin{aligned}\partial_{tt}u_2 - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t u_2 + \nu\eta\,\partial_{11}\partial_{22}u_2 + \partial_{11}\Delta^{-1}u_2 &= N_4,\\ \partial_{tt}\theta - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t\theta + \nu\eta\,\partial_{11}\,\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta &= N_5,\end{aligned} \tag{1.10}$$

with

$$\begin{aligned}N_4 &= (\partial_t - \eta\,\partial_{11})N_2 - \partial_1\,\partial_1\Delta^{-1}(u\cdot\nabla\theta),\\ N_5 &= -(\partial_t - \nu\,\partial_{22})(u\cdot\nabla\theta) - N_2.\end{aligned}$$

Combining (1.9) and (1.10) and rewriting them into the velocity vector form, we have converted (1.2) into the following new system:

$$\begin{cases}\partial_{tt}u - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t u + \nu\eta\,\partial_{11}\,\partial_{22}u + \partial_{11}\Delta^{-1}u = N_6,\\ \partial_{tt}\theta - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t\theta + \nu\eta\,\partial_{11}\,\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_5,\end{cases} \tag{1.11}$$

where

$$N_6 = (N_3, N_4) = -(\partial_t - \eta\,\partial_{11})\mathbb{P}(u\cdot\nabla u) + \nabla^{\perp}\partial_1\Delta^{-1}(u\cdot\nabla\theta)$$

with $\nabla^\perp = (\partial_2, -\partial_1)$. By taking the curl of the velocity equation, we can also convert (1.11) into a system of ω and θ,

$$\begin{cases} \partial_{tt}\omega - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t\omega + \nu\eta\,\partial_{11}\,\partial_{22}\omega + \partial_{11}\Delta^{-1}\omega = N_7, \\ \partial_{tt}\theta - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t\theta + \nu\eta\,\partial_{11}\,\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_5, \end{cases}$$

where

$$N_7 = -(\partial_t - \eta\,\partial_{11})(u \cdot \nabla\omega) - \partial_1(u \cdot \nabla\theta).$$

Amazingly, we have found that u, θ, and ω all satisfy the same damped degenerate wave equation only with different nonlinear terms. In comparison with the original system (1.2), the new system of wave type equations in (1.11) helps unearth all the smoothing and stabilization hidden in the original system. The velocity in (1.2) involves only vertical dissipation, but the wave structure actually implies that the temperature can stabilize the fluids by creating the horizontal regularization via the coupling and interaction.

How much regularity and stabilization can the wave structure help create? Our very first effort is devoted to understanding this natural question. We focus on the linearized system

$$\begin{cases} \partial_{tt}u - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t u + \nu\eta\,\partial_{11}\,\partial_{22}u + \partial_{11}\Delta^{-1}u = 0, \\ \partial_{tt}\theta - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t\theta + \nu\eta\,\partial_{11}\,\partial_{22}\,\theta + \partial_{11}\Delta^{-1}\theta = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.12}$$

To maximally extract the regularity and damping effects from the wave structure, we represent the solution of (1.12) explicitly in terms of kernel functions and the initial data. The two components u_1 and u_2 of the velocity field have slightly different explicit representations.

Proposition 1.1. *The solution of* (1.12) *can be explicitly represented as*

$$u_1(t) = K_1(t)u_{10} + K_2(t)\theta_0, \tag{1.13}$$

$$u_2(t) = K_1(t)u_{20} + K_3(t)\theta_0, \tag{1.14}$$

$$\theta(t) = K_4(t)u_{20} + K_5(t)\theta_0, \tag{1.15}$$

where K_1 through K_5 are Fourier multiplier operators with their symbols given by

$$K_1(\xi,t) = G_2(\xi,t) - \nu\xi_2^2 G_1(\xi,t), \tag{1.16}$$

$$K_2(\xi,t) = -\frac{\xi_1\xi_2}{|\xi|^2}G_1(\xi,t), \tag{1.17}$$

$$K_3(\xi,t) = \frac{\xi_1^2}{|\xi|^2}G_1(\xi,t), \tag{1.18}$$

$$K_4 = -G_1, \tag{1.19}$$

$$K_5(\xi,t) = G_2(\xi,t) - \eta\xi_1^2 G_1(\xi,t). \tag{1.20}$$

Here, G_1 and G_2 are two explicit symbols involving the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 + (\eta\xi_1^2 + \nu\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0$$

or

$$\lambda_1 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) - \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)},$$

$$\lambda_2 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) + \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}.$$

More precisely, when $\lambda_1 \neq \lambda_2$,

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}. \tag{1.21}$$

When $\lambda_1 = \lambda_2$,

$$G_1(\xi,t) = te^{\lambda_1 t}, \qquad G_2(\xi,t) = e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t}. \tag{1.22}$$

In order to understand the regularity and large-time behavior, we need to have precise upper bounds on the kernel functions K_1 through K_5. The behavior of these kernel functions depends crucially on the frequency ξ and is nonhomogeneous. In addition, the bounds for these kernel functions are anisotropic and are not uniform in different directions. The details of these upper bounds and how they are derived are provided in Proposition 2.2 in Section 2.

We are able to establish the precise large-time behavior of the solutions to (1.12) using the upper bounds for the kernel functions K_1 through K_5 in Proposition 2.2. To reflect the anisotropic behavior of the solutions, we need to employ anisotorpic Sobolev type spaces. For $s \geq 0$ and $\sigma \geq 0$, the anisotropic Sobolev space $\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_1|^{-2\sigma}\, |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2} < \infty.$$

Similarly, $\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_2|^{-2\sigma}\, |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2} < \infty.$$

In addition, we write $\mathring{H}^{s,-\sigma}(\mathbb{R}^2) = \mathring{H}_1^{s,-\sigma}(\mathbb{R}^2) \cap \mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ with the norm given by

$$\|f\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)} = \|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} + \|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)}.$$

Theorem 1.2. *Consider the linearized system in* (1.12) *with the initial data* u_0 *and* θ_0 *satisfying* $\nabla \cdot u_0 = 0$ *and*

$$\begin{aligned} u_0 &\in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-2,-\sigma}, \\ \theta_0 &\in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-1,-\sigma}, \end{aligned}$$

where $s \geq 0$ *and* $\sigma \geq 0$ *satisfy* $s + \sigma \geq 2$. *Then, the corresponding solution* (u, θ) *to* (1.12) *satisfies, for some constant* $C > 0$,

$$\begin{aligned} \|u_1(t)\|_{\mathring{H}^s} \leq{}& Ct^{-(1/2)(s+\sigma)}\|u_{10}\|_{\mathring{H}^{0,-\sigma}} + Ct^{-\sigma/2}\|u_{10}\|_{\mathring{H}^{s,-\sigma}} \\ &+ Ct^{-(1/2)(s+\sigma)+1}\|\theta_0\|_{\mathring{H}^{0,-\sigma}} + Ct^{-1/2-\sigma/2}\|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}, \\ \|u_2(t)\|_{\mathring{H}^s} \leq{}& Ct^{-(1/2)(s+\sigma)}\|u_{20}\|_{\mathring{H}^{0,-\sigma}} + Ct^{-\sigma/2}\|u_{20}\|_{\mathring{H}^{s,-\sigma}} \\ &+ Ct^{-(1/2)(s+\sigma)+1}\|\theta_0\|_{\mathring{H}^{0,-\sigma}} + Ct^{-1-\sigma/2}\|\theta_0\|_{\mathring{H}^{s,-\sigma}}, \\ \|\theta(t)\|_{\mathring{H}^s} \leq{}& Ct^{-(1/2)(s+\sigma)+1}\|u_{20}\|_{\mathring{H}^{0,-\sigma}} + Ct^{-\sigma/2}\|u_{20}\|_{\mathring{H}^{s-2,-\sigma}} \\ &+ Ct^{-(1/2)(s+\sigma)}\|\theta_0\|_{\mathring{H}^{0,-\sigma}} + Ct^{-\sigma/2}\|\theta_0\|_{\mathring{H}^{s,-\sigma}}, \end{aligned}$$

where $\mathring{H}^s$ *denotes the standard homogeneous Sobolev space with its norm defined by*

$$\|f\|_{\mathring{H}^s} = \||\xi|^s\,|\hat{f}(\xi)|\|_{L^2(\mathbb{R}^2)}.$$

Next, we further exploit the effects of stabilizing and regularization of the wave structure through the energy method. By forming suitable Lyapunov functional and computing their time evolution, we are able to show that the frequencies away from the two axes in the frequency space decay exponentially to zero as $t \to \infty$. To state our result more precisely, we define a frequency cutoff function, for $a_1 > 0$ and $a_2 > 0$,

$$\hat{\varphi}(\xi) = \hat{\varphi}(\xi_1, \xi_2) = \begin{cases} 0, & \text{if } |\xi_1| \leq a_1 \text{ or } |\xi_2| \leq a_2, \\ 1, & \text{otherwise.} \end{cases} \tag{1.23}$$

Theorem 1.3. *Let* $\nu > 0$ *and* $\eta > 0$. *Consider the linearized system in* (1.12) *or equivalently*

$$\begin{cases} \partial_t u_1 = \nu\,\partial_{22} u_1 - \Delta^{-1}\partial_1\,\partial_2\theta, \\ \partial_t u_2 = \nu\,\partial_{22} u_2 + \Delta^{-1}\partial_1\,\partial_1\theta, \\ \partial_t\theta = \eta\,\partial_{11}\theta - u_2, \\ (u_1, u_2, \theta)(x, 0) = (u_{01}, u_{02}, \theta_0). \end{cases}$$

Let (u, θ) *be the corresponding solution. The Fourier frequency piece of* (u, θ) *away from the two axes of the frequency space decays exponentially in time to zero. More precisely, if* $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$*, then there is constant* $C_0 = C_0(\nu, \eta, a_1, a_2)$ *such that, for all* $t \geq 0$*,*

$$\begin{aligned} &\|\partial_t(\varphi * u)(t)\|_{L^2}^2 + \|(\varphi * u)(t)\|_{H^1}^2 \\ &\qquad \leq C(\|\varphi * u_0\|_{H^2}^2 + \|\varphi * \theta_0\|_{L^2}^2)e^{-C_0 t}, \end{aligned} \tag{1.24}$$

$$\begin{aligned} &\|\partial_t(\varphi * \theta)(t)\|_{L^2}^2 + \|(\varphi * \theta)(t)\|_{H^1}^2 \\ &\qquad \leq C(\|\varphi * \theta_0\|_{H^2}^2 + \|\varphi * u_0\|_{L^2}^2)e^{-C_0 t}, \end{aligned} \tag{1.25}$$

where φ *is as defined in* (1.23) *and* $C = C(\nu, \eta, a_1, a_2) > 0$ *is a constant.*

We comment that the decay results in Theorems 1.2 and 1.3 reflect the properties of the linearized system. The nonlinear stability result presented below is completely independent of these decay results.

We now turn our attention to the main result of this paper, the nonlinear stability on (1.2). As we have explained before, the major obstacle is how to obtain a suitable upper bound on the nonlinear term from the momentum equation, namely (1.5). This is the main reason why the stability problem on the 2D Navier-Stokes equations with only one-directional dissipation remains open. However, for the coupled nonlinear system in (1.2), the smoothing and stabilizing effect of the temperature on the fluid velocity makes the nonlinear stability possible. In fact, we are able to prove the following theorem.

Theorem 1.4. *Consider* (1.2) *with* $\nu > 0$ *and* $\eta > 0$*. Assume the initial data* (u_0, θ_0) *is in* $H^2(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$*. Then, there exists* $\varepsilon = \varepsilon(\nu, \eta) > 0$ *such that, if* (u_0, θ_0) *satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (1.2) *has a unique global solution* (u, θ) *satisfying, for any* $t > 0$*,*

$$\begin{aligned} &\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \nu \int_0^t \|\partial_2 u\|_{H^2}^2 \, d\tau \\ &\qquad + \eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 \, d\tau + C(\nu, \eta) \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau \leq C\varepsilon^2, \end{aligned}$$

where $C(\nu, \eta) > 0$ *and* $C > 0$ *are constants.*

To prove Theorem 1.4, we need to exploit the extra regularization due to the wave structure in (1.11). In particular, the control on the time integral of the horizontal derivative of the velocity field, namely,

$$\int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 \, d\tau, \tag{1.26}$$

plays a crucial role in the proof. Clearly, the uniform boundedness of (1.26) is not a consequence of the vertical dissipation in the velocity equation but due to the interaction with the temperature equation. Besides understanding the time integrability in (1.26) from the wave structure derived before, there is another simple way to comprehend (1.26). This is due to the special coupling in the system (1.2), which allows us to transfer the time integrability from one function in the system to another. More precisely, as in (4.20), we represent $\partial_1 u_2$ in terms of the rest in the equation of θ:

$$\partial_1 u_2 = -\partial_t \partial_1 \theta - \partial_1 (u \cdot \nabla \theta) + \eta \partial_{111} \theta.$$

Then,

$$\|\partial_1 u_2\|_{L^2}^2 = -\int \partial_t \partial_1 \theta \, \partial_1 u_2 \, \mathrm{d}x - \int \partial_1 u_2 \, \partial_1 (u \cdot \nabla \theta) \, \mathrm{d}x + \eta \int \partial_1 u_2 \, \partial_{111} \theta \, \mathrm{d}x.$$

The time integrability of $\|\partial_1 u_2\|_{L^2}^2$ is then converted to the time integrability of other terms. This phenomenon of extra regularization and time integrability due to the coupling also shows up in some other models of partial differential equations such as the Oldroyd-B system (see [16, 25]). We use the bootstrapping argument to prove the boundedness of (1.26) and the stability of the solution simultaneously. A general statement on the bootstrapping principle can be found in [53, p. 21]. To achieve this goal, we first construct a suitable energy functional

$$E(t) = \max_{0 \le \tau \le t} (\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 \, \mathrm{d}\tau + 2\eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 \, \mathrm{d}\tau + \delta \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, \mathrm{d}\tau, \tag{1.27}$$

where $\delta > 0$ is a suitably selected parameter. We then show that $E(t)$ satisfies

$$E(t) \le CE(0) + CE(t)^{3/2}. \tag{1.28}$$

Our main efforts are devoted to proving (1.28). In particular, we need to estimate the difficult term (1.5). A suitable upper bound can now be achieved due to the inclusion of (1.26) in the energy function. Here, $\delta > 0$ is chosen to be sufficiently small so that some of the terms generated in the estimating of (1.26) can be majorized by the dissipative terms. We leave more technical details on how to bound (1.5) and other terms to Section 4. To take advantage of the anisotropic dissipation, the estimates are performed via anisotropic tools including an anisotropic triple product upper bound as stated in the following lemma taken from [6].

Lemma 1.5. *Assume that f, g, $\partial_2 g$, h, and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int_{\mathbb{R}^2} |fgh| \, \mathrm{d}x \le C \|f\|_{L^2} \|g\|_{L^2}^{1/2} \|\partial_2 g\|_{L^2}^{1/2} \|h\|_{L^2}^{1/2} \|\partial_1 h\|_{L^2}^{1/2}.$$

Once (1.28) is established, the bootstrapping argument then implies that, if $E(0)$ is sufficiently small, or equivalently

$$\|(u_0, \theta_0)\|_{H^2} \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains uniformly small for all time, namely,

$$E(t) \leq C\varepsilon^2$$

for a constant $C > 0$ and for all $t \geq 0$. Details on the application of the bootstrapping argument will be provided in the proof of Theorem 1.4 in Section 4.

Assessing the explicit decay rates for the nonlinear system (1.2) is a difficult problem. The explicit solution representation in Proposition 1.1 and the linear decay estimates in Theorem 1.2 serve as the first step in solving this problem. The plan is then to extend the representation formula to the nonlinear system via Duhamel's principle, and then to apply the bootstrapping argument. Even though the anisotropic Sobolev setting is difficult to bootstrap, this approach may lead to certain decay rates when we relax the requirement that the solution be in the same functional setting as the initial datum. This will be completed in a followup work.

Finally, we comment that, because of its importance in geophysics and astrophysics, the stability problem on the hydrostatic balance has recently attracted considerable interests. When the Boussinesq system does not involve full kinematic dissipation and thermal diffusion, the stability problem can be extremely difficult. Several recent works have made progress. Doering, Wu, Zhao, and Zheng [23] solved the stability problem on the 2D Boussinesq system with full velocity dissipation but without thermal diffusion in a bounded domain with stress-free boundary condition. A follow-up work by Tao, Wu, Zhao, and Zheng [52] was able to establish the precise large-time behavior of the stable solutions obtained in [23]. Castro, Cordoba and Lear [8] investigated the stability problem of the 2D Boussinesq system when the velocity involves a damping term, and obtained the asymptotic stability for a trip domain. We also mention that the study on the stability problem on the Boussinesq equations near the shear flow, another physically important steady state, has also gained momentum (see [20, 51, 65]).

The rest of this paper is naturally divided into three sections. Section 2 provides the proofs of Proposition 1.1 and Theorem 1.2. Section 3 proves Theorem 1.3, while Section 4 presents the proof of Theorem 1.4.

2. PROOFS OF PROPOSITION 1.1 AND THEOREM 1.2

This section is devoted to the proofs of Proposition 1.1 and Theorem 1.2. Proposition 1.1 represents the solution to the linearized system in (1.12) in terms of the initial data and several kernel functions. Its proof relies on a lemma that solves the degenerate damped wave equation explicitly. The decay estimates in Theorem 1.2 are based on the upper bounds for the kernel functions in the representation of solutions obtained in Proposition 1.1. The upper bounds are derived in Proposition 2.2 prior to the proof of Theorem 1.2.

Lemma 2.1. *Assume that f satisfies the damped degenerate wave type equation*

$$\begin{cases} \partial_{tt}f - (\nu\,\partial_{22} + \eta\,\partial_{11})\partial_t f + \eta\nu\,\partial_{11}\,\partial_{22}f + \partial_{11}\Delta^{-1}f = F, \\ f(x,0) = f_0(x), \quad (\partial_t f)(x,0) = f_1(x). \end{cases} \tag{2.1}$$

Then, f can be explicitly represented as

$$f(t) = G_1(t)f_1 + G_2(t)f_0 + \int_0^t G_1(t-\tau)F(\tau)\,\mathrm{d}\tau, \tag{2.2}$$

where G_1 and G_2 are two Fourier multiplier operators with their symbols given by

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}, \tag{2.3}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2 + (\eta\xi_1^2 + \nu\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0 \tag{2.4}$$

or

$$\lambda_1 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) - \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}, \tag{2.5a}$$

$$\lambda_2 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) + \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}. \tag{2.5b}$$

When $\lambda_1 = \lambda_2$, (2.2) remains valid if we replace G_1 and G_2 in (2.3) by their corresponding limit form, namely,

$$G_1(\xi,t) = \lim_{\lambda_2\to\lambda_1} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = te^{\lambda_1 t}$$

and

$$G_2(\xi,t) = \lim_{\lambda_2\to\lambda_1} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} = e^{\lambda_1 t} - \lambda_1 te^{\lambda_1 t}.$$

Proof. We first focus on the case when $F \equiv 0$. Since $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are the roots of the characteristic equation in (2.4), we can decompose the second-order differential operator as follows:

$$(\partial_t - \lambda_1(D))(\partial_t - \lambda_2(D))f = 0 \tag{2.6}$$

and

$$(\partial_t - \lambda_2(D))(\partial_t - \lambda_1(D))f = 0, \tag{2.7}$$

where $\lambda_1(D)$ and $\lambda_1(D)$ are the Fourier multiplier operators with their symbols given by $\lambda_1(\xi)$ and $\lambda_2(\xi)$, or

$$\begin{aligned}
\lambda_1(D) &= \frac{1}{2}(\nu\,\partial_{22} + \eta\,\partial_{11}) - \frac{1}{2}\sqrt{(\nu\,\partial_{22} + \eta\,\partial_{11})^2 - 4(\nu\eta\,\partial_{1122} + \partial_{11}\Delta^{-1})},\\
\lambda_2(D) &= \frac{1}{2}(\nu\,\partial_{22} + \eta\,\partial_{11}) + \frac{1}{2}\sqrt{(\nu\,\partial_{22} + \eta\,\partial_{11})^2 - 4(\nu\eta\,\partial_{1122} + \partial_{11}\Delta^{-1})}.
\end{aligned}$$

We can rewrite (2.6) and (2.7) into two systems:

$$\begin{cases} (\partial_t - \lambda_1(D))g = 0,\\ (\partial_t - \lambda_2(D))f = g, \end{cases} \tag{2.8}$$

and

$$\begin{cases} (\partial_t - \lambda_2(D))h = 0,\\ (\partial_t - \lambda_1(D))f = h. \end{cases} \tag{2.9}$$

By taking the difference of the second equations of (2.8) and (2.9), we obtain

$$(\lambda_1(D) - \lambda_2(D))f = g - h$$

or

$$f = (\lambda_1(D) - \lambda_2(D))^{-1}(g - h). \tag{2.10}$$

Solving the first equations of (2.8) and (2.9) yields

$$g(t) = g(0)e^{\lambda_1(D)t} = ((\partial_t f)(0) - \lambda_2(D)f(0))e^{\lambda_1(D)t} \tag{2.11}$$

and

$$h(t) = h(0)e^{\lambda_2(D)t} = ((\partial_t f)(0) - \lambda_1(D)f(0))e^{\lambda_2(D)t}, \tag{2.12}$$

where we have used second equations of (2.8) and (2.9) to obtain the initial data $g(0)$ and $h(0)$. Inserting (2.11) and (2.12) in (2.10) leads to

$$\begin{aligned}
f(t) &= (\lambda_1(D) - \lambda_2(D))^{-1}\Big((e^{\lambda_1(D)t} - e^{\lambda_2(D)t})(\partial_t f)(0)\\
&\qquad + (\lambda_1(D)e^{\lambda_2(D)t} - \lambda_2(D)e^{\lambda_1(D)t})f(0)\Big)\\
&= G_1 f_1 + G_2 f_0,
\end{aligned}$$

where

$$\begin{aligned} G_1 &= \frac{e^{\lambda_1(D)t} - e^{\lambda_2(D)t}}{\lambda_1(D) - \lambda_2(D)}, \\ G_2 &= \frac{\lambda_1(D)e^{\lambda_2(D)t} - \lambda_2(D)e^{\lambda_2(D)t}}{\lambda_1(D) - \lambda_2(D)}. \end{aligned}$$

When F in (2.1) is not identically zero, the formula in (2.2) is obtained by Duhamel's principle. This completes the proof of Lemma 2.1. □

We are now ready to prove Proposition 1.1.

Proof of Proposition 1.1. This is a direct consequence of Lemma 2.1. In fact, according to Lemma 2.1,

$$u(t) = G_2(t)u_0 + G_1(t)(\partial_t u)(x,0), \quad \theta(t) = G_2(t)\theta_0 + G_1(t)(\partial_t \theta)(x,0). \tag{2.13}$$

Since u and θ satisfy the original linearized equations

$$\begin{aligned} \partial_t u_1 &= \nu\, \partial_{22} u_1 - \partial_1 \partial_2 \Delta^{-1}\theta, \\ \partial_t u_2 &= \nu\, \partial_{22} u_2 + \partial_{11}\Delta^{-1}\theta, \\ \partial_t \theta &= \eta\, \partial_{11}\theta - u_2, \end{aligned}$$

we obtain

$$\begin{aligned} (\partial_t u_1)(x,0) &= \nu\, \partial_{22} u_{10} - \partial_1 \partial_2 \Delta^{-1}\theta_0, \\ (\partial_t u_2(x,0) &= \nu\, \partial_{22} u_{20} + \partial_{11}\Delta^{-1}\theta_0, \\ (\partial_t \theta)(x,0) &= \eta\, \partial_{11}\theta_0 - u_{20}. \end{aligned}$$

Inserting them in (2.13), we obtain

$$\begin{aligned} u_1(t) &= (G_2(t) + \nu\, \partial_{22} G_1) u_{10} - \partial_1 \partial_2 \Delta^{-1} G_1 \theta_0, \\ u_2(t) &= (G_2(t) + \nu\, \partial_{22} G_1) u_{20} + \partial_{11} \Delta^{-1} G_1 \theta_0, \\ \theta(t) &= -G_1 u_{20} + (G_2 + \eta\, \partial_{11} G_1)\theta_0, \end{aligned}$$

which are the representations in (1.13), (1.14), and (1.15). This completes the proof of Proposition 1.1. □

In order to prove Theorem 1.2, we need to understand the behavior of the kernel functions K_1 through K_5. Clearly, their behavior depends on the frequency ξ. To obtain a definite behavior for each kernel function, we need to divide the whole frequency space $\mathbb{R}^2$ into subdomains. The following proposition specifies these subdomains and the behavior of the kernel functions.

Proposition 2.2. *Assume the kernel functions* K_1 *through* K_5 *are given by* (1.16)–(1.20) *with* G_1 *and* G_2 *defined in* (1.21) *and* (1.22). *Set*

$$S_1 = \left\{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \nu\eta\xi_1^2\xi_2^2 + \xi_1^2|\xi|^{-2} \geq \frac{3}{16}(\nu\xi_2^2 + \eta\xi_1^2)^2 \right\},$$
$$S_2 = \mathbb{R}^2 \setminus S_1.$$

The kernel functions K_1 *through* K_5 *can then be bounded as follows:*

(a) *Let* $\xi \in S_1$. *Then,*

$$\operatorname{Re}\lambda_1 \leq -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2), \quad \operatorname{Re}\lambda_2 \leq -\frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2),$$

where Re *denotes the real part, and, for constants* $c_0 > 0$ *and* $C > 0$,

$$|K_1(\xi, t)|, |K_5(\xi, t)| \leq Ce^{-c_0|\xi|^2 t}, \tag{2.14}$$

$$|K_2(\xi, t)|, |K_3(\xi, t)|, |K_4(\xi, t)| \leq Cte^{-c_0|\xi|^2 t}. \tag{2.15}$$

(b) *Let* $\xi \in S_2$. *Then,*

$$\lambda_1 \leq -\frac{3}{4}(\nu\xi_2^2 + \eta\xi_1^2), \quad \lambda_2 \leq -\frac{\nu\eta\xi_1^2\xi_2^2 + \xi_1^2|\xi|^{-2}}{\nu\xi_2^2 + \eta\xi_1^2},$$

$$|K_1|,\ |K_5| \leq Ce^{-(3/4)(\nu\xi_2^2+\eta\xi_1^2)t} + Ce^{-((\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2))t} \tag{2.16}$$

and

$$\begin{aligned} |K_2| &\leq \frac{C|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0|\xi|^2 t} + \frac{C|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}, \\ |K_3| &\leq \frac{C|\xi_1|^2}{|\xi|^4}e^{-c_0|\xi|^2 t} + \frac{C|\xi_1|^2}{|\xi|^4}e^{-(c_0\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}, \\ |K_4| &\leq \frac{C}{|\xi|^2}e^{-c_0|\xi|^2 t} + \frac{C}{|\xi|^2}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}. \end{aligned} \tag{2.17}$$

Proof. To prove the bounds in (a), we further divide S_1 into two subsets:

$$S_{11} = \{\xi \in S_1 \mid (\nu\xi_2^2 + \eta\xi_1^2)^2 \geq 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2})\},$$
$$S_{12} = S_1 \setminus S_{11}.$$

For any $\xi \in S_{11}$,

$$0 \leq (\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) \leq \frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2)^2.$$

According to the formula for λ_1 and λ_2 in (2.5a)–(2.5b), λ_1 and λ_2 are real and satisfy

$$\lambda_1 \le -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2), \quad \lambda_2 \le -\frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2).$$

By the mean-value theorem, for a constant $C > 0$,

$$|G_1| = \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right| \le te^{-C|\xi|^2 t}. \tag{2.18}$$

Writing G_2 in (1.21) as

$$G_2 = e^{\lambda_1 t} - \lambda_1 G_1$$

and using the simple fact that $x^m e^{-x} \le C(m)$ for any $x \ge 0$ and $m \ge 0$, we can bound K_1 and K_5 as follows:

$$\begin{aligned} |K_1| &\le |G_2| + \nu|\xi_2^2|\,|G_1| \le e^{-c_0|\xi|^2 t} + C|\xi|^2 te^{-C|\xi|^2 t} + \nu|\xi_2^2|te^{-C|\xi|^2 t} \\ &\le Ce^{-c_0|\xi|^2 t}, \\ |K_5| &\le |G_2| + \eta|\xi_1^2|\,|G_1| \le Ce^{-c_0|\xi|^2 t}, \end{aligned}$$

where $C > 0$ and $c_0 > 0$ are constants. The bounds K_2, K_3 and K_4 follow directly from (2.18). For $\xi \in S_{12}$,

$$(\nu\xi_2^2 + \eta\xi_1^2)^2 < 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}),$$

and, as a consequence, λ_1 and λ_2 are complex numbers,

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2) - \frac{i}{2}\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}, \\ \lambda_2 &= -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2) + \frac{i}{2}\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}. \end{aligned}$$

Then,

$$\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2).$$

In addition,

$$\begin{aligned} |G_1| &= \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right| \\ &= e^{-(1/2)(\nu\xi_2^2+\eta\xi_1^2)t}\left|\frac{\sin(t\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2})}{\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}}\right| \\ &\le te^{-(1/2)(\nu\xi_2^2+\eta\xi_1^2)t}. \end{aligned}$$

The desired upper bounds for K_1 through K_5 then follow as before.

We now prove the bounds in (b). For $\xi \in S_2$,

$$(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}) \geq \frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2)^2. \tag{2.19}$$

Then, λ_1 and λ_2 are both real. Clearly, λ_1 satisfies

$$\lambda_1 \leq -\frac{3}{4}(\nu\xi_2^2 + \eta\xi_1^2). \tag{2.20}$$

To obtain the upper bound for λ_2, we try to make the terms in the representation of λ_2 have the same sign, and obtain

$$\begin{aligned} \lambda_2 &= -\frac{1}{2}\Big((\nu\xi_2^2 + \eta\xi_1^2) - \sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2})}\Big) \\ &= -2\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}}{\nu\xi_2^2 + \eta\xi_1^2 + \sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2})}} \\ &\leq -\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}}{\nu\xi_2^2 + \eta\xi_1^2}. \end{aligned} \tag{2.21}$$

It then follows from (2.19), (2.20), and (2.21) that

$$\begin{aligned} |G_1| &\leq \frac{1}{\sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2})}} \\ &\quad \times (e^{-(3/4)(\nu\xi_2^2+\eta\xi_1^2)t} + e^{-((\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2))t}) \\ &\leq \frac{2}{\nu\xi_2^2 + \eta\xi_1^2}(e^{-(3/4)(\nu\xi_2^2+\eta\xi_1^2)t} + e^{-((\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2))t}) \\ &\leq \frac{C}{|\xi|^2}e^{-c_0|\xi|^2t} + \frac{C}{|\xi|^2}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t} \\ &:= M(\xi,t), \end{aligned}$$

where $c_0 > 0$ is a constant. Therefore,

$$\begin{aligned} |K_2| &\leq \frac{C|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0|\xi|^2t} + \frac{C|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}, \\ |K_3| &\leq \frac{C|\xi_1|^2}{|\xi|^4}e^{-c_0\,|\xi|^2t} + \frac{C|\xi_1|^2}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t} \end{aligned}$$

and

$$|K_4| \leq \frac{C}{|\xi|^2}e^{-c_0|\xi|^2t} + \frac{C}{|\xi|^2}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}.$$

Note that K_1 is bounded by

$$\begin{aligned}|K_1| \le |G_2| + \nu|\xi_2^2|\,|G_1| \le e^{\lambda_1 t} &\le e^{\lambda_1 t} + |\lambda_1|\,|G_1| + \nu|\xi_2^2|\,|G_1| \\ &\le Ce^{-(3/4)(\nu\xi_2^2+\eta\xi_1^2)t} + Ce^{-(\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2)t}.\end{aligned}$$

Also, K_5 shares the same bound. This completes the proof of Proposition 2.2. □

To prove Theorem 1.2, we recall a lemma that provides an explicit decay rate for the heat kernel associated with a fractional Laplacian Λ^α ($\alpha \in \mathbb{R}$). Here, the fractional Laplacian operator can be defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi). \tag{2.22}$$

The proof of the lemma can be found in many references (see, e.g., [56]).

Lemma 2.3. *Let $\alpha \ge 0$, $\beta > 0$, and $1 \le q \le p \le \infty$. Then, there exists a constant C such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^p(\mathbb{R}^d)} \le Ct^{-\alpha/\beta-(d/\beta)(1/q-1/p)}\|f\|_{L^q(\mathbb{R}^d)}.$$

In addition to the fractional operator defined in (2.22), we also use the fractional operators Λ_i^σ with $i = 1,2$ defined by $\widehat{\Lambda_i^\sigma f}(\xi) = |\xi_i|^\sigma \hat{f}(\xi)$, with $\xi = (\xi_1,\xi_2)$. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Taking the $\mathring{H}^s$-norm of u_1 in (1.13), applying Plancherel's theorem, and dividing the spatial domain $\mathbb{R}^2$ as in Proposition 1.1, we obtain

$$\begin{aligned}\|u_1(t)\|_{\mathring{H}^s(\mathbb{R}^2)} &\le \|\Lambda^s K_1(t)u_0\|_{L^2(\mathbb{R}^2)} + \|\Lambda^s K_2(t)\theta_0\|_{L^2(\mathbb{R}^2)} \\ &\le C\big\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\big\|_{L^2(S_1)} + C\big\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\big\|_{L^2(S_2)} \\ &\quad + C\big\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\big\|_{L^2(S_1)} \\ &\quad + C\big\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\big\|_{L^2(S_2)}.\end{aligned}$$

To bound the terms on the righthand side, we invoke the upper bounds for K_1 and K_2 obtained in Proposition 2.2. By (2.14) in Proposition 2.2, Plancherel's theorem, and Lemma 2.3,

$$\begin{aligned}\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_1)} &\le C\big\||\xi|^s e^{-c_0|\xi|^2 t}\widehat{u_0}(\xi)\big\|_{L^2(S_1)} \\ &= C\big\||\xi|^s\,|\xi_1|^\sigma e^{-c_0|\xi|^2 t}|\xi_1|^{-\sigma}\widehat{u_0}(\xi)\big\|_{L^2(S_1)} \\ &\le C\big\||\xi|^{s+\sigma} e^{-c_0|\xi|^2 t}|\xi_1|^{-\sigma}\widehat{u_0}(\xi)\big\|_{L^2(S_1)} \\ &= C\|\Lambda^{s+\sigma} e^{c_0\Delta t}\Lambda_1^{-\sigma}u_0\|_{L^2(\mathbb{R}^2)} \\ &\le Ct^{-(1/2)(s+\sigma)}\|\Lambda_1^{-\sigma}u_0\|_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.23}$$

By (2.16) in Proposition 2.2,

$$
\begin{aligned}
\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_2)}
&\le C\||\xi|^s e^{-c_0|\xi|^2 t}\widehat{u_0}(\xi)\|_{L^2(S_2)} \\
&\quad + C\||\xi|^s e^{-((\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2))t}\widehat{u_0}(\xi)\|_{L^2(S_2)}.
\end{aligned}
$$

The first part can be bounded the same way as (2.23). To give a precise upper bound on the second part, we divide the consideration into two cases: $\xi \in S_{21}$ and $\xi \in S_{22}$, where

$$
\begin{aligned}
S_{21} &= \{\xi \in S_2 : |\xi_1| \ge |\xi_2|\}, \\
S_{22} &= \{\xi \in S_2 : |\xi_1| < |\xi_2|\},
\end{aligned}
$$

with S_2 being defined as in Proposition 2.2. For $\xi \in S_{21}$,

$$
-\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2} \le -C|\xi_2|^2 - C|\xi_1|^2\,|\xi|^{-4} \le -C|\xi_2|^2 \tag{2.24}
$$

and for $\xi \in S_{22}$,

$$
-\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2\,|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2} \le -C|\xi_1|^2 - C|\xi_1|^2\,|\xi|^{-4} \le -C|\xi_1|^2. \tag{2.25}
$$

Therefore,

$$
\begin{aligned}
&\||\xi|^s e^{-((\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2\,|\xi|^{-2})/(\nu\xi_2^2+\eta\xi_1^2))t}\widehat{u_0}(\xi)\|_{L^2(S_2)} \\
&\quad\le C\||\xi|^s e^{-C|\xi_2|^2 t}\widehat{u_0}(\xi)\|_{L^2(S_{21})} + C\||\xi|^s e^{-C|\xi_1|^2 t}\widehat{u_0}(\xi)\|_{L^2(S_{22})} \\
&\quad\le C\||\xi|^s\,|\xi_2|^{\sigma} e^{-C|\xi_2|^2 t}|\xi_2|^{-\sigma}\widehat{u_0}(\xi)\|_{L^2(S_{21})} \\
&\qquad + C\||\xi|^s\,|\xi_1|^{\sigma} e^{-C|\xi_1|^2 t}|\xi_1|^{-\sigma}\widehat{u_0}(\xi)\|_{L^2(S_{22})} \\
&\quad\le Ct^{-\sigma/2}\|u_0\|_{\mathring{H}^{s,-\sigma}}.
\end{aligned}
$$

We now estimate

$$
\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_1)}.
$$

Invoking (2.15) in Proposition 2.2 and proceeding as in (2.23), we have

$$
\begin{aligned}
\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_1)} &\le Ct\||\xi|^s e^{-c_0|\xi|^2 t}\widehat{\theta_0}(\xi)\|_{L^2(S_1)} \\
&\le Ct^{-(1/2)(s+\sigma)+1}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{2.26}
$$

We now turn to $\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_2)}$. By (2.17),

$$\||\xi|^s K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_2)} \le C\left\||\xi|^s\frac{\xi_1\xi_2}{|\xi|^4}e^{-c_0|\xi|^2t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_2)} + C\left\||\xi|^s\frac{|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_2)}. \tag{2.27}$$

The first part in (2.27) can be bounded as in (2.23) and (2.26),

$$\begin{aligned}\left\||\xi|^s\frac{\xi_1\xi_2}{|\xi|^4}e^{-c_0|\xi|^2t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_2)} &\le \||\xi|^{s-2}e^{-c_0|\xi|^2t}\widehat{\theta_0}(\xi)\|_{L^2(\mathbb{R}^2)}\\ &\le Ct^{-(1/2)(s+\sigma)+1}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

To estimate the second piece in (2.27), we invoke the simple fact that

$$x^m e^{-x} \le C(m)$$

valid for any $m \ge 0$ and $x \ge 0$, and proceed as in (2.24) and (2.25) to obtain

$$\begin{aligned}&|\xi|^s\frac{|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}\\ &\quad= |\xi|^{s-1}\frac{|\xi_2|}{|\xi|}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}t^{-1/2}\frac{|\xi_1|t^{1/2}}{|\xi|^2}e^{-c_0(\xi_1^2/|\xi|^4)t}\\ &\quad\le Ct^{-1/2}|\xi|^{s-1}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}\\ &\quad\le \begin{cases} Ct^{-1/2}|\xi|^{s-1}e^{-C\xi_2^2t} & \text{for } \xi \in S_{21},\\ Ct^{-1/2}|\xi|^{s-1}e^{-C\xi_1^2t} & \text{for } \xi \in S_{22}.\end{cases}\end{aligned}$$

Therefore, the second term in (2.27) can be bounded by

$$\begin{aligned}&\left\||\xi|^s\frac{|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_2)}\\ &\quad\le \left\||\xi|^s\frac{|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_{21})}\\ &\qquad+ \left\||\xi|^s\frac{|\xi_1|\,|\xi_2|}{|\xi|^4}e^{-c_0(\xi_1^2\xi_2^2/|\xi|^2)t}e^{-c_0(\xi_1^2/|\xi|^4)t}\widehat{\theta_0}(\xi)\right\|_{L^2(S_{22})}\\ &\quad\le Ct^{-1/2}\||\xi|^{s-1}e^{-C\xi_1^2t}\widehat{\theta_0}(\xi)\|_{L^2} + Ct^{-1/2}\||\xi|^{s-1}e^{-C\xi_2^2t}\widehat{\theta_0}(\xi)\|_{L^2}\\ &\quad\le Ct^{-1/2-\sigma/2}\|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}.\end{aligned}$$

We have completed the estimates of $\|u_1(t)\|_{\mathring{H}^s(\mathbb{R}^2)}$. Collecting the estimates yields

$$\begin{aligned}\|u_1(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \leq{}& Ct^{-(1/2)(s+\sigma)}\|\Lambda_1^{-\sigma}u_{10}\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-\sigma/2}\|u_{10}\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)}\\ &+ Ct^{-(1/2)(s+\sigma)+1}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-1/2-\sigma/2}\|\theta_0\|_{\mathring{H}^{s-1,-\sigma}(\mathbb{R}^2)}.\end{aligned}$$

Also, $\|u_2(t)\|_{\mathring{H}^s(\mathbb{R}^2)}$ can be estimated very similarly. Only the last piece is bounded slightly differently. Its upper bound is

$$\begin{aligned}\|u_2(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \leq{}& Ct^{-(1/2)(s+\sigma)}\|\Lambda_1^{-\sigma}u_{20}\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-\sigma/2}\|u_{20}\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)}\\ &+ Ct^{-(1/2)(s+\sigma)+1}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-1-\sigma/2}\|\theta_0\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)}.\end{aligned}$$

The estimate of $\|\theta(t)\|_{\mathring{H}^s(\mathbb{R}^2)}$ is also similar:

$$\begin{aligned}\|\theta(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \leq{}& Ct^{-(1/2)(s+\sigma)+1}\|\Lambda_1^{-\sigma}u_{20}\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-\sigma/2}\|u_{20}\|_{\mathring{H}^{s-2,-\sigma}(\mathbb{R}^2)}\\ &+ Ct^{-(1/2)(s+\sigma)}\|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}\\ &+ Ct^{-\sigma/2}\|\theta_0\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)}.\end{aligned}$$

This completes the proof of Theorem 1.2. □

3. PROOF OF THEOREM 1.3

This section proves Theorem 1.3. The proof makes use of the wave structure in (1.12) to construct a Lyapunov functional for the Fourier piece of the solution away from the axes in the frequency space. The construction involves a suitable combination of two energy inequalities.

Proof of Theorem 1.3. Let $\hat{\varphi}$ be the Fourier cutoff function defined in (1.23). Taking the convolution of φ with the velocity equation in (1.12) leads to

$$\begin{aligned}\partial_{tt}(\varphi * u) - (\eta\,\partial_{11} + \nu\,\partial_{22})\partial_t(\varphi * u)&\\ + \nu\eta\,\partial_{11}\,\partial_{22}(\varphi * u) + \partial_{11}\Delta^{-1}(\varphi * u) &= 0.\end{aligned} \tag{3.1}$$

Dotting (3.1) with $\partial_t(\varphi * u)$, we find

$$\begin{aligned}\frac{1}{2}\frac{dd}{\mathrm{d}t}(\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2)&\\ + \nu\|\partial_2\,\partial_t(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1\,\partial_t(\varphi * u)\|_{L^2}^2 &= 0,\end{aligned} \tag{3.2}$$

where we have written $\mathcal{R}_1 = \partial_1(-\Delta)^{-1/2}$, the standard notation for the Riesz transform. Dotting (3.1) with $\varphi * u$ yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1(\varphi * u)\|_{L^2}^2) + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2&\\ + \nu\eta\|\partial_{12}(\varphi * u)\|_{L^2}^2 + \int \partial_{tt}(\varphi * u)\cdot(\varphi * u)\,\mathrm{d}x &= 0.\end{aligned}$$

Writing

$$\int \partial_{tt}(\varphi * u)\cdot(\varphi * u)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_t(\varphi * u)\cdot(\varphi * u)\,\mathrm{d}x - \|\partial_t(\varphi * u)\|_{L^2}^2,$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1(\varphi * u)\|_{L^2}^2 + 2(\partial_t(\varphi * u),(\varphi * u)))&\\ + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \nu\eta\|\partial_{12}(\varphi * u)\|_{L^2}^2 - \|\partial_t(\varphi * u)\|_{L^2}^2 &= 0,\end{aligned}\tag{3.3}$$

where (f,g) denotes the L^2-inner product. Let $\lambda > 0$. Then, (3.2) $+\lambda$ (3.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) + 2B(t) = 0,\tag{3.4}$$

where

$$\begin{aligned}A(t) :=\ &\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2\\ &+ \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 + 2\lambda(\partial_t(\varphi * u),(\varphi * u)),\\ B(t) :=\ &\nu\|\partial_2\,\partial_t(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1\,\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2\\ &- \lambda\|\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2.\end{aligned}$$

Our immediate goal here is to show that, if we choose $\lambda = \lambda(\nu,\eta,a_1,a_2)$ suitably, then there is a constant $C_0 = C_0(\nu,\eta,a_1,a_2) > 0$ such that, for any $t \geq 0$,

$$B(t) \geq C_0A(t).\tag{3.5}$$

Recall that $a_1 > 0$ and $a_2 > 0$ are the parameters involved in the definition of the frequency cutoff function defined by (1.23). We now prove (3.5). By Plancherel's theorem,

$$\begin{aligned}\|\partial_2\,\partial_t(\varphi * u)\|_{L^2}^2 &= \int_{|\xi_1|\geq a_1,\,|\xi_2|\geq a_2} |\xi_2\,\partial_t(\hat{\varphi}\hat{u}(\xi,t))|^2\,\mathrm{d}\xi\\ &\geq a_2^2\|\partial_t(\varphi * u)\|_{L^2}^2.\end{aligned}\tag{3.6}$$

Similarly,

$$\|\partial_1 \partial_t(\varphi * u)\|_{L^2}^2 \geq a_1^2 \|\partial_t(\varphi * u)\|_{L^2}^2, \tag{3.7}$$

$$\|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_1^2 \|\partial_2(\varphi * u)\|_{L^2}^2, \tag{3.8}$$

$$\|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_2^2 \|\partial_1(\varphi * u)\|_{L^2}^2, \tag{3.9}$$

$$\|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_1^2 a_2^2 \|\varphi * u\|_{L^2}^2. \tag{3.10}$$

If $\lambda > 0$ satisfies

$$\lambda \leq \frac{1}{2}(\nu a_2^2 + \eta a_1^2),$$

then, by (3.6)–(3.10),

$$\begin{aligned}
B(t) &\geq (\nu a_2^2 + \eta a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 - \lambda\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu a_1^2\|\partial_2(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu a_2^2\|\partial_1(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu a_1^2 a_2^2\|\varphi * u\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 \\
&\geq \frac{1}{2}(\nu a_2^2 + \eta a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu a_1^2\|\partial_2(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu a_2^2\|\partial_1(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu a_1^2 a_2^2\|\varphi * u\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{1}{4}(\nu a_2^2 + \eta a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu a_1^2 a_2^2\|\varphi * u\|_{L^2}^2 \\
&\qquad \geq \frac{1}{2}\sqrt{\nu a_2^2 + \eta a_1^2}\sqrt{\lambda\eta\nu a_1^2 a_2^2}(\partial_t(\varphi * u), \varphi * u).
\end{aligned}$$

Therefore,

$$\begin{aligned}
B(t) &\geq \frac{1}{4}(\nu a_2^2 + \eta a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu a_1^2\|\partial_2(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{4}\lambda\eta\nu a_2^2\|\partial_1(\varphi * u)\|_{L^2}^2 \\
&\quad + \frac{1}{2}\sqrt{\nu a_2^2 + \eta a_1^2}\sqrt{\lambda\eta\nu a_1^2 a_2^2}(\partial_t(\varphi * u), \varphi * u).
\end{aligned}$$

If we choose C_0 as

$$C_0 = \frac{1}{4}\min\left\{(\nu a_2^2 + \eta a_1^2), \lambda, \eta a_1^2, \nu a_2^2, \frac{1}{\sqrt{\lambda}}\sqrt{\nu a_2^2 + \eta a_1^2}\sqrt{\eta\nu a_1^2 a_2^2}\right\},$$

then $B(t) \geq C_0 A(t)$, which is (3.5). Inserting (3.5) in (3.4) leads to

$$A(t) \leq A(0)e^{-C_0 t}. \tag{3.11}$$

To prove (1.24), we derive a lower bound for $A(t)$. By (3.9), (3.10), and the Cauchy-Schwarz inequality,

$$\begin{aligned}
A(t) \geq{}& \|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu a_1^2 a_2^2\|\varphi * u\|_{L^2}^2 \\
&+ \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 \\
&- \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 - 2\lambda^2\|\varphi * u\|_{L^2}^2 \\
={}& \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + (\eta\nu a_1^2 a_2^2 - 2\lambda^2)\|\varphi * u\|_{L^2}^2 \\
&+ \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2.
\end{aligned}$$

If λ is selected to satisfy

$$\eta\nu a_1^2 a_2^2 - 2\lambda^2 \geq \frac{1}{2}\eta\nu a_1^2 a_2^2 \quad \text{or} \quad \lambda \leq \frac{1}{2}\sqrt{\eta\nu}\, a_1 a_2,$$

then $A(t)$ is bounded below by

$$\begin{aligned}
A(t) \geq{}& \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \frac{1}{2}\eta\nu a_1^2 a_2^2\|\varphi * u\|_{L^2}^2 \\
&+ \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 \\
\geq{}& C(\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\varphi * u\|_{L^2}^2 + \|\nabla(\varphi * u)\|_{L^2}^2),
\end{aligned} \tag{3.12}$$

where $C = C(\nu, \eta, a_1, a_2) > 0$ is a constant. We now derive an upper bound for $A(0)$. Recalling that (u, θ) satisfies

$$\begin{aligned}
\partial_t u_1 &= \nu\,\partial_{22}u_1 - \Delta^{-1}\partial_1\partial_2\theta, \\
\partial_t u_2 &= \nu\,\partial_{22}u_2 + \Delta^{-1}\partial_1\partial_1\theta, \\
\partial_t\theta &= \eta\,\partial_{11}\theta - u_2,
\end{aligned}$$

we obtain

$$\begin{aligned}
\partial_t u_1(0) &= \nu\,\partial_{22}u_{01} - \Delta^{-1}\partial_1\partial_2\theta_0, \\
\partial_t u_2(0) &= \nu\,\partial_{22}u_{02} + \Delta^{-1}\partial_1\partial_1\theta_0,
\end{aligned}$$

and thus

$$||(\partial_t(\phi * u)(0)||_{L^2}^2 \le 2\nu^2||\partial_{22}(\varphi * u_0)||_{L^2}^2 + 2||\varphi * \theta_0||_{L^2}^2, \tag{3.13}$$

where we used the fact that Riesz transforms are bounded in L^q with $1 < q < \infty$ (see [50]):

$$\|\Delta^{-1}\partial_1\,\partial_2 f\|_{L^q} \le C\|f\|_{L^q}.$$

In addition, if we invoke the inequality

$$2\lambda(\partial_t(\varphi * u), (\varphi * u)) \le ||\partial_t(\phi * u)||_{L^2}^2 + \lambda^2||\varphi * u||_{L^2}^2,$$

we obtain the following upper bound for $A(0)$:

$$\begin{aligned} A(0) := {}& ||\partial_t(\varphi * u)(0)||_{L^2}^2 + ||\mathcal{R}_1(\varphi * u_0)||_{L^2}^2 \\ & \quad + \eta\nu||\partial_{12}(\varphi * u_0)||_{L^2}^2 + \lambda\nu||\partial_2(\varphi * u_0)||_{L^2}^2 \\ & \quad \lambda\,\eta||\partial_1(\varphi * u_0)||_{L^2}^2 + 2\lambda(\partial_t(\varphi * u)(0), (\varphi * u_0)), \\ \le {}& 4\nu^2||\partial_{22}(\varphi * u_0)||_{L^2}^2 + 4||\varphi * \theta_0||_{L^2}^2 \\ & \quad + (1 + \lambda^2)||\varphi * u_0||_{L^2}^2 + \eta\nu||\partial_{12}(\varphi * u_0)||_{L^2}^2 \\ & \quad + \lambda\nu||\partial_2(\varphi * u_0)||_{L^2}^2 + \lambda\eta||\partial_1(\varphi * u_0)||_{L^2}^2 \\ \le {}& C(||\varphi * u_0||_{H^2}^2 + ||\varphi * \theta_0||_{L^2}^2). \end{aligned} \tag{3.14}$$

Combining (3.11), (3.12), and (3.14), we find that

$$\begin{aligned} & ||\partial_t(\varphi * u)(t)||_{L^2}^2 + ||(\varphi * u)(t)||_{L^2}^2 + ||\nabla(\varphi * u)(t)||_{L^2}^2 \\ & \quad \le C(||\varphi * u_0||_{H^2}^2 + ||\varphi * \theta_0||_{L^2}^2)e^{-C_0 t}, \end{aligned}$$

which is (1.24). The proof for the exponential decay upper bound for θ in (1.25) is very similar. In fact, since θ satisfies the same wave equation as u, most of the lines for u remain valid when we replace u by θ and replace the bound in (3.13) by

$$||(\partial_t(\phi * \theta)(0)||_{L^2}^2 \le 2\eta^2||\partial_{11}(\varphi * \theta_0)||_{L^2}^2 + 2||\varphi * u_{02}||_{L^2}^2.$$

This completes the proof of Theorem 1.3. □

4. PROOF OF THEOREM 1.4

This section is devoted to the proof of Theorem 1.4. As was outlined in the Introduction, the proof uses the bootstrapping argument, and the major step is to establish the energy inequality

$$E(t) \le C_1 E(0) + C_2 E(t)^{3/2}, \tag{4.1}$$

where C_1 and C_2 are constants and $E(t)$ is the energy functional defined in (1.27), or

$$E(t) = \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2\,\mathrm{d}\tau + 2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2\,\mathrm{d}\tau + \delta\int_0^t \|\partial_1 u_2\|_{L^2}^2\,\mathrm{d}\tau, \tag{4.2}$$

with $\delta > 0$ to be specified later. We then apply the bootstrapping argument to (4.1) to get the desired stability result.

Proof of Theorem 1.4. We define $E(t)$ as in (4.2). Our main efforts are devoted to establishing (4.1). This process consists of two major parts. The first is to estimate the H^2-norm of $(u\theta)$, while the second is to estimate $\|\partial_1 u\|_{L^2}^2$ and its time integral.

For a divergence-free vector field u, namely $\nabla\cdot u = 0$, we have

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\Delta u\|_{L^2} = \|\nabla\omega\|_{L^2},$$

where $\omega = \nabla\times u$ is the vorticity. Therefore, the H^2-norm of u is equivalent to the sum of the L^2-norm of u, the L^2-norm of ω, and the L^2-norm of $\nabla\omega$. To estimate the L^2-norm of (u,θ), we take the inner product of (u,θ) with the first two equations in (1.2) to obtain

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2\,\mathrm{d}\tau + 2\eta\int_0^t \|\partial_1\theta(\tau)\|_{L^2}^2\,\mathrm{d}\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{4.3}$$

To estimate the L^2-norm of $(\omega,\nabla\theta)$, we resort to the vorticity equation combined with the equation of θ,

$$\partial_t\omega + u\cdot\nabla\omega = \nu\,\partial_{22}\omega + \partial_1\theta, \tag{4.4}$$

$$\partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\,\partial_{11}\theta. \tag{4.5}$$

Taking the inner product of $(\omega,\nabla\theta)$ with the equations of ω and $\nabla\theta$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|\partial_2\omega\|_{L^2}^2 + \eta\|\partial_1\nabla\theta\|_{L^2}^2 = I_1 + I_2, \tag{4.6}$$

where

$$I_1 = \int(\partial_1\theta\omega - \nabla u_2\cdot\nabla\theta)\,\mathrm{d}x,$$

$$I_2 = -\int \nabla\theta\cdot\nabla u\cdot\nabla\theta\,\mathrm{d}x.$$

It is easy to check that $I_1 = 0$. In fact, writing ω and u in terms of the stream function ψ, namely,

$$\omega = \Delta\psi \qquad \text{and} \quad u = \nabla^{\perp}\psi := (-\partial_2\psi, \partial_1\psi),$$

we have

$$\begin{aligned} I_1 &= \int (\partial_1\theta\omega - \nabla u_2 \cdot \nabla\theta)\,\mathrm{d}x = \int (\partial_1\theta\Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta)\,\mathrm{d}x \\ &= \int (-\theta\Delta\,\partial_1\psi + \Delta\,\partial_1\psi\theta)\,\mathrm{d}x = 0. \end{aligned}$$

To bound I_2, we write out the four terms in I_2 explicitly:

$$\begin{aligned} I_2 &= -\int (\partial_1 u_1(\partial_1\theta)^2 + \partial_1 u_2\,\partial_1\theta\,\partial_2\theta + \partial_2 u_1\,\partial_1\theta\,\partial_2\theta + \partial_2 u_2(\partial_2\theta)^2)\,\mathrm{d}x \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned}$$

The terms on the righthand side can be bounded as follows. The key point here is to obtain upper bounds that are time integrable. By Lemma 1.5,

$$\begin{aligned} |I_{21}| &\le C\|\partial_1 u_1\|_{L^2}\,\|\partial_1\theta\|_{L^2}^{1/2}\,\|\partial_2\,\partial_1\theta\|_{L^2}^{1/2}\,\|\partial_1\theta\|_{L^2}^{1/2}\,\|\partial_1\,\partial_1\theta\|_{L^2}^{1/2} \\ &\le C\|\partial_1 u_1\|_{L^2}\,\|\partial_1\theta\|_{L^2}\,\|\partial_1\nabla\theta\|_{L^2}, \\ |I_{22}| &\le C\|\partial_1\theta\|_{L^2}\,\|\partial_1 u_2\|_{L^2}^{1/2}\,\|\partial_2\,\partial_1 u_2\|_{L^2}^{1/2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_1\,\partial_2\theta\|_{L^2}^{1/2} \\ &\le C\|\partial_1 u_2\|_{L^2}^{1/2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_1\theta\|_{L^2}\,\|\partial_2\nabla u\|_{L^2}^{1/2}\,\|\partial_1\nabla\theta\|_{L^2}^{1/2}, \\ |I_{23}| &\le C\,\|\partial_2\theta\|_{L^2}\,\|\partial_1\theta\|_{L^2}^{1/2}\,\|\partial_2\,\partial_1\theta\|_{L^2}^{1/2}\,\|\partial_2 u_1\|_{L^2}^{1/2}\,\|\partial_1\partial_2 u_1\|_{L^2}^{1/2}. \end{aligned}$$

By the divergence-free condition $\nabla \cdot u = 0$,

$$\begin{aligned} I_{24} &= \int \partial_1 u_1(\partial_2\theta)^2\,\mathrm{d}x = -2\int u_1\,\partial_2\theta\,\partial_1\,\partial_2\theta\,\mathrm{d}x \\ &\le C\|\partial_1\,\partial_2\theta\|_{L^2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_1\partial_2\theta\|_{L^2}^{1/2}\,\|u_1\|_{L^2}^{1/2}\,\|\partial_2 u_1\|_{L^2}^{1/2} \\ &= C\|u_1\|_{L^2}^{1/2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_2 u_1\|_{L^2}^{1/2}\,\|\partial_1\nabla\theta\|_{L^2}^{3/2}. \end{aligned}$$

Clearly, the sum of the powers of the terms that contain the favorable derivatives (∂_1 on θ and ∂_2 on u) is 2 in each upper bound above. Therefore, each upper bound is time integrable. Collecting the upper bounds on I_2 and inserting them in (4.6), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + 2\nu\|\partial_2\nabla u\|_{L^2}^2 + 2\eta\|\partial_1\nabla\theta\|_{L^2}^2 \\ \le C(\|u\|_{H^1} + \|\nabla\theta\|_{L^2})(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2). \end{aligned} \tag{4.7}$$

Integrating (4.7) over $[0,t]$ and combining with (4.3), we obtain

$$\|(u,\theta)\|_{H^1}^2 + 2\nu\int_0^t \|\partial_2 u(s)\|_{H^1}^2\,\mathrm{d}s + 2\eta\int_0^t \|\partial_1\theta(s)\|_{H^1}^2\,\mathrm{d}s \tag{4.8}$$

$$\le \|(u_0,\theta_0)\|_{H^1}^2 + C\int_0^t (\|u\|_{H^1} + \|\nabla\theta\|_{L^2})(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2)\,\mathrm{d}\tau$$

$$\le E(0) + CE(t)^{3/2}. \tag{4.9}$$

We also notice that the H^1-estimate is actually self-contained. The upper bound in (4.8) depends only on the H^1-norm level quantities. A simple consequence of (4.8) is that any initial small H^1 initial data leads to a global H^1 weak solution. However, we do not know the uniqueness of H^1-level solutions. It does not appear possible to show that H^1-solutions are unique. When we evaluate the difference $(\tilde{u},\tilde{\theta})$ of two solutions $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$, the terms generated by the nonlinearity $\int \tilde{u}\cdot\nabla u^{(1)}\cdot\tilde{u}\,\mathrm{d}x$ and $\int \tilde{u}\cdot\nabla\theta^{(1)}\cdot\tilde{\theta}\,\mathrm{d}x$ are hard to deal with. When the solutions are only at the H^1-level, it does not appear possible to bound them suitably. We have attempted to gain the horizontal dissipative effect in the velocity to control the nonlinear terms above, but the process fails because of the generation of extra bad terms that cannot be controlled when we estimate the solution difference at the L^2-level. This is one of the reasons we are seeking global H^2-solutions.

To control the H^2-norm, it then suffices to bound the L^2-norm of $(\nabla\omega,\Delta\theta)$. Applying ∇ to the first equation of (4.4) and dotting with $\nabla\omega$, and applying Δ to the second equation of (4.5) and dotting with $\Delta\theta$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2) + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\partial_1\Delta\theta\|_{L^2}^2 = J_1 + J_2 + J_3, \tag{4.10}$$

where

$$J_1 = \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\Delta\theta)\,\mathrm{d}x,$$

$$J_2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,\mathrm{d}x,$$

$$J_3 = -\int \Delta\theta\cdot\Delta(u\cdot\nabla\theta)\,\mathrm{d}x.$$

First, we verify that $J_1 = 0$. In fact, since $u_2 = \partial_1\psi$ and $\Delta\psi = \omega$, we have

$$\begin{aligned} J_1 &= \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\Delta\theta)\,\mathrm{d}x = \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta\,\partial_1\psi\Delta\theta)\,\mathrm{d}x \\ &= \int(\nabla\partial_1\theta\cdot\nabla\omega - \partial_1\omega\Delta\theta)\,\mathrm{d}x = \int(\nabla\partial_1\theta\cdot\nabla\omega + \partial_1\nabla\omega\cdot\nabla\theta)\,\mathrm{d}x \\ &= \int \partial_1(\nabla\theta\cdot\nabla\omega)\,\mathrm{d}x = 0. \end{aligned}$$

We now estimate J_3 and then J_2. The effort is still devoted to obtaining an upper bound that is time integrable for each term. After integration by parts,

$$\begin{aligned} J_3 &= -\int \Delta\theta\Delta u_1\,\partial_1\theta\,\mathrm{d}x - \int \Delta\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x \\ &\quad - 2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta\,\mathrm{d}x - 2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta\,\mathrm{d}x \\ &:= J_{31} + J_{32} + J_{33} + J_{34}. \end{aligned}$$

By Lemma 1.5,

$$\begin{aligned} |J_{31}| &\le C\|\partial_1\theta\|_{L^2}\,\|\Delta\|_{L^2}^{1/2}\,\|\partial_1\Delta\theta\|_{L^2}^{1/2}\,\|\Delta u_1\|_{L^2}^{1/2}\,\|\partial_2\Delta u_1\|_{L^2}^{1/2} \\ &\le C(\|\Delta\theta\|_{L^2} + \|\Delta u_1\|_{L^2})\|\partial_1\theta\|_{H^2}^{3/2}\,\|\partial_2\Delta u_1\|_{L^2}^{1/2}. \end{aligned} \tag{4.11}$$

The bound on the righthand side is time integrable. To bound J_{32}, we further decompose it into two terms:

$$\begin{aligned} J_{32} &= -\int \Delta\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x \\ &= -\int \partial_1\,\partial_1\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x - \int \partial_2\,\partial_2\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x \\ &= -\int \partial_1\,\partial_1\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x + \frac{1}{2}\int \Delta\,\partial_2 u_2(\partial_2\theta)^2\,\mathrm{d}x \\ &= -\int \partial_1\,\partial_1\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x - \frac{1}{2}\int \Delta\,\partial_1 u_1(\partial_2\theta)^2\,\mathrm{d}x \\ &= -\int \partial_1\,\partial_1\theta\Delta u_2\,\partial_2\theta\,\mathrm{d}x + \int \Delta u_1\,\partial_2\theta\,\partial_1\,\partial_2\theta\,\mathrm{d}x. \end{aligned}$$

Therefore, by Lemma 1.5,

$$\begin{aligned} |J_{32}| &\le C\|\partial_1\,\partial_1\theta\|_{L^2}\,\|\Delta u_2\|_{L^2}^{1/2}\,\|\partial_2\Delta u_2\,\|_{L^2}^{1/2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_1\,\partial_2\theta\|_{L^2}^{1/2} \\ &\qquad + C\|\partial_1\,\partial_2\theta\|_{L^2}\,\|\partial_2\theta\|_{L^2}^{1/2}\,\|\partial_1\,\partial_2\theta\|_{L^2}^{1/2} \\ &\qquad \times \|\Delta u_1\|_{L^2}^{1/2}\,\|\partial_2\Delta u_1\|_{L^2}^{1/2} \\ &\le C(\|\partial_2\theta\|_{L^2} + \|\Delta u\|_{L^2})\,\|\partial_1\nabla\theta\|_{L^2}^{3/2}\,\|\partial_2\Delta u\|_{L^2}^{1/2}. \end{aligned} \tag{4.12}$$

We note that J_{33} can be bounded as follows:

$$\begin{aligned} |J_{33}| &\le C\|\partial_1\nabla\theta\|_{L^2}\,\|\Delta\theta\|_{L^2}^{1/2}\,\|\partial_1\Delta\theta\|_{L^2}^{1/2}\,\|\nabla u_1\|_{L^2}^{1/2}\,\|\partial_2\nabla u_1\|_{L^2}^{1/2} \\ &\le C(\|\Delta\theta\|_{L^2} + \|\nabla u_1\|_{L^2})\|\partial_1\theta\|_{H^2}^{3/2}\,\|\partial_2\nabla u_1\|_{L^2}^{1/2}. \end{aligned} \tag{4.13}$$

By integration by parts,

$$\begin{aligned}
J_{34} &= -2\int (\partial_1 u_2\, \partial_1\, \partial_2\theta\Delta\theta + \partial_2 u_2\, \partial_2\, \partial_2\theta\Delta\theta)\, \mathrm{d}x \\
&= -2\int \partial_1 u_2\, \partial_1\, \partial_2\theta\Delta\theta\, \mathrm{d}x + 2\int \partial_1 u_1\, \partial_2\, \partial_2\theta\Delta\theta\, \mathrm{d}x \\
&= -2\int \partial_1 u_2\, \partial_1\, \partial_2\theta\Delta\theta\, \mathrm{d}x - 2\int u_1\, \partial_1\, \partial_2\, \partial_2\theta\Delta\theta\, \mathrm{d}x \\
&\qquad - 2\int u_1\, \partial_2\, \partial_2\theta\, \partial_1\Delta\theta\, \mathrm{d}x \\
&:= J_{341} + J_{342} + J_{343}.
\end{aligned}$$

The terms on the right can be bounded as follows:

$$\begin{aligned}
|J_{341}| &\le C\|\partial_1\, \partial_2\theta\|_{L^2}\, \|\partial_1 u_2\|_{L^2}^{1/2}\, \|\partial_2\, \partial_1 u_2\|_{L^2}^{1/2}\, \|\Delta\theta\|_{L^2}^{1/2}\, \|\partial_1\Delta\theta\|_{L^2}^{1/2} \\
&\le C(\|\Delta\theta\|_{L^2} + \|\partial_1 u_2\|_{L^2})\|\partial_1\theta\|_{H^2}^{3/2}\, \|\partial_2\nabla u_2\|_{L^2}^{1/2}, \\
|J_{342}| &\le C\|\partial_1\, \partial_2\, \partial_2\theta\|_{L^2}\, \|\Delta\theta\|_{L^2}^{1/2}\, \|\partial_1\Delta\theta\|_{L^2}^{1/2}\, \|u_1\|_{L^2}^{1/2}\, \|\partial_2 u_1\|_{L^2}^{1/2} \\
&\le C(\|\Delta\theta\|_{L^2} + \|u_1\|_{L^2})\|\partial_1\theta\|_{H^2}^{3/2}\, \|\partial_2 u_1\|_{L^2}^{1/2}, \\
|J_{343}| &\le C\|\partial_1\Delta\theta\|_{L^2}\, \|\partial_2\, \partial_2\theta\|_{L^2}^{1/2}\, \|\partial_1\, \partial_2\, \partial_2\theta\|_{L^2}^{1/2}\, \|u_1\|_{L^2}^{1/2}\, \|\partial_2 u_1\|_{L^2}^{1/2} \\
&\le C(\|\Delta\theta\|_{L^2} + \|u_1\|_{L^2})\|\partial_1\theta\|_{H^2}^{3/2}\, \|\partial_2 u_1\|_{L^2}^{1/2}.
\end{aligned}$$

Combining these estimates yields

$$|J_{34}| \le C(\|\theta\|_{H^2} + \|u\|_{H^2})\|\partial_1\theta\|_{H^2}^{3/2}\, \|\partial_2 u\|_{H^2}^{1/2}. \tag{4.14}$$

Putting (4.11), (4.12), (4.13), and (4.14) together, we obtain

$$|J_3| \le C(\|\theta\|_{H^2} + \|u\|_{H^2})\|\partial_1\theta\|_{H^2}^{3/2}\, \|\partial_2 u\|_{H^2}^{1/2}. \tag{4.15}$$

We now turn to the estimate of J_2. As was explained in the Introduction, we need the help of the extra regularization term

$$\int_0^t \|\partial_1 u_2\|_{L^2}^2\, \mathrm{d}\tau. \tag{4.16}$$

To make full use of the anisotropic dissipation, we further write J_2 as

$$\begin{aligned} J_2 = &-\int \partial_1 u_1 (\partial_1 \omega)^2 \, \mathrm{d}x - \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, \mathrm{d}x \\ &- \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, \mathrm{d}x - \int \partial_2 u_2 (\partial_2 \omega)^2 \, \mathrm{d}x \\ = &\int \partial_2 u_2 (\partial_1 \omega)^2 \, \mathrm{d}x - \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\ &- \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, \mathrm{d}x - \int \partial_2 u_2 (\partial_2 \omega)^2 \, \mathrm{d}x \\ := &\, J_{21} + J_{22} + J_{23} + J_{24}. \end{aligned}$$

To bound the first two terms, we need to make use of the term in (4.16). By integration by parts and Lemma 1.5,

$$\begin{aligned} J_{21} &= -2 \int u_2 \, \partial_1 \omega \, \partial_2 \, \partial_1 \omega \, \mathrm{d}x \\ &\leq C \|\partial_2 \, \partial_1 \omega\|_{L^2} \, ||\partial_1 \omega||_{L^2}^{1/2} \, ||\partial_2 \, \partial_1 \omega||_{L^2}^{1/2} \, ||u_2||_{L^2}^{1/2} \, ||\partial_1 u_2||_{L^2}^{1/2} \\ &\leq C(\|u_2\|_{L^2} + \|\partial_1 \omega\|_{L^2}) ||\partial_2 \, \partial_1 \omega||_{L^2}^{3/2} \, ||\partial_1 u_2||_{L^2}^{1/2}. \end{aligned}$$

By Lemma 1.5,

$$\begin{aligned} |J_{22}| &\leq C \, \|\partial_1 u_2\|_{L^2} \, ||\partial_1 \omega||_{L^2}^{1/2} \, ||\partial_2 \, \partial_1 \omega||_{L^2}^{1/2} \, ||\partial_2 \omega||_{L^2}^{1/2} \, ||\partial_1 \, \partial_2 \omega||_{L^2}^{1/2} \\ &\leq C \|\nabla \omega\|_{L^2} \, \|\partial_2 \, \partial_1 \omega\|_{L^2} \, \|\partial_1 u_2\|_{L^2}, \\ |J_{23}| &\leq C \|\partial_2 u_1\|_{L^2} \, ||\partial_1 \omega||_{L^2}^{1/2} \, ||\partial_2 \, \partial_1 \omega||_{L^2}^{1/2} \, ||\partial_2 \omega||_{L^2}^{1/2} \, ||\partial_1 \, \partial_2 \omega||_{L^2}^{1/2} \\ &\leq C \|\nabla \omega\|_{L^2} \, \|\partial_2 \, \partial_1 \omega\|_{L^2} \, \|\partial_2 u_1\|_{L^2}, \\ |J_{24}| &\leq C \|\partial_2 u_2\|_{L^2} \, ||\partial_2 \omega||_{L^2}^{1/2} \, ||\partial_1 \, \partial_2 \omega||_{L^2}^{1/2} \, ||\partial_2 \omega||_{L^2}^{1/2} \, ||\partial_2 \, \partial_2 \omega||_{L^2}^{1/2} \\ &\leq C \|\nabla \omega\|_{L^2} \, \|\partial_2 \nabla \omega\|_{L^2} \, \|\partial_2 u_2\|_{L^2}. \end{aligned}$$

Therefore,

$$|J_2| \leq C \|u\|_{H^2} (||\partial_2 \nabla \omega||_{L^2}^2 + ||\partial_1 u_2||_{L^2}^2 + ||\partial_2 u_1||_{L^2}^2). \tag{4.17}$$

Inserting $J_1 = 0$, (4.15) and (4.17) in (4.10), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} (||\Delta u||_{L^2}^2 + ||\Delta \theta||_{L^2}^2) + 2\nu ||\partial_2 \Delta u||_{L^2}^2 + 2\eta ||\partial_1 \Delta \theta||_{L^2}^2 \\ &\quad \leq C(\|\theta\|_{H^2} + \|u\|_{H^2}) ||\partial_1 \theta||_{H^2}^{3/2} \, ||\partial_2 u||_{H^2}^{1/2} \\ &\quad + C \|u\|_{H^2} (||\partial_2 \nabla \omega||_{L^2}^2 + ||\partial_1 u_2||_{L^2}^2 + ||\partial_2 u_1||_{L^2}^2). \end{aligned} \tag{4.18}$$

Integrating (4.18) over the time interval $[0,t]$ yields

$$
\begin{aligned}
(4.19)\quad & \|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_2\Delta u\|_{L^2}^2\,\mathrm{d}\tau \\
&\quad + 2\eta\int_0^t \|\Delta\,\partial_1\theta\|_{L^2}^2\,\mathrm{d}\tau \\
&\leq \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 + C\int_0^t (\|\theta\|_{H^2} + \|u\|_{H^2})\|\partial_1\theta\|_{H^2}^{3/2}\,\|\partial_2 u\|_{H^2}^{1/2}\,\mathrm{d}\tau \\
&\quad + C\int_0^t \|u\|_{H^2}(\|\partial_2\nabla\omega\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2)\,\mathrm{d}\tau \\
&\leq E(0) + CE(t)^{3/2}.
\end{aligned}
$$

The next major step is to bound the last piece in $E(t)$ defined by (4.2), namely,

$$\int_0^t \|\partial_1 u_2\|_{L^2}^2\,\mathrm{d}\tau.$$

We make use of the equation of θ. By the equation of θ,

$$\partial_1 u_2 = -\partial_t\,\partial_1\theta - \partial_1(u\cdot\nabla\theta) + \eta\,\partial_{111}\theta. \tag{4.20}$$

Multiplying (4.20) with $\partial_1 u_2$ and then integrating over $\mathbb{R}^2$ yields

$$
\begin{aligned}
\|\partial_1 u_2\|_{L^2}^2 &= -\int \partial_t\,\partial_1\theta\,\partial_1 u_2\,\mathrm{d}x - \int \partial_1 u_2\,\partial_1(u\cdot\nabla\theta)\,\mathrm{d}x \\
&\quad + \eta\int \partial_1 u_2\,\partial_{111}\theta\,\mathrm{d}x \\
&:= K_1 + K_2 + K_3.
\end{aligned}
$$

Even though the estimate of K_3 appears to be easy, the term with unfavorable derivative $\partial_1 u_2$ will be absorbed by the lefthand side:

$$|K_3| \leq \eta\|\partial_1 u_2\|_{L^2}\,\|\partial_{111}\theta\|_{L^2} \leq \frac{1}{2}\|\partial_1 u_2\|_{L^2}^2 + C\|\partial_1\theta\|_{H^2}^2. \tag{4.21}$$

We shift the time derivative in K_1,

$$K_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1\theta\,\partial_1 u_2\,\mathrm{d}x + \int \partial_1\theta\,\partial_1\,\partial_t u_2\,\mathrm{d}x := K_{11} + K_{12}. \tag{4.22}$$

Invoking the equation for the second component of the velocity, we have

$$\begin{aligned} K_{12} &= -\int \partial_1\,\partial_1\theta\,\partial_t u_2\,\mathrm{d}x \\ &= -\int \partial_{11}\theta(-(u\cdot\nabla)u_2 - \partial_2 p + \nu\,\partial_{22}u_2 + \theta)\,\mathrm{d}x \\ &= \int \partial_{11}\theta(u\cdot\nabla)u_2\,\mathrm{d}x + \int \partial_{11}\theta\,\partial_2 p\,\mathrm{d}x \\ &\quad - \nu\int \partial_{11}\theta\,\partial_{22}u_2\,\mathrm{d}x - \int \partial_{11}\theta\theta\,\mathrm{d}x. \end{aligned}$$

We further replace the pressure term. Applying the divergence operator to the velocity equation yields

$$p = -\Delta^{-1}\nabla\cdot(u\cdot\nabla u) + \Delta^{-1}\partial_2\theta.$$

Therefore,

$$\begin{aligned} K_{12} &= \int \partial_{11}\theta(u\cdot\nabla)u_2\,\mathrm{d}x + \int \partial_{11}\theta(-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u))\,\mathrm{d}x \\ &\quad - \nu\int \partial_{11}\theta\,\partial_{22}u_2\,\mathrm{d}x - \int \partial_{11}\theta\,\partial_{11}\Delta^{-1}\theta\,\mathrm{d}x \\ &:= K_{121} + K_{122} + K_{123} + K_{124}. \end{aligned}$$

By the boundedness of the double Riesz transform (see, e.g., [50])

$$\|\partial_{11}\Delta^{-1}f\|_{L^q} \le C\|f\|_{L^q}, \quad 1 < q < \infty,$$

we have

$$K_{124} = \int \partial_1\theta\,\partial_{11}\Delta^{-1}\partial_1\theta\,\mathrm{d}x \le C\|\partial_1\theta\|_{L^2}^2.$$

Note that K_{123} can be easily bounded: $|K_{123}| \le C\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2}$. By integration by parts and the boundedness of the double Riesz transform,

$$\begin{aligned} K_{122} &= -\int \partial_1\theta\,\partial_{12}\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\,\mathrm{d}x \\ &\le \|\partial_1\theta\|_{L^2}\,\|\Delta^{-1}\partial_{12}\nabla\cdot(u\cdot\nabla u)\|_{L^2} \\ &\le C\|\partial_1\theta\|_{L^2}\,\|\partial_2(u\cdot\nabla u)\|_{L^2} \\ &\le C\|\partial_1\theta\|_{L^2}\,\|\partial_2 u\cdot\nabla u + u\cdot\nabla\partial_2 u\|_{L^2} \\ &\le C\|\partial_1\theta\|_{L^2}(\|\partial_2 u\|_{L^4}\,\|\nabla u\|_{L^4} + \|u\|_\infty\,\|\nabla\partial_2 u\|_{L^2}) \\ &\le C\|\partial_1\theta\|_{L^2}\,\|\partial_2 u\|_{H^1}\,\|\nabla u\|_{H^1} + C\|\partial_1\theta\|_{L^2}\,\|u\|_{H^2}\,\|\nabla\partial_2 u\|_{L^2}. \end{aligned}$$

To bound K_{121}, we further split it:

$$\begin{aligned} K_{121} &= \int \partial_{11}\theta(u_1\,\partial_1 u_2 + u_2\,\partial_2 u_2)\,\mathrm{d}x \\ &= \int \partial_{11}\theta u_1\,\partial_1 u_2\,\mathrm{d}x + \int \partial_{11}\theta u_2\,\partial_2 u_2\,\mathrm{d}x. \end{aligned}$$

By Lemma 1.5,

$$\begin{aligned} |K_{121}| &\le C\|\partial_{11}\theta\|_{L^2}\,||u_1||_{L^2}^{1/2}\,||\partial_1 u_1||_{L^2}^{1/2}\,||\partial_1 u_2||_{L^2}^{1/2}\,||\partial_2\,\partial_1 u_2||_{L^2}^{1/2} \\ &\qquad + C\|u_2\|_{L^\infty}\,\|\partial_{11}\theta\|_{L^2}\,\|\partial_2 u_2\|_{L^2} \\ &\le C\|u\|_{H^1}\,\|\partial_2 u\|_{H^1}\,\|\partial_{11}\theta\|_{L^2} + C\|u\|_{H^2}\,\|\partial_2 u\|_{L^2}\,\|\partial_{11}\theta\|_{L^2}. \end{aligned}$$

We have thus obtained an upper bound for K_{12}:

$$\begin{aligned} |K_{12}| &\le C||\partial_1\theta||_{L^2}^2 + C\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2} \\ &\qquad + C\|u\|_{H^2}\,\|\partial_2 u\|_{H^1}\,\|\partial_1\theta\|_{H^1}. \end{aligned} \tag{4.23}$$

It remains to bound K_2. We decompose K_2 into four terms:

$$\begin{aligned} K_2 = &-\int \partial_1 u_2\,\partial_1 u_1\,\partial_1\theta\,\mathrm{d}x - \int \partial_1 u_2 u_1\,\partial_1\,\partial_1\theta\,\mathrm{d}x \\ &-\int \partial_1 u_2\,\partial_1 u_2\,\partial_2\theta\,\mathrm{d}x - \int \partial_1 u_2 u_2\,\partial_1\,\partial_2\theta\,\mathrm{d}x. \end{aligned}$$

By Lemma 1.5,

$$\begin{aligned} |K_2| &\le C\|\partial_1 u_2\|_{L^2}\,||\partial_2 u_2||_{L^2}^{1/2}\,||\partial_1\,\partial_2 u_2||_{L^2}^{1/2}\,||\partial_1\theta||_{L^2}^{1/2}\,||\partial_1\,\partial_1\theta||_{L^2}^{1/2} \\ &\qquad + C||u_1||_{L^2}^{1/2}\,||\partial_1 u_1||_{L^2}^{1/2}\,||\partial_1 u_2||_{L^2}^{1/2}\,||\partial_2\,\partial_1 u_2||_{L^2}^{1/2}\,\|\partial_1\,\partial_1\theta\|_{L^2} \\ &\qquad + C\|\partial_1 u_2\|_{L^2}\,||\partial_1 u_2||_{L^2}^{1/2}\,||\partial_2\,\partial_1 u_2||_{L^2}^{1/2}\,||\partial_2\theta||_{L^2}^{1/2}\,||\partial_2\,\partial_1\theta||_{L^2}^{1/2} \\ &\qquad + C\|\partial_1\,\partial_2\theta\|_{L^2}\,||u_2||_{L^2}^{1/2}\,||\partial_1 u_2||_{L^2}^{1/2}\,||\partial_1 u_2||_{L^2}^{1/2}\,||\partial_2\,\partial_1 u_2||_{L^2}^{1/2} \\ &\le C\|u\|_{H^1}(||\partial_2 u||_{H^1}^2 + ||\partial_1\theta||_{H^1}^2) \\ &\qquad + C(\|u\|_{H^2} + \|\theta\|_{H^2})(||\partial_1 u_2||_{L^2}^2 + ||\partial_1\theta||_{H^1}^2). \end{aligned} \tag{4.24}$$

Combining (4.21), (4.22), (4.23), and (4.24), we find

$$\begin{aligned} \frac{1}{2}||\partial_1 u_2||_{L^2}^2 \le C||\partial_1\theta||_{H^2}^2 &- \frac{\mathrm{d}}{\mathrm{d}t}\int \partial_1\theta\,\partial_1 u_2\,\mathrm{d}x \\ &+ C\|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2} + C\|u\|_{H^2}(||\partial_2 u||_{H^1}^2 + ||\partial_1\theta||_{H^1}^2) \\ &+ C(\|u\|_{H^2} + \|\theta\|_{H^2})(||\partial_1 u_2||_{L^2}^2 + ||\partial_1\theta||_{H^1}^2). \end{aligned}$$

Integrating over $[0,t]$ yields

$$
\begin{aligned}
&\int_0^t \|\partial_1 u_2\|_{L^2}^2 \,\mathrm{d}\tau \\
&\quad \le C\int_0^t \|\partial_1\theta\|_{H^2}^2 \,\mathrm{d}\tau - 2\int \partial_1\theta\,\partial_1 u_2 \,\mathrm{d}x + 2\int \partial_1\theta_0\,\partial_1 u_{02}\,\mathrm{d}x \\
&\qquad + C\int_0^t \|\partial_{11}\theta\|_{L^2}\,\|\partial_{22}u_2\|_{L^2}\,\mathrm{d}\tau \\
&\qquad + C\int_0^t \|u\|_{H^2}(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2)\,\mathrm{d}\tau \\
&\qquad + C\int_0^t (\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^1}^2)\,\mathrm{d}\tau \\
&\quad \le C\int_0^t \|\partial_1\theta\|_{H^2}^2 \,\mathrm{d}\tau + C\int_0^t \|\partial_2 u\|_{H^2}^2\,\mathrm{d}\tau + C(\|u\|_{H^1}^2 + \|\theta\|_{H^1}^2) \\
&\qquad + C(\|u_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2) + CE(t)^{3/2}.
\end{aligned}
\tag{4.25}
$$

We then combine the H^1-bound in (4.9), the homogeneous H^2-bound in (4.19), and the bound for the extra regularization term in (4.25). We need to eliminate the quadratic terms on the righthand side of (4.25) by the corresponding terms on the lefthand side, so we need to multiply both sides of (4.25) by a suitably small coefficient δ. Thus, (4.9) + (4.19) + δ (4.25) gives

$$
\begin{aligned}
&\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2\,\mathrm{d}\tau \\
&\qquad + 2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2\,\mathrm{d}\tau + \delta\int_0^t \|\partial_1 u_2\|_{L^2}^2 \\
&\quad \le E(0) + CE(t)^{3/2} + C\delta(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2) \\
&\qquad + C\delta(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2) \\
&\qquad + C\delta\int_0^t \|\partial_2 u\|_{H^2}^2\,\mathrm{d}\tau + C\delta\int_0^t \|\partial_1\theta\|_{H^2}^2\,\mathrm{d}\tau + C\delta E(t)^{3/2}.
\end{aligned}
\tag{4.26}
$$

If $\delta > 0$ is chosen to be sufficiently small, say

$$
C\delta \le \frac{1}{2}, \quad C\delta \le \nu,\ C\delta \le \eta,
$$

then (4.26) is reduced to

$$
E(t) \le C_1 E(0) + C_2 E(t)^{3/2},
\tag{4.27}
$$

where C_1 and C_2 are positive constants. An application of the bootstrapping argument to (4.27) then leads to the desired stability result. In fact, if the initial data (u_0, θ_0) is sufficiently small,

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon := \frac{1}{4\sqrt{C_1}C_2},$$

then (4.27) allows us to show that $\|(u(t), \theta(t))\|_{H^2} \le \sqrt{2C_1}\varepsilon$. The bootstrapping argument starts with the ansatz that, for $t < T$,

$$E(t) \le \frac{1}{4C_2^2}, \tag{4.28}$$

and shows that

$$E(t) \le \frac{1}{8C_2^2} \quad \forall\, t \le T. \tag{4.29}$$

Then, the bootstrapping argument would imply that $T = \infty$ and that (4.29) actually holds for all t. Note that (4.29) is an easy consequence of (4.27) and (4.28). Inserting (4.28) in (4.27) yields

$$\begin{aligned} E(t) &\le C_1 E(0) + C_2 E(t)^{3/2} \\ &\le C_1\varepsilon^2 + C_2 \frac{1}{2C_2} E(t). \end{aligned}$$

That is,

$$\frac{1}{2}E(t) \le C_1\varepsilon^2 \quad \text{or} \quad E(t) \le 2C_1 \frac{1}{16C_1C_2^2} = \frac{1}{8C_2^2} = 2C_1\varepsilon^2,$$

which is (4.29). This establishes the global stability.

Finally, we briefly explain the uniqueness. It is not difficult to see that the solutions to (1.2) at this regularity level must be unique. Assume that $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ are two solutions of (1.2) with one of them in the H^2-regularity class, say, $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T; H^2)$. The difference $(\tilde{u}, \tilde{p}, \tilde{\theta})$ with

$$\tilde{u} = u^{(2)} - u^{(1)}, \quad \tilde{p} = p^{(2)} - p^{(1)} \quad \text{and} \quad \tilde{\theta} = \theta^{(2)} - \theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t \tilde{u} + u^{(2)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(1)} + \nabla \tilde{p} = \nu\, \partial_{22}\tilde{u} + \tilde{\theta}\mathbf{e}_2, \\ \partial_t \tilde{\theta} + u^{(2)} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \theta^{(1)} + \widetilde{u_2} = \eta\, \partial_{11}\tilde{\theta}, \\ \nabla \cdot \tilde{u} = 0, \\ \tilde{u}(x, 0) = 0, \quad \tilde{\theta}(x, 0) = 0. \end{cases} \tag{4.30}$$

We estimate the difference $(\tilde{u}, \tilde{p}, \tilde{\theta})$ in $L^2(\mathbb{R}^2)$. Dotting (4.30) by $(\tilde{u}, \tilde{\theta})$ and applying the divergence free condition, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||(\tilde{u}, \tilde{\theta})||_{L^2}^2 + \nu||\partial_2\tilde{u}||_{L^2}^2 + \eta||\partial_1\tilde{\theta}||_{L^2}^2 \\ = -\int \tilde{u} \cdot \nabla u^{(1)} \cdot \tilde{u}\,\mathrm{d}x - \int \tilde{u} \cdot \nabla \theta^{(1)} \cdot \tilde{\theta}\,\mathrm{d}x.\end{aligned}$$

By Lemma 1.5, the uniformly global bound for $\|(u^{(1)}, \theta^{(1)})\|_{H^2}$, and Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||(\tilde{u}, \tilde{\theta})||_{L^2}^2 + \nu||\partial_2\tilde{u}||_{L^2}^2 + \eta||\partial_1\tilde{\theta}||_{L^2}^2 \\ &\quad\le C\|\tilde{u}\|_{L^2}\, ||\tilde{u}||_{L^2}^{1/2}\, ||\partial_2\tilde{u}||_{L^2}^{1/2}\, ||\nabla u^{(1)}||_{L^2}^{1/2}\, ||\partial_1\nabla u^{(1)}||_{L^2}^{1/2} \\ &\qquad + C\|\tilde{\theta}\|_{L^2}\, ||\tilde{u}||_{L^2}^{1/2}\, ||\partial_2\tilde{u}||_{L^2}^{1/2}\, ||\nabla\theta^{(1)}||_{L^2}^{1/2}\, ||\partial_1\nabla\theta^{(1)}||_{L^2}^{1/2} \\ &\quad\le C||\tilde{u}||_{L^2}^{3/2}\, ||\partial_2\tilde{u}||_{L^2}^{1/2} + C||\tilde{u}||_{L^2}^{1/2}\, ||\partial_2\tilde{u}||_{L^2}^{1/2}\, \|\tilde{\theta}\|_{L^2} \\ &\quad\le \frac{\nu}{2}||\partial_2\tilde{u}||_{L^2}^2 + C||(\tilde{u}, \tilde{\theta})||_{L^2}^2.\end{aligned}$$

It then follows from Gronwall's inequality that

$$\|\tilde{u}(t)\|_{L^2} = \|\tilde{\theta}(t)\|_{L^2} = 0.$$

That is, these two solutions coincide. This, then, completes the proof of Theorem 1.4. ◻

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REFERENCES

[1] D. ADHIKARI, C. CAO, H. SHANG, J. WU, X. XU, and Z. YE, *Global regularity results for the 2D Boussinesq equations with partial dissipation*, J. Differential Equations **260** (2016), no. 2, 1893–1917. `http://dx.doi.org/10.1016/j.jde.2015.09.049`. MR3419749

[2] D. ADHIKARI, C. CAO, and J. WU, *The 2D Boussinesq equations with vertical viscosity and vertical diffusivity*, J. Differential Equations **249** (2010), no. 5, 1078–1088. `http://dx.doi.org/10.1016/j.jde.2010.03.021`. MR2652164

[3] ———, *Global regularity results for the 2D Boussinesq equations with vertical dissipation*, J. Differential Equations **251** (2011), no. 6, 1637–1655. `http://dx.doi.org/10.1016/j.jde.2011.05.027`. MR2813893

[4] D. ADHIKARI, C. CAO, J. WU, and X. XU, *Small global solutions to the damped two-dimensional Boussinesq equations*, J. Differential Equations **256** (2014), no. 11, 3594–3613. `http://dx.doi.org/10.1016/j.jde.2014.02.012`. MR3186840

[5] N. BOARDMAN, R. JI, H. QIU, and J. WU, *Uniqueness of weak solutions to the Boussinesq equations without thermal diffusion*, Commun. Math. Sci. **17** (2019), no. 6, 1595–1624. `http://dx.doi.org/10.4310/CMS.2019.v17.n6.a5`. MR4052752

[6] C. CAO and J. WU, *Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion*, Adv. Math. **226** (2011), no. 2, 1803–1822. `http://dx.doi.org/10.1016/j.aim.2010.08.017`. MR2737801

[7] ———, *Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal. **208** (2013), no. 3, 985–1004. `http://dx.doi.org/10.1007/s00205-013-0610-3`. MR3048599

[8] Á. CASTRO, D. CÓRDOBA, and D. LEAR, *On the asymptotic stability of stratified solutions for the 2D Boussinesq equations with a velocity damping term*, Math. Models Methods Appl. Sci. **29** (2019), no. 7, 1227–1277. `http://dx.doi.org/10.1142/S0218202519500210`. MR3974167

[9] D. CHAE, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math. **203** (2006), no. 2, 497–513. `http://dx.doi.org/10.1016/j.aim.2005.05.001`. MR2227730

[10] D. CHAE, P. CONSTANTIN, and J. WU, *An incompressible 2D didactic model with singularity and explicit solutions of the 2D Boussinesq equations*, J. Math. Fluid Mech. **16** (2014), no. 3, 473–480. `http://dx.doi.org/10.1007/s00021-014-0166-5`. MR3247363

[11] D. CHAE and H. S. NAM, *Local existence and blow-up criterion for the Boussinesq equations*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 5, 935–946. `http://dx.doi.org/10.1017/S0308210500026810`. MR1475638

[12] D. CHAE and J. WU, *The 2D Boussinesq equations with logarithmically supercritical velocities*, Adv. Math. **230** (2012), no. 4–6, 1618–1645. `http://dx.doi.org/10.1016/j.aim.2012.04.004`. MR2927350

[13] K. CHOI, A. KISELEV, and Y. YAO, *Finite time blow up for a 1D model of 2D Boussinesq system*, Comm. Math. Phys. **334** (2015), no. 3, 1667–1679. `http://dx.doi.org/10.1007/s00220-014-2146-2`. MR3312447

[14] P. CONSTANTIN and C. R. DOERING, *Heat transfer in convective turbulence*, Nonlinearity **9** (1996), no. 4, 1049–1060. `http://dx.doi.org/10.1088/0951-7715/9/4/013`. MR1399486

[15] P. CONSTANTIN, V. VICOL, and J. WU, *Analyticity of Lagrangian trajectories for well posed inviscid incompressible fluid models*, Adv. Math. **285** (2015), 352–393. `http://dx.doi.org/10.1016/j.aim.2015.05.019`. MR3406503

[16] P. CONSTANTIN, J. WU, J. ZHAO, and Y. ZHU, *High Reynolds number and high Weissenberg number Oldroyd-B model with dissipation*, J. Evol. Equ. **21** (2021), no. 3, Special Issue in honor of the 60th birthday of Professor Matthias Hieber, 2787–2806. `http://dx.doi.org/10.1007/s00028-020-00616-8`. MR4350254

[17] Y. DAI, W. HU, J. WU, and B. XIAO, *The Littlewood-Paley decomposition for periodic functions and applications to the Boussinesq equations*, Anal. Appl. (Singap.) **18** (2020), no. 4, 639–682. `http://dx.doi.org/10.1142/S0219530519500234`. MR4109537

[18] R. DANCHIN and M. PAICU, *Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data*, Comm. Math. Phys. **290** (2009), no. 1, 1–14. `http://dx.doi.org/10.1007/s00220-009-0821-5`. MR2520505

[19] ———, *Global existence results for the anisotropic Boussinesq system in dimension two*, Math. Models Methods Appl. Sci. **21** (2011), no. 3, 421–457. `http://dx.doi.org/10.1142/S0218202511005106`. MR2782720

[20] W. DENG, J. WU, and P. ZHANG, *Stability of Couette flow for 2D Boussinesq system with vertical dissipation*, J. Funct. Anal. **281** (2021), no. 12, Paper No. 109255, 40 pp. `http://dx.doi.org/10.1016/j.jfa.2021.109255`. MR4322283

[21] S. A. DENISOV, *Double exponential growth of the vorticity gradient for the two-dimensional Euler equation*, Proc. Amer. Math. Soc. **143** (2015), no. 3, 1199–1210. `http://dx.doi.org/10.1090/S0002-9939-2014-12286-6`. MR3293735

[22] C. R. DOERING and J. D. GIBBON, *Applied Analysis of the Navier-Stokes Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995. `http://dx.doi.org/10.1017/CBO9780511608803`. MR1325465

[23] C. R. DOERING, J. WU, K. ZHAO, and X. ZHENG, *Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion*, Phys. D **376/377** (2018), 144–159. `http://dx.doi.org/10.1016/j.physd.2017.12.013`. MR3815212

[24] T. M. ELGINDI and I.-J. JEONG, *Finite-time singularity formation for strong solutions to the Boussinesq system*, Ann. PDE **6** (2020), no. 1, Paper No. 5, 50 pp. `http://dx.doi.org/10.1007/s40818-020-00080-0`. MR4098032

[25] T. M. ELGINDI and F. ROUSSET, *Global regularity for some Oldroyd-B type models*, Comm. Pure Appl. Math. **68** (2015), no. 11, 2005–2021. `http://dx.doi.org/10.1002/cpa.21563`. MR3403757

[26] T. M. ELGINDI and K. WIDMAYER, *Sharp decay estimates for an anisotropic linear semigroup and applications to the surface quasi-geostrophic and inviscid Boussinesq systems*, SIAM J. Math. Anal. **47** (2015), no. 6, 4672–4684. `http://dx.doi.org/10.1137/14099036X`. MR3431131

[27] L. HE, *Smoothing estimates of 2d incompressible Navier-Stokes equations in bounded domains with applications*, J. Funct. Anal. **262** (2012), no. 7, 3430–3464. `http://dx.doi.org/10.1016/j.jfa.2012.01.017`. MR2885958

[28] T. HMIDI, S. KERAANI, and F. ROUSSET, *Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation*, J. Differential Equations **249** (2010), no. 9, 2147–2174. `http://dx.doi.org/10.1016/j.jde.2010.07.008`. MR2718654

[29] ———, *Global well-posedness for Euler-Boussinesq system with critical dissipation*, Comm. Partial Differential Equations **36** (2011), no. 3, 420–445. `http://dx.doi.org/10.1080/03605302.2010.518657`. MR2763332

[30] T. Y. HOU and C. LI, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst. **12** (2005), no. 1, 1–12. `http://dx.doi.org/10.3934/dcds.2005.12.1`. MR2121245

[31] W. HU, I. KUKAVICA, and M. ZIANE, *Persistence of regularity for the viscous Boussinesq equations with zero diffusivity*, Asymptot. Anal. **91** (2015), no. 2, 111–124. `http://dx.doi.org/10.3233/asy-141261`. MR3305763

[32] W. HU, Y. WANG, J. WU, B. XIAO, and J. YUAN, *Partially dissipative 2D Boussinesq equations with Navier type boundary conditions*, Phys. D **376/377** (2018), 39–48. `http://dx.doi.org/10.1016/j.physd.2017.07.003`. MR3815202

[33] Q. JIU, C. MIAO, J. WU, and Z. ZHANG, *The two-dimensional incompressible Boussinesq equations with general critical dissipation*, SIAM J. Math. Anal. **46** (2014), no. 5, 3426–3454. `http://dx.doi.org/10.1137/140958256`. MR3267159

[34] Q. JIU, J. WU, and W. YANG, *Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation*, J. Nonlinear Sci. **25** (2015), no. 1, 37–58. `http://dx.doi.org/10.1007/s00332-014-9220-y`. MR3302123

[35] D. KC, D. REGMI, L. TAO, and J. WU, *Generalized 2D Euler-Boussinesq equations with a singular velocity*, J. Differential Equations **257** (2014), no. 1, 82–108. `http://dx.doi.org/10.1016/j.jde.2014.03.012`. MR3197241

[36] A. KISELEV and V. ŠVERÁK, *Small scale creation for solutions of the incompressible two-dimensional Euler equation*, Ann. of Math. (2) **180** (2014), no. 3, 1205–1220. `http://dx.doi.org/10.4007/annals.2014.180.3.9`. MR3245016

[37] A. KISELEV and C. TAN, *Finite time blow up in the hyperbolic Boussinesq system*, Adv. Math. **325** (2018), 34–55. `http://dx.doi.org/10.1016/j.aim.2017.11.019`. MR3742585

[38] M.-J. LAI, R. PAN, and K. ZHAO, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal. **199** (2011), no. 3, 739–760. `http://dx.doi.org/10.1007/s00205-010-0357-z`. MR2771665

[39] A. LARIOS, E. LUNASIN, and E. S. TITI, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differential Equations **255** (2013), no. 9, 2636–2654. `http://dx.doi.org/10.1016/j.jde.2013.07.011`. MR3090072

[40] J. LI, H. SHANG, J. WU, X. XU, and Z. YE, *Regularity criteria for the 2D Boussinesq equations with supercritical dissipation*, Commun. Math. Sci. **14** (2016), no. 7, 1999–2022. `http://dx.doi.org/10.4310/CMS.2016.v14.n7.a10`. MR3549359

[41] J. LI and E. S. TITI, *Global well-posedness of the 2D Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal. **220** (2016), no. 3, 983–1001. `http://dx.doi.org/10.1007/s00205-015-0946-y`. MR3466839

[42] A. J. MAJDA, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. `http://dx.doi.org/10.1090/cln/009`. MR1965452

[43] A. J. MAJDA and A. L. BERTOZZI, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002. MR1867882

[44] C. MIAO and L. XUE, *On the global well-posedness of a class of Boussinesq-Navier-Stokes systems*, NoDEA Nonlinear Differential Equations Appl. **18** (2011), no. 6, 707–735. `http://dx.doi.org/10.1007/s00030-011-0114-5`. MR2861261

[45] J. PEDLOSKY, *Geophysical Fluid Dynamics*, Springer, New York, 1987.

[46] A. SARRIA and J. WU, *Blowup in stagnation-point form solutions of the inviscid 2d Boussinesq equations*, J. Differential Equations **259** (2015), no. 8, 3559–3576. `http://dx.doi.org/10.1016/j.jde.2015.04.029`. MR3369255

[47] M. E. SCHONBEK, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **88** (1985), no. 3, 209–222. `http://dx.doi.org/10.1007/BF00752111`. MR775190

[48] M. E. SCHONBEK and M. WIEGNER, *On the decay of higher-order norms of the solutions of Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), no. 3, 677–685. `http://dx.doi.org/10.1017/S0308210500022976`. MR1396285

[49] A. STEFANOV and J. WU, *A global regularity result for the 2D Boussinesq equations with critical dissipation*, J. Anal. Math. **137** (2019), no. 1, 269–290. `http://dx.doi.org/10.1007/s11854-018-0073-4`. MR3938005

[50] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095

[51] L. TAO and J. WU, *The 2D Boussinesq equations with vertical dissipation and linear stability of shear flows*, J. Differential Equations **267** (2019), no. 3, 1731–1747. `http://dx.doi.org/10.1016/j.jde.2019.02.020`. MR3945615

[52] L. TAO, J. WU, K. ZHAO, and X. ZHENG, *Stability near hydrostatic equilibrium to the 2D Boussinesq equations without thermal diffusion*, Arch. Ration. Mech. Anal. **237** (2020), no. 2, 585–630. `http://dx.doi.org/10.1007/s00205-020-01515-5`. MR4097325

[53] T. TAO, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. `http://dx.doi.org/10.1090/cbms/106`. MR2233925

[54] B. WEN, N. DIANATI, E. LUNASIN, G. P. CHINI, and C. R. DOERING, *New upper bounds and reduced dynamical modeling for Rayleigh-Bénard convection in a fluid saturated porous layer*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), no. 5, 2191–2199. `http://dx.doi.org/10.1016/j.cnsns.2011.06.039`. MR2863089

[55] J. WU, *The 2D Boussinesq equations with partial or fractional dissipation*, Lectures on the Analysis of Nonlinear Partial Differential Equations. Part 4, Morningside Lect. Math., vol. 4, Int. Press, Somerville, MA, 2016, pp. 223–269. MR3525876

[56] ———, *Dissipative quasi-geostrophic equations with L^p data*, Electron. J. Differential Equations (2001), No. 56, 13. MR1846672

[57] J. WU and X. XU, *Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation*, Nonlinearity **27** (2014), no. 9, 2215–2232. `http://dx.doi.org/10.1088/0951-7715/27/9/2215`. MR3247077

[58] J. WU, X. XU, L. XUE, and Z. YE, *Regularity results for the 2D Boussinesq equations with critical or supercritical dissipation*, Commun. Math. Sci. **14** (2016), no. 7, 1963–1997. `http://dx.doi.org/10.4310/CMS.2016.v14.n7.a9`. MR3549358

[59] J. WU, X. XU, and Z. YE, *The 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion*, J. Math. Pures Appl. (9) **115** (2018), 187–217 (English, with English and French summaries). `http://dx.doi.org/10.1016/j.matpur.2018.01.006`. MR3808344

[60] X. XU, *Global regularity of solutions of 2D Boussinesq equations with fractional diffusion*, Nonlinear Anal. **72** (2010), no. 2, 677–681. `http://dx.doi.org/10.1016/j.na.2009.07.008`. MR2579335

[61] W. YANG, Q. JIU, and J. WU, *Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation*, J. Differential Equations **257** (2014), no. 11, 4188–4213. `http://dx.doi.org/10.1016/j.jde.2014.08.006`. MR3264420

[62] ———, *The 3D incompressible Boussinesq equation with fractional partial dissipation*, Commun. Math. Sci. **16** (2018), no. 3, 617–633. `http://dx.doi.org/10.4310/CMS.2018.v16.n3.a2`. MR3853900

[63] Z. YE and X. XU, *Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation*, J. Differential Equations **260** (2016), no. 8, 6716–6744. `http://dx.doi.org/10.1016/j.jde.2016.01.014`. MR3460229

[64] K. ZHAO, *2D inviscid heat conductive Boussinesq equations on a bounded domain*, Michigan Math. J. **59** (2010), no. 2, 329–352. `http://dx.doi.org/10.1307/mmj/1281531460`. MR2677625

[65] C. ZILLINGER, *On enhanced dissipation for the Boussinesq equations*, J. Differential Equations **282** (2021), 407–445. `http://dx.doi.org/10.1016/j.jde.2021.02.029`. MR4219325

[66] A. ZLATOŠ, *Exponential growth of the vorticity gradient for the Euler equation on the torus*, Adv. Math. **268** (2015), 396–403. `http://dx.doi.org/10.1016/j.aim.2014.08.012`. MR3276599

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