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Mild Ill-Posedness in L^∞ for 2D Resistive MHD Equations Near a Background Magnetic Field

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The global well-posedness on the 2D resistive MHD equations without kinematic dissipation remains an outstanding open problem. This is a critical problem. Any L^p-norm of the vorticity ω with $1 \le p < \infty$ has been shown to be bounded globally (in time), but whether the L^∞-norm of ω is globally bounded remains elusive. The global boundedness of $\|\omega\|_{L^\infty}$ yields the resolution of the aforementioned open problem. This paper examines the L^∞-norm of ω from a different perspective. We construct a sequence of initial data near a special steady state to show that the L^∞-norm of ω is actually mildly ill-posed.

1 Introduction

The magnetohydrodynamic (MHD) system governs the motion of electrically conducting fluids in a magnetic field such as plasmas, liquid metals, and electrolytes and has a wide range of applications in astrophysics, geophysics, cosmology, and engineering (see, e.g., [4, 11, 25]). The MHD system is a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of the electromagnetism. The coupling and interaction between the magnetic field and the fluid enable the MHD system to

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model many more phenomena than the Navier–Stokes and the Euler equations. One outstanding feature of the MHD systems is the various wave phenomena they describe.

This paper focuses on the 2D resistive MHD equations,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \quad x \in \mathbb{R}^2,\ t > 0, \\ b_t + u \cdot \nabla b = \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\ \ b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the fluid velocity, $b = b(x,t)$ the magnetic field, and $p = p(x,t)$ the pressure. Equation (1.1) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence (see [25]). Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system such as solar flares and northern lights.

Two fundamental issues on (1.1) have recently attracted considerable interest. The 1st one is the global well-posedness problem. A lot of efforts have been devoted to this difficult problem, even though it remains open (see, e.g., [1, 6, 8–10, 13, 15–18, 31–34]). The 2nd one is the stability problem on perturbations near a background magnetic field. A background magnetic field, say

$$u^{(0)} = (0,0), \quad b^{(0)} = (1,0),$$

constitutes a special class of steady-state solutions. The perturbation near the background magnetic field, still denoted by (u,b), obeys a resistive MHD system with two extra terms,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b + \partial_1 b, \\ b_t + u \cdot \nabla b = \Delta b + b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{1.2}$$

The study of the stability problem on (1.2) has been motivated by the observed physical phenomenon that the magnetic field can stabilize the electrically conducting fluids. There have been substantial recent developments on the stability problem on the MHD equations near a background magnetic field (see, e.g., [3, 5, 7, 14, 20–22, 27–30, 35]). These two problems remain open. This paper examines these problems from the perspective of mild ill-posedness and intends to shed some light on these open problems.

We describe some of the progress that has been made on these open problems and explain the remaining main obstacles that have been preventing us from completely solving these problems. The results presented in this paper may help gain a better understanding of these difficulties. First of all, any solution (u, b) of (1.1) or (1.2) emanating from an initial data $(u_0, b_0) \in H^1$ admits uniform global H^1-bound,

$$\|(u(t), b(t))\|_{H^1} \le \|(u_0, b_0)\|_{H^1}\, e^{c\,\|(u_0,b_0)\|_{L^2}^2}.$$

As a consequence, (1.1) or (1.2) always possesses a global H^1-weak solution (see, e.g., [9, 18]). However, the uniqueness of the H^1-weak solutions remains an open problem. Strong or classical solutions are unique but are not known to be global (in time). Extensive efforts have been devoted to the global well-posedness problem on (1.1). Existing results revealed the criticality of this problem. If the Laplacian dissipation Δb in (1.1) is replaced by the hyper-dissipation $-(-\Delta)^{\beta} b$ with any $\beta > 1$, then the resulting system is globally well-posedness [10, 17]. If we keep Δb in (1.1) but add fractional dissipation $-(-\Delta)^{\alpha} u$ or even logarithmic dissipation $-\log(2-\Delta)u$, then the slightly dissipated MHD system always possesses a unique global classical solution [13, 32, 34]. These results illustrate the criticality of the dissipation Δb.

Another type of criticality is reflected on the L^∞-norm of the vorticity $\omega = \nabla \times u$. As established in [15] via the maximal regularity of the heat operator, any L^q-norm of ω with $1 \le q < \infty$ is bounded,

$$\|\omega(t)\|_{L^q} \le c(q, t),$$

where the upper bound $c(q, t)$ depends on q, the initial data, and t. However, the L^∞-bound of ω is missing. Whether or not $\|\omega(t)\|_{L^\infty}$ is bounded for all time remains an open problem. This is the main obstacle in solving the global well-posedness problem on (1.1) as well as on (1.2). The mild ill-posedness result obtained in this paper suggests that attempts to establish a global bound for $\|\omega\|_{L^\infty}$ may fail. A different approach of avoiding the control of $\|\omega\|_{L^\infty}$ appears to be necessary in order to solve this open well-posedness problem.

This paper focuses on the MHD system (1.2). The goal is to understand the growth behavior of $\|\omega(t)\|_{L^\infty}$ by exploiting the structure of the equation governing the vorticity. When (u, b) satisfies (1.2), the corresponding vorticity obeys

$$\omega_t + u \cdot \nabla\omega = b_1 \Delta b_2 - b_2 \Delta b_1 + \Delta b_2. \tag{1.3}$$

By regrouping the terms on the right-hand side, (1.3) can be rewritten as

$$\omega_t + u \cdot \nabla\omega = \mathcal{R}_1^2 \omega + H + L,$$

where $\mathcal{R}_1 = \partial_1 \Lambda^{-1} = \partial_1 (-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform, and H and L are given by

$$\begin{aligned} H &= b_1(\Delta b_2 + b \cdot \nabla u_2 + \partial_1 u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1 + \partial_1 u_1) \\ &\quad + (\Delta b_2 + b \cdot \nabla u_2 + \partial_1 u_2), \\ L &= -b_1(b \cdot \nabla u_2 + \partial_1 u_2) + b_2(b \cdot \nabla u_1 + \partial_1 u_1) - b \cdot \nabla u_2. \end{aligned}$$

As we shall show later, H and L represent regular terms. Since the Riesz transforms, a class of standard singular integral operators, are not bounded on L^∞, the boundedness of $\|\omega\|_{L^\infty}$ remains unknown. This work is partially inspired by a recent work of Elgindi and Masmoudi [12] on the 2D Euler-like equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega = \mathcal{R}_2^2 \omega, \quad x \in \mathbb{R}^2, \ t > 0, \\ u = \nabla^\perp \Delta^{-1} \omega, \end{cases} \tag{1.4}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$ and $\mathcal{R}_2 = \partial_2 \Lambda^{-1}$ is the Riesz transform. We explore the growth behavior of $\|\omega(t)\|_{L^\infty}$ associated with the MHD system (1.2) and are able to establish the mild ill-posedness on the local solution of (1.2). We provide a rigorous definition of this concept proposed by Elgindi and Masmoudi [12].

Definition 1.1. Let X and Y be two Banach spaces and $Y \hookrightarrow X$. A Cauchy problem

$$\begin{cases} v_t = N(v), \\ v(0) = v_0 \end{cases}$$

is mildly ill-posed in a space X if there exists a constant $c > 0$ such that for any $\varepsilon, \delta > 0$, there exists $v_0 \in Y$, with $\|v_0\|_X \leqslant \varepsilon$ for which there exists a unique solution $v(t) \in L^\infty([0, T]; Y)$ for some $T > 0$, but $\|v(t)\|_X \geqslant c$ for some $0 < t < \delta$.

We remark that T in the definition above could depend on ε. We are able to establish the mild ill-posedness of (1.2), as stated in the following theorem.

Theorem 1.1. There exists a sequence of initial data $\{(u_0^N, b_0^N)\}_{N=1}^{\infty}$ with $\mathrm{div} u_0^N = \mathrm{div} b_0^N = 0$ satisfying

$$u_0^N \in B^{\frac{3}{2}}_{4,1}, \quad b_0^N = b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1},$$
$$\|u_0^N\|_{H^1} \leqslant \frac{c}{\sqrt{N}}, \quad \|\omega_0^N\|_{L^\infty} \leqslant \frac{c}{\sqrt{N}}, \quad \|b_0\|_{H^1 \cap B^{\frac{1}{2}}_{4,1}} \leqslant \delta, \quad \|\omega_0^N\|_{B^{\frac{1}{2}}_{4,1}} \leqslant C_1\sqrt{N},$$

where $\omega_0^N = \nabla \times u_0^N$, $\delta > 0$ is a small constant and $C_1 > 0$ is a constant independent of N. Let (u^N, b^N) be the corresponding local solution of (1.2). Then (u^N, b^N) and $\omega^N = \nabla \times u^N$ satisfy

$$u^N \in L^\infty\left(\left[0, \frac{C_2}{\sqrt{N}}\right]; H^1 \cap B^{\frac{3}{2}}_{4,1}\right), \quad b^N \in \left(\left[0, \frac{C_2}{\sqrt{N}}\right]; H^1\right) \cap L^1\left(\left[0, \frac{C_2}{\sqrt{N}}\right]; B^{\frac{5}{2}}_{4,1})\right),$$
$$\|\omega^N(t)\|_{L^\infty([0, \frac{C_2}{\sqrt{N}}]; L^\infty)} \geqslant C_3,$$

where C_2 and C_3 are universal constants independent of N.

Theorem 1.1 asserts that we can construct a sequence of initial data $\{(u_0^N, b_0^N)\}$ such that the initial H^1-norm, the L^∞-norm of u_0^N as well as the initial L^∞-norm of ω_0^N are all small and approach zero as $N \to \infty$, but the vorticity ω^N of the corresponding solution (u^N, b^N) actually grows in the L^∞-norm and becomes bounded below by a constant uniform in N. A special consequence of Theorem 1.1 is the following instability result on the vorticity in the L^∞-norm. We recall the equations of the vorticity ω and the current density $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j + \partial_1 j, \\ \partial_t j + u \cdot \nabla j - \Delta j = b \cdot \nabla \omega + Q(\nabla u, \nabla b) + \partial_1 \omega, \end{cases} \tag{1.5}$$

where

$$Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

Corollary 1.2. The zero solution of (1.5) is unstable with respect to L^∞-norm.

We explain the main lines in the proof of Theorem 1.1. We start with the local existence and uniqueness in a suitable functional setting.

Proposition 1.3. Assume the initial data (u_0, b_0) with $\mathrm{div} u_0^N = \mathrm{div} b_0^N = 0$ satisfy

$$u_0 \in H^1 \cap B^{\frac{3}{2}}_{4,1}, \quad \omega_0 = \nabla \times u_0 \in L^\infty, \quad b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1}, \quad \nabla \cdot u_0 = \nabla \cdot b_0 = 0.$$

Then (1.2) has a unique solution $(u, b) \in L^\infty(0, T; B^{\frac{3}{2}}_{4,1}) \times L^\infty(0, T; B^{\frac{1}{2}}_{4,1})$ for some $T > 0$. In addition, the following estimate holds,

$$\|u\|_{B^{\frac{3}{2}}_{4,1}} \leqslant \frac{c(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)}{1 - c(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)t}, \tag{1.6}$$

with $ct(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1) < \frac{1}{2}$. Alternatively,

$$\|u\|_{B^{\frac{3}{2}}_{4,1}} \leqslant \frac{(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0)e^{(c+c_0)t}}{1 - \frac{c}{c+c_0}(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0)(e^{(c+c_0)t} - 1)}, \tag{1.7}$$

where c_0 is a constant depending only on $\|u_0\|_{H^1}$, $\|\omega_0\|_{L^\infty}$ and $\|b_0\|_{H^1 \cap B^{\frac{1}{2}}_{4,1}}$.

The next step is to prove several global *a priori* bounds. In particular, the terms in H and L are shown to be bounded globally.

Proposition 1.4. Assume $u_0 \in H^1$, $\omega_0 = \nabla \times u_0 \in L^\infty$ and $\nabla \cdot u_0 = 0$. Assume $b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1}$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.2). Then, for any $q \in (1, 4/3)$ and $t > 0$,

$$\|\omega\|_{L^\infty_t L^4_x} \leqslant c_0(1 + r(t))e^{(c_0+c)r(t)}, \quad \|u\|_{L^\infty_t B^{\frac{1}{2}}_{4,1}} \leqslant c_0(1 + r(t))e^{(c_0+c)r(t)},$$

$$\|\Delta b\|_{L^q_t L^4_x} \leqslant c_0 r(t) e^{(c_0+c)r(t)}, \quad \int_0^t \|\nabla b\|_{B^{\frac{1}{2}}_{4,1}} \,\mathrm{d}\tau \leqslant c_0 + c_0 r(t) e^{(c_0+c)r(t)},$$

$$\int_0^t \|\Delta b + b \cdot \nabla u + \partial_1 u\|_{B^{\frac{1}{2}}_{4,1}} \,\mathrm{d}\tau \leqslant c_0 + c_0 r(t) e^{(c_0+c)r(t)},$$

where $r(t)$ depends only on q and t (in the fashion like t^γ with $\gamma > 0$).

In addition, to prepare for the construction of the sequence of the initial data, we construct a sequence of functions with special properties. The precise construction and the special properties are stated in the following proposition.

Proposition 1.5. There exists a sequence of functions $f_N \in B^{\frac{1}{2}}_{4,1}$ such that the following holds:

$$\|\Lambda^{-1} f_N\|_{L^2} \le c, \tag{1.8}$$

$$\|f_N\|_{L^2} \le c, \tag{1.9}$$

$$\|f_N\|_{L^\infty} \le c, \tag{1.10}$$

$$\|\mathcal{R} f_N\|_{L^\infty} \ge c' N, \tag{1.11}$$

$$\|f_N\|_{B^{\frac{1}{2}}_{4,1}} \le cN, \tag{1.12}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$, $\mathcal{R} = \mathcal{R}_1^2$, or $\mathcal{R}_1 \mathcal{R}_2$ with $\mathcal{R}_j = \partial_j \Lambda^{-1}$ $(j = 1, 2)$ being the standard Riesz transform and c and c' are constants independent of N.

The rest of this paper is divided into five sections. Section 2 provides a list of facts to be used in the proofs of the propositions and the theorem stated above. Section 3 presents a detailed construction of a sequence of functions with special properties. The construction of the initial data in the proof of Theorem 1.1 makes use of this special sequence. Section 4 proves Proposition 1.3, the local existence theory on the system in (1.2). Section 5 is devoted to the proof of several global *a priori* bounds on solutions of (1.2) stated in Proposition 1.4. Section 6 contains the proof of our main results stated in Theorem 1.1.

2 Preparations

This section prepares several facts to be used in the proofs of the propositions and the theorem. The 1st lemma recounts the maximal regularity of the heat operator (see, e.g., [19, p.64]).

Lemma 2.1. Let $0 < T \le \infty$. Assume $f \in L^p((0, T), L^q(\mathbb{R}^d))$ with $1 < p, q < \infty$. Define

$$Af(t) = \int_0^t e^{(t-s)\Delta} \Delta f(s)\, ds,$$

then, for some constant $c > 0$,

$$\|Af\|_{L^p((0,T),L^q(\mathbb{R}^d))} \leqslant c\,\|f\|_{L^p((0,T),L^q(\mathbb{R}^d))}\,.$$

The 2nd lemma states that the bi-Lipscitz map preserves the regularity in the Besov space $B^0_{\infty,1}$. The definition of Besov spaces $B^s_{p,q}$ can be found in many books and papers (see, e.g., [2, 19, 24]).

Lemma 2.2 ([26, Theorem 4.2]). Let $\phi : \mathbb{R}^d \xrightarrow{onto} \mathbb{R}^d$ be the volume-preserving bi-Lipschitz homeomorphism. If $h \in B^0_{\infty,1}$, then

$$\|h \circ \phi^{-1}\|_{B^0_{\infty,1}} \leqslant c(1 + \log(\|\phi\|_{\mathrm{Lip}}\|\phi^{-1}\|_{\mathrm{Lip}}))\|h\|_{B^0_{\infty,1}},$$

where

$$\|\phi\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}$$

and c is a constant depending only on d.

The next lemma provides an estimate in a Besov setting for a special commutator.

Lemma 2.3 (see [12]). Let $d \geq 2$, $0 < a < 1$, and $1 < \rho < \infty$. Let Φ be the volume-preserving bi-Lipschitz mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$. Define the following commutator:

$$[R, \Phi]\omega = R(\omega \circ \Phi) - R(\omega) \circ \Phi,$$

where R is Riesz transform. Then $[R, \Phi] : B^a_{\rho,1} \to B^a_{\rho,1}$ is bounded, and

$$\|[R, \Phi]\omega\|_{B^a_{p,1}} \leq c\max\{\|\Phi - Id\|_{\mathrm{Lip}}, \|\Phi^{-1} - Id\|_{\mathrm{Lip}}\}\|\omega\|_{B^a_{p,1}},$$

where Id is the identity matrix and c is a constant depending only upon $\|\Phi\|_{Lip}$, $\|\Phi^{-1}\|_{Lip}$, the dimension d, and the transform R.

The following lemma bounds the flow map in terms of the Lipschitz norm of the velocity field.

Lemma 2.4 (see [2, 24]). Let v be a smooth time-dependent vector field with bounded 1st-order space derivatives. Let Φ be the flow map induced by v,

$$\begin{cases} \dot{\Phi}(x,t) = v(\Phi(x,t),t), \\ \Phi(x,0) = x. \end{cases}$$

Then

$$\Phi(x,t) = x + \int_0^t v(\Phi(x,\tau),\tau)\,\mathrm{d}\tau$$

and Φ is a C^1 diffeomorphism over $\mathbb{R}^d$ for all $t \in R^+$. In addition, we have

$$\|\Phi^{\pm}\|_{\mathrm{Lip}} \leqslant e^{\int_0^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau}.$$

Lemma 2.5 (see [2, Corollary 2.86]). For any $s > 0$ and $(p,q) \in [1,\infty]^2$, the space $L^\infty \cap B^s_{p,q}$ is an algebra, and

$$\|uv\|_{B^s_{p,q}} \leqslant \frac{c^{s+1}}{s}(\|u\|_{L^\infty}\|v\|_{B^s_{p,q}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,q}}),$$

where c is a constant.

To state the next lemma, we introduce a few notations. Δ_k is the Fourier localization operator in the inhomogeneous Littlewood–Paley decomposition, namely $\mathrm{Id} = \sum_k \Delta_k$. We use S_k to denote the identity approximation operator,

$$S_k = \sum_{-1 \leqslant k_1 \leqslant k-1} \Delta_{k_1}.$$

In addition, we write

$$R_k = (S_{k-1}v \cdot \nabla)\Delta_k u - \Delta_k(v \cdot \nabla u).$$

Lemma 2.6 (see [24]). Let u, v be two vector functions with $\nabla \cdot v = 0$. Assume $1 \leqslant p \leqslant p_1 \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $p' = (1 - \frac{1}{p})^{-1}$, and $s > -1 - \min(\frac{d}{p_1}, \frac{d}{p'})$. Then,

$$\begin{cases} 2^{ks}\|R_k\|_{L^p(\mathbb{R}^d)} \leqslant ch_k\|\nabla v\|_{B^{\frac{d}{p_1}}_{p_1,\infty}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}\|u\|_{B^s_{p,q}(\mathbb{R}^d)}, & \text{if } s < 1 + \frac{d}{p_1}, \\ 2^{ks}\|R_k\|_{L^p(\mathbb{R}^d)} \leqslant ch_k\|\nabla v\|_{B^{s-1}_{p_1,q}(\mathbb{R}^d)}\|u\|_{B^s_{p,q}(\mathbb{R}^d)}, & \text{if } s > 1 + \frac{d}{p_1}, \text{ or } s = 1 + \frac{d}{p_1}, q = 1, \end{cases}$$

where $c = c(d,p,p_1,s,q)$, and $\sum (h_k^q)^{\frac{1}{q}} = 1$.

3 Construction of a Special Function Sequence

This section provides the construction of a special sequence of functions as a proof of Proposition 1.5. The construction of the initial data in the proof of the main theorem relies on the sequence constructed here.

Proof. Proof of Proposition 1.5 We first remark that it suffices to prove this proposition for $\mathcal{R} = \mathcal{R}_1 \mathcal{R}_2$, since $\mathcal{R}_1^2$ becomes $2\mathcal{R}_1\mathcal{R}_2 - Id$ under a rotation of $\frac{\pi}{2}$. We set

$$f(x, y) = \chi_{[-1,1]^2}, \qquad \widehat{f}(\xi_1, \xi_2) = 4\frac{\sin \xi_1 \sin \xi_2}{\xi_1 \xi_2}$$

and define

$$\widehat{f_N}(\xi) = \chi_{\{\xi \mid \frac{1}{2}|\xi_1| \leqslant |\xi_2| \leqslant \frac{\sqrt{3}}{2}|\xi_1|\} \bigcap \{\xi \mid 1 \leqslant |\xi_1| \leqslant 2^N\}} \widehat{f}(\xi),$$

where χ is the characteristic function. Clearly, f_N belongs to $\dot{H}^s$ for all $s \geqslant -1$. It is easy to see that f_N satisfies (1.8) and (1.9). Now we prove that f_N satisfies (1.10). In the process, we use the following fact repeatedly,

$$\sup_{a,b} \left| \int_a^b \frac{\sin y}{y} \, dy \right| < c.$$

By the Fourier inversion formula, we have

$$|f_N|_{L^\infty} \leqslant c \sup_{x_1, x_2} \left| \int_1^{2^N} \left(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_1)\sin(\xi_2)}{\xi_1 \xi_2} \cos(x_1\xi_1)\cos(x_2\xi_2) \, d\xi_2 \right) d\xi_1 \right|.$$

Since $\cos(x_1\xi_1)$ and $\cos(x_2\xi_2)$ are even functions,

$$\begin{aligned} |f_N|_{L^\infty} \leqslant c \sup_{x_1 \geqslant 0, x_2 \geqslant 0} & \left| \int_1^{2^N} \frac{\sin(\xi_1 + x_1\xi_1) - \sin(\xi_1 - x_1\xi_1)}{2\xi_1} \right. \\ & \left. \times \left(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 + x_2\xi_2) - \sin(\xi_2 - x_2\xi_2)}{2\xi_2} \, d\xi_2 \right) d\xi_1 \right| \\ \leqslant c \sum_{k=1}^{4} f_{N_k}, & \end{aligned}$$

where

$$f_{N_1} = \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\Big(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 + x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_2} = \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1}\Big(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 + x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_3} = \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\Big(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_4} = \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1}\Big(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|.$$

By integration by parts,

$$\begin{aligned} f_{N_1} &= \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\Big(\int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 + x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big| \\ &= \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_1^{2^N} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\Big(\frac{\cos(\frac{\sqrt{3}}{2}(1+x_2)\xi_1)}{\frac{\sqrt{3}}{2}(1+x_2)\xi_1} - \frac{\cos(\frac{1}{2}(1+x_2)\xi_1)}{\frac{1}{2}(1+x_2)\xi_1} \\ &\qquad + \int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\cos(\xi_2 + x_2\xi_2)}{(1+x_2)\xi_2^2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big| \\ &\leqslant c. \end{aligned}$$

Similarly, we have $f_{N_2} \leqslant c$. To estimate f_{N_3}, we exchange the order of integration and divide f_{N_3} into three parts,

$$\begin{aligned} f_{N_3} \leq{} & \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\int_1^{2\xi_2} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Big| \\ & + \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_{\frac{\sqrt{3}}{2}}^{2^{N-1}}\int_{\frac{2\xi_2}{\sqrt{3}}}^{2\xi_2} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Big| \\ & + \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_{2^{N-1}}^{\sqrt{3}2^{N-1}}\int_{\frac{2\xi_2}{\sqrt{3}}}^{2^N} \frac{\sin(\xi_1 + x_1\xi_1)}{\xi_1}\frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Big| \\ \leqslant{} & c + \sup_{x_1\geqslant 0, x_2\geqslant 0}\Big|\int_{\frac{\sqrt{3}}{2}}^{2^{N-1}} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2}\Big(\frac{\cos(2(1+x_1)\xi_2)}{2(1+x_1)\xi_2} - \frac{\cos(\frac{2}{\sqrt{3}}(1+x_1)\xi_2)}{\frac{2}{\sqrt{3}}(1+x_1)\xi_2} \\ & \qquad + \int_{\frac{2\xi_2}{\sqrt{3}}}^{2\xi_2} \frac{\cos(\xi_1 + x_1\xi_1)}{(1+x_1)\xi_1^2}\,\mathrm{d}\xi_1\Big)\,\mathrm{d}\xi_2\Big| \\ \leqslant{} & c. \end{aligned}$$

Here we have used the fact that

$$\sup_{x_1\geqslant 0,x_2\geqslant 0}\Big|\int_{\frac12}^{\frac{\sqrt3}{2}}\int_1^{2\xi_2}\frac{\sin(\xi_1+x_1\xi_1)}{\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Big|\le c,$$
$$\sup_{x_1\geqslant 0,x_2\geqslant 0}\Big|\int_{2^{N-1}}^{\sqrt3 2^{N-1}}\int_{\frac{2\xi_2}{\sqrt3}}^{2^N}\frac{\sin(\xi_1+x_1\xi_1)}{\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Big|\le c.$$

It is much more difficult to bound f_{N_4}. We decompose the domain of $\{x_1\ge 0, x_2\ge 0\}$ into several pieces,

$$f_{N_4}\leqslant\sum_{k=1}^{6}f_{N_{4k}},$$

where

$$f_{N_{41}}=\sup_{|1-x_1|\geqslant\frac12,x_2\geqslant 0}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_{42}}=\sup_{x_1\geqslant 0,|1-x_2|\geqslant\frac12}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_{43}}=\sup_{|1-x_1|\leqslant 10\cdot 2^{-N},x_2\geqslant 0}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_{44}}=\sup_{x_1\geqslant 0,|1-x_2|\leqslant 10\cdot 2^{-N}}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_{45}}=\sup_{\kappa_1(x_1,x_2)}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|,$$

$$f_{N_{46}}=\sup_{\kappa_2(x_1,x_2)}\Big|\int_1^{2^N}\frac{\sin(\xi_1-x_1\xi_1)}{\xi_1}\Big(\int_{\frac12\xi_1}^{\frac{\sqrt3}{2}\xi_1}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\,\mathrm{d}\xi_2\Big)\,\mathrm{d}\xi_1\Big|.$$

Here

$$\kappa_1(x_1,x_2)=\{(x_1,x_2)|10\cdot 2^{-N}\leqslant|1-x_1|\leqslant\frac12,10\cdot 2^{-N}\leqslant|1-x_2|\leqslant\frac12,$$
$$|1-x_1|\leqslant|1-x_2|\},$$
$$\kappa_2(x_1,x_2)=\{(x_1,x_2)|10\cdot 2^{-N}\leqslant|1-x_1|\leqslant\frac12,10\cdot 2^{-N}\leqslant|1-x_2|\leqslant\frac12,$$
$$|1-x_1|\geqslant|1-x_2|\}.$$

Similar to the estimates for f_{N_3} and f_{N_1}, we have

$$f_{N_{41}} \leqslant c \quad \text{and} \quad f_{N_{42}} \leqslant c.$$

Clearly,

$$f_{N_{43}} \leqslant c \sup_{|1-x_1|\leqslant 10\cdot 2^{-N}} \Big| \int_1^{2^N} \frac{|\sin(\xi_1 - x_1\xi_1)|}{\xi_1} \, \mathrm{d}\xi_1 \Big| \leqslant c$$

and

$$f_{N_{44}} \leqslant c \sup_{x_1\geqslant 0, |1-x_2|\leqslant 10\cdot 2^{-N}} \Big| \int_1^{2^N} \frac{|\sin(\xi_1 - x_1\xi_1)|}{\xi_1} \cdot (1-x_2)\xi_1 \, \mathrm{d}\xi_1 \Big| \leqslant c.$$

It remains to estimate $f_{N_{45}}$ and $f_{N_{46}}$. Splitting the interval and then integrating by parts, we have

$$\begin{aligned}
f_{N_{45}} &= \sup_{\kappa_1(x_1,x_2)} \Big| \Big(\int_1^{\frac{1}{|1-x_1|}} + \int_{\frac{1}{|1-x_1|}}^{2^N} \Big) \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2} \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \Big| \\
&\leqslant c \sup_{\kappa_1(x_1,x_2)} \Big| \int_1^{\frac{1}{|1-x_1|}} \frac{|\sin(\xi_1 - x_1\xi_1)|}{\xi_1} \, \mathrm{d}\xi_1 \Big| + \sup_{\kappa_1(x_1,x_2)} \Big| \int_{\frac{1}{|1-x_1|}}^{2^N} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \\
&\quad \times \Big(\frac{\cos(\frac{\sqrt{3}}{2}(1-x_2)\xi_1)}{\frac{\sqrt{3}}{2}(1-x_2)\xi_1} - \frac{\cos(\frac{1}{2}(1-x_2)\xi_1)}{\frac{1}{2}(1-x_2)\xi_1} + \int_{\frac{1}{2}\xi_1}^{\frac{\sqrt{3}}{2}\xi_1} \frac{\cos(\xi_2 - x_2\xi_2)}{(1-x_2)\xi_2^2} \, \mathrm{d}\xi_2 \Big) \, \mathrm{d}\xi_1 \Big| \\
&\leqslant c \sup_{\kappa_1(x_1,x_2)} \Big| \int_{|1-x_1|}^1 \frac{|\sin \xi_1|}{\xi_1} \, \mathrm{d}\xi_1 \Big| + c \sup_{\kappa_1(x_1,x_2)} \Big| \int_{\frac{1}{|1-x_1|}}^{2^N} \frac{|\sin(\xi_1 - x_1\xi_1)|}{(1-x_2)\xi_1^2} \, \mathrm{d}\xi_1 \Big| \\
&\leqslant c + c \sup_{\kappa_1(x_1,x_2)} \Big| \frac{1-x_1}{1-x_2} \Big| \leqslant c.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_{N_{46}} &\le \sup_{\kappa_2(x_1,x_2)} \Big| \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \int_1^{2\xi_2} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \Big| \\
&\quad + \sup_{\kappa_2(x_1,x_2)} \Big| \int_{\frac{\sqrt{3}}{2}}^{2^{N-1}} \int_{\frac{2\xi_2}{\sqrt{3}}}^{2\xi_2} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \Big| \\
&\quad + \sup_{\kappa_2(x_1,x_2)} \Big| \int_{2^{N-1}}^{\sqrt{3}2^{N-1}} \int_{\frac{2\xi_2}{\sqrt{3}}}^{2^N} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \Big| \\
&\leqslant c + \sup_{\kappa_2(x_1,x_2)} \Big| \Big(\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{|1-x_2|}} + \int_{\frac{1}{|1-x_2|}}^{2^{N-1}} \Big) \int_{\frac{2\xi_2}{\sqrt{3}}}^{2\xi_2} \frac{\sin(\xi_1 - x_1\xi_1)}{\xi_1} \frac{\sin(\xi_2 - x_2\xi_2)}{\xi_2} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \Big| \\
&\leqslant c + c \sup_{\kappa_2(x_1,x_2)} \Big| \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{|1-x_2|}} \frac{|\sin(\xi_2 - x_2\xi_2)|}{\xi_2} \, \mathrm{d}\xi_2 \Big|
\end{aligned}$$

$$+\sup_{\kappa_2(x_1,x_2)}\Big|\int_{\frac{1}{|1-x_2|}}^{2^{N-1}}\frac{\sin(\xi_2-x_2\xi_2)}{\xi_2}\Big(\frac{\cos(2(1-x_1)\xi_2)}{2(1-x_1)\xi_2}-\frac{\cos(\frac{2}{\sqrt{3}}(1-x_1)\xi_2)}{\frac{2}{\sqrt{3}}(1-x_1)\xi_2}$$

$$+\int_{\frac{2\xi_2}{\sqrt{3}}}^{2\xi_2}\frac{\cos(\xi_1-x_1\xi_1)}{(1-x_1)\xi_1^2}\,\mathrm{d}\xi_1\Big)\,\mathrm{d}\xi_2\Big|$$

$$\leqslant c.$$

Thus, we have obtained $f_{N_4}\leqslant c$. Combining the estimates for f_{N_1} through f_{N_4}, we have established

$$\|f_N\|_{L^\infty}\leqslant c.$$

Next we show that f_N satisfies (1.11). By the Fourier inversion formula,

$$\begin{aligned}&\mathcal{R}_1\mathcal{R}_2f_N(x,y)\\&=4\int\chi_{\{\xi|\frac12|\xi_1|\leqslant|\xi_2|\leqslant\frac{\sqrt3}{2}|\xi_1|\}\bigcap\{\xi|1\leqslant|\xi_1|\leqslant2^N\}}\frac{\sin(\xi_1)\sin(\xi_2)}{\xi_1^2+\xi_2^2}\sin(x\xi_1)\sin(y\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{R}_1\mathcal{R}_2f_N(1,1)&=4\int\chi_{\{\xi|\frac12|\xi_1|\leqslant|\xi_2|\leqslant\frac{\sqrt3}{2}|\xi_1|\}\bigcap\{\xi|1\leqslant|\xi_1|\leqslant2^N\}}\frac{\sin^2(\xi_1)\sin^2(\xi_2)}{\xi_1^2+\xi_2^2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\&\geq\sin^2(\frac12)\,\sin^2(1)\int\chi_{\{\xi|\frac12|\xi_1|\leqslant|\xi_2|\leqslant\frac{\sqrt3}{2}|\xi_1|\}\bigcap\{\xi|1\leqslant|\xi_1|\leqslant2^N\}}\frac{1}{\xi_1^2+\xi_2^2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\&\geq cN\end{aligned}$$

and (1.11) then follows. Finally, we show (1.12). For $k\leqslant cN$,

$$\begin{aligned}\|\nabla\Delta_kf_N\|_{L^2}&\leqslant c2^k\|\Delta_kf_N\|_{L^2}\\&=c2^k\Big(\int|\varphi(2^k\xi)\chi_{\{\xi|\frac12|\xi_1|\leqslant|\xi_2|\leqslant\frac{\sqrt3}{2}|\xi_1|\}\bigcap\{\xi|1\leqslant|\xi_1|\leqslant2^N\}}\frac{\sin\xi_1\sin\xi_2}{\xi_1\xi_2}|^2\,\mathrm{d}\xi\Big)^{\frac12}\\&\leqslant c.\end{aligned}$$

For $k\geqslant cN$,

$$\|\nabla\Delta_kf_N\|_{L^2}=0.$$

Therefore,

$$\|f_N\|_{B^{\frac12}_{4,1}}\leqslant\|f_N\|_{B^1_{2,1}}\leqslant\sum_{k=-1}^{cN}\|\nabla\Delta_kf_N\|_{L^2}\leqslant cN.$$

This finishes the proof of Proposition 1.5. ■

4 Proof of the Local Existence Result

This section proves Proposition 1.3, the local existence, and uniqueness result on (1.2). To prove this result, we first provide several global bounds.

The 1st one is the L^2-bound, which follows directly from the equations in (1.2).

Lemma 4.1. Assume $u_0, b_0 \in L^2$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.2). Then, for any $t > 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\nabla b\|_{L^2}^2 \,\mathrm{d}\tau \leqslant \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \leqslant c_0.$$

The H^1-norm of (u, b) is also bounded globally and uniformly in time. For simplicity, we resort to the equations of the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j + \partial_1 j, \\ \partial_t j + u \cdot \nabla j - \Delta j = b \cdot \nabla \omega + Q(\nabla u, \nabla b) + \partial_1 \omega, \end{cases} \tag{4.1}$$

where

$$Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

The proof of the uniform H^1-bound in the following lemma is simple and can be found in [9, 15, 18].

Lemma 4.2. Assume $u_0, b_0 \in H^1$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.2). Then, for any $t > 0$,

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\nabla j\|_{L^2}^2 \,\mathrm{d}\tau \leqslant (\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2) e^{c(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)} \leqslant c_0.$$

Lemma 4.3. Assume $u_0 \in H^1$, $b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1}$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.2). Then, for any $t > 0$,

$$\|b\|_{L^\infty_t B^{\frac{1}{2}}_{4,1}} \leqslant c_0, \quad \|b\|_{L^\infty_t L^\infty_x} \leqslant c_0.$$

Proof. Applying Δ_k to the equation of b in (1.2) yields

$$\Delta_k b_t - \Delta\Delta_k b = -\Delta_k(u\cdot\nabla b) + \Delta_k(b\cdot\nabla u) + \partial_1\Delta_k u. \tag{4.2}$$

We write (4.2) in the integral form

$$\begin{aligned}\Delta_k b(t) &= e^{t\Delta}\Delta_k b_0 - \int_0^t e^{(t-\tau)\Delta}\,\Delta_k(u\cdot\nabla b)\,\mathrm{d}\tau\\ &\quad + \int_0^t e^{(t-\tau)\Delta}\Delta_k(b\cdot\nabla u)\,\mathrm{d}\tau + \int_0^t e^{(t-\tau)\Delta}\partial_1\Delta_k u\,\mathrm{d}\tau.\end{aligned} \tag{4.3}$$

Taking the L^4-norm and using the fact that, for any integer $k\geqslant 0$ and $1\leqslant q\leqslant\infty$,

$$\|e^{t\Delta}\Delta_k f\|_{L^q} \leqslant c e^{-c\,2^{2k}t}\|\Delta_k f\|_{L^q},$$

we have

$$\begin{aligned}\|\Delta_k b\|_{L^4} \leqslant\ & c e^{-c2^{2k}t}\|\Delta_k b_0\|_{L^4} + c\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k(u\cdot\nabla b)\|_{L^4}\,\mathrm{d}\tau\\ & + c\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k(b\cdot\nabla u)\|_{L^4}\,\mathrm{d}\tau + c\int_0^t e^{-c2^{2k}(t-\tau)}\|\partial_1\Delta_k u\|_{L^4}\,\mathrm{d}\tau.\end{aligned}$$

Multiplying by $2^{\frac{1}{2}k}$, summing over any integer $k\geqslant 0$, applying Sobolev's inequalities and invoking Lemmas 4.1 and 4.2, we obtain

$$\begin{aligned}\|b\|_{B^{\frac12}_{4,1}} \leqslant\ & c\|b\|_{L^4} + c\sum_k e^{-c2^{2k}t}2^{\frac12 k}\|\Delta_k b_0\|_{L^4} + c\sum_k\int_0^t e^{-c2^{2k}(t-\tau)}2^{\frac12 k}(2^k\|ub\|_{L^4})\,\mathrm{d}\tau\\ & + c\sum_k\int_0^t e^{-c2^{2k}(t-\tau)}2^{\frac12 k}(2^k\|u\|_{L^4})\,\mathrm{d}\tau\\ \leqslant\ & c_0 + c\|b_0\|_{B^{\frac12}_{4,1}} + c\|u\|_{L^\infty_t L^8_x}\|b\|_{L^\infty_t L^8_x} + c\|u\|_{L^\infty_t L^4_x}\\ \leqslant\ & c_0.\end{aligned}$$

Due to the embedding $B^{\frac12}_{4,1}\hookrightarrow L^\infty$, we have

$$\|b\|_{L^\infty_t L^\infty_x} \leqslant c_0.$$

This completes the proof of Lemma 4.3. ■

We are now ready to prove Proposition 1.3.

Proof of Proposition 1.3. The key part of the proof is the local *a priori* bound on (u, b). Once these bounds are established, a complete proof then follows from a standard procedure (see, e.g., [23]). For the sake of conciseness, we shall only prove the local bounds.

Applying $\Delta_k \nabla\times$ and Δ_k to equations (1.2) and (1.2), respectively, yields

$$\begin{cases} \Delta_k \omega_t + (S_{k-1}u\cdot\nabla)\Delta_k\omega = (S_{k-1}u\cdot\nabla)\Delta_k\omega - \Delta_k(u\cdot\nabla\omega) \\ \qquad\qquad + \Delta_k\nabla\times(b\cdot\nabla b) + \partial_1\Delta_k\nabla\times b, \\ \Delta_k b_t + (S_{k-1}u\cdot\nabla)\Delta_k b - \Delta\Delta_k b = (S_{k-1}u\cdot\nabla)\Delta_k b - \Delta_k(u\cdot\nabla b) \\ \qquad\qquad + \Delta_k(b\cdot\nabla u) + \partial_1\Delta_k u. \end{cases} \tag{4.4}$$

Taking the L^4-norm of the equation of ω in (4.4), we have

$$\|\Delta_k\omega\|_{L^4} \leqslant \|\Delta_k\omega_0\|_{L^4} + \int_0^t \|(S_{k-1}u\cdot\nabla)\Delta_k\omega - \Delta_k(u\cdot\nabla\omega)\|_{L^4}\,\mathrm{d}\tau$$
$$+ \int_0^t \|\Delta_k\nabla\times(b\cdot\nabla b)\|_{L^4}\,\mathrm{d}\tau + \int_0^t \|\partial_1\Delta_k\nabla\times b\|_{L^4}\,\mathrm{d}\tau.$$

By Lemmas 2.5, 2.6, 4.1, and 4.2,

$$\|u\|_{B^{\frac{3}{2}}_{4,1}} \leqslant c\|\Delta_{-1}u\|_{L^4} + c\|\omega\|_{B^{\frac{1}{2}}_{4,1}} \tag{4.5}$$
$$\leqslant c\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + c\int_0^t \|u\|^2_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau + c\int_0^t \|b\|_{B^{\frac{1}{2}}_{4,1}}\|b\|_{B^{\frac{5}{2}}_{4,1}}\,\mathrm{d}\tau + c\int_0^t \|b\|_{B^{\frac{5}{2}}_{4,1}}\,\mathrm{d}\tau.$$

Similarly, for $k \geqslant 0$, we have

$$\|\Delta_k b\|_{L^4} \leqslant ce^{-c2^{2k}t}\|\Delta_k b_0\|_{L^4}$$
$$+ c\int_0^t e^{-c2^{2k}(t-\tau)}\|(S_{k-1}u\cdot\nabla)\Delta_k b - \Delta_k(u\cdot\nabla b)\|_{L^4}\,\mathrm{d}\tau$$
$$+ c\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k(b\cdot\nabla u)\|_{L^4}\,\mathrm{d}\tau + c\int_0^t e^{-c2^{2k}(t-\tau)}\|\partial_1\Delta_k u\|_{L^4}\,\mathrm{d}\tau$$

and

$$
\begin{aligned}
\|b\|_{B^{\frac{5}{2}}_{4,1}} \leqslant c\|b\|_{L^4} + c\sum_k e^{-c2^{2k}t} 2^{\frac{5}{2}k}\|\Delta_k b_0\|_{L^4} \\
+ c\sum_k \int_0^t e^{-c2^{2k}(t-\tau)} 2^{\frac{5}{2}k}(c_k 2^{-\frac{1}{2}k}\|u\|_{B^{\frac{3}{2}}_{4,1}} \|b\|_{B^{\frac{1}{2}}_{4,1}})\,\mathrm{d}\tau \\
+ c\sum_k \int_0^t e^{-c2^{2k}(t-\tau)} 2^{\frac{5}{2}k}(c_k 2^{-\frac{1}{2}k}\|u\|_{B^{\frac{3}{2}}_{4,1}})\,\mathrm{d}\tau,
\end{aligned}
$$

where $c_k = 2^{\frac{1}{2}k}\|\Delta_k b\|_{L^4}/\|b\|_{B^{\frac{1}{2}}_{4,1}}$ and, by the definition of the norm in $\|b\|_{B^{\frac{1}{2}}_{4,1}}$, we have

$$
\sum_k c_k = 1.
$$

By Lemma 4.3,

$$
\begin{aligned}
\int_0^t \|b\|_{B^{\frac{5}{2}}_{4,1}}\,\mathrm{d}\tau &\leqslant c\|b_0\|_{B^{\frac{1}{2}}_{4,1}} + c\int_0^t \|b\|_{L^4}\,\mathrm{d}\tau + c\int_0^t \|u\|_{B^{\frac{3}{2}}_{4,1}}\|b\|_{B^{\frac{1}{2}}_{4,1}}\,\mathrm{d}\tau + c\int_0^t \|u\|_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau \\
&\leqslant c_0 + c_0 t + (c+c_0)\int_0^t \|u\|_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau. \qquad (4.6)
\end{aligned}
$$

Combining (4.5) and (4.6), we have

$$
\begin{aligned}
\|u\|_{B^{\frac{3}{2}}_{4,1}} &\leqslant c\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + c_0 t + c\int_0^t \|u\|^2_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau + (c+c_0)\int_0^t \|u\|_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau \\
&\leqslant c\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + c\int_0^t \|u\|^2_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau + (c+c_0)t \\
&\leqslant c\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + (c+c_0) + c\int_0^t \|u\|^2_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau.
\end{aligned}
$$

By the above differential inequality, we get

$$
\|u\|_{B^{\frac{3}{2}}_{4,1}} \leqslant \frac{c(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)}{1 - c(\|u_0\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)t},
$$

which is (1.6). To prove (1.7), we set

$$G(t)=\|u_0\|_{B^{\frac{3}{2}}_{4,1}}+c_0+c_0t+c\int_0^t\|u\|^2_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau+(c+c_0)\int_0^t\|u\|_{B^{\frac{3}{2}}_{4,1}}\,\mathrm{d}\tau.$$

It is then clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}G=c\|u\|^2_{B^{\frac{3}{2}}_{4,1}}+(c+c_0)\|u\|_{B^{\frac{3}{2}}_{4,1}}+c_0\leqslant cG^2+(c+c_0)G,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-(c+c_0)t}G)\leqslant ce^{(c+c_0)t}(e^{-(c+c_0)t}G)^2.$$

Noticing that $G_0=\|u_0\|_{B^{\frac{3}{2}}_{4,1}}+c_0$, we have

$$\|u\|_{B^{\frac{3}{2}}_{4,1}}\le G(t)\leqslant\frac{(\|u_0\|_{B^{\frac{3}{2}}_{4,1}}+c_0)e^{(c+c_0)t}}{1-\frac{c}{c+c_0}(\|u_0\|_{B^{\frac{3}{2}}_{4,1}}+c_0)(e^{(c+c_0)t}-1)}.$$

This completes the proof of Proposition 1.3. ■

5 Global *A Priori* Estimates

This section presents the proof of the global *a priori* bounds stated in Proposition 1.4.

Proof of Proposition 1.4. We start by writing the equation of b in (1.1) in the integral form,

$$b(t)=e^{t\Delta}b_0+\int_0^t e^{(t-s)\Delta}(b\cdot\nabla u-u\cdot\nabla b)(s)\,\mathrm{d}s+\int_0^t e^{(t-s)\Delta}\partial_1u(s)\,\mathrm{d}s. \tag{5.1}$$

Let $q\in(1,4/3)$. Applying Lemma 2.1 to (5.1) yields

$$\|\Delta b\|_{L^q_tL^4_x}\leqslant c\,t^{\frac{1}{q}-\frac{3}{4}}\|\Lambda^{\frac{1}{2}}b_0\|_{L^4}+c\|b\cdot\nabla u-u\cdot\nabla b\|_{L^q_tL^4_x}+c\|\nabla u\|_{L^q_tL^4_x}. \tag{5.2}$$

It follows from the vorticity equation in (4.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^q_{L^4}\leqslant(\|b\|_{L^\infty}+c)\|\nabla j\|_{L^4}\|\omega\|^{q-1}_{L^4}\leqslant(c_0+c)\|\nabla j\|_{L^4}\|\omega\|^{q-1}_{L^4}.$$

Thus,

$$\begin{aligned}
\|\omega\|_{L^4}^q &\leqslant \|\omega_0\|_{L^4}^q + (c_0+c)\int_0^t \|\nabla j\|_{L^4}\|\omega\|_{L^4}^{q-1}\,\mathrm{d}\tau \\
&\leqslant c_0 + (c_0+c)\int_0^t (\|\Delta b\|_{L^4}^q + \|\omega\|_{L^4}^q)\,\mathrm{d}\tau \\
&\leqslant c_0 + (c_0+c)t^{1-\frac{3}{4}q}\|b_0\|^q_{B^{\frac{1}{2}}_{4,1}} + (c_0+c)\int_0^t (\|b\cdot\nabla u - u\cdot\nabla b\|_{L^4}^q + \|\omega\|_{L^4}^q)\,\mathrm{d}\tau \\
&\leqslant c_0 + c_0 t^{1-\frac{3}{4}q} + (c_0+c)\int_0^t (\|b\|_{L^\infty}^q\|\omega\|_{L^4}^q \\
&\qquad\qquad + (\|u_0\|_{L^2}^q + \|\omega\|_{L^4}^q)\|\nabla b\|_{L^4}^q + \|\omega\|_{L^4}^q)\,\mathrm{d}\tau \\
&\leqslant c_0 + c_0 t^{1-\frac{3}{4}q} + c_0\int_0^t (\|\nabla b\|_{L^2}^q + \|\nabla j\|_{L^2}^q)\,\mathrm{d}\tau \\
&\qquad + (c_0+c)\int_0^t (1+\|\nabla b\|_{L^4}^q)\|\omega\|_{L^4}^q\,\mathrm{d}\tau \\
&\leqslant c_0 + c_0(t^{1-\frac{3}{4}q} + t + t^{1-\frac{1}{2}q}) + (c_0+c)\int_0^t (1+\|\nabla j\|_{L^2}^q)\|\omega\|_{L^4}^q\,\mathrm{d}\tau \\
&= c_0(1+r(t)) + (c_0+c)\int_0^t (1+\|\nabla j\|_{L^2}^q)\|\omega\|_{L^4}^q\,\mathrm{d}\tau,
\end{aligned}$$

where we have used Lemmas 4.1, 4.2, and 4.3. By Gronwall's inequality and Lemma 4.2,

$$\begin{aligned}
\|\omega\|_{L^4} &\leqslant c_0(1+r(t)) \\
&\quad + c_0(1+r(t))(c_0+c)\Big(\int_0^t (1+\|\nabla j\|_{L^2}^q)\,\mathrm{d}\tau\Big)^{\frac{1}{q}} e^{(c_0+c)\int_0^t (1+\|\nabla j\|_{L^2}^q)\,\mathrm{d}\tau} \\
&\leqslant c_0(1+r(t))e^{(c_0+c)r(t)}.
\end{aligned}$$

Using Sobolev type embedding inequalities, we obtain

$$\|u\|_{B^{\frac{1}{2}}_{4,1}} \leqslant c\|u\|^{\frac{1}{2}}_{B^0_{4,\infty}}\|\omega\|^{\frac{1}{2}}_{B^0_{4,\infty}} \leqslant c\|u\|^{\frac{1}{2}}_{L^4}\|\omega\|^{\frac{1}{2}}_{L^4} \leqslant c_0(1+r(t))e^{(c_0+c)r(t)}.$$

It then follows from (5.2) that

$$\|\Delta b\|_{L^q_t L^4_x} \leqslant c_0 r(t) e^{(c_0+c)r(t)}.$$

Taking the gradient of (4.3), then taking the L^4-norm, multiplying by $2^{\frac{1}{2}k}$ and summing over all integers $k \geq 0$, we have

$$\begin{aligned}\int_0^t \|\nabla b\|_{B^{\frac{1}{2}}_{4,1}}\,\mathrm{d}\tau \leqslant{}& c\int_0^t \|b\|_{L^4}\,\mathrm{d}\tau + c\int_0^t \sum_k e^{-c2^{2k}\tau}2^{\frac{1}{2}k}\|\Delta_k\nabla b_0\|_{L^4}\,\mathrm{d}\tau\\ &+ c\int_0^t\sum_k\int_0^\tau e^{-c2^{2k}(\tau-s)}2^{\frac{1}{2}k}(\|\Delta_k\nabla(u\cdot\nabla b)\|_{L^4}\\ &+\|\Delta_k\nabla(b\cdot\nabla u)\|_{L^4})\,\mathrm{d}s\,\mathrm{d}\tau\\ &+ c\int_0^t\sum_k\int_0^\tau e^{-c2^{2k}(\tau-s)}2^{\frac{1}{2}k}\|\partial_1\nabla u\|_{L^4}\,\mathrm{d}s\,\mathrm{d}\tau\\ \leqslant{}& c_0t + c\|b_0\|_{L^4} + c\int_0^t(\|u\cdot\nabla b\|_{L^4}+\|b\cdot\nabla u\|_{L^4}+\|\nabla u\|_{L^4})\,\mathrm{d}\tau\\ \leqslant{}& c_0 + c_0t + c\int_0^t(\|u\|_{L^\infty}\|\nabla b\|_{L^4}+\|b\|_{L^\infty}\|\nabla u\|_{L^4}+\|\nabla u\|_{L^4})\,\mathrm{d}\tau\\ \leqslant{}& c_0 + c_0 r(t)e^{(c_0+c)r(t)}.\end{aligned}$$

Applying $b\cdot\nabla$ and ∂_1 to the equation of u in (1.2), applying Δ to the equation of b in (1.2), multiplying the equation of b by ∇u and adding the resulting equations, we obtain

$$(\Delta b + b\cdot\nabla u + \partial_1 u)_t - \Delta(\Delta b + b\cdot\nabla u + \partial_1 u) = g, \tag{5.3}$$

where

$$\begin{aligned} g ={}& -b\cdot\nabla(u\cdot\nabla u) + b\cdot\nabla(b\cdot\nabla b) - (u\cdot\nabla b)\cdot\nabla u + (b\cdot\nabla u)\cdot\nabla u\\ &+\Delta b\cdot\nabla u - b\cdot\nabla(\nabla p) - \Delta(u\cdot\nabla b) + b\cdot\nabla\partial_1 b + \partial_1 u\cdot\nabla u\\ &-\partial_1(u\cdot\nabla u) - \nabla\partial_1 p + \partial_1(b\cdot\nabla b) + \partial_1^2 b.\end{aligned}$$

Applying Δ_k to (5.3) gives

$$\Delta_k(\Delta b + b\cdot\nabla u + \partial_1 u)_t - \Delta\Delta_k(\Delta b + b\cdot\nabla u + \partial_1 u) = \Delta_k g. \tag{5.4}$$

Rewriting (5.4) into integral form and then taking the L^4-norm yield

$$\|\Delta_k(\Delta b + b\cdot\nabla u + \partial_1 u)\|_{L^4} \leqslant ce^{-c2^{2k}t}\|\Delta_k(\Delta b + b\cdot\nabla u + \partial_1 u)_0\|_{L^4} + c\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k g\|_{L^4}\,\mathrm{d}\tau,$$

for $k \geqslant 0$. Therefore,

$$\begin{aligned}
&\int_0^t \|\Delta b + b\cdot\nabla u + \partial_1 u\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau\\
&\leqslant c\int_0^t (\|b\|_{L^4} + \|u\|_{L^8}\|b\|_{L^8} + \|u\|_{L^4})\,\mathrm{d}\tau + c\int_0^t \sum_k e^{-c2^{2k}\tau}2^{\frac12 k}\|\Delta_k(\Delta b + b\cdot\nabla u + \partial_1 u)_0\|_{L^4}\,\mathrm{d}\tau\\
&\quad + c\int_0^t \sum_k \int_0^\tau e^{-c2^{2k}(\tau-s)}2^{\frac12 k}\|\Delta_k g\|_{L^4}\,\mathrm{d}s\,\mathrm{d}\tau\\
&\leqslant c(\|b_0\|_{B^{\frac12}_{4,1}} + \|b_0 u_0\|_{L^4} + \|u_0\|_{L^4}) + c_0 t\\
&\quad + c\int_0^t (\|b(u\cdot\nabla u)\|_{L^4} + \|b(b\cdot\nabla b)\|_{L^4} + \|(u\cdot\nabla b)\cdot\nabla u\|_{L^2}\\
&\qquad + \|(b\cdot\nabla u)\cdot\nabla u\|_{L^2} + \|\Delta b u\|_{L^4}) + \|b\nabla p\|_{L^4} + \|b\partial_1 b\|_{L^4}\\
&\qquad + \|u\nabla u\|_{L^4} + \|\nabla p\|_{L^4} + \|b\cdot\nabla b\|_{L^4} + \|\partial_1 b\|_{L^4})\,\mathrm{d}\tau\\
&\quad + c\int_0^t \|u\cdot\nabla b\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau\\
&\leqslant c_0 + c_0 t + c\int_0^t (\|b\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^4} + \|b\|_{L^\infty}^2\|\nabla b\|_{L^4} + \|u\|_{L^\infty}\|\nabla b\|_{L^4}\|\nabla u\|_{L^4}\\
&\qquad + \|b\|_{L^\infty}\|\nabla u\|_{L^4}^2 + \|u\|_{L^\infty}\|\Delta b\|_{L^4} + \|b\|_{L^\infty}\|\nabla b\|_{L^4}\\
&\qquad + \|u\|_{L^\infty}\|\nabla u\|_{L^4} + \|\nabla b\|_{L^4})\,\mathrm{d}\tau\\
&\quad + c\int_0^t \|u\|_{B^{\frac12}_{4,1}}\|\nabla b\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau\\
&\leqslant c_0 + c_0 r(t)\,e^{(c_0+c)r(t)},
\end{aligned}$$

where we have used Lemmas 2.5, 4.1, and 4.2. This completes the proof of Proposition 1.4. ∎

6 Proof of Theorem 1.1

This section proves Theorem 1.1. For the sake of clarity, we first prove a weaker version of Theorem 1.1 and then continue to prove Theorem 1.1 itself. The weaker version is stated in the following theorem.

Theorem 6.1. There exists a sequence of initial data $\{(u_0^N, b_0^N)\}_{N=1}^{\infty}$ with $\mathrm{div} u_0^N = \mathrm{div} b_0^N = 0$ satisfying

$$u_0^N \in B^{\frac{3}{2}}_{4,1}, \quad b_0^N = b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1},$$
$$\|u_0^N\|_{H^1} \leqslant \frac{c}{N}, \quad \|\omega_0^N\|_{L^\infty} \leqslant \frac{c}{N}, \quad \|b_0\|_{H^1\cap B^{\frac{1}{2}}_{4,1}} \leqslant \delta, \quad \|\omega_0^N\|_{B^{\frac{1}{2}}_{4,1}} \leqslant C_1,$$

where $\omega_0^N = \nabla \times u_0^N$, $\delta > 0$ is a small constant and $C_1 > 0$ is a constant independent of N. Let (u^N, b^N) be the corresponding local solution of (1.2). Then (u^N, b^N) and $\omega^N = \nabla \times u^N$ satisfy

$$u^N \in L^\infty([0, C_2]; H^1 \cap B^{\frac{3}{2}}_{4,1}), \quad b^N \in L^\infty([0, C_2]; H^1 \cap B^{\frac{1}{2}}_{4,1}) \cap L^1([0, C_2]; B^{\frac{5}{2}}_{4,1})),$$
$$\|\omega^N(t)\|_{L^\infty([0,C_4];L^\infty)} \geqslant C_3, \quad C_4 \leqslant C_2,$$

where C_2, C_3, and C_4 are universal constants independent of N.

Proof of Theorem 6.1. First, we reformulate the vorticity equation of (4.1). Recall that

$$\omega_t + u \cdot \nabla\omega = b \cdot \nabla j + \partial_1 j.$$

It is easy to see that

$$\partial_1 j = \Delta b_2, \quad \partial_2 j = -\Delta b_1.$$

By making suitable combinations and regrouping the terms on the right-hand side, we obtain

$$\begin{aligned}
&\omega_t + u \cdot \nabla\omega \\
&= b_1\Delta b_2 - b_2\Delta b_1 + \Delta b_2 \\
&= b_1(\Delta b_2 + b \cdot \nabla u_2 + \partial_1 u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1 + \partial_1 u_1) + (\Delta b_2 + b \cdot \nabla u_2 + \partial_1 u_2) \\
&- b_1(b \cdot \nabla u_2 + \partial_1 u_2) + b_2(b \cdot \nabla u_1 + \partial_1 u_1) - b \cdot \nabla u_2 - \partial_1 u_2 \\
&= H + L - \partial_1 u_2,
\end{aligned}$$

where

$$H = b_1(\Delta b_2 + b\cdot\nabla u_2 + \partial_1 u_2) - b_2(\Delta b_1 + b\cdot\nabla u_1 + \partial_1 u_1) + (\Delta b_2 + b\cdot\nabla u_2 + \partial_1 u_2),$$

$$L = -b_1(b\cdot\nabla u_2 + \partial_1 u_2) + b_2(b\cdot\nabla u_1 + \partial_1 u_1) - b\cdot\nabla u_2.$$

We further rewrite

$$-\partial_1 u_2 = -\partial_1\partial_1\Delta^{-1}\omega = \mathcal{R}_1^2\omega := \mathcal{R}\omega$$

where $\mathcal{R}_1 = \partial_1\Lambda^{-1} := \partial_1(-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform. Consider the flow map Φ induced by the velocity u,

$$\begin{cases} \dot\Phi(x,t) = u(\Phi(x,t),t), \\ \Phi(x,0) = x. \end{cases}$$

Then ω satisfies

$$\begin{aligned}(\omega\circ\Phi)_t &= \mathcal{R}(\omega)\circ\Phi + H\circ\Phi + L\circ\Phi \\ &= \mathcal{R}(\omega\circ\Phi) + [\mathcal{R},\Phi]\omega + H\circ\Phi + L\circ\Phi,\end{aligned}$$

where

$$[\mathcal{R},\Phi]\omega = \mathcal{R}(\omega)\circ\Phi - \mathcal{R}(\omega\circ\Phi).$$

By Duhamel's principle,

$$\omega\circ\Phi = e^{\mathcal{R}t}\omega_0 + \int_0^t e^{\mathcal{R}(t-\tau)}[\mathcal{R},\Phi]\omega(\tau)\,\mathrm{d}\tau + \int_0^t e^{\mathcal{R}(t-\tau)}(H\circ\Phi + L\circ\Phi)(\tau)\,\mathrm{d}\tau.$$

Due to $B^{\frac{1}{2}}_{4,1} \hookrightarrow B^0_{\infty,1} \hookrightarrow L^\infty$, we have

$$\begin{aligned}\|\omega\|_{L^\infty} \geqslant \|e^{\mathcal{R}t}\omega_0\|_{L^\infty} &- c\int_0^t \|e^{\mathcal{R}(t-\tau)}[\mathcal{R},\Phi]\omega(\tau)\|_{B^{\frac{1}{2}}_{4,1}}\,\mathrm{d}\tau \\ &- c\int_0^t \|e^{\mathcal{R}(t-\tau)}(H\circ\Phi + L\circ\Phi)(\tau)\|_{B^0_{\infty,1}}\,\mathrm{d}\tau.\end{aligned}$$

Noting that

$$e^{\mathcal{R}t} = I + t\mathcal{R} + t^2\sum_{n=2}^{\infty}\frac{t^{n-2}\mathcal{R}^n}{n!},$$

and $\mathcal{R}$ is bounded on $B^{\frac{1}{2}}_{4,1}$, we have, for $t\in[0,1]$,

$$\int_0^t \|e^{\mathcal{R}(t-\tau)}[\mathcal{R},\Phi]\omega(\tau)\|_{B^{\frac{1}{2}}_{4,1}}\,\mathrm{d}\tau \leqslant c\int_0^t \|[\mathcal{R},\Phi]\omega(\tau)\|_{B^{\frac{1}{2}}_{4,1}}\,\mathrm{d}\tau.$$

Similarly, for $t \in [0, 1]$, we have

$$\begin{aligned}
&\int_0^t \|e^{\mathcal{R}(t-\tau)}(H\circ\Phi + L\circ\Phi)(\tau)\|_{B^0_{\infty,1}}\,\mathrm{d}\tau \\
&\leqslant c\int_0^t (\|(H\circ\Phi + L\circ\Phi)(\tau)\|_{L^4} + \|(H\circ\Phi + L\circ\Phi)(\tau)\|_{B^0_{\infty,1}})\,\mathrm{d}\tau \\
&\leqslant c\int_0^t (\|H\|_{L^4} + \|L\|_{L^4})\,\mathrm{d}\tau + c(1+\int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau)\int_0^t (\|H\|_{B^0_{\infty,1}} + \|L\|_{B^0_{\infty,1}})\,\mathrm{d}\tau \\
&\leqslant c(1+\int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau)\int_0^t (\|H\|_{B^{\frac12}_{4,1}} + \|L\|_{B^{\frac12}_{4,1}})\,\mathrm{d}\tau,
\end{aligned}$$

where we use Lemmas 2.2 and 2.4. Thanks to Lemmas 2.5, 4.3 and Propostion 1.4, we have

$$\begin{aligned}
\int_0^t \|H\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau &\leqslant c\int_0^t (1+\|b\|_{B^{\frac12}_{4,1}})\|\Delta b + b\cdot\nabla u + \partial_1 u\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau \\
&\leqslant c_0 + c_0 r(t)e^{(c_0+c)r(t)}, \\
\int_0^t \|L\|_{L^{\frac12}_{4,1}}\,\mathrm{d}\tau &\leqslant c\int_0^t \|b\|_{B^{\frac12}_{4,1}}(\|b\|_{B^{\frac12}_{4,1}}+1)\|\omega\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau \\
&\leqslant c_0\int_0^t \|\omega\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau.
\end{aligned}$$

By further revoking

$$\|e^{\mathcal{R}t}\omega_0\|_{L^\infty} \geq \|t\mathcal{R}(\omega_0)+\omega_0\|_{L^\infty} - ct^2\|\omega_0\|_{B^{\frac12}_{4,1}},$$

we obtain

$$\begin{aligned}
\|\omega\|_{L^\infty} \quad\geqslant\quad & \|t\mathcal{R}(\omega_0)+\omega_0\|_{L^\infty} - ct^2\|\omega_0\|_{B^{\frac12}_{4,1}} - c\int_0^t \|[\mathcal{R},\Phi]\omega(\tau)\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau \\
& - c_0(1+t\|\nabla u\|_{L^\infty_t L^\infty_x})(1+r(t)e^{(c_0+c)r(t)} + \int_0^t \|\omega\|_{B^{\frac12}_{4,1}}\,\mathrm{d}\tau).
\end{aligned}$$

Taking advantage of Lemmas 2.3 and 2.4, we obtain, for t small enough,

$$\begin{aligned}
\|\omega\|_{L^\infty} \geqslant\ & t\|\mathcal{R}(\omega_0)\|_{L^\infty} - \|\omega_0\|_{L^\infty} - ct^2\|\omega_0\|_{B^{\frac12}_{4,1}} \\
& - ct^2\|\nabla u\|_{L^\infty_t L^\infty_x} e^{ct\|\nabla u\|_{L^\infty_t L^\infty_x}}\|\omega\|_{L^\infty_t B^{\frac12}_{4,1}} \\
& - c_0(1+t\|\nabla u\|_{L^\infty_t L^\infty_x})\Big(1+r(t)e^{(c_0+c)r(t)} + t\|\omega\|_{L^\infty_t B^{\frac12}_{4,1}}\Big).
\end{aligned} \tag{6.1}$$

Now we consider the solutions of (1.2) corresponding to a special sequence of initial data. We recall the special sequence f_N constructed in Proposition 1.5 and define the initial data $\{(u_0^N, b_0^N)\}_{N=1}^{\infty}$ by

$$\omega_0^N = \frac{f_N}{N}, \quad u_0^N = \nabla^{\perp}\Delta^{-1}\omega_0^N, \quad b_0^N = b_0,$$

where $b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1}$ and is sufficiently small. According to Proposition 1.5,

$$\|u_0^N\|_{H^1} \le \frac{c}{N}, \quad \|\omega_0^N\|_{L^\infty} \le \frac{c}{N}, \quad \|u_0^N\|_{B^{\frac{3}{2}}_{4,1}} \le c.$$

The local well-posedness result in Proposition 1.3 asserts that the corresponding local solution (u^N, b^N) satisfies, for $t > 0$ sufficiently small,

$$\|u^N(t)\|_{B^{\frac{3}{2}}_{4,1}} \leqslant \frac{c(\|u_0^N\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)}{1 - c(\|u_0^N\|_{B^{\frac{3}{2}}_{4,1}} + c_0 + 1)t} \le \frac{c}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1.$$

Invoking the embedding inequalities $\|\nabla u\|_{L^\infty} \leqslant c\|\omega\|_{B^{\frac{1}{2}}_{4,1}} \leqslant c\|u\|_{B^{\frac{3}{2}}_{4,1}}$, we obtain

$$\|\nabla u^N\|_{L^\infty} \le \frac{c}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1, \qquad \|\omega^N\|_{B^{\frac{1}{2}}_{4,1}} \le \frac{c}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1.$$

It then follows from (6.1) that ω^N satisfies

$$\begin{aligned}
\|\omega^N\|_{L^\infty} \geqslant{}& \frac{t}{N}\|\mathcal{R}(f_N)\|_{L^\infty} - \frac{1}{N}\|f_N\|_{L^\infty} - \frac{ct^2}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} \\
&- ct^2\left(\frac{1}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)e^{ct\left(\frac{1}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)}\left(\frac{1}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right) \\
&- c_0\left(1 + t(\frac{1}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1)\right) \\
&\quad \times \left(1 + r(t)e^{(c_0+c)r(t)} + t\left(\frac{1}{N}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)\right) \\
\geqslant{}& ct - c\frac{1}{N} - ct^2 - ct^2(c_0+1)e^{ct(c_0+1)}(c_0+1) \\
&- c_0(1 + t(c_0+1))(1 + r(t)e^{(c_0+c)r(t)} + t(c_0+1)),
\end{aligned}$$

where c_0 is a small constant depending on $\|u_0^N\|_{H^1}$, $\|\omega_0^N\|_{L^\infty}$, $\|b_0\|_{H^1 \cap B^{\frac{1}{2}}_{4,1}}$ and c is a constant independent of N. But $\|u_0^N\|_{H^1} \leqslant \frac{c}{N}$, $\|\omega_0^N\|_{L^\infty} \leqslant \frac{c}{N}$, so c_0 can be chosen to depend only on $\|b_0\|_{H^1 \cap B^{\frac{1}{2}}_{4,1}}$. Therefore, for t sufficiently small and N sufficiently large,

$$\|\omega^N\|_{L^\infty} \geqslant (c - c_0)t - ct^2 - c_0.$$

When $t > 0$ is sufficiently small, we have

$$\|\omega^N\|_{L^\infty} \geqslant \alpha,$$

where α is a constant independent of N. This completes the proof of Theorem 6.1. ■

By modifying the proof of Theorem 6.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. We choose the initial data slightly differently. We take

$$\omega_0^N = \frac{f_N}{\sqrt{N}}, \qquad b_0^N = b_0, \quad N = 1, 2, \cdots,$$

where $b_0 \in H^1 \cap B^{\frac{1}{2}}_{4,1}$ is taken to be sufficiently small. Let (u^N, b^N) be the corresponding local solution given by Proposition 1.3, and let ω^N be the corresponding vorticity. Due to the change in the choice of ω_0^N, the corresponding estimate of ω^N also changes,

$$\begin{aligned}
\|\omega^N\|_{L^\infty} \geqslant\ & \frac{t}{\sqrt{N}}\|\mathcal{R}(f_N)\|_{L^\infty} - \frac{1}{\sqrt{N}}\|f_N\|_{L^\infty} - \frac{ct^2}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} \\
& - ct^2\left(\frac{1}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right) e^{ct\left(\frac{1}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)} \\
& \times \left(\frac{1}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right) \\
& - c_0\left(1 + t\left(\frac{1}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)\right) \\
& \times \left(1 + r(t)e^{(c_0+c)r(t)} + t\left(\frac{1}{\sqrt{N}}\|f_N\|_{B^{\frac{1}{2}}_{4,1}} + c_0 + 1\right)\right) \\
\geqslant\ & c\sqrt{N}t - c\frac{1}{\sqrt{N}} - ct^2\sqrt{N} - ct^2\sqrt{N}e^{c\sqrt{N}t}\sqrt{N} \\
& - c_0(1 + \sqrt{N}t)(1 + r(t)e^{(c_0+c)r(t)} + \sqrt{N}t),
\end{aligned}$$

where c_0 is a small constant depending on $\|u_0^N\|_{H^1}$, $\|\omega_0^N\|_{L^\infty}$, $\|b_0\|_{H^1\cap B^{\frac{1}{2}}_{4,1}}$ and c is a constant independent of N. But $\|u_0^N\|_{H^1} \leqslant \frac{c}{\sqrt{N}}$, $\|\omega_0^N\|_{L^\infty} \leqslant \frac{c}{\sqrt{N}}$, so c_0 can be chosen to depend only on $\|b_0\|_{H^1\cap B^{\frac{1}{2}}_{4,1}}$. For small t and large N enough, we obtain

$$\|\omega^N\|_{L^\infty} \geqslant (c - c_0)\sqrt{N}t - ct^2 N - c_0.$$

Choosing $t \leqslant \frac{C_5}{\sqrt{N}}$ (C_5 is a fixed small constant), we have

$$\|\omega^N\|_{L^\infty} \geqslant \alpha,$$

where α is a constant independent of N. This completes the proof of Theorem 1.1. ■

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