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To cite this article: Ruihong Ji *et al* 2023 *Nonlinearity* **36** 3219

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IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **36** (2023) 3219–3237 https://doi.org/10.1088/1361-6544/acd160

3D anisotropic Navier–Stokes equations in $\mathbb{T}^2 \times \mathbb{R}$: stability and large-time behaviour

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Received 31 October 2021; revised 7 March 2023
Accepted for publication 28 April 2023
Published 11 May 2023

Recommended by Dr Tasso J Kaper

Abstract
The study on the large-time behaviour of solutions to the 3D incompressible anisotropic Navier–Stokes (ANS) equations is very recent. Powerful tools designed for the Navier–Stokes equations with full Laplacian dissipation such as the Fourier splitting method no longer apply to the case when there is only horizontal dissipation. For the whole space $\mathbb{R}^3$, as $t \to \infty$, solutions of the ANS equations converge to the trivial solution and the convergence rate is algebraic. This paper is devoted to the case when the spatial domain Ω is $\mathbb{T}^2 \times \mathbb{R}$. Our results reveal that the large-time behaviour for $\mathbb{T}^2 \times \mathbb{R}$ is quite different from that for $\mathbb{R}^3$. We show that any small initial velocity field $u_0 \in H^2(\Omega)$ leads to a unique global solution u that remains small in $H^2(\Omega)$. More importantly, as $t \to \infty$, the velocity field u converges to a nontrivial steady state. The first two components of the steady state are given by the horizontal average of the first two components of u_0 while the third component vanishes. In addition, this convergence is exponentially fast.

Keywords: anisotropic dissipation, decay rates, Navier–Stokes equations, stability

Mathematics subject classification numbers: 35A05, 35Q35, 76D03

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1. Introduction

The goal of this paper is to understand the stability and the precise large-time behaviour of solutions to the 3D Navier–Stokes (NS) equations with only horizontal dissipation

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla P + \nu\Delta_h u, \quad x\in\Omega,\ t>0, \\ \nabla\cdot u = 0, \quad x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the velocity field of the fluid, $P = P(x,t)$ the pressure, and $\nu > 0$ the kinematic viscosity. The spatial domain is taken to be $\Omega = \mathbb{T}^2\times\mathbb{R}$ with $\mathbb{T}^2 = [0,1]^2$ being the 2D periodic box. Here $\triangle_h = \partial_1^2 + \partial_2^2$ and, for notational convenience, we write ∂_i for ∂_{x_i} with $i = 1,2,3$. In addition, we use $\nabla_h := (\partial_1,\partial_2)$ for the horizontal gradient and $u_h = (u_1,u_2)$ for the horizontal velocity components.

The anisotropic Navier–Stokes (ANS) equations arise in the modeling of laminar and turbulent flows in Ekman layers ([3], [17, chapter 4]) as well as in various centrifuge studies (see, e.g. [2]). Ekman layers are boundary layers in which there is a balance between the viscous force and the Coriolis acceleration. They are typically quite thin in the vertical direction. In these thin layers, the horizontal diffusion of velocity dominates the vertical diffusion and (1.1) is relevant.

There have been substantial recent developments on the 3D ANS equations in the whole space $\mathbb{R}^3$. Significant progress has been on the well-posedness of (1.1) in various Sobolev and Besov spaces (see, e.g. [3, 4, 13, 15, 18, 19, 29, 30]). But no study has been done on the ANS equations in the domain focused in this paper. The approach for the whole space case $\mathbb{R}^3$ may not work for the domain $\mathbb{T}^2\times\mathbb{R}$. The estimates on the nonlinear term are actually quite different. A detailed explanation is given later. In addition, our investigation reveals that the large-time behaviour of solutions for the domain $\mathbb{T}^2\times\mathbb{R}$ is significantly different from the $\mathbb{R}^3$ case. It is hoped that the results of this paper will help us gain a more complete understanding of the ANS equations.

As we know, the global existence and regularity problem on the 3D NS equations with a general smooth initial data remains an outstanding open problem. The ANS equations in (1.1) are less regularized than the NS equations due to the lack of vertical dissipation. Many fundamental issues on (1.1) such as the global regularity problem are not well understood. The main difficulty is due to the fact that the dissipation is insufficient to control the nonlinearity for general solutions.

When the initial data is small, even the anisotropic dissipation may dominate since the nonlinearity is quadratic and may be even smaller. The Sobolev space is a very natural functional setting that allows us to quantity the size of the solutions. Here we choose the Sobolev space $H^2(\Omega)$. To understand the global existence and regularity problem, we consider the evolution of the Sobolev norm of the solution, namely $\|u(t)\|_{H^2}$. The energy method is a powerful tool for this purpose. It starts with taking up to the second-order weak derivatives of (1.1) and then taking the inner product with the corresponding derivatives of u. This yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^2}^2 + \nu\|\nabla_h u(t)\|_{H^2}^2 = -J,$$

where

$$J := \sum_{\alpha_1+\alpha_2+\alpha_3\leqslant 2}\int_{\mathbb{T}^2\times\mathbb{R}} \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}(u\cdot\nabla u)\cdot\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} u\,dx.$$

Obviously J comes from the nonlinearity. The whole issue is then how to obtain a suitable upper bound on J. Due to the lack of dissipation in the vertical direction, the most difficult parts in J are those with many vertical derivatives such as

$$J_1 := \int_{\mathbb{T}^2\times\mathbb{R}} \partial_3^2(u\cdot\nabla u)\cdot\partial_3^2 u\,dx.$$

Some of the vertical derivatives may be converted to horizontal derivatives via the divergence-free condition,

$$\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2).$$

Unfortunately there are terms in J_1 for which such conversions are not possible and one such term is

$$J_{11} := \int_{\mathbb{T}^2\times\mathbb{R}} \partial_3^2 u_h\cdot\nabla_h u_h\cdot\partial_3^2 u_h\,dx, \tag{1.2}$$

J_{11} involves a product of three terms (triple product) and a natural way of estimating it is to first apply Hölder's inequality followed by Sobolev's embedding inequalities.

In the case of full Laplacian dissipation, we have smoothing and control in all directions and the standard Sobolev inequalities suffice. When the dissipation is anisotropic, the lack of vertical regularization makes it harder to control J_{11}. This motivated the developments of anisotropic upper bounds for triple products. When the spatial domain is the whole space $\mathbb{R}^3$, the anisotropic upper bound reads

$$\begin{aligned}\left|\int_{\mathbb{R}^3} fgh\,dx\right| \leqslant C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_2 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\\ \times\|h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_3 h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.\end{aligned} \tag{1.3}$$

The bound on the right-hand side of (1.3) only requires partial derivatives $\partial_1 f$, $\partial_2 g$ and $\partial_3 h$. In comparison, the classical upper bounds require full gradient of these functions. Therefore, (1.3) is much sharper than the classical triple product estimates and removes unnecessary requirements on the derivatives.

Mathematically (1.3) is proven by first bounding the triple product by anisotropic Hölder inequalities (Hölder inequalities with different indices in different directions) and then invoking the following 1D Sobolev inequality on $\mathbb{R}$,

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{1.4}$$

A detailed proof of (1.3) can be found in [27]. The 2D version of (1.3) was obtained in [7]. Intuitively (1.3) can be understood as follows. The triple product would be bounded when the functions f, g nd h have enough high integrability. This requirement is fulfilled if f is essentially bounded in x_1, g in x_2 and h in x_3. Sobolev's inequality then translates these conditions into partial derivatives.

As we know, Sobolev's inequality for bounded domains usually contains an extra integrability term of the function itself. For example, the 1D Sobolev inequality for the periodic domain $\mathbb{T}$ (a special bounded domain) assumes the form,

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant \sqrt{2}\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})}, \tag{1.5}$$

which contains the extra term $\|f\|_{L^2(\mathbb{T})}$. As a consequence, when $\Omega=\mathbb{T}^2\times\mathbb{R}$, (1.3) needs to be changed to

$$\left|\int_\Omega fgh\,dx\right| \leqslant C\|f\|_{L^2(\Omega)}^{\frac12}\left(\|f\|_{L^2(\Omega)}+\|\partial_1 f\|_{L^2(\Omega)}\right)^{\frac12} \times\|g\|_{L^2(\Omega)}^{\frac12}\left(\|g\|_{L^2(\Omega)}+\|\partial_2 g\|_{L^2(\Omega)}\right)^{\frac12}\|h\|_{L^2(\Omega)}^{\frac12}\|\partial_3 h\|_{L^2(\Omega)}^{\frac12}. \tag{1.6}$$

A rigorous proof of (1.6) can be achieved by first applying anisotropic Hölder's inequality and then (1.5). Intuitively the triple products on Ω can be bounded in terms of directional high integrability, which can be further controlled by directional derivatives along with low integrability terms. The 2D version of this type of inequalities have been used in [10, 11].

The difference between (1.3) for $\mathbb{R}^3$ and (1.6) for Ω generates a different upper bound on the nonlinearity in the case of $\mathbb{R}^3$ from the one for Ω. In fact, applying (1.3) to (1.2) in the whole space $\mathbb{R}^3$ yields

$$\begin{aligned}|J_{11}| &\leqslant C\|\partial_3^2 u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\partial_1\partial_3^2 u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\nabla_h u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\partial_3\nabla_h u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\\ &\quad\times\|\partial_3^2 u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\partial_2\partial_3^2 u_h\|_{L^2(\mathbb{R}^3)}^{\frac12}\\ &\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2,\end{aligned}$$

which is the desired upper bound. When we apply (1.6), we no longer obtain the desired upper bound $C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2$ due to the presence of $\left(\|f\|_{L^2(\Omega)}+\|\partial_1 f\|_{L^2(\Omega)}\right)^{\frac12}$ instead of $\|\partial_1 f\|_{L^2(\Omega)}^{\frac12}$. Therefore the approach for the whole space $\mathbb{R}^3$ no longer applies to the domain $\Omega=\mathbb{T}^2\times\mathbb{R}$.

This paper presents some new ideas. One strategy is to split u into two parts

$$u=\overline{u}+\widetilde{u}, \tag{1.7}$$

where $\overline{u}$ denotes the horizontal average, namely

$$\overline{u}(x_3,t)=\int_{\mathbb{T}^2} u(x_h,x_3,t)\,dx_h.$$

$\widetilde{u}=u-\overline{u}$ is the corresponding oscillation part. This decomposition is orthogonal in any $H^m(\mathbb{T}^2\times\mathbb{R})$ for integer $m\geqslant 0$, that is,

$$\sum_{|\alpha|=m}\int_{\mathbb{T}^2\times\mathbb{R}}\partial^\alpha\overline{u}\cdot\partial^\alpha\widetilde{u}\,dx=0,\qquad \|u\|_{\dot H^m}^2=\|\overline{u}\|_{\dot H^m}^2+\|\widetilde{u}\|_{\dot H^m}^2.$$

We remark that the decomposition in (1.7) has been used before for the case when only one direction is periodic and the average is taken over a single variable (see, e.g. [8, 10, 11, 20, 26]). Here the average is taken over the 2D periodic domain or torus. In addition, due to its zero horizontal average, $\widetilde{u}$ has some crucial properties that we need. For example, $\widetilde{u}$ obeys a strong version of the Poincaré type inequality,

$$\|\widetilde{u}\|_{L^2(\Omega)}\leqslant C\|\nabla_h\widetilde{u}\|_{L^2(\Omega)}, \tag{1.8}$$

where the full gradient in the standard Poincaré type inequality is replaced by ∇_h. The strong Poincaré type inequality in (1.8) has mostly been used in the case when the right-hand side of (1.8) involves partial derivatives in a single direction (see, e.g. [10, 11, 20]). This decomposition allows us to separate u into two pieces with quite different properties. It will be extensively used in the estimates of the nonlinear parts, especially in J_{11}. Then we can write

$$
\begin{aligned}
J_{11} &= \int_\Omega \partial_3^2(\bar{u}_h+\widetilde{u}_h)\cdot\nabla_h(\bar{u}_h+\widetilde{u}_h)\cdot\partial_3^2(\bar{u}_h+\widetilde{u}_h)\,dx \\
&= \int_\Omega \partial_3^2\bar{u}_h\cdot\nabla_h\widetilde{u}_h\cdot\partial_3^2\widetilde{u}_h\,dx + \int_\Omega \partial_3^2\widetilde{u}_h\cdot\nabla_h\widetilde{u}_h\cdot\partial_3^2\bar{u}_h\,dx + \int_\Omega \partial_3^2\widetilde{u}_h\cdot\nabla_h\widetilde{u}_h\cdot\partial_3^2\widetilde{u}_h\,dx, \qquad (1.9)
\end{aligned}
$$

where we have used the fact $\nabla_h\bar{u}_h = 0$ and

$$\int_\Omega \partial_3^2\bar{u}_h\cdot\nabla_h\widetilde{u}_h\cdot\partial_3^2\bar{u}_h\,dx = 0.$$

In addition, we also make use of the following anisotropic inequality, for $\Omega = \mathbb{T}^2\times\mathbb{R}$,

$$
\begin{aligned}
\left|\int_\Omega fgh\,dx\right| &\leqslant C\left(\|f\|_{L^2(\Omega)}+\|\nabla_h f\|_{L^2(\Omega)}\right)^{\frac12}\left(\|f\|_{L^2(\Omega)}+\|\partial_3 f\|_{L^2(\Omega)}\right)^{\frac12} \\
&\quad\times\|g\|_{L^2(\Omega)}^{\frac12}\left(\|g\|_{L^2(\Omega)}+\|\nabla_h g\|_{L^2(\Omega)}\right)^{\frac12}\|h\|_{L^2(\Omega)}. \qquad (1.10)
\end{aligned}
$$

Cao *et al* [5] have proved and used this inequality when they study the Hasegawa-Mima model in a periodic domain. We can check that (1.10) is still valid for the domain $\Omega = \mathbb{T}^2\times\mathbb{R}$. The anisotropic upper bound inequality in (1.10) can be easily understood intuitively. As aforementioned, the triple product is bounded if the functions involved have enough anisotropic integrability. This would be the case if f is in the Lebesgue space L^4 in terms of the horizontal two variables and L^∞ in terms of the vertical variable, and g is in the Lebesgue space L^4 in terms of the horizontal two variables. Sobolev's inequality states that L^4-norm in 2D can be controlled by the L^2-norm of half-derivative, and L^∞-norm in 1D can be more or less controlled by the L^2-norm of half-derivative. This fact gives the terms on the right-hand side of (1.10). Combining with (1.8), (1.10) is then reduced to

$$
\begin{aligned}
\left|\int_\Omega \widetilde{f}\widetilde{g}h\,dx\right| &\leqslant C\|\nabla_h\widetilde{f}\|_{L^2(\Omega)}\left(\|\widetilde{f}\|_{L^2(\Omega)}+\|\partial_3\widetilde{f}\|_{L^2(\Omega)}\right)^{\frac12} \\
&\quad\times\|\nabla_h\widetilde{g}\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)}, \qquad (1.11)
\end{aligned}
$$

where $\widetilde{f}$ and $\widetilde{g}$ denote the oscillation parts of f and g, respectively. Applying (1.11) to the terms in (1.9) would lead to the desired upper bound

$$C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.$$

This explains the major differences between the approaches for two different domains $\mathbb{R}^2$ and $\mathbb{T}^2\times\mathbb{R}$. Implementing the strategy outlined above and applying the bootstrapping argument, we are able to establish the following well-posedness and stability result.

Theorem 1.1. *Let $\Omega = \mathbb{T}^2\times\mathbb{R}$. Assume the initial data $u_0\in H^2(\Omega)$ with $\nabla\cdot u_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu) > 0$ such that, if*

$$\|u_0\|_{H^2}\leqslant\varepsilon, \qquad (1.12)$$

then (1.1) has a unique global solution

$$u\in L^\infty(0,\infty;H^2(\Omega)). \qquad (1.13)$$

In addition, for an absolute constant $C_0 > 0$,

$$\sup_{\tau\in[0,t]}\|u(\tau)\|_{H^2}^2 + \nu\int_0^t\|\nabla_h u(\tau)\|_{H^2}^2\,d\tau\leqslant C_0^2\varepsilon^2,$$

for any $t > 0$.

Our second goal is to understand the large-time behaviour of solutions obtained in theorem 1.1. As we shall reveal, this behaviour relies crucially on the spatial domain. The large-time behaviour for $\Omega = \mathbb{T}^2 \times \mathbb{R}$ will be quite different from that for the whole space $\mathbb{R}^3$.

Large-time behaviour has been a prominent topic in the study of many PDE models. For the 3D NS equations with full Laplacian dissipation, a set of effective approaches such as the Fourier splitting method of Schonbek have been created to capture their large-time behaviour (see, e.g. [21–24]). However, when the dissipation is anisotropic and only in the horizontal direction, the methods designed for the full dissipation case no longer apply. Several very recent papers have succeeded in establishing the large-time behaviour of solutions to the ANS equations in the whole space $\mathbb{R}^3$. Ji *et al* [14] showed that, if the initial data u_0 obeys $\nabla \cdot u_0 = 0$ and, for $\sigma \in [3/4, 1)$,

$$\|u_0\|_{H^4(\mathbb{R}^3)} + \|u_0\|_{\dot{H}^{-\sigma,0}(\mathbb{R}^3)} + \|\partial_3 u_0\|_{\dot{H}^{-\sigma,0}(\mathbb{R}^3)} \leqslant \varepsilon,$$

for some sufficiently small $\varepsilon > 0$, then the corresponding solution u to (1.1) has the following decay rates:

$$\|u(t)\|_{L^2(\mathbb{R}^3)} + \|\partial_3 u(t)\|_{L^2(\mathbb{R}^3)} \leqslant C\varepsilon(1+t)^{-\frac{\sigma}{2}}, \quad \|\nabla_h u(t)\|_{L^2(\mathbb{R}^3)} \leqslant C\varepsilon(1+t)^{-\frac{1+\sigma}{2}}.$$

Here the norm $\|f\|_{\dot{H}^{s,s_1}(\mathbb{R}^3)}$ is defined by

$$\|f\|_{\dot{H}^{s,s_1}(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s_1} |\widehat{f}(\xi)|^2 \, d\xi\right]^{\frac{1}{2}}.$$

The work of Xu and Zhang [28] weakens the regularity assumption of [14] on the initial data by employing anisotropic Besov spaces. They also observed the remarkable enhanced dissipation in the third component of the velocity, which decays faster than the first two components! In addition, Fujii [12] considered solutions of (1.1) in a subspace of $H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $s \geqslant 5$, and established algebraic decay rates in $L^q(\mathbb{R}^3)$ for $1 \leqslant q \leqslant \infty$. Large-time asymptotic behaviour is also established. Fujii [12] reaffirmed the enhanced dissipation in the third velocity component.

When the spatial domain Ω is $\mathbb{T}^2 \times \mathbb{R}$, the large-time behaviour of solutions to (1.1) is different. Here the initial data u_0 is assumed to be in $H^2(\Omega)$ only. The corresponding solution u of (1.1) may not even decay in $L^2(\Omega)$. Our idea here is to decompose u into its horizontal average $\overline{u}$ and the oscillation part $\widetilde{u}$, as in (1.7). We can show that $\widetilde{u}$ decays exponentially to zero in $H^1(\Omega)$ as $t \to \infty$. As a consequence, the ANS equations in (1.1) converges to an 1D system involving the horizontal average of the first two velocity components $\overline{u}_1$ and $\overline{u}_2$, and with the horizontal average of the third component $\overline{u}_3$ remaining a constant. More precisely, we obtain the following result.

Theorem 1.2. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. Assume $u_0 \in H^2(\Omega)$ satisfies $\nabla \cdot u_0 = 0$ and*

$$\|u_0\|_{H^2(\Omega)} \leqslant \varepsilon,$$

for sufficiently small $\varepsilon > 0$. Let u be the corresponding solution of (1.1) obtained in theorem 1.1. Then the H^1-norm of the oscillation part $\widetilde{u}$ decays exponentially in time, for a constant $C_1 > 0$,

$$\|\widetilde{u}(t)\|_{H^1} \leqslant \|\widetilde{u}_0\|_{H^1} e^{-C_1 t}, \tag{1.14}$$

for all $t > 0$. As a consequence, (1.1) converges to

$$\begin{cases} \partial_t \overline{u}_1 + \overline{u}_{03}\,\partial_3 \overline{u}_1 = 0, \\ \partial_t \overline{u}_2 + \overline{u}_{03}\,\partial_3 \overline{u}_2 = 0, \\ \overline{u}_3(x_3,t) = \overline{u}_{03}, \end{cases} \tag{1.15}$$

which, together with $\overline{u}_{03} = 0$ due to the incompressibility and functional setting, leads to the large-time asymptotics

$$u(x_1,x_2,x_3,t) = (\overline{u}_{01}(x_3),\overline{u}_{02}(x_3),0) + O(e^{-Ct}), \tag{1.16}$$

where $\overline{u}_{01}$, $\overline{u}_{02}$ and $\overline{u}_{03}$ are the horizontal averages of the initial velocity components.

Equation (1.16) states that the velocity field u converges to a nontrivial steady state solution. The first two components of the steady state are given by the horizontal average of the initial horizontal velocity u_{0h} while the third component vanishes. In addition, the convergence to the steady state is exponentially fast. This confirms our claim that the large-time behaviour for $\Omega = \mathbb{T}^2 \times \mathbb{R}$ is quite different from that for $\mathbb{R}^3$. In the cases of $\mathbb{R}^3$, solutions of the ANS converge to the trivial solution and the convergence rate is algebraic.

We briefly explain why the third component of the steady state vanishes. Due to the incompressibility condition $\nabla \cdot u = 0$, we have $\nabla \cdot \overline{u} = 0$. Since $\partial_1 \overline{u}_1 = 0$ and $\partial_2 \overline{u}_2 = 0$, we have

$$\partial_3 \overline{u}_3 = 0.$$

That is, $\overline{u}_3$ depends only on t. But $\overline{u}_3 \in L^2(\mathbb{T}^2 \times \mathbb{R})$ forces $\overline{u}_3 = 0$. It is clear that the solution of (1.15) is given by

$$(\overline{u}_{01}(x_3 - \overline{u}_{03}t), \overline{u}_{02}(x_3 - \overline{u}_{03}t), \overline{u}_{03}).$$

The first two components are propagating waves with the constant speed $\overline{u}_{03}$ and the initial profiles given by the horizontal average of the horizontal components. When $\overline{u}_{03} = 0$, the solution becomes the steady-state given in (1.16).

To prove (1.14), we first derive the system governing the oscillation part $\widetilde{u}$,

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_3 \partial_3 \overline{u} = -\nabla \widetilde{p} + \nu \Delta_h \widetilde{u}, \\ \nabla \cdot \widetilde{u} = 0. \end{cases}$$

Applying the decomposition in (1.7), Poincaré's inequality (1.8) and various anisotropic inequalities, we estimate $\|\widetilde{u}\|_2^2$ and $\|\nabla \widetilde{u}\|_2^2$ to obtain the following inequality

$$\frac{d}{dt}\|\widetilde{u}(t)\|_{H^1}^2 + (2\nu - C\|u\|_{H^2})\|\nabla_h \widetilde{u}\|_{H^1}^2 \leqslant 0. \tag{1.17}$$

When the initial norm satisfies $\|u_0\|_{H^2} \leqslant \varepsilon$ for sufficiently small $\varepsilon > 0$, we have $\|u(t)\|_{H^2} \leqslant C_0\varepsilon$ and

$$2\nu - C\|u\|_{H^2} \geqslant \nu.$$

Applying the strong Poincaré's inequality to (1.17) then yields the desired exponential decay.

Finally we state a more general version of theorems 1.1 and 1.2. This general result can be similarly shown as the theorems.

Corollary 1.1. *Consider (1.1) with $\nu > 0$. Let $m \geqslant 2$ be an integer. Assume $u_0 \in H^m(\Omega)$ with $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon > 0$ such that, if*

$$\|u_0\|_{H^m(\Omega)} \leqslant \varepsilon,$$

then (1.1) has a unique global solution $u \in L^\infty(0,\infty;H^m(\Omega))$ satisfying

$$\sup_{\tau\in[0,t]} \|u(\tau)\|_{H^m}^2 + \nu \int_0^t \|\nabla_h u(\tau)\|_{H^m}^2 \, d\tau \leqslant C^2\varepsilon^2.$$

Furthermore, then the H^m-norm of the oscillation part $\widetilde{u}$ of u decays exponentially in time

$$\|\widetilde{u}(t)\|_{H^{m-1}} \leqslant C\|\widetilde{u}_0\|_{H^{m-1}} e^{-Ct},$$

for some constant $C > 0$ and for all $t > 0$.

The rest of this paper is divided into three sections. Section 2 presents several anisotropic inequalities and some fine properties related to the orthogonal decomposition. Section 3 proves theorem 1.1 while section 4 is devoted to establishing theorem 1.2.

2. Orthogonal decomposition and anisotropic inequalities

This section presents properties associated with the orthogonal decomposition in (1.7), a strong version of the Poincaré's inequality and various anisotropic Sobolev inequalities. These properties and inequalities will be used in the proofs of theorems 1.1 and 1.2.

Let $\mathbb{T}^2 = [0,1]^2$ be a 2D periodic box. Let $f = f(x_h, x_3)$ with $(x_h, x_3) \in \mathbb{T}^2 \times \mathbb{R}$ be integrable in x_h on $\mathbb{T}^2$. Define

$$\bar{f}(x_3) = \int_{\mathbb{T}^2} f(x_h, x_3) \, dx_h. \tag{2.1}$$

We decompose f into $\bar{f}$ and the corresponding oscillation portion $\widetilde{f}$ with

$$\widetilde{f} = f - \bar{f}. \tag{2.2}$$

The properties stated in the following lemma follows directly from the definitions of $\bar{f}$ and $\widetilde{f}$.

Lemma 2.1. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. Assume that $f = f(x_h, x_3)$ defined on Ω is sufficiently regular. Let $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1) and (2.2), respectively.*

(1) The following basic properties hold,

$$\overline{\partial_j f} = \partial_j \bar{f} = 0,\ j = 1,2; \quad \overline{\partial_3 f} = \partial_3 \bar{f}, \quad \bar{\widetilde{f}} = 0.$$

(2) Partial derivatives commute with the bar and the tilde operates, namely

$$\overline{\partial^\alpha f} = \partial^\alpha \bar{f}, \quad \widetilde{\partial^\alpha f} = \partial^\alpha \widetilde{f},$$

for any partial derivative ∂^α. As a consequence, if $\nabla \cdot u = 0$, then

$$\nabla \cdot \bar{u} = 0, \quad \nabla \cdot \widetilde{u} = 0.$$

(3) $\bar{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^m(\Omega)$ for any integer $m \geqslant 0$, namely

$$(\bar{f}, \widetilde{f})_{\dot{H}^m} := \sum_{|\alpha|=m} \int_\Omega \partial^\alpha \bar{f} \cdot \partial^\alpha \widetilde{f} dx = 0, \qquad \|f\|_{\dot{H}^m}^2 = \|\bar{f}\|_{\dot{H}^m}^2 + \|\widetilde{f}\|_{\dot{H}^m}^2.$$

More generally, for any $f, g \in \dot{H}^m(\Omega)$ with integer $m \geqslant 0$,

$$(\bar{f}, \widetilde{g})_{\dot{H}^m} = 0.$$

The following lemma states a strong version of the Poincaré's inequality for the oscillation portion $\widetilde{f}$.

Lemma 2.2. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. Assume $f \in \dot{H}^1(\Omega)$ and let $\widetilde{f}$ be the corresponding oscillation part. Then there is a constant $C > 0$ such that*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leqslant C\|\nabla_h \widetilde{f}\|_{L^2(\Omega)}.$$

We need several anisotropic Sobolev inequalities in order to deal with the 3D ANS inequalities. The 1D Sobolev inequalities play a crucial role in the derivation of the 3D anisotropic inequalities. For the 1D function defined on the whole line $\mathbb{R}$, $f \in H^1(\mathbb{R})$ implies

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{2.3}$$

For a function defined on the 1D periodic box $\mathbb{T}$, the Sobolev inequality is different. In fact, for $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant C\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \left(\|f\|_{L^2(\mathbb{T})} + \|f'\|_{L^2(\mathbb{T})}\right)^{\frac{1}{2}}. \tag{2.4}$$

However, the oscillation piece $\widetilde{f}$ still obeys the same inequality as in (2.3). If $f \in H^1(\mathbb{T})$ and $\widetilde{f}$ is the corresponding oscillation piece, then

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \leqslant C\|\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \|\widetilde{f}'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}. \tag{2.5}$$

As a consequence of (2.3), the anisotropic inequality in $\mathbb{R}^3$ is given by

$$\begin{aligned}\left|\int_{\mathbb{R}^3} fgh\,dx\right| &\leqslant C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \\ &\quad\times \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_2 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_3 h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.\end{aligned} \tag{2.6}$$

Due to (2.4) and (2.5), the anisotropic Sobolev inequalities stated in the following lemma hold.

Lemma 2.3. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. Assume $f, \partial_1 f, g, \partial_2 g, h, \partial_3 h \in L^2(\Omega)$. Then*

$$\begin{aligned}\left|\int_\Omega fgh\,dx\right| &\leqslant C\|f\|_{L^2(\Omega)}^{\frac{1}{2}}\left(\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\quad\times \|g\|_{L^2(\Omega)}^{\frac{1}{2}}\left(\|g\|_{L^2(\Omega)} + \|\partial_2 g\|_{L^2(\Omega)}\right)^{\frac{1}{2}} \|h\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_3 h\|_{L^2(\Omega)}^{\frac{1}{2}},\end{aligned}$$

where the last part has only $\|\partial_3 h\|_{L^2(\Omega)}^{\frac{1}{2}}$ instead of $\left(\|h\|_{L^2(\Omega)} + \|\partial_3 h\|_{L^2(\Omega)}\right)^{\frac{1}{2}}$ due to the whole line setting for x_3. When f and g are replaced by their corresponding oscillation pieces $\widetilde{f}$ and $\widetilde{g}$, then the inequality resembles the whole space case,

$$\left|\int_\Omega \widetilde{f}\widetilde{g}h\,dx\right| \leqslant C\|\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_1\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\widetilde{g}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_2\widetilde{g}\|_{L^2(\Omega)}^{\frac{1}{2}}\|h\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_3 h\|_{L^2(\Omega)}^{\frac{1}{2}}. \tag{2.7}$$

We will also make use of the anisotropic inequality stated in the following lemma. The periodic version of this inequality was shown and used by Cao *et al* [5] when they study the Hasegawa–Mima model in a periodic domain. We have found that this inequality remains valid for the domain $\Omega = \mathbb{T}^2 \times \mathbb{R}$.

Lemma 2.4. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. There exists a constant $C > 0$ such that, for any $f \in H^1(\Omega)$ and $g, \nabla_h g, h \in L^2(\Omega)$, we have*

$$\int_\Omega |fgh| dx \leqslant C(\|f\|_{L^2} + \|\nabla_h f\|_{L^2})^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_3 f\|_{L^2})^{\frac{1}{2}} \times \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2} + \|\nabla_h g\|_{L^2})^{\frac{1}{2}} \|h\|_{L^2}.$$

A special consequence of lemma 2.4 and the Poincaré's inequality in lemma 2.2 is the anisotropic inequality when one or two functions involved are the oscillation parts.

Corollary 2.5. *Let $\Omega = \mathbb{T}^2 \times \mathbb{R}$. There exists a constant $C > 0$ such that, for any $f \in H^1(\Omega)$ and $g, \nabla_h g, h \in L^2(\Omega)$, we have*

$$\int_\Omega |\widetilde{f} gh| dx \leqslant C \|\nabla_h \widetilde{f}\|_{L^2}^{\frac{1}{2}} (\|\widetilde{f}\|_{L^2} + \|\partial_3 \widetilde{f}\|_{L^2})^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2} + \|\nabla_h g\|_{L^2})^{\frac{1}{2}} \|h\|_{L^2}, \tag{2.8}$$

and

$$\int_\Omega |\widetilde{f} \widetilde{g} h| dx \leqslant C \|\nabla_h \widetilde{f}\|_{L^2}^{\frac{1}{2}} (\|\widetilde{f}\|_{L^2} + \|\partial_3 \widetilde{f}\|_{L^2})^{\frac{1}{2}} \|\nabla_h \widetilde{g}\|_{L^2} \|h\|_{L^2}, \tag{2.9}$$

here $\widetilde{f}$ and $\widetilde{g}$ denote the oscillation parts of f and g, respectively.

3. Proof of theorem 1.1

This section proves theorem 1.1. The local (in time) existence part can be shown via a rather standard procedure such as Friedrichs' method of cutoff in Fourier space (see, e.g. [1]). Therefore our focus is on the global *a priori* bound on the solution in $H^2(\Omega)$. We employ the bootstrapping argument. A rigorous statement of the abstract bootstrapping principle can be found in T. Tao's book (see [25]).

Proof of theorem 1.1. The centerpiece of the proof is the energy inequality, for a constant $C > 0$ and any $t > 0$,

$$E(t) \leqslant E(0) + CE(t)^{\frac{3}{2}}, \tag{3.1}$$

where $E(t)$ denotes the energy functional

$$E(t) = \sup_{\tau \in [0,t]} \|u(\tau)\|_{H^2}^2 + \nu \int_0^t \|\nabla_h u(\tau)\|_{H^2}^2 d\tau.$$

To prove (3.1), we estimate $\|u\|_{H^2}$. The L^2 bound is immediate,

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla_h u(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2. \tag{3.2}$$

It remains to bound the $\dot{H}^2$-norm of u. Applying ∇^2 to (1.1) and then dotting by $\nabla^2 u$, we find

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2 u\|_{L^2}^2 + \nu \|\nabla^2 \nabla_h u\|_{L^2}^2 = I_1 + I_2, \tag{3.3}$$

where the pressure term vanishes due to $\nabla \cdot u = 0$, and

$$I_1 = -\sum_{i=1}^2 \int_\Omega \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx, \quad I_2 = -\int_\Omega \partial_3^2 (u \cdot \nabla u) \cdot \partial_3^2 u \, dx.$$

Invoking the decomposition $u = \overline{u} + \widetilde{u}$ and using lemma 2.1, we have

$$
\begin{aligned}
I_1 &= -\sum_{i=1}^{2} \int_\Omega \partial_i^2 u \cdot \nabla u \cdot \partial_i^2 u \, dx - 2 \sum_{i=1}^{2} \int_\Omega \partial_i u \cdot \nabla \partial_i u \cdot \partial_i^2 u \, dx \\
&= -\sum_{i=1}^{2} \int_\Omega \partial_i^2 \widetilde{u} \cdot \nabla u \cdot \partial_i^2 \widetilde{u} \, dx - 2 \sum_{i=1}^{2} \int_\Omega \partial_i \widetilde{u} \cdot \nabla \partial_i \widetilde{u} \cdot \partial_i^2 \widetilde{u} \, dx,
\end{aligned}
$$

where we have used $\partial_i \overline{u} = 0$ for $i = 1, 2$. By (2.7) in lemma 2.3,

$$
\begin{aligned}
|I_1| &\leqslant C \sum_{i=1}^{2} \|\partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \sum_{i=1}^{2} \|\partial_i \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_i \nabla \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i \nabla \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\leqslant C \|u\|_{H^2} \|\nabla_h u\|_{H^2}^2.
\end{aligned} \tag{3.4}
$$

We further decompose I_2 into two terms,

$$
I_2 = -\int_\Omega \partial_3^2 (u_h \cdot \nabla_h u + u_3 \partial_3 u) \cdot \partial_3^2 u \, dx := I_{21} + I_{22}.
$$

Writing $u = \overline{u} + \widetilde{u}$ and using lemma 2.1, we have

$$
\begin{aligned}
I_{21} &= -\sum_{k=1}^{2} C_2^k \int_\Omega \partial_3^k (\overline{u}_h + \widetilde{u}_h) \cdot \partial_3^{2-k} \nabla_h \widetilde{u} \cdot \partial_3^2 (\overline{u} + \widetilde{u}) \, dx \\
&= -\sum_{k=1}^{2} C_2^k \Bigg(\int_\Omega \partial_3^k \overline{u}_h \cdot \partial_3^{2-k} \nabla_h \widetilde{u} \cdot \partial_3^2 \widetilde{u} \, dx - \int_\Omega \partial_3^k \widetilde{u}_h \cdot \partial_3^{2-k} \nabla_h \widetilde{u} \cdot \partial_3^2 \overline{u} \, dx \\
&\quad - \int_\Omega \partial_3^k \widetilde{u}_h \cdot \partial_3^{2-k} \nabla_h \widetilde{u} \cdot \partial_3^2 \widetilde{u} \, dx \Bigg) \\
&:= I_{211} + I_{212} + I_{213},
\end{aligned}
$$

where C_2^k denotes the binomial coefficient, and we have used the fact that

$$
\int_\Omega \partial_3^2 \overline{u}_h \cdot \nabla_h \widetilde{u} \cdot \partial_3^2 \overline{u} \, dx = \int_{\mathbb{R}} \partial_3^2 \overline{u}_h \cdot \left(\int_{\mathbb{T}^2} \nabla_h \widetilde{u} \, dx_h \right) \cdot \partial_3^2 \overline{u} \, dx_3 = 0.
$$

By lemma 2.5, we have

$$
\begin{aligned}
I_{211} &\leqslant C \sum_{k=1}^{2} \|\nabla_h \partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2}^{\frac{1}{2}} (\|\partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2} + \|\partial_3 \partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2})^{\frac{1}{2}} \\
&\quad \times \|\nabla_h \partial_3^2 \widetilde{u}\|_{L^2} \|\partial_3^k \overline{u}_h\|_{L^2} \\
&\leqslant C \|u\|_{H^2} \|\nabla_h u\|_{H^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
I_{212} &\leqslant C \sum_{k=1}^{2} \|\nabla_h \partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2}^{\frac{1}{2}} (\|\partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2} + \|\partial_3 \partial_3^{2-k} \nabla_h \widetilde{u}\|_{L^2})^{\frac{1}{2}} \\
&\quad \times \|\nabla_h \partial_3^k \widetilde{u}_h\|_{L^2} \|\partial_3^2 \overline{u}\|_{L^2} \\
&\leqslant C \|u\|_{H^2} \|\nabla_h u\|_{H^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{213} &\leqslant C\sum_{k=1}^{2}\|\nabla_h\partial_3^{2-k}\nabla_h\widetilde{u}\|_{L^2}^{\frac{1}{2}}(\|\partial_3^{2-k}\nabla_h\widetilde{u}\|_{L^2}+\|\partial_3\partial_3^{2-k}\nabla_h\widetilde{u}\|_{L^2})^{\frac{1}{2}}\\
&\quad\times\|\nabla_h\partial_3^k\widetilde{u}_h\|_{L^2}\|\partial_3^2\widetilde{u}\|_{L^2}\\
&\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

We now turn to the estimates of I_{22},

$$
\begin{aligned}
I_{22} &= -\sum_{k=1}^{2}C_2^k\int_\Omega\partial_3^k u_3\partial_3^{2-k}\partial_3 u\cdot\partial_3^2 u dx\\
&=\sum_{k=1}^{2}C_2^k\int_\Omega\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\partial_3^{3-k}u\cdot\partial_3^2 u dx\\
&=\sum_{k=1}^{2}C_2^k\int_\Omega\left(\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\partial_3^{3-k}\widetilde{u}\cdot\partial_3^2 u+\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\partial_3^{3-k}\overline{u}\cdot\partial_3^2\widetilde{u}\right)dx\\
&:= I_{221}+I_{222}.
\end{aligned}
$$

By lemma 2.5, we have

$$
\begin{aligned}
I_{221} &\leqslant C\|\nabla_h\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2}^{\frac{1}{2}}(\|\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2}+\|\partial_3\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2})^{\frac{1}{2}}\\
&\quad\times\|\nabla_h\partial_3^{3-k}\widetilde{u}\|_{L^2}\|\partial_3^2 u\|_{L^2}\\
&\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

and

$$
\begin{aligned}
I_{222} &\leqslant C\|\nabla_h\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2}^{\frac{1}{2}}(\|\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2}+\|\partial_3\partial_3^{k-1}\nabla_h\cdot\widetilde{u}_h\|_{L^2})^{\frac{1}{2}}\\
&\quad\times\|\nabla_h\partial_3^2\widetilde{u}\|_{L^2}\|\partial_3^{3-k}\overline{u}\|_{L^2}\\
&\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

Combining all the estimates for I_2, we have

$$
I_2\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2. \tag{3.5}
$$

Equations (3.4) and (3.5) yield

$$
I_1+I_2\leqslant C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2. \tag{3.6}
$$

It then follows from (3.3) that

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2}^2+2\nu\int_0^t\|\nabla^2\nabla_h u\|_{L^2}^2\,d\tau &\leqslant C\int_0^t\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2\,d\tau\\
&\leqslant C\sup_{\tau\in[0,t]}\|u(\tau)\|_{H^2}\int_0^t\|\nabla_h u(\tau)\|_{H^2}^2\,d\tau.
\end{aligned}
$$

Adding this inequality to (3.2) gives

$$
E(t)\leqslant E(0)+CE(t)^{\frac{3}{2}}.
$$

A bootstrapping argument implies that, there is $\varepsilon>0$, such that, if $E(0)<\varepsilon^2$, then

$$
E(t)\leqslant C_0^2\varepsilon^2,
$$

for an absolute constant $C_0 > 0$ and for all $t > 0$. This finishes the global stability and *a priori* bound part.

We now prove the uniqueness part of theorem 1.1. We show that if two solutions $(u^{(1)}, P^{(1)})$ and $(u^{(2)}, P^{(2)})$ of (1.1) are in the regularity class $L^\infty(0,T;H^2)$, then they must coincide. Their difference (u^*, P^*) with

$$u^* = u^{(1)} - u^{(2)},\ P^* = P^{(1)} - P^{(2)},$$

satisfies

$$\begin{cases} \partial_t u^* + u^{(1)} \cdot \nabla u^* + u^* \cdot \nabla u^{(2)} = -\nabla P^* + \nu\, \Delta_h u^*, \quad x \in \Omega,\ t > 0, \\ \nabla \cdot u^* = 0, \end{cases} \tag{3.7}$$

Taking the inner product of u^* with (3.7) yields

$$\frac{d}{dt}\|u^*\|_{L^2}^2 + 2\nu\|\nabla_h u^*\|_{L^2}^2 = -\int_\Omega u^* \cdot \nabla u^{(2)} \cdot u^* dx. \tag{3.8}$$

Since the dissipation is only in the horizontal direction, we need an anisotropic upper bound for the term on the right-hand side. By Hölder's, Sobolev's and Minkowski's inequalities, we get

$$\begin{aligned} -\int_\Omega u^* \cdot \nabla u^{(2)} \cdot u^* dx &\leqslant \|u^*\|_{L_h^4 L_{x_3}^2}^2 \|\nabla u^{(2)}\|_{L_h^2 L_{x_3}^\infty} \\ &\leqslant C\|u^*\|_{L_h^4 L_{x_3}^2}^2 \|\nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \\ &\leqslant C\|u^*\|_{L^2}\|\nabla_h u^*\|_{L^2}\|\nabla u^{(2)}\|_{L^2}^{\frac{1}{2}}\|\partial_3 \nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \\ &\leqslant \nu\|\nabla_h u^*\|_{L^2}^2 + C(\nu)\|u^{(2)}\|_{H^2}^2\|u^*\|_{L^2}^2. \end{aligned}$$

Incorporating this upper bound in (3.8) yields

$$\frac{d}{dt}\|u^*\|_{L^2}^2 + \nu\|\nabla_h u^*\|_{L^2}^2 \leqslant C(\nu)\|u^{(2)}\|_{H^2}^2\|u^*\|_{L^2}^2,$$

which leads to the uniqueness due to $u^{(2)} \in L^\infty(0,T;H^2)$. This completes the proof of theorem 1.1. □

4. Proof of theorem 1.2

This section is devoted to the proof of theorem 1.2. We will use the fact stated in the following lemma. This lemma was proven and used in [9].

Lemma 4.1. *Let $f = f(y)$ with $y \in \mathbb{R}$ be a nonnegative continuous function. Assume f is integrable on $\mathbb{R}$,*

$$\int_\mathbb{R} f(y)\, dy < \infty. \tag{4.1}$$

If f is uniformly continuous on $\mathbb{R}$, then

$$f(y) \to 0 \quad as\ |y| \to \infty.$$

Proof of theorem 1.2. The proof is naturally divided into two parts. The first part verifies that the H^1-norm of the oscillation part $\widetilde{u}$ decays exponentially for large time. The second part establishes the large-time asymptotics for u.

Taking the horizontal average of (1.1) yields

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \overline{u_3 \partial_3 \overline{u}} = \begin{pmatrix} 0 \\ 0 \\ -\partial_3 \overline{p} \end{pmatrix}, \\ \partial_3 \overline{u}_3 = 0, \end{cases} \tag{4.2}$$

where we have invoked $\overline{\Delta_h u} = 0$ and

$$\overline{u \cdot \nabla u} = \overline{u \cdot \nabla \widetilde{u}} + \overline{u \cdot \nabla \overline{u}} = \overline{u \cdot \nabla \widetilde{u}} + \overline{u_3 \partial_3 \overline{u}},$$

due to $\partial_1 \overline{u} = \partial_2 \overline{u} = 0$. Taking the difference of (1.1) and (4.2), we find

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_3 \partial_3 \overline{u} = -\nabla \widetilde{p} + \nu \Delta_h \widetilde{u}, \\ \nabla \cdot \widetilde{u} = 0. \end{cases} \tag{4.3}$$

Here we have used

$$\begin{aligned} u \cdot \nabla u - (\overline{u \cdot \nabla \widetilde{u}} + \overline{u_3 \partial_3 \overline{u}}) &= u \cdot \nabla \widetilde{u} - \overline{u \cdot \nabla \widetilde{u}} + u \cdot \nabla \overline{u} - \overline{u_3 \partial_3 \overline{u}} \\ &= \widetilde{u \cdot \nabla \widetilde{u}} + u_3 \partial_3 \overline{u} - \overline{u}_3 \partial_3 \overline{u} \\ &= \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_3 \partial_3 \overline{u}. \end{aligned}$$

The L^2-estimate gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widetilde{u}(t)\|_{L^2}^2 + \nu \|\nabla_h \widetilde{u}\|_{L^2}^2 &= -\int_\Omega \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx - \int_\Omega \widetilde{u}_3 \partial_3 \overline{u} \cdot \widetilde{u} dx \\ &:= A_1 + A_2. \end{aligned} \tag{4.4}$$

Using $\nabla \cdot u = 0$ and a property in lemma 2.1, we have

$$A_1 = -\int_\Omega \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = -\int_\Omega u \cdot \nabla \widetilde{u} \cdot \widetilde{u} dx + \int_\Omega \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = 0.$$

The second part above is zero due to $\int_\Omega \bar{f} \cdot \widetilde{g} \, dx = 0$ or a direct verification,

$$\int_\Omega \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = \int_{\mathbb{R}} \overline{u \cdot \nabla \widetilde{u}}(x_3) \int_{\mathbb{T}^2} \widetilde{u} dx_h dx_3 = \int_{\mathbb{R}} \overline{u \cdot \nabla \widetilde{u}} \cdot \overline{\widetilde{u}} dx_3 = 0.$$

By lemmas 2.2 and 2.5,

$$\begin{aligned} A_2 &= -\int_\Omega \widetilde{u}_3 \partial_3 \overline{u} \cdot \widetilde{u} dx \\ &\leqslant C(\|\partial_3 \overline{u}\|_{L^2} + \|\nabla_h \partial_3 \overline{u}\|_{L^2})^{\frac{1}{2}} (\|\partial_3 \overline{u}\|_{L^2} + \|\partial_3 \partial_3 \overline{u}\|_{L^2})^{\frac{1}{2}} \|\nabla_h \widetilde{u}\|_{L^2} \|\widetilde{u}_3\|_{L^2} \\ &\leqslant C \|u\|_{H^2} \|\nabla_h \widetilde{u}\|_{L^2}^2. \end{aligned}$$

It then follows from (4.4) that

$$\frac{d}{dt} \|\widetilde{u}(t)\|_{L^2}^2 + (2\nu - C\|u\|_{H^2}) \|\nabla_h \widetilde{u}\|_{L^2}^2 \leqslant 0. \tag{4.5}$$

By theorem 1.1, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{H^2} \leqslant \varepsilon$, then $\|u(t)\|_{H^2} \leqslant C_0\varepsilon$ and

$$2\nu - C\|u\|_{H^2} \geqslant \nu. \tag{4.6}$$

Equations (4.5) and (4.6) and Poincaré type inequality in lemma 2.2 leads to the desired exponential decay for $\|\widetilde{u}\|_{L^2}$,

$$\|\widetilde{u}(t)\|_{L^2} \leqslant \|\widetilde{u}_0\|_{L^2} e^{-C_1 t}, \tag{4.7}$$

where $C_1 = C_1(\nu) > 0$.

We now prove the exponential decay of $\|\nabla\widetilde{u}(t)\|_2$. Applying ∇ to (4.3) yields

$$\partial_t \nabla\widetilde{u} + \nabla(\widetilde{u\cdot\nabla\widetilde{u}}) + \nabla(\widetilde{u}_3\partial_3\overline{u}) = -\nabla\nabla\widetilde{p} + \nu\Delta_h\nabla\widetilde{u}. \tag{4.8}$$

Taking the L^2 inner product of (4.8) with $\nabla\widetilde{u}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \nu\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2 \\
&\quad = d - \int_\Omega \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx - \int_\Omega \nabla(\widetilde{u}_3\partial_3\overline{u})\cdot\nabla\widetilde{u}dx \\
&\quad := B_1 + B_2.
\end{aligned} \tag{4.9}$$

By lemma 2.1, B_1 can be further written as

$$\begin{aligned}
B_1 &= -\int_\Omega \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx \\
&= -\int_\Omega \nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}dx + \int_\Omega \nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx \\
&= -\int_\Omega \nabla(u_h\cdot\nabla_h\widetilde{u} + u_3\partial_3\widetilde{u})\cdot\nabla\widetilde{u}dx \\
&:= B_{11} + B_{12}.
\end{aligned}$$

Using lemma 2.5, B_{11} can be bounded by

$$\begin{aligned}
B_{11} &= -\int_\Omega \nabla u_h\cdot\nabla_h\widetilde{u}\cdot\nabla\widetilde{u}dx \\
&\leqslant C(\|\nabla u_h\|_{L^2} + \|\nabla_h\nabla u_h\|_{L^2})^{\frac{1}{2}}(\|\nabla u_h\|_{L^2} + \|\partial_3\nabla u_h\|_{L^2})^{\frac{1}{2}}\|\nabla_h\nabla_h\widetilde{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2} \\
&\leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
B_{12} &= -\int_\Omega \nabla u_3\partial_3\widetilde{u}\cdot\nabla\widetilde{u}dx \\
&\leqslant C(\|\nabla u_3\|_{L^2} + \|\nabla_h\nabla u_3\|_{L^2})^{\frac{1}{2}}(\|\nabla u_3\|_{L^2} + \|\partial_3\nabla u_3\|_{L^2})^{\frac{1}{2}}\|\nabla_h\partial_3\widetilde{u}\|_{L^2}\|\nabla\widetilde{u}\|_2 \\
&\leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Therefore, B_1 is bounded by

$$|B_1| \leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2.$$

To bounded B_2, we first write it explicitly as

$$
\begin{aligned}
B_2 &= -\int_\Omega \nabla(\widetilde{u}_3\partial_3\overline{u})\cdot\nabla\widetilde{u}dx \\
&= -\int_\Omega \nabla\widetilde{u}_3\cdot\nabla\widetilde{u}\cdot\partial_3\overline{u}dx - \int_\Omega \widetilde{u}_3\partial_{31}\overline{u}\cdot\partial_1\widetilde{u}dx \\
&\quad -\int_\Omega \widetilde{u}_3\partial_{32}\overline{u}\cdot\partial_2\widetilde{u}dx - \int_\Omega \widetilde{u}_3\partial_{33}\overline{u}\cdot\partial_3\widetilde{u}dx \\
&:= B_{21}+\cdots+B_{24}.
\end{aligned}
$$

By the simple property that $\partial_1\overline{u}=\partial_2\overline{u}=0$,

$$
B_{22} = -\int_\Omega \widetilde{u}_3\,\partial_{31}\overline{u}\cdot\partial_1\widetilde{u}dx = 0, \quad B_{23} = -\int_\Omega \widetilde{u}_3\,\partial_{32}\overline{u}\cdot\partial_2\widetilde{u}dx = 0.
$$

By lemmas 2.2 and 2.5,

$$
\begin{aligned}
B_{21} &= -\int_\Omega \nabla\widetilde{u}_3\cdot\nabla\widetilde{u}\cdot\partial_3\overline{u}dx \\
&\leqslant C(\|\partial_3\overline{u}\|_{L^2}+\|\nabla_h\partial_3\overline{u}\|_{L^2})^{\frac{1}{2}}(\|\partial_3\overline{u}\|_{L^2}+\|\partial_3\partial_3\overline{u}\|_{L^2})^{\frac{1}{2}}\|\nabla\nabla_h\widetilde{u}_3\|_{L^2}\|\nabla\widetilde{u}\|_{L^2} \\
&\leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
B_{24} &= -\int_\Omega \widetilde{u}_3\partial_{33}\overline{u}\cdot\partial_3\widetilde{u}dx \\
&\leqslant C\|\nabla_h\widetilde{u}_3\|_{L^2}^{\frac{1}{2}}(\|\widetilde{u}_3\|_{L^2}+\|\partial_3\widetilde{u}_3\|_{L^2})^{\frac{1}{2}}\|\partial_3\nabla_h\widetilde{u}\|_{L^2}\|\partial_{33}\overline{u}\|_{L^2} \\
&\leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

Thus

$$
|B_2| \leqslant C\|u\|_{H^2}\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2.
$$

Inserting the estimates for B_1 through B_2 in (4.9), we obtain

$$
\frac{d}{dt}\|\nabla\widetilde{u}(t)\|_{L^2}^2 + (2\nu - C\|u\|_{H^2})\|\nabla_h\nabla\widetilde{u}\|_{L^2}^2 \leqslant 0.
$$

As in the estimate of $\|\widetilde{u}\|_{L^2}$, choosing ε sufficiently small and invoking the Poincaré's type inequality in lemma 2.2, we obtain the exponential decay result for $\|\nabla\widetilde{u}\|_{L^2}$,

$$
\|\nabla\widetilde{u}(t)\|_{L^2} \leqslant \|\nabla\widetilde{u}_0\|_{L^2}e^{-C_1 t}. \tag{4.10}
$$

Combining estimates (4.7) and (4.10), we obtain the desired decay result.

The second part of the proof is devoted to the large-time asymptotics of u. Taking the horizontal average of (1.1) and using the facts that $\partial_1\overline{u}=\partial_2\overline{u}=0$, we have

$$
\begin{cases}
\partial_t\overline{u} + \overline{\partial_3(u_3u)} = \begin{pmatrix} 0 \\ 0 \\ -\partial_3\overline{p} \end{pmatrix}, \\
\partial_3\overline{u}_3 = 0.
\end{cases} \tag{4.11}
$$

$\partial_3 \overline{u}_3 = 0$ implies that $\overline{u}_3$ is independent of x_3. We first simplify the equations of the first two components in (4.11). By lemma 2.1,

$$\begin{aligned}
\overline{\partial_3(u_3 u_1)} &= \overline{\partial_3((\overline{u}_3 + \widetilde{u}_3)(\overline{u}_1 + \widetilde{u}_1))} \\
&= \overline{\partial_3(\overline{u}_3 \overline{u}_1)} + \overline{\partial_3(\widetilde{u}_3 \overline{u}_1)} + \overline{\partial_3(\overline{u}_3 \widetilde{u}_1)} + \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)} \\
&= \partial_3(\overline{u}_3 \overline{u}_1) + \partial_3(\overline{\widetilde{u}}_3 \overline{u}_1) + \partial_3(\overline{u}_3 \overline{\widetilde{u}}_1) + \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)} \\
&= \overline{u}_3 \partial_3 \overline{u}_1 + \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)},
\end{aligned}$$

due to $\overline{\widetilde{u}}_1 = \overline{\widetilde{u}}_3 = 0$ and the fact that $\overline{u}_3$ is independent of x_3. Therefore, the equations of the first two components are given by

$$\begin{cases}
\partial_t \overline{u}_1 + \overline{u}_3 \partial_3 \overline{u}_1 + \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)} = 0, \\
\partial_t \overline{u}_2 + \overline{u}_3 \partial_3 \overline{u}_2 + \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_2)} = 0.
\end{cases} \tag{4.12}$$

Now we show, for a constant $C > 0$,

$$\left|\overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)}\right| \leqslant C\varepsilon^2 e^{-2C_1 t}, \qquad \left|\overline{\partial_3(\widetilde{u}_3 \widetilde{u}_2)}\right| \leqslant C\varepsilon^2 e^{-2C_1 t}. \tag{4.13}$$

It suffices to prove the first inequality in (4.13). The proof for the second one is similar. To simplify the notation, we set

$$F(x_3, t) := \overline{\partial_3(\widetilde{u}_3 \widetilde{u}_1)}, \quad x_3 \in \mathbb{R}.$$

Since $u \in H^2(\Omega)$, we can easily check that F as a function of x_3 is in $H^1(\mathbb{R})$, namely

$$F \in H^1(\mathbb{R}).$$

By the 1D Sobolev embedding, for any $\alpha \in [0, \frac{1}{2})$,

$$F \in H^1(\mathbb{R}) \quad \hookrightarrow \quad C^\alpha(\mathbb{R}),$$

where C^α denotes the standard Hölder space. In particular, F is uniformly continuous on $\mathbb{R}$. Next we check that

$$\varepsilon^{-2} e^{2C_1 t} F \in L^1(\mathbb{R}). \tag{4.14}$$

In fact, by the exponential decay upper bound on $\|\widetilde{u}(t)\|_{H^1}$,

$$\begin{aligned}
\varepsilon^{-2} e^{2C_1 t} \int_{\mathbb{R}} |F(x_3, t)|\, dx_3 &= \varepsilon^{-2} e^{2C_1 t} \int_{\mathbb{R}} \left| \int_{\mathbb{T}^2} \partial_3(\widetilde{u}_3 \widetilde{u}_1)\, dx_h \right| dx_3 \\
&\leqslant \varepsilon^{-2} e^{2C_1 t} \int_{\mathbb{R}} \int_{\mathbb{T}^2} (|\widetilde{u}_1 \partial_3 \widetilde{u}_3| + |\widetilde{u}_3 \partial_3 \widetilde{u}_1|)\, dx_h dx_3 \\
&\leqslant \varepsilon^{-2} e^{2C_1 t} \|\widetilde{u}(t)\|_{L^2} \|\nabla \widetilde{u}(t)\|_{L^2} \\
&\leqslant C,
\end{aligned}$$

which verifies (4.14). It then follows from lemma 4.1 that

$$\varepsilon^{-2} e^{2C_1 t} |F(x_3, t)| \to 0 \quad \text{as } |x_3| \to \infty.$$

Especially, $\varepsilon^{-2} e^{2C_1 t} F$ is uniformly bounded on $\mathbb{R}$ or

$$\|F(\cdot, t)\|_{L^\infty(\mathbb{R})} \leqslant C\varepsilon^2 e^{-2C_1 t},$$

This verifies (4.13). Therefore, for large time $t > 0$, (4.12) is majorized by

$$\begin{cases} \partial_t \overline{u}_1 + \overline{u}_3 \partial_3 \overline{u}_1 = 0, \\ \partial_t \overline{u}_2 + \overline{u}_3 \partial_3 \overline{u}_2 = 0. \end{cases} \tag{4.15}$$

Since $\overline{u}_3$ is a constant, the transport equations in (4.15) are solved by

$$\overline{u}_1(x_3,t) = \overline{u}_{01}(x_3 - \overline{u}_3 t), \quad \overline{u}_2(x_3,t) = \overline{u}_{02}(x_3 - \overline{u}_3 t).$$

The fact that the constant function $\overline{u}_3$ is in $L^2(\mathbb{R})$ leads to $\overline{u}_3 = 0$. Then $\overline{u}_1(x_3,t)$ and $\overline{u}_2(x_3,t)$ asymptotically approach $\overline{u}_{01}(x_3)$ and $\overline{u}_{02}(x_3)$, respectively. This completes the proof of theorem 1.2. □

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

R Ji was supported by the National Natural Science Foundation of China (NSFC) under Grant No. 12001065 and Creative Research Groups of the Natural Science Foundation of Sichuan under Grant No. 2023NSFSC1984. J Wu was partially supported by NSF Grant No. DMS 2104682 and the AT&T Foundation at Oklahoma State University.

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References

[1] Bahouri H, Chemin J Y and Danchin R 2011 *Fourier Analysis and Nonlinear Partial Differential Equations* (Berlin: Springer)
[2] Beardsley R C, Saunders K D, Warn-Varnas A C and Harding J M 1979 An experimental and numerical study of the secular spin-up of a thermally stratified rotating fluid *J. Fluid Mech.* **93** 161–84
[3] Chemin J, Desjardins B, Gallagher I and Grenier E 2006 *Mathematical Geophysics. An Introduction to Rotating Fluids and the Navier-Stokes Equations* (*Oxford Lecture Series in Mathematics and its Applications* vol 32) (Oxford: The Clarendon/Oxford University Press)
[4] Chemin J and Zhang P 2007 On the global well-posedness to the 3-D incompressible anisotropic Navier-Stokes equations *Commun. Math. Phys.* **272** 529–66
[5] Cao C, Guo Y and Titi E S 2018 Global strong solutions for the three-dimensional Hasegawa-Mima model with partial dissipation *J. Math. Phys.* **59** 071503
[6] Cao C, Regmi D and Wu J 2013 The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion *J. Differ. Equ.* **254** 2661–81
[7] Cao C and Wu J 2011 Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion *Adv. Math.* **226** 1803–22
[8] Deng W, Wu J and Zhang P 2021 Stability of couette flow for 2D boussinesq system with vertical dissipation *J. Funct. Anal.* **281** 109255

[9] Doering C R, Wu J, Zhao K and Zheng X 2018 Long time behavior of the two-dimensional boussinesq equations without buoyancy diffusion *Physica* D **376/377** 144–59
[10] Dong B, Wu J, Xu X and Zhu N 2021 Stability and exponential decay for the 2D anisotropic Navier-Stokes equations with horizontal dissipation *J. Math. Fluid Mech.* **23** 100
[11] Dong B, Wu J, Xu X and Zhu N 2021 Stability and exponential decay for the 2D anisotropic boussinesq equation with horizontal dissipation *Calc. Var. Partial Differ. Equ.* **60** 116
[12] Fujii M 2021 Large time behavior of solutions to the 3D anisotropic Navier-Stokes equation (arXiv:2108.11940v1)
[13] Iftimie D 1999 The resolution of the Navier-Stokes equations in anisotropic spaces *Rev. Mat. Iberoam.* **15** 1–36
[14] Ji R, Wu J and Yang W 2021 Stability and optimal decay for the 3D Navier-Stokes equations with horizontal dissipation *J. Differ. Equ.* **290** 57–77
[15] Liu Y, Paicu M and Zhang P 2020 Global well-posedness of 3-D anisotropic Navier-Stokes system with small unidirectional derivative *Arch. Ration. Mech. Anal.* **238** 805–43
[16] Liu Y and Zhang P 2020 Global solutions of 3-D Navier-Stokes system with small unidirectional derivative *Arch. Ration. Mech. Anal.* **235** 1405–44
[17] Pedlosky J 1987 *Geophysical Fluid Dynamics* 2nd edn (Berlin: Springer)
[18] Paicu M 2005 Équation anisotrope de Navier-Stokes dans des espaces critiques *Rev. Mat. Iberoam.* **21** 179–235
[19] Paicu M and Zhang P 2019 Global strong solutions to 3-D Navier-Stokes system with strong dissipation in one direction *Sci. China Math.* **62** 1175–204
[20] Paicu M and Zhu N 2021 Global regularity for the 2D MHD and tropical climate model with horizontal dissipation *J. Nonlinear Sci.* **31** 99
[21] Schonbek M 1985 L^2 decay for weak solutions of the Navier-Stokes equations *Arch. Ration. Mech. Anal.* **88** 209–22
[22] Schonbek M and Wiegner M 1996 On the decay of higher-order norms of the solutions of Navier-Stokes equations *Proc. R. Soc. Edinburgh* A **126** 677–85
[23] Schonbek M and Schonbek T 2005 Moments and lower bounds in the far-field of solutions to quasi-geostrophic flows *Discrete Contin. Dyn. Syst.* **13** 1277–304
[24] Schonbek M, Schonbek T and Süli E 1996 Large-time behaviour of solutions to the magnetohydrodynamics equations *Math. Ann.* **304** 717–56
[25] Tao T 2006 Nonlinear Dispersive Equations Local and Global Analysis *CBMS Regional Conf. Series in Mathematics* (Providence, RI: American Mathematical Society)
[26] Wu J and Zhang Q 2021 Stability and optimal decay for a system of 3D anisotropic Boussinesq equations *Nonlinearity* **34** 5456–84
[27] Wu J and Zhu Y 2021 Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium *Adv. Math.* **377** 107466
[28] Xu L and Zhang P 2021 Enhanced dissipation for the third component of 3D anisotropic Navier-Stokes equations (arXiv:2107.06453v1)
[29] Zhang T 2009 Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space *Commun. Math. Phys.* **287** 211–24
[30] Zhang T 2010 Erratum to: Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space *Commun. Math. Phys.* **295** 877–84