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Global well-posedness and time decay for 2D Oldroyd-B-type fluids in periodic domains with dissipation in the velocity equation only

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ABSTRACT

There have been substantial recent developments on the stability problem concerning the Oldroyd-B model of the incompressible non-Newtonian fluids, especially when the system involves only partial dissipation. One particular case is when there is only velocity dissipation, and no damping or dissipation in the equation of the non-Newtonian stress tensor τ. Yi Zhu was able to obtain the global stability for the 3D Oldroyd-B model in the Sobolev setting by employing time-weighted Sobolev spaces (Zhu, 2018). However, her approach can not be extended to the 2D whole space case due to the criticality of the time-weight. This paper presents the global stability and the large-time behavior of solutions to the 2D Oldroyd-B model with only dissipation in a periodic domain. The proof of this result overcomes the difficulty due to the lack of dissipation in τ by exploiting the special wave structure obeyed by the velocity u and $\mathbb{P}\nabla\cdot\tau$ (the projection of the divergence of τ). The enhanced dissipation in u and $\mathbb{P}\nabla\cdot\tau$ allows us to gain enough regularity and stabilizing property to control the growth of u and τ. In fact we are also able to show that the H^1-norm of ∇u and $\mathbb{P}\nabla\cdot\tau$ decays exponentially in time.

1. Introduction

Due to its physical applications and mathematical significance, the Oldroyd-B model of the incompressible non-Newtonian fluids has recently attracted considerable interests. The Oldroyd-B model governs the motion of viscoelastic fluids such as a solvent with particles suspended in it. More details on its derivation and applications can be found in [1–3]. Mathematically the Oldroyd-B model consists of the equation for the fluid (usually the Navier–Stokes equation for viscous fluids and the Euler equation for inviscid ones) with a forcing term and the evolution of the non-Newtonian stress tensor. More precisely, the standard incompressible

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Oldroyd-B equations can be written as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\,\Delta u - \nabla P = \nu_1 \nabla\cdot\tau, \\ \partial_t \tau + (u\cdot\nabla)\tau + \alpha\tau - \eta\,\Delta\tau + Q(\tau,\nabla u) = \nu_2 D(u), \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where u denotes the velocity field of the fluid, P the scalar pressure and τ the non-Newtonian stress tensor, represented by a symmetric matrix. The parameters $\mu,\eta,\alpha,\nu_1,\nu_2$ are nonnegative constants and ν_1,ν_2 are called the coupling parameters. The nonlinear term $Q(\tau,\nabla u)$ is typical in models for viscoelastic models and is a bilinear form given by

$$Q = \tau W(u) - W(u)\tau + b(D(u)\tau + \tau D(u)), \tag{1.2}$$

where $b\in[-1,1]$ is a constant, $W(u)=\frac{1}{2}(\nabla u - (\nabla u)^\top)$ is the vorticity tensor and $D(u)=\frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the deformation tensor. If $b=0$, the system is called corotational.

The first mathematical results were published in a series of papers starting by [4] in 1987, dealing with general differential models for viscoelastic fluids, including the Oldroyd-B model. The wellposedness problem is one of the most fundamental issues on Oldroyd-B model. The special coupling structure between the velocity u and the symmetric tensor τ in (1.1) makes the Oldroyd-B well-posedness problem significant and challenging. The global existence and regularity for the 3D Oldroyd-B with general large initial data is beyond reach at this moment. The 2D Oldroyd-B with full dissipation has been shown to always possess global classical solutions by Constantin and Kliegl [5]. The global existence of weak solutions is known only for the corotational Oldroyd-B [6]. Two earlier papers of Guillopé and Saut [7,8] establish the local existence and uniqueness of strong solutions to the Oldroyd-B-type fluid in a bounded domain in 2D or 3D, and the global existence of strong periodic solutions in the case of small coupling parameters and initial data. The results described in [4,7] have been improved by Molinet and Talhouk [9], where it is shown that the smallness of the coupling parameters for existence and uniqueness of solutions is not necessary. Moreover, these authors show in [10] the existence and uniqueness of solutions to the full physical problem in 2D and 3D in the case of a small Weissenberg number (which means α large), on the same time interval of existence of the solutions to the Navier–Stokes equation.

More recent efforts focus on the Oldroyd-B models with only partial dissipation, and with or without damping in the equation of τ. Significant progress has been made on the small data global well-posedness, stability and large-time behavior. One array of results are for the Oldroyd-B model without velocity dissipation, namely (1.1) with $\mu=0$. The work of Elgindi and Rousset [11] focused on the 2D Oldroyd-B equations with damping and with stress tensor dissipation. Global solutions in the Sobolev space $H^s(\mathbb{R}^2)$ with $s>2$ are obtained for small initial data, and for general data when $Q=0$. The 3D Oldroyd-B equation without velocity dissipation, or (1.1) with $\mu=0$, $\alpha>0$ and $\eta>0$, was shown in [12] to always possess small global solutions if they are initially so. We remark that the damping term in the equation of τ plays an important role in proving the global results of [11,12]. Constantin, Wu, Zhao and Zhu in a recent work [13] considered the general d-dimensional Oldroyd-B model with only stress tensor dissipation $(-\Delta)^\beta\tau$ and without damping, namely $\mu=\alpha=0$. They established the global existence and stability of small solutions in Sobolev space $H^s(\mathbb{R}^d)$ with $d\ge 2$ and $s>1+\frac{d}{2}$ if the fractional power β satisfies $\frac{1}{2}\le\beta\le 1$. Wu and Zhao [14] were able to prove the global existence and stability of the same system as in [13] but in the hybrid critical Besov spaces. Very recently Wang, Wu, Xu and Zhong [15] investigated the large-time behavior of the global solutions of [13] and obtained sharp decay rates.

A list of results have also been obtained for the Oldroyd-B models without the dissipation in the equations of τ. The work of Chemin and Masmoudi [16] dealt with the Oldroyd-B model (1.1) with $\mu>0$ and $\alpha>0$, and established regularity criteria in the Sobolev setting and the local well-posedness for general large solutions and the global well-posedness with small initial data and small ν_1 and ν_2. Lei, Masmoudi and Zhou [17] were able to improve the criteria of [16]. Zi, Fang and Zhang [18] and Wan [19] weakened the initial assumptions of [16]. Moreover, Fang and Zi [20] established the global well-posedness for a class of large initial data. We remark that all these results require the presence of the damping term, namely $\alpha > 0$. The omission of the damping term poses challenges on the well-posedness problem. Several recent papers have succeeded in obtaining the small data global well-posedness or large-time behavior without damping (see, e.g., [21–25]). The work of Zhu [21] is the first one to prove the small data global well-posedness in the Sobolev setting for the 3D Oldroyd-B system with only standard Laplacian dissipation, namely (1.1) with $\mu > 0$, $\alpha = \eta = 0$. Suitable time-weighted energy functionals are constructed to overcome the difficulty caused by the lack of damping and dissipation in the equation of τ. Chen and Hao [22] obtained the small data global well-posedness in a critical Besov space. A similar result to that of [22] was also shown by Zhai [25]. A recent work of Wu and Zhao [24] was able to establish the small data global well-posedness in critical Besov spaces when the standard Laplacian is replaced by more general fractional operator $(-\Delta)^\beta u$ with $\frac{1}{2} \le \beta \le 1$. Wan [23] obtained sharp decay rates for the global solutions of Zhu [21]. In addition to the results described above, there are many other important work and some of them are listed in the references (see, e.g., [26–37]).

Two problems remain open. One is the small data global well-posedness in the Sobolev setting for the 2D Oldroyd-B model with only velocity dissipation, namely (1.1) with $\mu > 0$ and $\eta = \alpha = 0$, although the small data global well-posedness problem in the Besov pace setting has been obtained [22,24], as aforementioned. Zhu [21] dealt with the 3D Oldroyd-B model by constructing time-weighted energy functionals in Sobolev spaces, but the approach of Zhu cannot be extended to the 2D case due to the criticality of the time weight. It appears no suitable time-weighted functionals can be constructed to control the nonlinear terms in the equation of τ. On one hand, it is natural to include the time-weighted norms

$$\sup_{0\le s\le t} (1+s)\,\|u(s)\|_{H^1(\mathbb{R}^2)}^2 + \int_0^t (1+s)\,\|\nabla u(s)\|_{H^1(\mathbb{R}^2)}^2\,ds$$

in the energy functional, but on the other hand the terms associated with the nonlinearity in the equation of τ cannot be controlled by these time weighted norms due to the unboundedness of the integral

$$\int_0^\infty (1+t)^{-1}\,dt = \infty.$$

The time weight $1+s$ is chosen according to the decay rate of $\|u(s)\|_{L^2}$ when the initial data is in the Sobolev space with negative index of order -1. The second open problem is to establish the small data global well-posedness in the Sobolev spaces for the 3D Oldroyd-B model with fractional dissipation $(-\Delta)^\beta u$ for $\beta < 1$. The Besov setting result has been obtained in [24]. However, when the Sobolev spaces are used, the fractional dissipation is not sufficient to control the nonlinear term $Q(\tau, \nabla u)$.

This paper focuses on the initial-value problem for the 2D Oldroyd-B model with only velocity dissipation in a periodic domain $\Omega = \mathbb{T}^2$,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\,\Delta u - \nabla P = \nu_1 \nabla\cdot\tau,\ x\in\Omega,\ t>0, \\ \partial_t \tau + (u\cdot\nabla)\tau + Q(\tau,\nabla u) = \nu_2 D(u), \\ \nabla\cdot u = 0, \\ u(x,0) = u_0,\ \tau(x,0) = \tau_0. \end{cases} \tag{1.3}$$

The goal here is to establish the small data global well-posedness in the Sobolev space $H^2(\Omega)$. In addition, we obtain explicit large-time decay rates for the derivatives of u and for $\mathbb{P}\nabla\cdot\tau$, where $\mathbb{P}$ denotes the Helmholtz-Leray projection onto divergence-free vector fields, namely,

$$\mathbb{P} : L^2(\Omega) \to L^2_\sigma(\Omega),$$

where

$$L^2_\sigma(\Omega) = \{v \in L^2(\Omega)\,|\,\nabla\cdot v = 0 \text{ and } v \text{ is a periodic function on } \Omega\}.$$

More precisely, the following theorem holds.

Theorem 1.1. *Consider (1.3) with $\mu > 0$ and $\nu_1 = \nu_2 > 0$. Assume $(u_0, \tau_0) \in H^2(\Omega)$ with $\nabla \cdot u_0 = 0$ and τ_0 being a symmetric matrix. Then there exists a sufficiently small $\delta > 0$ depending on μ, ν_1 and ν_2 such that if*

$$\|u_0\|_{H^2(\Omega)} + \|b_0\|_{H^2(\Omega)} \le \delta, \tag{1.4}$$

then the 2D incompressible Oldroyd-B model (1.3) admits a unique global solution $(u, \tau) \in C\big([0,\infty); H^2(\Omega)\big)$ satisfying, for some uniform constant C_0 and for any $t > 0$,

$$\|u(t)\|^2_{H^2(\Omega)} + \|\tau(t)\|^2_{H^2(\Omega)} + 2\mu \int_0^t \|\nabla u(s)\|^2_{H^2(\Omega)}\, ds + \int_0^t \|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1(\Omega)} ds \le C_0\delta^2. \tag{1.5}$$

Furthermore, the following decay rate holds,

$$\|\nabla u(t)\|_{H^1(\Omega)} + \|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1(\Omega)} \le Ce^{-C_1 t}. \tag{1.6}$$

for some constants $C > 0$ and $C_1 > 0$.

Remark 1.1. *If we additionally assume u_0 has mean zero, i.e. $\int_\Omega u_0 dx = 0$, then, by Poincaré inequality, we also have the decay estimate on u,*

$$\|u(t)\|_{L^2(\Omega)} \le Ce^{-C_1 t}.$$

We now explain the proof of Theorem 1.1. Due to the lack of dissipation or damping in the equation of τ, direct H^2-energy estimates would fail to bound the nonlinear terms in the equation of τ. To overcome this difficulty, we need to employ time-weighted norms, which would help bound the nonlinear terms. For example, when we estimate the H^2-norm of τ, we need to bound the following integral associated with the nonlinear term $u\cdot\nabla\tau$,

$$I_2 := \int_0^t \int_\Omega \nabla^2(u(x,s)\cdot\nabla\tau(x,s))\cdot\nabla^2\tau(x,s)\, dx\, ds$$

As in the proof of Lemma 3.2, this term can be bounded by

$$|I_2| \le C\int_0^t \|\nabla^2 u(s)\|_{H^1} \|\nabla\tau(s)\|^2_{H^1} ds \le \sup_{0\le s\le t} \|\nabla\tau(s)\|^2_{H^1} \int_0^t \|\nabla^2 u(s)\|_{H^1} ds.$$

However, the time integral $\int_0^t \|\nabla^2 u(s)\|_{H^1} ds$ is not known to be bounded. Time-weighted Sobolev norms would help. In fact,

$$\begin{aligned} |I_2| &\le C \sup_{0\le s\le t} \|\nabla\tau(s)\|^2_{H^1} \int_0^t \|\nabla^2 u(s)\|_{H^1} ds \\ &\le C \sup_{0\le s\le t} \|\nabla\tau(s)\|^2_{H^1} \left(\int_0^t (1+s)^2 \|\nabla^2 u(s)\|^2_{H^1} ds\right)^{\frac{1}{2}}, \end{aligned} \tag{1.7}$$

which would be bounded if the time-weighed norm on the right is bounded. The natural issue then is to control the time-weighted norm in (1.7). By the equation of u in (1.3) or

$$\partial_t u + \mathbb{P}(u\cdot\nabla)u - \mu\,\Delta u = \nu_1 \mathbb{P}\nabla\cdot\tau, \tag{1.8}$$

we need to bound the time integral of time-weighted norms of $\mathbb{P}\nabla\cdot\tau$. This does not appear to be possible due to the lack of dissipation and damping in the equation of τ. In order to overcome this difficulty, we make use of the special structure for u and $\mathbb{P}\nabla\cdot\tau$ observed by Zhu [21]. By differentiating (1.8) and the equation of τ in (1.3) in time, and making several substitutions, we find that u and $\mathbb{P}\nabla\cdot\tau$ satisfy the following nonlinear wave equations

$$\begin{cases} \partial_{tt} u - \mu\Delta\partial_t u - \frac{1}{2}\nu_1\,\nu_2\,\Delta u = N_1, \\ \partial_{tt}(\mathbb{P}\nabla\cdot\tau) - \mu\Delta\partial_t(\mathbb{P}\nabla\cdot\tau) - \frac{1}{2}\nu_1\,\nu_2\,\Delta(\mathbb{P}\nabla\cdot\tau) = N_2, \end{cases} \tag{1.9}$$

where N_1 and N_2 represent the nonlinear terms,

$$N_1 = -\partial_t \mathbb{P}(u\cdot\nabla u) - \nu_1 \mathbb{P}\nabla\cdot(u\cdot\nabla)\tau - \nu_1\mathbb{P}\nabla\cdot Q,$$
$$N_2 = -\frac{1}{2}\nu_2 \Delta\mathbb{P}((u\cdot\nabla)u) + (-\partial_t + \mu\Delta)(\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau + \mathbb{P}\nabla\cdot Q).$$

Although the regularity of solutions to (1.9) depends on the initial data, (1.9) contains more regularizing terms than its original counterparts in (1.3). These extra terms are due to the coupling and interaction in (1.3). In particular, $\mathbb{P}\nabla\cdot\tau$ is indeed dissipative. By constructing energy functionals that suitably pair the time-weighted norms of u with those of $\mathbb{P}\nabla\cdot\tau$, we are able to establish closed inequalities. More precisely, we introduce the following energy functionals

$$\mathcal{E}_1(t) = \sup_{0\le s\le t}\left(\|u(s)\|_{H^2}^2 + \|\tau(s)\|_{H^2}^2\right) + \int_0^t \Big(\|\nabla u(s)\|_{H^2}^2 + \|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2\Big)\,ds, \tag{1.10}$$
$$\mathcal{E}_2(t) = \sup_{0\le s\le t}(1+s)^2\Big(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2\Big)$$
$$\qquad + \int_0^t (1+s)^2\Big(\|\nabla^2 u(s)\|_{H^1}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2\Big)\,ds. \tag{1.11}$$

Our main efforts are devoted to proving that

$$\mathcal{E}(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t)$$

satisfies

$$\mathcal{E}(t) \le C\,\mathcal{E}(0) + C\,\mathcal{E}^{\frac{3}{2}}(t). \tag{1.12}$$

A bootstrapping argument (see, e.g., [38, p. 21]) applied to (1.12) then implies the desired stability result in Theorem 1.1. The proof of (1.12) is lengthy and accomplished by several lemmas. In particular, the damped wave structure in (1.9) is exploited to control the time integral terms in (1.10) and (1.11),

$$\int_0^t \|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2\,ds, \quad \int_0^t (1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2\,ds.$$

In addition, Poincaré type inequalities are used repeatedly to facilitate the estimates. To establish the exponential decay rate in (1.6), we include an inner product term to take advantage of the dissipative effects in $\mathbb{P}\nabla\cdot\tau$. This process allows us to simultaneously obtain the time integrability of $\|\nabla^2 u\|_{L^2}^2$ and $\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2$ and thus leads to the desired inequality for exponential decay.

The rest of this paper is organized as follows. Section 2 puts forward several simple facts to be used in the proof of Theorem 1.1. Section 3 proves the global existence and stability part of Theorem 1.1. The main efforts are devoted to the proof of (1.12). To make the lengthy proof easy to understand, we divide the proof into two main lemmas. Section 4 is devoted to the decay estimate (1.6) in Theorem 1.1. By establishing two new estimates and combining with the global existence result part, we obtain the desired exponential time decay rate.

2. Preliminary

This section presents several simple facts to be used in the proof of Theorem 1.1. First, we recall the Helmholtz-Leray decomposition, for any $\nabla\cdot\tau\in L^2(\Omega)$, there exists a unique φ (up to a constant) such that

$$\nabla\cdot\tau = \mathbb{P}\nabla\cdot\tau + \nabla\varphi. \tag{2.1}$$

Then the following two lemmas hold.

Lemma 2.1. *For any* $\tau \in H^2(\Omega)$,

$$\|\nabla\varphi\|_{H^1(\Omega)} \leq C\|\nabla\tau\|_{H^1(\Omega)} \tag{2.2}$$

for some constant $C > 0$.

Proof. Due to the orthogonality of $\mathbb{P}\nabla\cdot\tau$ with $\nabla\varphi$,

$$\|\nabla\varphi\|^2_{L^2(\Omega)} = \|\nabla\cdot\tau\|^2_{L^2(\Omega)} - \|\mathbb{P}\nabla\cdot\tau\|^2_{L^2(\Omega)}. \tag{2.3}$$

Applying the operator $\nabla\cdot$ to the equality (2.1) and using $\nabla\cdot\mathbb{P}(\nabla\cdot\tau) = 0$, we have

$$\nabla\cdot(\nabla\cdot\tau) = \Delta\varphi.$$

Then it is clear that

$$\|\Delta\varphi\|_{L^2(\Omega)} \leq \|\nabla^2\tau\|_{L^2(\Omega)},$$

which, together with (2.3), implies (2.2). □

Lemma 2.2. *For smooth* u *and* τ*, the following decomposition holds*

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) = \mathbb{P}(\nabla u\cdot\nabla\tau) + \mathbb{P}((u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)) + \mathbb{P}(\nabla u\cdot\nabla\varphi). \tag{2.4}$$

Proof. To prove (2.4), we can write via (2.1)

$$\begin{aligned}\nabla\cdot(u\cdot\nabla\tau) &= (\nabla u\cdot\nabla)\tau + (u\cdot\nabla)(\nabla\cdot\tau)\\ &= (\nabla u\cdot\nabla)\tau + (u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau) + (u\cdot\nabla)\nabla\varphi.\end{aligned} \tag{2.5}$$

Then, applying the projection operator $\mathbb{P}$ to (2.5) yields

$$\begin{aligned}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) &= \mathbb{P}(\nabla u\cdot\nabla)\tau + \mathbb{P}(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau) + \mathbb{P}(u\cdot\nabla)\nabla\varphi\\ &= \mathbb{P}(\nabla u\cdot\nabla)\tau + \mathbb{P}(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau) + \mathbb{P}(\nabla u\cdot\nabla)\varphi,\end{aligned}$$

where we have used $\mathbb{P}\nabla(u\cdot\nabla\varphi) = 0$. This completes the proof of Lemma 2.2. □

Throughout the rest of this paper, we assume $\mu = \nu_1 = \nu_2 = 1$, without loss of generality. In addition, we write $\int f(x)\,dx$ for the integral over $\Omega = \mathbb{T}^2$. The norms $\|g\|_{L^p(\Omega)}$ and $\|h\|_{H^s(\Omega)}$ are abbreviated as $\|g\|_{L^p}$ and $\|h\|_{H^s}$, respectively. The constants $C > 0$ in the paper are absolutely constants and may vary from line to line.

3. The global well-posedness

This section proves the well-posedness part in Theorem 1.1. We apply the bootstrapping argument. The main effort is devoted to establishing *a priori* estimate stated in the following proposition.

Proposition 3.1. *Let* (u,τ) *be the solution of* (1.3) *with initial data* (u_0,τ_0) *satisfying* $\operatorname{div} u_0 = 0$ *and* $(\tau_0)_{ij} = (\tau_0)_{ji}$ *in* Ω*. Define* $\mathcal{E}_1(t)$ *and* $\mathcal{E}_2(t)$ *as in* (1.10) and (1.11)*, respectively. Set*

$$\mathcal{E}(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t).$$

Then there exists a constant C_2 *such that for any* $t \geq 0$,

$$\mathcal{E}(t) \leq C_2\Big(\mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(t)\Big). \tag{3.1}$$

The global well-posedness part of Theorem 1.1 follows as a consequence of the bootstrapping argument applied to (3.1) in Proposition 3.1.

Proof of the global well-posedness. We show that, if δ in (1.4) is taken to be sufficiently small, then the global uniform bound in (1.5) holds for all time. In fact, if δ satisfies

$$\delta^2 \le \frac{1}{48C_2^3},$$

then (1.5) holds with $C_0 = 6C_2$. This is obtained by applying the bootstrapping argument to (3.1). We start by assuming that

$$\mathcal{E}(t) \le M := \Big(\frac{1}{2C_2}\Big)^2.$$

Then (3.1) implies that

$$\mathcal{E}(t) \le C_2\,\mathcal{E}(0) + \frac{1}{2}\mathcal{E}(t)$$

or

$$\mathcal{E}(t) \le 2C_2\,\mathcal{E}(0) \le 2C_2 \;\cdot 3\big(\|u_0\|_{H^2}^2 + \|\tau_0\|_{H^2}^2\big) \le 6C_2\delta^2 \le \frac{1}{8C_2^2} = \frac{1}{2}M.$$

The bootstrapping argument then concludes that, for all $t \ge 0$,

$$\mathcal{E}(t) \le 6C_2\delta^2,$$

which, in particular, implies (1.5). Combining (1.5) with the standard local well-posedness theory leads to the desired global well-posedness. □

Now we turn to the proof of Proposition 3.1. The proof consists of two main parts. The first part bounds $\mathcal{E}_1(t)$ while the second part bounds $\mathcal{E}_2(t)$. For the sake of clarity, we state each part as a lemma.

Lemma 3.2. *For some constant $C_3 > 0$.*

$$\mathcal{E}_1(t) \le C_3\mathcal{E}_1(0) + C_3\mathcal{E}_1^{\frac{3}{2}}(t) + C_3\mathcal{E}_1(t)\mathcal{E}_2^{\frac{1}{2}}(t). \tag{3.2}$$

Lemma 3.3. *For some constant $C_4 > 0$,*

$$\mathcal{E}_2(t) \le C_4\mathcal{E}_1(0) + C_4\mathcal{E}_1(t) + C_4\mathcal{E}_1^{\frac{1}{2}}(t)\mathcal{E}_2(t). \tag{3.3}$$

There are two key points for the proof. One is that we will take advantage of the regularizing and stabilizing properties offered by the wave structure of u and $\mathbb{P}\nabla\cdot\tau$. The other is that we will apply the Poincaré inequality to $\|\nabla u\|_{L^2}$ and $\|\mathbb{P}\nabla\cdot\tau\|_{L^2}$ based on the fact $\int_\Omega \nabla u\,dx = 0$ and $\int_\Omega \mathbb{P}\nabla\cdot\tau\,dx = 0$ to overcome the difficulty of the weak decay of these terms. We now prove the two lemmas.

Proof of Lemma 3.2. The proof is divided into two steps.

Step 1. This step estimates the first three terms in $\mathcal{E}_1(t)$. Applying the operator ∇^k for $k = 0, 1, 2$ to (1.3) and taking the L^2 inner product of the resulting equations with $(\nabla^k u, \nabla^k\tau)$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|u(t)\|_{H^2}^2 + \|\tau(t)\|_{H^2}^2\big) + \|\nabla u(t)\|_{H^2}^2 &= -\sum_{k=0}^{2}\int \nabla^k(u\cdot\nabla u)\cdot\nabla^k u dx\\ -\sum_{k=0}^{2}\int \nabla^k(u\cdot\nabla\tau)\cdot\nabla^k\tau dx &- \sum_{k=0}^{2}\int \nabla^k Q\cdot\nabla^k\tau dx := I_1 + I_2 + I_3,\end{aligned} \tag{3.4}$$

where we have used

$$\int \nabla^k(\nabla P)\cdot\nabla^k u dx = 0, \quad \int \nabla^k(\nabla\cdot\tau)\cdot\nabla^k u dx + \int \nabla^k D(u)\cdot\nabla^k\tau dx = 0.$$

In fact, by integration by parts and the symmetry of τ,

$$\begin{aligned}
&\int \nabla^k(\nabla\cdot\tau)\cdot\nabla^k u dx + \int \nabla^k D(u)\cdot\nabla^k\tau dx\\
&= \int \nabla^k\partial_j\tau_{ij}\nabla^k u_i dx + \frac{1}{2}\int \nabla^k(\partial_j u_i + \partial_i u_j)\nabla^k\tau_{ij} dx\\
&= -\int \nabla^k\tau_{ij}\nabla^k\partial_j u_i dx + \frac{1}{2}\int \Big(\nabla^k\partial_j u_i\nabla^k\tau_{ij} + \nabla^k\partial_i u_j\nabla^k\tau_{ji}\Big) dx\\
&= -\int \nabla^k\tau_{ij}\nabla^k\partial_j u_i dx + \int \nabla^k\tau_{ij}\nabla^k\partial_j u_i dx = 0.
\end{aligned}$$

We bound the terms on the right hand side of (3.4). We rewrite I_1 into three terms by integration by parts, and then use Sobolev imbedding inequalities

$$\|v\|_{L^4} \le C\|v\|_{H^1}, \quad \|v\|_{L^\infty} \le C\|v\|_{H^2} \tag{3.5}$$

to obtain

$$\begin{aligned}
I_1 &= -\int(\nabla u\cdot\nabla)u\cdot\nabla u dx - \int(\nabla^2 u\cdot\nabla)u\cdot\nabla^2 u dx - 2\int(\nabla u\cdot\nabla)\nabla u\cdot\nabla^2 u dx\\
&\le \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 + 3\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^2 + C\|\nabla u\|_{H^2}\|\nabla^2 u\|_{L^2}^2 \le C\|\nabla u\|_{H^1}\|\nabla u\|_{H^2}^2.
\end{aligned} \tag{3.6}$$

Due to $\int \nabla u dx = 0$, the Poincaré inequality

$$\|\nabla u\|_{L^2} \le C\|\nabla^2 u\|_{L^2} \tag{3.7}$$

holds. By (3.7) and a similar argument as above,

$$\begin{aligned}
I_2 &= -\int(\nabla u\cdot\nabla)\tau\cdot\nabla\tau dx - \int(\nabla^2 u\cdot\nabla)\tau\cdot\nabla^2\tau dx - 2\int(\nabla u\cdot\nabla)\nabla\tau\cdot\nabla^2\tau dx\\
&\le \|\nabla u\|_{L^2}\|\nabla\tau\|_{L^4}^2 + \|\nabla^2 u\|_{L^4}\|\nabla\tau\|_{L^4}\|\nabla^2\tau\|_{L^2} + 2\|\nabla u\|_{L^\infty}\|\nabla^2\tau\|_{L^2}^2\\
&\le C\|\nabla^2 u\|_{L^2}\|\nabla\tau\|_{H^1}^2 + C\|\nabla^2 u\|_{H^1}\|\nabla\tau\|_{H^1}\|\nabla^2\tau\|_{L^2} \le C\|\nabla^2 u\|_{H^1}\|\nabla\tau\|_{H^1}^2.
\end{aligned} \tag{3.8}$$

Recalling the definition of Q in (1.2), and invoking (3.5) and (3.7), we find

$$\begin{aligned}
I_3 &= -\int Q\cdot\tau dx + \int Q\cdot\Delta\tau dx - \int\nabla^2 Q\cdot\nabla^2\tau dx\\
&\le C\int|\nabla u|\,|\tau|^2 dx + C\int|\nabla u|\,|\tau||\Delta\tau| dx\\
&\quad + C\int\big(|\nabla u|\,|\nabla^2\tau| + |\nabla^2 u|\,|\nabla\tau| + |\nabla^3 u|\,|\tau|\big)|\nabla^2\tau| dx\\
&\le C\|\nabla u\|_{L^2}\|\tau\|_{L^4}^2 + C\|\nabla u\|_{L^2}\|\tau\|_{L^\infty}\|\Delta\tau\|_{L^2}\\
&\quad + C\big(\|\nabla u\|_{L^\infty}\|\nabla^2\tau\|_{L^2} + \|\nabla^2 u\|_{L^4}\|\nabla\tau\|_{L^4} + \|\nabla^3 u\|_{L^2}\|\tau\|_{L^\infty}\big)\|\nabla^2\tau\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\tau\|_{H^1}^2 + C\|\nabla u\|_{L^2}\|\tau\|_{H^2}\|\Delta\tau\|_{L^2}\\
&\quad + C\big(\|\nabla u\|_{H^2}\|\nabla^2\tau\|_{L^2} + \|\nabla^2 u\|_{H^1}\|\nabla\tau\|_{H^1} + \|\nabla^3 u\|_{L^2}\|\tau\|_{H^2}\big)\|\nabla^2\tau\|_{L^2}\\
&\le C\|\nabla^2 u\|_{H^1}\|\tau\|_{H^2}^2.
\end{aligned} \tag{3.9}$$

Inserting (3.6), (3.8) and (3.9) in (3.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u(t)\|_{H^2}^2+\|\tau(t)\|_{H^2}^2\big)+\|\nabla u(t)\|_{H^2}^2\le C\|\nabla u\|_{H^1}\|\nabla u\|_{H^2}^2+C\|\nabla^2 u\|_{H^1}\|\tau\|_{H^2}^2. \tag{3.10}$$

Then integrating (3.10) over $[0,t]$ yields

$$\begin{aligned}
&\big(\|u(t)\|_{H^2}^2+\|\tau(t)\|_{H^2}^2\big)+2\int_0^t\|\nabla u(s)\|_{H^2}^2ds\\
&\le\big(\|u_0\|_{H^2}^2+\|\tau_0\|_{H^2}^2\big)+C\int_0^t\|\nabla u(s)\|_{H^1}\|\nabla u(s)\|_{H^2}^2ds\\
&\quad+C\int_0^t\|\nabla^2 u(s)\|_{H^1}\|\tau(s)\|_{H^2}^2ds\\
&\le\big(\|u_0\|_{H^2}^2+\|\tau_0\|_{H^2}^2\big)+C\sup_{0\le s\le t}\|\nabla u(s)\|_{H^1}\int_0^t\|\nabla u(s)\|_{H^2}^2ds\\
&\quad+C\sup_{0\le s\le t}\|\tau(s)\|_{H^2}^2\int_0^t\|\nabla^2 u(s)\|_{H^1}ds\\
&\le\mathcal{E}_1(0)+C\mathcal{E}_1^{\frac{3}{2}}(t)+C\mathcal{E}_1(t)\mathcal{E}_2^{\frac{1}{2}}(t),
\end{aligned}\tag{3.11}$$

where we have used the fact $\int_0^t\|\nabla^2 u(s)\|_{H^1}ds\le\mathcal{E}_2^{\frac{1}{2}}(t)$, which can be deduced by Hölder's inequality,

$$\begin{aligned}
&\int_0^t\|\nabla^2 u(s)\|_{H^1}ds\le\int_0^t(1+s)\|\nabla^2 u(s)\|_{H^1}(1+s)^{-1}ds\\
&\le\Big(\int_0^t(1+s)^2\|\nabla^2 u(s)\|_{H^1}^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t(1+s)^{-2}ds\Big)^{\frac{1}{2}}\le\mathcal{E}_2^{\frac{1}{2}}(t).
\end{aligned}\tag{3.12}$$

Step 2. This step estimates $\int_0^t\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2ds$. Applying the Leray-Helmholtz projection operator $\mathbb{P}$ to the velocity equation and the divergence operator $\nabla\cdot$ to the second Eq. (1.3), we have

$$\begin{cases}\partial_t u+\mathbb{P}(u\cdot\nabla)u-\Delta u=\mathbb{P}\nabla\cdot\tau,\\ \nabla\cdot\partial_t\tau+\nabla\cdot(u\cdot\nabla)\tau+\nabla\cdot Q(\tau,\nabla u)=\frac{1}{2}\Delta u,\end{cases}\tag{3.13}$$

where we have used $\nabla\cdot D(u)=\frac{1}{2}\Delta u$. Applying ∇^k $(k=0,1)$ to (3.13), dotting the resulting equations by $(\nabla^k\mathbb{P}\nabla\cdot\tau,\nabla^k u)$ and integrating over Ω yield

$$\begin{aligned}
&\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2\\
&=\sum_{k=0}^1\Big(\int\nabla^k\partial_t u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx+\int\nabla^k\mathbb{P}(u\cdot\nabla)u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx-\int\nabla^k\Delta u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx\Big),\\
&-\frac{1}{2}\|\nabla u\|_{H^1}^2\\
&=\sum_{k=0}^1\Big(\int\nabla^k\nabla\cdot\partial_t\tau\cdot\nabla^k u dx+\int\nabla^k\nabla\cdot(u\cdot\nabla)\tau\cdot\nabla^k u dx+\int\nabla^k(\nabla\cdot Q)\cdot\nabla^k u dx\Big).
\end{aligned}$$

Adding two equations above leads to

$$\begin{aligned}
\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2=&\sum_{k=0}^1\frac{d}{dt}\int\nabla^k u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx+\frac{1}{2}\|\nabla u\|_{H^1}^2\\
&+\sum_{k=0}^1\int\nabla^k\mathbb{P}(u\cdot\nabla)u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{1} \int \nabla^k \Delta u \cdot \nabla^k \mathbb{P}\nabla \cdot \tau dx + \sum_{k=0}^{1} \int \nabla^k \nabla \cdot (u \cdot \nabla)\tau \cdot \nabla^k u dx \\
&+ \sum_{k=0}^{1} \int \nabla^k \nabla \cdot Q \cdot \nabla^k u dx \\
=& \sum_{k=0}^{1} \frac{d}{dt} \int \nabla^k u \cdot \nabla^k \mathbb{P}\nabla \cdot \tau dx + \frac{1}{2}\|\nabla u\|_{H^1}^2 + I_4 + I_5 + I_6 + I_7. \qquad (3.14)
\end{aligned}
$$

Invoking $\mathbb{P}\mathbb{P}v = \mathbb{P}v$, Hölder's inequality and Sobolev's inequality, we have

$$
\begin{aligned}
I_4 =& \int (u \cdot \nabla)u \cdot \mathbb{P}\nabla \cdot \tau dx - \int \nabla(u \cdot \nabla)u \cdot \nabla\mathbb{P}\nabla \cdot \tau dx \\
\leq& \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\mathbb{P}\nabla \cdot \tau\|_{L^2} + \Big(\|\nabla u\|_{L^4}^2 + \|u\|_{L^\infty}\|\nabla^2 u\|_{L^2}\Big)\|\nabla\mathbb{P}\nabla \cdot \tau\|_{L^2} \\
\leq& C\|u\|_{H^2}\|\nabla u\|_{L^2}\|\mathbb{P}\nabla \cdot \tau\|_{L^2} + C\Big(\|\nabla u\|_{H^1}^2 + \|u\|_{H^2}\|\nabla^2 u\|_{L^2}\Big)\|\nabla\mathbb{P}\nabla \cdot \tau\|_{L^2} \\
\leq& C\|u\|_{H^2}\|\nabla u\|_{H^1}\|\mathbb{P}\nabla \cdot \tau\|_{H^1}. \qquad (3.15)
\end{aligned}
$$

Similarly, I_6 and I_7 can be estimated as

$$
\begin{aligned}
I_6 =& -\int (u \cdot \nabla)\tau \cdot \nabla u dx - \int \nabla \cdot (u \cdot \nabla)\tau \cdot \Delta u dx \\
\leq& \|u\|_{L^\infty}\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^2} + \Big(\|\nabla u\|_{L^4}\|\nabla\tau\|_{L^4} + \|u\|_{L^\infty}\|\nabla^2\tau\|_{L^2}\Big)\|\Delta u\|_{L^2} \\
\leq& C\Big(\|u\|_{H^2}\|\nabla\tau\|_{L^2} + \|\nabla u\|_{H^1}\|\nabla\tau\|_{H^1} + \|u\|_{H^2}\|\nabla^2\tau\|_{L^2}\Big)\|\nabla^2 u\|_{L^2} \\
\leq& C\|u\|_{H^2}\|\nabla\tau\|_{H^1}\|\nabla^2 u\|_{L^2}, \qquad (3.16) \\
I_7 =& \int \nabla \cdot Q \cdot u dx - \int \nabla \cdot Q \cdot \Delta u dx \\
\leq& C\int \Big(|\nabla\tau|\ |\nabla u| + |\tau|\ |\nabla^2 u|\Big)\big(|u| + |\Delta u|\big)dx \\
\leq& C\Big(\|\nabla\tau\|_{L^4}\|\nabla u\|_{L^4} + \|\tau\|_{L^\infty}\|\nabla^2 u\|_{L^2}\Big)\|u\|_{H^2} \\
\leq& C\Big(\|\nabla\tau\|_{H^1}\|\nabla u\|_{H^1} + \|\tau\|_{H^2}\|\nabla^2 u\|_{L^2}\Big)\|u\|_{H^2} \\
\leq& C\|\tau\|_{H^2}\|u\|_{H^2}\|\nabla^2 u\|_{L^2}, \qquad (3.17)
\end{aligned}
$$

where we have used the Poincaré inequality (3.7) in (3.17). Clearly, I_5 is bounded by

$$
I_5 \leq \frac{1}{2}\|\Delta u\|_{H^1}^2 + \frac{1}{2}\|\mathbb{P}\nabla \cdot \tau\|_{H^1}^2. \qquad (3.18)
$$

As a consequence of (3.15)–(3.18), we derive

$$
\begin{aligned}
\|\mathbb{P}\nabla \cdot \tau\|_{H^1}^2 \leq& 2\sum_{k=0}^{1} \frac{d}{dt} \int \nabla^k u \cdot \nabla^k \mathbb{P}\nabla \cdot \tau dx + 2\|\nabla u\|_{H^2}^2 \\
&+ C\Big(\|\nabla u\|_{H^1}\|\mathbb{P}\nabla \cdot \tau\|_{H^1} + \|\tau\|_{H^2}\|\nabla^2 u\|_{L^2}\Big)\|u\|_{H^2}. \qquad (3.19)
\end{aligned}
$$

Integrating (3.19) over $[0, t]$, combining with Hölder's inequality, Young's inequality, $\|\mathbb{P}\nabla \cdot \tau\|_{L^2} \leq \|\nabla \cdot \tau\|_{L^2}$ and (3.12), we obtain

$$
\int_0^t \| \mathbb{P}\nabla\cdot\tau(s) \|_{H^1}^2\ ds = 2\int_0^t \|\nabla u(s)\|_{H^2}^2 ds
$$

$$
\begin{aligned}
&+2\sum_{k=0}^{1}\Big(\int \nabla^k u\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx-\int \nabla^k u_0\nabla^k\mathbb{P}\nabla\cdot\tau_0 dx\Big)\\
&+C\int_0^t\Big(\|\nabla u(s)\|_{H^1}\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}+\|\tau(s)\|_{H^2}\|\nabla^2u(s)\|_{L^2}\Big)\|u(s)\|_{H^2}ds\\
\le{}& 2\int_0^t\|\nabla u(s)\|_{H^1}^2ds+2(\|u\|_{H^1}\|\mathbb{P}\nabla\cdot\tau\|_{H^1}+\|u_0\|_{H^1}\|\mathbb{P}\nabla\cdot\tau_0\|_{H^1})\\
&+C\sup_{0\le s\le t}\|u(s)\|_{H^2}\int_0^t\|\nabla u(s)\|_{H^1}\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}ds\\
&+C\sup_{0\le s\le t}\|u(s)\|_{H^2}\|\tau(s)\|_{H^2}\int_0^t\|\nabla^2u(s)\|_{H^1}ds\\
\le{}& 2(\|u\|_{H^1}^2+\|\nabla\tau\|_{H^1}^2)+2\int_0^t\|\nabla u(s)\|_{H^1}^2ds\\
&+C\mathcal{E}_1^{\frac32}(t)+C\mathcal{E}_1(t)\mathcal{E}_2^{\frac12}(t)+2\mathcal{E}_1(0).
\end{aligned}\tag{3.20}
$$

Therefore, multiplying (3.20) by $\frac14$ and then adding it to (3.11) yield the desired estimates (3.2). This completes the proof of Lemma 3.2. □

We now turn to the proof of Lemma 3.3.

Proof of Lemma 3.3. We divide the proof into two parts: the estimate of the first three terms in $\mathcal{E}_2(t)$ and the estimate on the last term $\int_0^t(1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2ds$.

Step 1. First, applying the differential operator ∇^{k+1} with $k=0,1$ to $(1.3)_1$ and the operator $\nabla^k\mathbb{P}\nabla\cdot$ with $k=0,1$ to $(1.3)_2$, and then taking the L^2 inner product of the resulting equations with $(\nabla^{k+1}u,\nabla^k\mathbb{P}\nabla\cdot\tau)$, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\Big)+\|\nabla^2u(t)\|_{L^2}^2=-\sum_{k=0}^{1}\int\nabla^{k+1}(u\cdot\nabla u)\cdot\nabla^{k+1}udx,\\
&-2\sum_{k=0}^{1}\int\nabla^k\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx-2\sum_{k=0}^{1}\int\nabla^k\mathbb{P}\nabla\cdot Q\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx,
\end{aligned}\tag{3.21}
$$

where we have used

$$
\begin{aligned}
&\sum_{k=0}^{1}\int\nabla^{k+1}(\nabla P)\cdot\nabla^{k+1}udx=0,\\
&\sum_{k=0}^{1}\int\nabla^{k+1}(\nabla\cdot\tau)\cdot\nabla^{k+1}udx+2\sum_{k=0}^{1}\int\nabla^k\mathbb{P}\nabla\cdot D(u)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx=0,
\end{aligned}
$$

which can be proved by integration by parts, $\nabla\cdot D(u)=\frac12\Delta u$ and $\mathbb{P}\Delta u=\Delta u$. Therefore, multiplying (3.21) by time weight $(1+t)^2$ yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}(1+t)^2\big(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\big)+(1+t)^2\|\nabla^2u(t)\|_{H^1}^2\\
&=(1+t)\big(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\big)-\sum_{k=0}^{1}(1+t)^2\int\nabla^{k+1}(u\cdot\nabla u)\cdot\nabla^{k+1}udx\\
&-2\sum_{k=0}^{1}(1+t)^2\int\nabla^k\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx
\end{aligned}
$$

$$
\begin{aligned}
&- 2\sum_{k=0}^{1}(1+t)^2\int \nabla^k\mathbb{P}(\nabla\cdot Q)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx\\
&:= J_1+J_2+J_3+J_4. \qquad (3.22)
\end{aligned}
$$

For J_2, it follows from (3.6) that

$$
\begin{aligned}
J_2 &= -(1+t)^2\Big(\int(\nabla u\cdot\nabla)u\cdot\nabla u dx+\int(\nabla^2u\cdot\nabla)u\cdot\nabla^2u dx\\
&\qquad +2\int(\nabla u\cdot\nabla)\nabla u\cdot\nabla^2 u dx\Big)\\
&\le C(1+t)^2\|\nabla u\|_{H^1}\|\nabla^2u\|_{H^1}^2. \qquad (3.23)
\end{aligned}
$$

The estimate of J_3 is more subtle. By Lemma 2.2, we first decompose J_2 into the following three parts

$$
\begin{aligned}
J_3 =&-2(1+t)^2\sum_{k=0}^{1}\Big(\int\nabla^k\mathbb{P}(\nabla u\cdot\nabla)\tau\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx\\
&-\int\nabla^k\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx+\int\nabla^k\mathbb{P}(\nabla u\cdot\nabla)\varphi\cdot\nabla^k\mathbb{P}\nabla\cdot\tau dx\Big)\\
:=&J_{31}+J_{32}+J_{33}.
\end{aligned}
$$

By $\mathbb{P}\mathbb{P}v=\mathbb{P}v$, Hölder's inequality, Sobolev's inequality and Poincaré inequality,

$$
\begin{aligned}
J_{31} &= -2(1+t)^2\int\Big((\nabla u\cdot\nabla)\tau\cdot\mathbb{P}\nabla\cdot\tau+\nabla(\nabla u\cdot\nabla\tau)\cdot\nabla\mathbb{P}\nabla\cdot\tau\Big)dx\\
&\le 2(1+t)^2\Big(\|\nabla u\|_{L^4}\|\nabla\tau\|_{L^4}\|\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^2\tau\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\qquad +\|\nabla^2u\|_{L^4}\|\nabla\tau\|_{L^4}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big)\\
&\le C(1+t)^2\Big(\|\nabla u\|_{H^1}\|\nabla\tau\|_{H^1}\|\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u\|_{H^2}\|\nabla^2\tau\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\qquad +\|\nabla^2u\|_{H^1}\|\nabla\tau\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big)\\
&\le C(1+t)^2\|\nabla\tau\|_{H^1}\|\nabla^2u\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}. \qquad (3.24)
\end{aligned}
$$

Here we have used the Poincaré inequality $\|\mathbb{P}\nabla\cdot\tau\|_{L^2}\le C\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}$ due to $\int\mathbb{P}\nabla\cdot\tau dx=0$. By the fact $\mathbb{P}\mathbb{P}v=\mathbb{P}v$ again together with the integration by parts and $\|\mathbb{P}v\|_{L^2}\le\|v\|_{L^2}$,

$$
\begin{aligned}
J_{32} &= -2(1+t)^2\int(\nabla u\cdot\nabla)(\mathbb{P}\nabla\cdot\tau)\cdot\nabla\mathbb{P}\nabla\cdot\tau dx\\
&\le 2(1+t)^2\|\nabla u\|_{L^\infty}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\le C(1+t)^2\|\nabla u\|_{H^2}\|\nabla^2\tau\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\le C(1+t)^2\|\nabla^2 u\|_{H^1}\|\nabla^2\tau\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}. \qquad (3.25)
\end{aligned}
$$

Invoking the estimate $\|\nabla\varphi\|_{H^1(\Omega)}\le C\|\nabla\tau\|_{H^1(\Omega)}$ in Lemma 2.1 and following a similar argument as the one for J_{31}, we have

$$
\begin{aligned}
J_{33} &= -2(1+t)^2\int\Big((\nabla u\cdot\nabla)\varphi\cdot\mathbb{P}\nabla\cdot\tau+\nabla(\nabla u\cdot\nabla\varphi)\cdot\nabla\mathbb{P}\nabla\cdot\tau\Big)dx\\
&\le 2(1+t)^2\Big(\|\nabla u\|_{L^4}\|\nabla\varphi\|_{L^4}\|\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^2\varphi\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\qquad +\|\nabla^2u\|_{L^4}\|\nabla\varphi\|_{L^4}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big)\\
&\le C(1+t)^2\Big(\|\nabla u\|_{H^1}\|\nabla\varphi\|_{H^1}\|\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u\|_{H^2}\|\nabla^2\varphi\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|\nabla^2 u\|_{H^1}\|\nabla\varphi\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big) \\
&\le C(1+t)^2\|\nabla\tau\|_{H^1}\|\nabla^2 u\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}. \qquad (3.26)
\end{aligned}
$$

Combining the upper bounds in (3.24), (3.25) and (3.26) leads to

$$
J_3 \le C(1+t)^2\|\nabla\tau\|_{H^1}\|\nabla^2 u\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}. \qquad (3.27)
$$

By $\mathbb{P}\mathbb{P}v = \mathbb{P}v$, Hölder's inequality, Sobolev's inequality and Poincaré inequality, J_4 can be bounded by

$$
\begin{aligned}
J_4 &= 2(1+t)^2\int\Big(Q\cdot\nabla\mathbb{P}\nabla\cdot\tau - \nabla(\nabla\cdot Q)\cdot\nabla\mathbb{P}\nabla\cdot\tau\Big)dx \\
&\le 2(1+t)^2(\|\tau\|_{L^\infty}\|\nabla u\|_{L^2} + \|\tau\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\nabla\tau\|_{L^4}\|\nabla^2 u\|_{L^4} \\
&\qquad + \|\nabla^2\tau\|_{L^2}\|\nabla u\|_{L^\infty})\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2} \\
&\le C(1+t)^2(\|\tau\|_{H^2}\|\nabla u\|_{L^2} + \|\tau\|_{H^2}\|\nabla^3 u\|_{L^2} + \|\nabla\tau\|_{H^1}\|\nabla^2 u\|_{H^1} \\
&\qquad + \|\nabla^2\tau\|_{L^2}\|\nabla u\|_{H^2})\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2} \\
&\le C(1+t)^2\|\tau\|_{H^2}\|\nabla^2 u\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}. \qquad (3.28)
\end{aligned}
$$

Inserting (3.23), (3.27) and (3.28) into (3.22), and integrating over $[0,t]$, we get

$$
\begin{aligned}
&(1+t)^2\big(\|\nabla u(t)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\big) + 2\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2 ds \\
&\le \|\nabla u_0\|_{H^1}^2 + 2\|\nabla\tau_0\|_{H^1}^2 + 2\int_0^t (1+s)\big(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2\big)ds \\
&\quad + C\int_0^t (1+s)^2\|\nabla u(s)\|_{H^1}\|\nabla^2 u(s)\|_{H^1}^2 ds \\
&\quad + C\int_0^t (1+s)^2\|\tau(s)\|_{H^2}\|\nabla^2 u(s)\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2} ds. \qquad (3.29)
\end{aligned}
$$

The integral terms on the right-hand side can be further bounded. By Poincaré's inequality and Hölder's inequality,

$$
\begin{aligned}
&\int_0^t (1+s)\Big(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2\Big)ds \\
&\le C\int_0^t (1+s)\big(\|\nabla u(s)\|_{H^1}\|\nabla^2 u(s)\|_{L^2} + 2\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}\big)ds \\
&\le C\Big(\int_0^t \|\nabla u(s)\|_{H^1}^2 ds\Big)^{\frac12}\Big(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{L^2}^2 ds\Big)^{\frac12} \\
&+ C\Big(\int_0^t \|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2 ds\Big)^{\frac12}\Big(\int_0^t (1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2 ds\Big)^{\frac12} \\
&\le C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2^{\frac12}(t). \qquad (3.30)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^t (1+s)^2\|\nabla u(s)\|_{H^1}\|\nabla^2 u(s)\|_{H^1}^2 ds \\
&\le \sup_{0\le s\le t}\|\nabla u(s)\|_{H^1}\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2 ds \\
&\le \mathcal{E}_1^{\frac12}(t)\,\mathcal{E}_2(t), \qquad (3.31)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t (1+s)^2\|\tau(s)\|_{H^2}\|\nabla^2 u(s)\|_{H^1}\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}ds\\
&\le \sup_{0\le s\le t}\|\tau(s)\|_{H^2}\Big(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2 ds\Big)^{\frac12}\Big(\int_0^t (1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2 ds\Big)^{\frac12}\\
&\le \mathcal{E}_1^{\frac12}(t)\,\mathcal{E}_2(t),
\end{aligned}
\tag{3.32}
$$

Inserting (3.30), (3.31) and (3.32) in (3.29) leads to

$$
\begin{aligned}
&(1+t)^2\Big(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\Big)+2\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2 ds\\
&\le C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2^{\frac12}(t)+C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2(t)+2\mathcal{E}_1(0).
\end{aligned}
\tag{3.33}
$$

Step 2. This step establishes a suitable estimate for $\int_0^t (1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2 ds$. First, by (3.14),

$$
\begin{aligned}
\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 =&\frac{d}{dt}\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx+\frac12\|\Delta u\|_{L^2}^2\\
&+\int\nabla\mathbb{P}(u\cdot\nabla)u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx-\int\nabla\Delta u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx\\
&+\int\nabla\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla u dx+\int\nabla\nabla\cdot Q\cdot\nabla u dx.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&(1+t)^2\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2=\frac{d}{dt}(1+t)^2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx+\frac12(1+t)^2\|\Delta u\|_{L^2}^2\\
&-\ 2(1+t)\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx+(1+t)^2\int\nabla\mathbb{P}(u\cdot\nabla)u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx\\
&-\ (1+t)^2\int\nabla\Delta u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx+(1+t)^2\int\nabla\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla u dx\\
&+\ (1+t)^2\int\nabla\nabla\cdot Q\cdot\nabla u dx\\
&=\frac{d}{dt}(1+t)^2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx+\frac12(1+t)^2\|\Delta u\|_{L^2}^2+J_4+J_5+J_6+J_7+J_8.
\end{aligned}
$$

Next we estimate J_4-J_8. First, for the integrals of linear terms J_4 and J_6, Hölder's inequality implies

$$
J_4+J_6\le 2(1+t)\|\nabla u\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}+(1+t)^2\|\nabla\Delta u\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}.
$$

Following the estimate (3.15) for I_4 and (3.17) for I_7, we have

$$
J_5+J_8\le C(1+t)^2\|u\|_{H^2}\|\nabla^2 u\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}+(1+t)^2\|\tau\|_{H^2}\|\nabla^2 u\|_{L^2}^2,
$$

where we have used $\|\nabla u\|\le C\|\nabla^2 u\|$. Finally, we deal with J_7. The estimate is more elaborate. We first divide it into three terms according to Lemma 2.2,

$$
\begin{aligned}
J_7=&-(1+t)^2\int\nabla\cdot(u\cdot\nabla)\tau\cdot\Delta u dx=(1+t)^2\int\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau\cdot\Delta u dx\\
=&(1+t)^2\Big(\int\mathbb{P}(\nabla u\cdot\nabla)\tau\cdot\Delta u dx-\int\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau)\cdot\Delta u dx\\
&+\int\mathbb{P}(\nabla u\cdot\nabla)\varphi\cdot\Delta u dx\Big)\\
=&(1+t)^2\Big(\int(\nabla u\cdot\nabla)\tau\cdot\Delta u dx-\int(u\cdot\nabla)\mathbb{P}\nabla\cdot\tau\cdot\Delta u dx\\
&+\int(\nabla u\cdot\nabla)\varphi\cdot\Delta u dx\Big).
\end{aligned}
$$

By Hölder's inequality, Sobolev's inequality and Poincaré's inequality,

$$\begin{aligned}J_7 &\le (1+t)^2\Big(\|\nabla u\|_{L^4}\|\nabla\tau\|_{L^4}+\|u\|_{L^\infty}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u\|_{L^4}\|\nabla\varphi\|_{L^4}\Big)\|\Delta u\|_{L^2}\\ &\le C(1+t)^2\Big(\|\nabla u\|_{H^1}\|\nabla\tau\|_{H^1}+\|u\|_{H^2}\|\nabla\mathbb{P}\cdot\nabla\tau\|_{L^2}+\|\nabla u\|_{H^1}\|\nabla\varphi\|_{H^1}\Big)\|\Delta u\|_{L^2}\\ &\le C(1+t)^2(\|\nabla\tau\|_{H^1}+\|u\|_{H^2})\Big(\|\nabla^2 u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\Big)\end{aligned}$$

where we have used $\|\nabla\varphi\|_{H^1}\le C\|\nabla\tau\|_{H^1}$ by Lemma 2.1. Collecting all the estimates for J_4-J_8 and integrating the resulted inequality on $[0,t]$, we obtain

$$\begin{aligned}&\int_0^t(1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2ds\le\frac12\int_0^t(1+s)^2\|\Delta u(s)\|_{L^2}^2ds\\ &\quad+\Big((1+t)^2\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot\tau dx-\int\nabla u_0\cdot\nabla\mathbb{P}\nabla\cdot\tau_0dx\Big)\\ &\quad+\int_0^t\Big(2(1+s)\|\nabla u(s)\|_{L^2}+(1+s)^2\|\nabla\Delta u(s)\|_{L^2}\Big)\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}ds\\ &\quad+C\int_0^t(1+s)^2(\|\tau(s)\|_{H^2}+\|u(s)\|_{H^2})\big(\|\nabla^2u(s)\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_2^2\big)ds\\ &\quad:=K_1+K_2+K_3+K_4.\end{aligned}$$

By Hölder's inequality, Young's inequality and $\|\mathbb{P}v\|_{L^2}\le\|v\|_{L^2}$,

$$\begin{aligned}K_2&\le(1+t)^2\|\nabla u\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla u_0\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau_0\|_{L^2}\\ &\le\frac12(1+t)^2\big(\|\nabla u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big)+\|\nabla u_0\|_{L^2}\|\nabla^2\tau_0\|_{L^2}\\ &\le\frac12(1+t)^2\big(\|\nabla u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big)+\mathcal{E}_1(0),\end{aligned}$$

Similarly,

$$\begin{aligned}K_3&\le\Big[2\Big(\int_0^t\|\nabla u(s)\|_{L^2}^2ds\Big)^{\frac12}+\Big(\int_0^t(1+s)^2\|\nabla^3u(s)\|_{L^2}^2ds\Big)^{\frac12}\Big]\Big(\int_0^t(1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\Big)^{\frac12}\\ &\le2\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2^{\frac12}(t)+\frac12\Big(\int_0^t(1+s)^2\|\nabla^3u(s)\|_{L^2}^2ds+\int_0^t(1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2ds\Big)\end{aligned}$$

and

$$\begin{aligned}K_4&\le C\sup_{0\le s\le t}(\|\tau(s)\|_{H^2}+\|u(s)\|_{H^2})\int_0^t(1+s)^2\Big(\|\nabla^2u(s)\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2\Big)ds\\ &\le C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2(t).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\int_0^t(1+s)^2\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2ds\le(1+t)^2\big(\|\nabla u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big)\\ &\quad+\int_0^t(1+s)^2\|\nabla^2u(s)\|_{H^1}^2ds+4\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2^{\frac12}(t)+C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2(t)+2\mathcal{E}_1(0),\end{aligned}$$

which, together with (3.33), implies

$$\mathcal{E}_2(t)\le C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2^{\frac12}(t)+C\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2(t)+C\mathcal{E}_1(0).$$

for some constant $C>0$. Consequently, there exists $C_4>0$ such that

$$\mathcal{E}_2(t)\le C_4\mathcal{E}_1(t)+C_4\mathcal{E}_1^{\frac12}(t)\mathcal{E}_2(t)+C_4\mathcal{E}_1(0).$$

This completes the proof of Lemma 3.3. □

We combine Lemmas 3.2 and 3.3 to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. The estimates (3.2) and (3.3) together with Young's inequality give the desired a *prior* estimate

$$\mathcal{E}(t) \leq C_2\mathcal{E}(0) + C_2\mathcal{E}^{\frac{3}{2}}(t).$$

This completes the proof of Proposition 3.1. □

4. The decay estimates

The section is devoted to proving the time decay rate (1.6) in Theorem 1.1. The idea is to obtain a self-contained inequality of the form

$$\frac{d}{dt}X(t) + C_1\, X(t) \leq 0, \tag{4.1}$$

where

$$X(t) := \|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau).$$

Here (f, g) denotes the L^2-product of f and g. The point of including the inner product term in X is to extract the dissipation in $\mathbb{P}\nabla\cdot\tau(t)$ as revealed by the wave equations in (1.9). To make the proof simple and easy to understand, we divide the proof of (4.1) into two lemmas.

Lemma 4.1. *Let (u, τ) be the solution of (1.3), as obtained by the first part of Theorem 1.1. Then, for some constant $C > 0$, we have*

$$\frac{d}{dt}\big(\|\nabla u(t)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2\big) + 2\|\nabla^2 u(t)\|_{H^1}^2 \leq C(\|\nabla u\|_{H^1} + \|\tau\|_{H^2})\big(\|\nabla^2 u\|_{H^1}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big). \tag{4.2}$$

Proof of Lemma 4.1. It follows from (3.21) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2\Big) + \|\nabla^2 u\|_{H^1}^2 = -\sum_{k=0}^{1}\int \nabla^{k+1}(u\cdot\nabla u)\cdot\nabla^{k+1}u\,dx,\\ &-2\sum_{k=0}^{1}\int \nabla^k\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla^k\mathbb{P}\nabla\cdot\tau\,dx - 2\sum_{k=0}^{1}\int \nabla^k\mathbb{P}\nabla\cdot Q\cdot\nabla^k\mathbb{P}\nabla\cdot\tau\,dx\\ &:= K_1 + K_2 + K_3.\end{aligned}$$

Invoking the estimate (3.6) for I_1, we have

$$K_1 \leq C\|\nabla u\|_{H^1}\|\nabla^2 u\|_{H^1}^2.$$

From (3.27) for J_3,

$$K_2 \leq C\|\nabla\tau\|_{H^1}(\|\nabla^2 u\|_{H^1}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

Also, by (3.28),

$$K_3 \leq C\|\tau\|_{H^2}(\|\nabla^2 u\|_{H^1}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2) + \|\nabla^2 u\|_{H^1}^2 \leq C(\|\nabla u\|_{H^1} + \|\tau\|_{H^2})(\|\nabla^2 u\|_{H^1}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

This completes the proof of Lemma 4.1. □

The second lemma bounds the inner product term.

Lemma 4.2. *Let (u,τ) be the solution of (1.3), as obtained by the first part of Theorem 1.1. Then, for some constant $C>0$, we have*

$$-\frac{d}{dt}(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)+\frac{1}{2}\|\nabla\mathbb{P}\nabla\cdot\tau(t)\|_{L^2}^2-\frac{1}{2}\|\Delta u\|_{H^1}^2 \leq C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla^2u(t)\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2). \tag{4.3}$$

Proof of Lemma 4.2. Consider the functional $(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)$. Applying the operator $\mathbb{P}$ to $(1.3)_1$ and $\mathbb{P}\nabla\cdot$ to $(1.3)_2$, respectively, we have

$$\begin{cases}\partial_t u+\mathbb{P}(u\cdot\nabla)u-\Delta u=\mathbb{P}\nabla\cdot\tau,\\ \mathbb{P}\nabla\cdot\partial_t\tau+\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau+\mathbb{P}\nabla\cdot Q(\tau,\nabla u)=\frac{1}{2}\Delta u,\end{cases}$$

Then a direct calculation leads to

$$\begin{aligned}&-\frac{d}{dt}(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-\frac{1}{2}\|\Delta u\|_{L^2}^2\\ &=\int\nabla\mathbb{P}(u\cdot\nabla u)\cdot\nabla\mathbb{P}\nabla\cdot\tau\,dx-\int\nabla\Delta u\cdot\nabla\mathbb{P}\nabla\cdot\tau\,dx\\ &\quad+\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\,dx+\int\nabla u\cdot\nabla\mathbb{P}\nabla\cdot Q\,dx\\ &:=K_4+K_5+K_6+K_7.\end{aligned}$$

Invoking the bounds in (3.15) and (3.17), we have

$$K_4+K_7\leq C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla^2u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

Clearly,

$$K_5\leq\frac{1}{2}(\|\nabla\Delta u(t)\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

K_6 can be similarly estimated as J_7,

$$K_6\leq C(\|u\|_{H^2}+\|\nabla\tau\|_{H^1})(\|\nabla^2u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

In summary, we obtain

$$-\frac{d}{dt}(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)+\frac{1}{2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-\frac{1}{2}\|\Delta u\|_{H^1}^2 \leq C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla^2u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

This completes the proof of Lemma 4.2. □

With the lemmas at our disposal, we are ready to prove the decay estimate.

Proof of the decay rate in Theorem 1.1. Adding (4.2) and (4.3) yields

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla u\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2-(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)\big)+\frac{1}{2}\big(\|\nabla^2u\|_{H^1}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big)\\ &\leq C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla^2u\|_{H^1}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)\\ &\leq C\delta(\|\nabla^2u\|_{H^1}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2),\end{aligned}$$

where we have used the global well-posedness result (1.5). If δ is sufficiently small, then for some constant $C>0$,

$$\frac{d}{dt}\big(\|\nabla u\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2-(\nabla u,\nabla\mathbb{P}\nabla\cdot\tau)\big)+C\big(\|\nabla^2u\|_{H^1}^2+\|\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\big)\leq 0 \tag{4.4}$$

Due to the Poincaré inequality, we have

$$\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 \le C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

In addition, by Hölder's inequality, Young's inequality and Poincaré's inequality,

$$-(\nabla u, \nabla\mathbb{P}\nabla\cdot\tau) \le C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

Therefore, we obtain

$$\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau) \le C(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).$$

Then by (4.4), there exists a constant $C_1 > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau)\big)\\ &\quad + 2C_1\big(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau)\big) \le 0.\end{aligned} \tag{4.5}$$

We notice that

$$\begin{aligned}&\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau)\\ &\ge \|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)\\ &\ge C(\|\nabla u\|_{H^1}^2 + \|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2).\end{aligned}$$

Hence, by (4.5),

$$\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - (\nabla u, \nabla\mathbb{P}\nabla\cdot\tau) \le Ce^{-2C_1 t}.$$

Therefore,

$$\|\nabla u\|_{H^1}^2 + \|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 \le Ce^{-2C_1 t}.$$

This completes the proof of the decay rate in Theorem 1.1. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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