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Oldroyd-B Model with High Weissenberg Number and Fractional Velocity Dissipation

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Abstract

This paper focuses on a high Weissenberg number Oldroyd-B model of complex fluids with fractional frequency velocity dissipation. Mathematically the fluid velocity u satisfies the Navier–Stokes equations with fractional dissipation $(-\Delta)^{\alpha}u$ while the equation of the non-Newtonian tensor τ involves no diffusion or damping mechanism. The aim here is to solve the small-data global well-posedness and stability problem with the least amount of dissipation and minimal regularity requirement. We are able to establish the desired well-posedness and stability result in a hybrid homogeneous Besov setting for any fractional power in the range $1/2 \le \alpha \le 1$. To deal with the difficulties due to the weak velocity dissipation and the lack of diffusion or damping in the τ-equation, we exploit the coupling and interaction of this Oldroyd-B model to reveal the hidden wave structure and make extensive use of the associated smoothing and stabilizing effect.

Keywords Incompressible Oldroyd-B model · Global solution · Hybrid Besov space

Mathematics Subject Classification 35B35 · 76A05 · 76A10 · 76D03

1 Introduction

We consider the d-dimensional incompressible Oldroyd-B model of the non-Newtonianflows

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$$\begin{cases} u_t + u\cdot\nabla u + \nu\Lambda^{2\alpha}u + \nabla p = \mu_1\nabla\cdot\tau, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ \tau_t + u\cdot\nabla\tau + a\tau + Q(\tau,\nabla u) = \mu_2 D(u),\\ \nabla\cdot u = 0,\\ u(0,x) = u_0(x), \quad \tau(0,x) = \tau_0(x), \end{cases} \tag{1.1}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$, $0 \leqslant \alpha \leqslant 1$, and ν, μ_1, a, and μ_2 are non-negative constants. The fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\,\widehat{f}(\xi).$$

Trivially, when $\alpha = 1$, (1.1) reduces to the Oldroyd-B model with the standard Laplacian dissipation. Here $u(t,x)$ denotes the velocity, $p(t,x)$ the pressure, and $\tau(t,x)$ denotes a d-by-d symmetric matrix representing the non-Newtonian part of the stress tensor. The term $D(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big)$ is the deformation tensor and the term Q is taken to be

$$Q(\tau,\nabla u) = \tau W(u) - W(u)\tau - b\big(D(u)\tau + \tau D(u)\big),$$

where $b \in [-1, 1]$ is a real parameter and $W(u) = \frac{1}{2}\big(\nabla u - (\nabla u)^\top\big)$ is the vorticity tensor. The Oldroyd-B model is a prototypical model for viscoelastic fluids such as a solvent with particles suspended in it. The fractional frequency-dependent dissipation is physically relevant for the dynamics of anomalous solvent. More background information on the Oldroyd-B model can be found in [3, 14, 33].

The goal of this paper is to establish the sharpest possible small-data global well-posedness in the following two senses. The first is to allow the Oldroyd-B system to have the least amount of dissipation. The second is to obtain a unique solution in functional settings with the lowest possible regularities. Here we seek solutions in critical Besov spaces. To serve this purpose, we consider the following Oldroyd-B model without damping mechanism, namely $a = 0$,

$$\begin{cases} u_t + u\cdot\nabla u + \nu\Lambda^{2\alpha}u + \nabla p = \mu_1\nabla\cdot\tau, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ \tau_t + u\cdot\nabla\tau + Q(\tau,\nabla u) = \mu_2 D(u),\\ \nabla\cdot u = 0,\\ u(0,x) = u_0(x), \quad \tau(0,x) = \tau_0(x). \end{cases} \tag{1.2}$$

The equation of τ is now inviscid and without damping. Physically a corresponds to the reciprocal of the Weissenberg number. In the case of high Weissenberg number, $a \approx 0$ and the damping term $a\tau$ can be ignored. Therefore, (1.2) is a model for the case of high Weissenberg number. To reduce the regularity requirements on the initial data, we choose the critical homogeneous Besov spaces. Strictly speaking, there is no exact scaling invariance for the Oldroyd-B model (1.2). However, neglecting the coupling term $\nabla\cdot\tau$, if $(u(t,x), \tau(t,x), p(t,x))$ satisfies (1.2), then for any $\lambda > 0$,

$$(u_\lambda(t,x), \tau_\lambda(t,x), p_\lambda(t,x)) \triangleq (\lambda^{2\alpha-1}u(\lambda^{2\alpha}t,\lambda x), \tau(\lambda^{2\alpha}t,\lambda x), \lambda^{4\alpha-2}p(\lambda^2 t,\lambda x))$$

also satisfies (1.2). Following the studies on the incompressible Navier–Stokes equations (see, e.g., [1, 4–6, 22]) and the settings for the Oldroyd-B model (see, e.g., [7, 8, 35, 40]), we take the functional setting for (u, τ) of (1.2) to be the following critical homogeneous Besov spaces

$$u \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}, \quad \tau \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}.$$

The definition of homogeneous Besov spaces can be found in Sect. 2.

We are able to establish the following small-data global well-posedness and stability result on (1.2) for any $\frac{1}{2} \leq \alpha \leq 1$. More precisely, we prove the following theorem.

Theorem 1.1 *Let $d \geqslant 2$, $s \geqslant 0$ and $\nu, \mu_1, \mu_2 > 0$. Assume either $\frac{1}{2} < \alpha \leq 1$ or $\alpha = \frac{1}{2}$ and $\nu^2 \geqslant C\mu_1\mu_2$ for some suitable constant $C > 0$. Then there exists a small constant $\varepsilon > 0$ such that if $\tau_0 \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}$, $u_0 \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}$ with $\nabla \cdot u_0 = 0$, and*

$$\|u_0\|_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}} \leq \varepsilon,$$

then the system (1.2) has a unique global solution (u, τ) satisfying

$$u \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1});$$
$$\tau \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}), \quad \mathbb{P}\nabla \cdot \tau \in L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1} + \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}).$$

Here $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the Leray projection onto divergence-free vector fields. Furthermore, if $\tau_0 \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}+s}_{2,1}$, $u_0 \in \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}+1+s-2\alpha}_{2,1}$, and their norms are smaller than $\varepsilon = \varepsilon(s)$, then we have

$$u \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}+1+s-2\alpha}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1} \cap \dot{B}^{\frac{d}{2}+1+s}_{2,1});$$
$$\tau \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1} \cap \dot{B}^{\frac{d}{2}+s}_{2,1}), \quad \mathbb{P}\nabla \cdot \tau \in L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1+s-2\alpha}_{2,1}).$$

Remark 1.2 Thanks to (3.16), if there exists a constant C such that $\nu^2 \geqslant C\mu_1\mu_2$, then Theorem 1.1 holds when $\alpha = \frac{1}{2}$.

The requirement on the index α appears to be sharp. When $\alpha < \frac{1}{2}$, some of the nonlinear terms such as $Q(\tau, \nabla u)$ can no longer be suitably bounded even when we resort to the enhanced dissipation (explained later). Therefore, the dissipation requirement imposed here is minimal.

We briefly describe several closely related recent work and explain the differences between those results and Theorem 1.1. Zhu [39] studied the solutions of the 3D Oldroyd-B model without damping and with full Laplacian dissipation, namely (1.2) with $\alpha = 1$, and established the small-data global well-posedness in the Sobolev setting $H^3(\mathbb{R}^3)$. When $\alpha = 1$, Chen–Hao [8] were able to construct global solutions in critical Besov spaces for small initial data. Theorem 1.1 here allows α to be in the range $\frac{1}{2} \le \alpha \le 1$. In order to handle this reduced dissipation, many of the estimates are much more delicate and many of the tools such as Proposition 2.9 have to be employed.

The Oldroyd-B models have been intensively studied due to their physical applications and mathematical significance. Besides the papers already mentioned above, many other important results on the well-posedness problem and related issues are also available for various Oldroyd-B models. We shall not describe these results and interested readers may want to consult the references (see, e.g., [2, 7, 9–14, 17–21, 23–29, 31, 32, 34–38, 40]).

We explain the main ideas in the proof of Theorem 1.1. Due to the lack of dissipation or damping in the equation of τ, direct energy estimates will not work. We explore potential enhanced dissipation in the system by taking advantage of its structure. Inspired by the papers of Constantin–Wu–Zhao–Zhu [14] and Wu–Zhao [35], we exploit the coupling and interaction between u and τ. [14] and [35] studied the small-data global well-posedness of the following generalized Oldroyd-B model

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu_1 \nabla \cdot \tau, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \tau_t + u \cdot \nabla \tau + \nu \Lambda^{2\alpha} \tau + Q(\tau, \nabla u) = \mu_2 D(u), \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \quad \tau(0, x) = \tau_0(x). \end{cases} \tag{1.3}$$

These papers observed that (1.3) has a wave structure. More precisely, if we apply the Leray projection $\mathbb{P}$ to eliminate the pressure term and consider the linearized system

$$\begin{cases} u_t = \mu_1 \Lambda(\Lambda^{-1}\mathbb{P}\nabla \cdot \tau), \\ (\Lambda^{-1}\mathbb{P}\nabla \cdot \tau)_t = \nu \Lambda^{2\alpha}(\Lambda^{-1}\mathbb{P}\nabla \cdot \tau) + \frac{\mu_2}{2}\Lambda u. \end{cases}$$

Differentiating in t and making suitable substitutions lead to the decoupled wave equations

$$\begin{cases} u_{tt} + \nu(-\Delta)^\alpha u_t - \frac{1}{2}\mu_1\mu_2 \Delta u = 0, \\ (\mathbb{P}\nabla \cdot \tau)_{tt} + \nu(-\Delta)^\alpha (\mathbb{P}\nabla \cdot \tau)_t - \frac{1}{2}\mu_1\mu_2 \Delta(\mathbb{P}\nabla \cdot \tau) = 0. \end{cases} \tag{1.4}$$

Going through a similar process with (1.2) yields exactly the same structure as (1.4). This suggests that similar well-posedness results may be shown for both (1.2) and (1.3). The wave structure in (1.4) reveals the hidden smoothing and stabilizing effect on the fluid velocity. This effect is incorporated in the proof of Theorem 1.1. As aforementioned, $\alpha \ge \frac{1}{2}$ appears to be sharp. The enhanced dissipation in the wave structure does not help reduce the requirement on α. Even when the equation of u involves the dissipation $(-\Delta)^\alpha$, the control of terms in $Q(\tau, \nabla u)$ requires $\alpha \ge \frac{1}{2}$.

The bootstrapping argument is used here to prove the small-data global existence part of Theorem 1.1. A crucial step is to construct a suitable energy functional. To do so, we first analyze the behavior of the linearized system. The associated linearized system of (1.2) is given by

$$\begin{cases} u_t + \nu\Lambda^{2\alpha}u - \mu_1\Lambda(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) = 0, \\ (\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)_t + \frac{\mu_2}{2}\Lambda u = 0. \end{cases} \tag{1.5}$$

For notational convenience, we set $\Lambda^{-1}\mathbb{P}\nabla\cdot\tau = \tilde{\tau}$. Then

$$(u(t), \tilde{\tau}(t))^\top = e^{A(\Lambda)t}(u(0), \tilde{\tau}(0))^\top,$$

where

$$A(\Lambda) = \begin{pmatrix} -\nu\Lambda^{2\alpha} & \mu_1\Lambda \\ -\frac{\mu_2}{2}\Lambda & 0 \end{pmatrix}. \tag{1.6}$$

The features of the solution to system (1.5) are very different in low and high frequencies by simply examining the eigenvalues of $A(\xi)$ defined in (1.6),

$$\lambda_+ = -\frac{\nu|\xi|^{2\alpha} + \sqrt{\nu^2|\xi|^{4\alpha} - 2\mu_1\mu_2|\xi|^2}}{2},$$
$$\lambda_- = -\frac{\nu|\xi|^{2\alpha} - \sqrt{\nu^2|\xi|^{4\alpha} - 2\mu_1\mu_2|\xi|^2}}{2}.$$

(1.5) can be diagonalized by the change of functions

$$\hat{v}^+(\xi) = \lambda_+\hat{u} - \mu_1|\xi|\hat{\tilde{\tau}},$$
$$\hat{v}^-(\xi) = \lambda_+\hat{\tilde{\tau}} - \frac{\mu_2}{2}|\xi|\hat{u}.$$

For α in the range $\frac{1}{2} \le \alpha < 1$, as λ_+ and $\lambda_- \sim -\frac{\nu}{2}|\xi|^{2\alpha}$ for $\xi \to 0$, v^+ and v^- both behave like the heat kernel operator $e^{-\frac{\nu}{2}t\Lambda^{2\alpha}}$ for the low frequencies. Moreover, noticing that the eigenvalue λ_+ roughly corresponds to $\tilde{\tau}$ and λ_- to u, we deduce that u and $\tilde{\tau}$ have parabolic behavior like $e^{-\frac{\nu}{2}t\Lambda^{2\alpha}}$ for the low frequencies. Similarly, as $\lambda_+ \sim -\nu|\xi|^{2\alpha}$ and $\lambda_- \sim -\frac{\mu_1\mu_2}{2\nu}|\xi|^{2-2\alpha}$ for $\xi \to \infty$, v^+ and v^- behave like the heat kernel operators $e^{-\nu t\Lambda^{2\alpha}}$ and $e^{-\frac{\mu_1\mu_2}{2\nu}t\Lambda^{2-2\alpha}}$, respectively, for the high frequencies. Because the eigenvalue λ_+ roughly corresponds to u and λ_- to $\tilde{\tau}$, u, and $\tilde{\tau}$ have the parabolic smoothing effect that behaves like heat kernel operator $e^{-\nu t\Lambda^{2\alpha}}$ and $e^{-\frac{\mu_1\mu_2}{2\nu}t\Lambda^{2-2\alpha}}$ for the high frequencies, which is different from the case $\alpha = 1$. In the cases when $\alpha = 1$, for high frequencies, u has a smoothing effect like $e^{\nu t\Delta}$, and $\tilde{\tau}$ only has a damping effect.

Due to the different behavior in the two cases $\frac{1}{2} \le \alpha < 1$ and $\alpha = 1$, some of the non-linear terms are dealt with differently. When $\frac{1}{2} \le \alpha < 1$, u has a weaker parabolic smoothing effect, although $\tilde{\tau}$ has a stronger one. This will cause difficulties.

For example, to estimate $Q(\tau, \nabla u)$, we need higher regularity for u. However, u only behaves like the heat operator $e^{-\nu t\Lambda^{2\alpha}}$.

According to the analysis above, the low and the high frequencies of u and $\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$ enjoy different parabolic smoothing properties. This suggests that we employ the hybrid Besov spaces, which allow different regularity indices for different frequencies. We now define the components of the total energy functional. First we define the functionals for u and $\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$ by taking into account of the enhanced dissipation revealed in the wave structure (1.4). The low frequency part is given by

$$
\begin{aligned}
E_0^l(t) \triangleq\ & \sup_t \|u\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} \\
& + \int_0^t \|u\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t'
\end{aligned} \tag{1.7}
$$

and the high frequency part by

$$
\begin{aligned}
E_0^h(t) \triangleq\ & \sup_t \|u\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\
& + \int_0^t \|u\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} \mathrm{d}t'.
\end{aligned} \tag{1.8}
$$

Here $\|\ .\ \|^l$ and $\|\ .\ \|^h$ denote the low part and the high part of corresponding hybrid Besov norm (see Sect. 2.3). We also need to define the following energy functionals in terms of u and τ

$$
\begin{aligned}
E^l(t) \triangleq\ & \sup_t \|u\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t \|\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} \\
& + \int_0^t \|u\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t',
\end{aligned} \tag{1.9}
$$

and

$$
\begin{aligned}
E^h(t) \triangleq\ & \sup_t \|u\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t \|\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}}} \\
& + \int_0^t \|u\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} \mathrm{d}t'.
\end{aligned} \tag{1.10}
$$

The total energy functional $E(t)$ is a summation of E_0^l, E_0^h, E^l, and E^h defined in (1.7), (1.8), (1.9), and (1.10), respectively. Since E_0^l and E_0^h can be majorized by E^l and E^h, respectively, it suffices to include E^l and E^h. Therefore, we set

$$
E(t) = E^l + E^h. \tag{1.11}
$$

To prove the existence part of Theorem 1.1, we verify the *a priori* estimate

$$E(t) \leq C_1 E_0 + C_2 E^2(t), \tag{1.12}$$

where

$$E_0 = \|u_0\|_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}. \tag{1.13}$$

Once (1.12) is established, a simple bootstrapping argument implies that, if

$$E_0 \leq \varepsilon$$

for sufficiently small $\varepsilon = \varepsilon(C_1, C_2) > 0$, then, for any $t > 0$,

$$E(t) \leq C\varepsilon. \tag{1.14}$$

for a constant $C > 0$. In particular, this upper bound yields the desired global bound for the hybrid norms of (u, τ) and the global stability.

Proving (1.12) is a long process and makes use of the enhanced dissipation revealed by the wave structure. The enhanced dissipation allows us to bound the time integral parts of $E(t)$ suitably. The enhanced dissipation is realized by constructing suitable Lyapunov function of u and $\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$. In addition, we also estimate E^l and E^h through (1.2) and then make a combination with those of E_0^l and E_0^h. This process is extremely tedious and we leave more technical details in Sect. 3.

In the case when $\alpha = 1$, the uniqueness can be directly shown by performing the L^2-estimate on the system satisfies by $(\delta u, \delta\tau)$, the difference between two solutions (u_1, τ_1) and (u_2, τ_2).

However, when $\frac{1}{2} \leq \alpha < 1$, the uniqueness can no longer be achieved by the L^2-estimate. The reason is due to the term $Q(\tau_1, \nabla\delta u)$, which requires the control of $\|\nabla\delta u\|_{L^2}$. But the dissipation in the velocity equation can only absorb $\|\Lambda^\alpha\delta u\|_{L^2}$ for $\frac{1}{2} \leq \alpha < 1$. Therefore, direct energy estimate on the difference in L^2 would not work. The new idea here is to make use of the enhanced dissipation as in the proof of the existence part. This process is also very long and the details are left for Sect. 4.

To prove the higher regularity part of Theorem 1.1, it suffices to bound the high frequency part. For this purpose, we define the higher order energy for high frequencies

$$\begin{aligned} E_0^{'h}(t) \triangleq\ & \sup_t \|u\|^h_{\dot{B}^{\frac{d}{2}+2-2\alpha}_{2,1}} + \sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} \\ & + \int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+2}_{2,1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}+2-2\alpha}_{2,1}} \mathrm{d}t', \end{aligned} \tag{1.15}$$

and

$$E'^h(t) \triangleq \sup_t \|u\|^h_{\dot{B}^{\frac{d}{2}+2-2\alpha}_{2,1}} + \sup_t \|\tau\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+2}_{2,1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}+2-2\alpha}_{2,1}} \mathrm{d}t'. \tag{1.16}$$

We then show that, via the enhanced dissipation.

$$E_0'^h(t) + E'^h(t) \le C_1 E_0' + C_2((E'^h_0(t) + E'^h(t))E(t) + E^2(t)). \tag{1.17}$$

According to (1.14), we can take the initial norm to be sufficiently small such that

$$C_2\, E(t) \le \frac{1}{2}.$$

Then (1.17) yields the desired higher regularity. The proof of (1.17) is presented in Sect. 5.

The rest of this paper is structured as follows. Section 2 recalls some facts about the homogeneous Littlewood–Paley theory, Besov spaces and hybrid Besov spaces. Section 3 proves the *a priori* bound in (1.12). Section 4 proves the global existence and uniqueness of the solution to the system (1.2). Section 5 focuses on the higher regularity part of Theorem 1.1.

2 Littlewood–Paley Theory and Besov Spaces

2.1 Littlewood–Paley Decomposition

We recall the dyadic partition of unity and the definition of the homogeneous Littlewood–Paley decomposition (see, e.g., [1, 30]). Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a radial function supported in $\mathcal{C} = \{\xi \in \mathbb{R}^d,\ \frac{5}{6} \le |\xi| \le \frac{12}{5}\}$ and satisfy

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$

Setting $h(x) = \mathcal{F}^{-1}\big(\varphi(\xi)\big)$, we define the dyadic blocks as follows, for $j \in \mathbb{Z}$,

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)\mathrm{d}y,$$
$$\dot{S}_j u = \sum_{j' \leqslant j-1} \dot{\Delta}_{j'} u.$$

Definition 2.1 We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{j\to-\infty} \dot{S}_j u = 0 \ \text{ in } \ \mathcal{S}'.$$

Then the so-called homogeneous Littlewood–Paley decomposition is defined as

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u, \qquad \text{for} \quad u \in \mathcal{S}'_h.$$

With our choice of φ, we have

$$\dot{\Delta}_j \dot{\Delta}_k u = 0 \ \text{ if } \ |j-k| \geq 2, \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k u) = 0 \ \text{ if } \ |j-k| \geq 4.$$

The following lemma provides Bernstein-type inequalities for fractional derivatives. Throughout the rest of this paper, C stands for a constant that may change from line to line. We use $\widehat{u}$ and $\mathcal{F}(u)$ to denote the Fourier transform of u and use $c_j > 0$ to denote a sequence satisfying $\sum_{j\in\mathbb{Z}} c_j \leq 1$.

Lemma 2.1 *Let* $\beta \geqslant 0$. *Let* $1 \leq p \leq q \leq +\infty$.

(1) *If* f *satisfies*

$$\text{supp}\hat{f} \subseteq \{|\xi| \leq l2^j\},$$

then we have

$$\|\Lambda^\beta f\|_{L^q} \leq C2^{j|\beta|+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

(2) *If* f *satisfies*

$$\text{supp}\hat{f} \subseteq \{l_1 2^j \leq |\xi| \leq l_2 2^j\},$$

then we have

$$C_1 2^{\beta j}\|f\|_{L^q} \leqslant \|\Lambda^\beta f\|_{L^q} \leq C_2 2^{\beta j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

Here C, C_1, C_2 *are constants independent of* f *and* j.

2.2 Homogeneous Besov Space

Definition 2.2 Let $u \in \mathcal{S}'(\mathbb{R}^d)$, $s \in \mathbb{R}$, and $1 \leq p, r \leq \infty$. Let

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j u\|_{L^p}\}_j\|_{l^r}.$$

We then define the space $\dot{B}^s_{p,r} \triangleq \{u \in \mathcal{S}'_h,\ \|u\|_{\dot{B}^s_{p,r}} < \infty\}$.

Remark 2.2 The definition of the space $\dot{B}^s_{p,r}$ does not depend on the choice of φ.

2.3 Hybrid Besov Spaces

We now introduce the hybrid Besov spaces.

Definition 2.3 [16] Let $s, t \in \mathbb{R}$ and set

$$\|u\|_{\dot{B}^{s,t}} = \sum_{j\leq 0} 2^{js}\|\dot{\Delta}_j u\|_{L^2} + \sum_{j>0} 2^{jt}\|\dot{\Delta}_j u\|_{L^2}.$$

We then define the space $\dot{B}^{s,t} \triangleq \{u \in \mathcal{S}'_h,\ \|u\|_{\dot{B}^{s,t}} < \infty\}$.

We will use the definitions:

$$\|u\|^{l}_{\dot{B}^{s}_{2,1}} \triangleq \sum_{j\leq 0} 2^{js}\|\dot{\Delta}_j u\|_{L^2}; \quad \|u\|^{h}_{\dot{B}^{s}_{2,1}} \triangleq \sum_{j>0} 2^{js}\|\dot{\Delta}_j u\|_{L^2}.$$

For $r \in [1, \infty]$, the norm of the space $L^r_T(\dot{B}^{s,t})$ is defined by

$$\|u\|_{L^r_T(\dot{B}^{s,t})} = \left(\int_0^T \|u\|^r_{\dot{B}^{s,t}}\,\mathrm{d}t\right)^{\frac{1}{r}}$$

with the usual change if $r = \infty$. Furthermore, the norm of the time-space Besov space $\widetilde{L}^r_T(\dot{B}^{s,t})$ is defined by

$$\|u\|_{\widetilde{L}^r_T(\dot{B}^{s,t})} = \sum_{j\leq 0} 2^{js}\|\dot{\Delta}_j u\|_{L^r_T L^2} + \sum_{j>0} 2^{js}\|\dot{\Delta}_j u\|_{L^r_T L^2}.$$

Remark 2.3 (i) It is easy to check that $\widetilde{L}^1_T(\dot{B}^{s,t}) = L^1_T(\dot{B}^{s,t})$ and $\widetilde{L}^r_T(\dot{B}^{s,t}) \subseteq L^r_T(\dot{B}^{s,t})$ for $r \geqslant 1$ by the Minkowski inequality.
(ii) Proposition 2.6 below remains valid for products in the time-space Besov spaces.

The following proposition is a direct consequence of the definition of Besov spaces (see [16] for more details).

Proposition 2.4 (i) *We have* $\dot{B}^{s,s} = \dot{B}^{s}_{2,1}$.
(ii) *If* $s \leq t$ *then* $\dot{B}^{s,t} = \dot{B}^{s}_{2,1} \cap \dot{B}^{t}_{2,1}$. *Otherwise,* $\dot{B}^{s,t} = \dot{B}^{s}_{2,1} + \dot{B}^{t}_{2,1}$.
(iii) *If* $s_1 \leq s_2$ *and* $t_1 \geq t_2$, *then* $\dot{B}^{s_1,t_1} \hookrightarrow \dot{B}^{s_2,t_2}$.

2.4 Paraproducts and Product Estimates in Hybrid Besov Spaces

We continue to review more information on Besov spaces and hybrid Besov spaces. Especially product and triple product estimates in these spaces are provided. We start by recalling the paraproduct decomposition

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where the homogeneous paraproduct of v by u is given by

$$\dot{T}_u v \triangleq \sum_q \dot{S}_{q-1} u \dot{\Delta}_q v,$$

and the homogeneous remainder of u and v by

$$\dot{R}(u, v) \triangleq \sum_q \dot{\Delta}_q u \tilde{\dot{\Delta}}_q v, \quad \text{and} \quad \tilde{\dot{\Delta}}_q = \dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1}.$$

One useful property of the homogeneous Besov spaces is the Besov embedding.

Proposition 2.5 *Assume $s, s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, r, r_1, r_2 \le +\infty$. Then we have the following properties:*

(i) *If $p_1 \le p_2$, $r_1 \le r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-\frac{d}{p_1}+\frac{d}{p_2}}_{p_2,r_2}$.*
(ii) *If $s_1 \ne s_2$ and $\theta \in (0, 1)$, then*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}.$$

(iii) *$\dot{H}^s \approx \dot{B}^s_{2,2}$ and*

$$\frac{1}{C^{|s|+1}} \|u\|_{\dot{B}^s_{2,2}} \le \|u\|_{\dot{H}^s} \le C^{|s|+1} \|u\|_{\dot{B}^s_{2,2}}.$$

Proposition 2.6 *If $s > 0$, then $\dot{B}^s_{2,1} \cap L^\infty$ (especially $\dot{B}^{\frac{d}{2}}_{2,1}$) is an algebra. That is, if $u \in L^\infty \cap \dot{B}^s_{2,1}$ and $v \in L^\infty \cap \dot{B}^s_{2,1}$ with $s > 0$, then $uv \in L^\infty \cap \dot{B}^s_{2,1}$ and*

$$\|uv\|_{\dot{B}^s_{2,1}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{2,1}} + \|v\|_{L^\infty} \|u\|_{\dot{B}^s_{2,1}}.$$

If $u \in \dot{B}^{s_1}_{2,1}$ and $v \in \dot{B}^{s_2}_{2,1}$ with $s_1, s_2 \le \frac{d}{2}$ and $s_1 + s_2 > 0$, then $uv \in \dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}$ and

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,1}}.$$

The two propositions above can be found in [1]. The following estimates in hybrid Besov spaces are very useful and their proofs can be found in [16, 35].

Proposition 2.7 *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ and $s_1 \le \frac{d}{2}$ and $s_2 \le \frac{d}{2}$. Then the following estimate holds*

$$\|\dot{T}_u v\|_{\dot{B}^{s_1+t_1-\frac{d}{2}, s_2+t_2-\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1,s_2}} \|v\|_{\dot{B}^{t_1,t_2}}.$$

If $\min(s_1 + t_1, s_2 + t_2) > 0$, *then*

$$\|\dot{R}(u, v)\|_{\dot{B}^{s_1+t_1-\frac{d}{2}, s_2+t_2-\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1,s_2}} \|v\|_{\dot{B}^{t_1,t_2}}.$$

If $u \in L^\infty$,

$$\|\dot{T}_u v\|_{\dot{B}^{t_1,t_2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{t_1,t_2}},$$

and, if $\min(t_1, t_2) > 0$, *then*

$$\|\dot{R}(u, v)\|_{\dot{B}^{t_1,t_2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{t_1,t_2}}.$$

Remark 2.8 When $d \geq 2$, we have $\|uv\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|v\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}$ $(\frac{1}{2} \leq \beta \leq 1)$.

2.5 Triple Product Estimates in Hybrid Besov Spaces

The following triple product estimates will be used frequently and the proof can be found in [16, 35].

Proposition 2.9 *Let* u *be a vector with* $\nabla \cdot u = 0$ *and* F *be a homogeneous smooth function of degree* m. *Suppose that* $-1-\frac{d}{2} < s_1, t_1, s_2, t_2 \leq 1+\frac{d}{2}$ *and* $r_1, r_2 > -1-\frac{d}{2}$. *The following estimates hold*

$$\begin{aligned}
&\left|(F(D)\dot{\Delta}_p(u\cdot\nabla v), F(D)\dot{\Delta}_p v)\right| \\
&\quad \lesssim 2^{(m-s_1)p} c_p \|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1}_{2,1}}\|\dot{\Delta}_p F(D)v\|_{L^2}, \\
&\left|(F(D)\dot{\Delta}_p(u\cdot\nabla v), F(D)\dot{\Delta}_p v)\right| \\
&\quad \lesssim c_p 2^{pm} 2^{-p\psi^{s_1,s_2}(p)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}\|F(D)\dot{\Delta}_p v\|_{L^2}, \\
&\left|(F(D)\dot{\Delta}_p(u\cdot\nabla v), F(D)\dot{\Delta}_p v)\right| \\
&\quad \lesssim c_p 2^{p(m-r_2)}(\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{r_1,r_2}} + \|v\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|u\|_{\dot{B}^{r_1,r_2}}) \\
&\qquad \times\|F(D)\dot{\Delta}_p v\|_{L^2} \quad \text{for } p > 0, \\
&\left|(F(D)\dot{\Delta}_p(u\cdot\nabla v), \dot{\Delta}_p w) + (\dot{\Delta}_p(u\cdot\nabla w), F(D)\dot{\Delta}_p v)\right| \\
&\quad \lesssim c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}(2^{pm}2^{-p\psi^{s_1,s_2}(p)}\|v\|_{\dot{B}^{s_1,s_2}}\|\dot{\Delta}_p w\|_{L^2} \\
&\qquad +2^{-p\psi^{t_1,t_2}(p)}\|w\|_{\dot{B}^{t_1,t_2}}\|F(D)\dot{\Delta}_p v\|_{L^2}),
\end{aligned}$$

where the function $\psi^{\alpha,\beta}(p)$ *is defined as* $\psi^{\alpha,\beta}(p) = \alpha$ *if* $p \leq 0$, $\psi^{\alpha,\beta}(p) = \beta$, *if* $p > 0$, *and* $\sum_{p\in\mathbb{Z}} c_p \leq 1$.

We also need the following proposition, A proof can be found in [39].

Proposition 2.10 *For any smooth tensor* $[\tau^{i,j}]_{d\times d}$ *and* d *dimensional vector* u, *it always holds that*

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) = \mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau) + \mathbb{P}(\nabla u\cdot\nabla\tau) - \mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau),$$

where the ith component of $\nabla u \cdot \nabla \tau$ *is*

$$[\nabla u \cdot \nabla \tau]^i = \sum_j \partial_j u \cdot \nabla \tau^{i,j},$$

and also

$$[\nabla u \cdot \nabla \Delta^{-1} \nabla \cdot \nabla \cdot \tau]^i = \partial_i u \cdot \nabla \Delta^{-1} \nabla \cdot \nabla \cdot \tau.$$

3 A Priori Estimates

This section presents the proof of the key energy estimate (1.12). To achieve this, we overcome two main difficulties. The first is that there is no diffusion or damping term in τ. The wave structure described in the introduction will be used to deal with this difficulty. We construct Lyapunov functionals that contain the norms and suitable inner products. The second is that there is only fractional dissipation in the velocity equation rather than the standard Laplacian dissipation. This is dealt with by making full use of suitable combinations and the fractional Laplacian.

More precisely, we prove the estimates in the following proposition.

Proposition 3.1 *Assume that* (u, τ) *is a solution to the system* (1.2) *on* $[0, T)$. *Then, there exist two positive constants* C_1, C_2 *independent of* T *such that*

$$E(t) \le C_1 E_0 + C_2 E^2(t). \tag{3.1}$$

To prove Proposition 3.1, we first establish two important Lemmas. The first lemma provides an upper bound for $E_0^l(t)+E_0^h(t)$ while the second lemma is for $E^l(t)+E^h(t)$.

Lemma 3.2 *Let* (u, τ) *be a solution of* (1.2) *on* $[0, T)$. *Then there exist two constant* C_1, C_2 *independent of* T *such that*

$$\begin{aligned} E_0^l(t) + E_0^h(t) \le C_1 E_0 + C_2 \int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)} |G_j|/(\|\dot{\Delta}_j u\|_{L^2} \\ + \|\dot{\Delta}_j \Lambda^{-1} \mathbb{P} \nabla \cdot \tau\|_{L^2}) \mathrm{d}t' \\ + C_2 \int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)} \Big(|H_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\Lambda^{2\alpha-2} \dot{\Delta}_j \mathbb{P}\nabla \cdot \tau\|_{L^2}) \\ + |I_j|/\|\dot{\Delta}_j u\|_{L^2} \Big) \mathrm{d}t', \end{aligned}$$

where

$$
\begin{aligned}
G_j =& -\frac{\mu_2}{2}(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u) - \mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \\
&\cdot (u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)) \\
&+ K_1(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla\cdot\tau) \\
&+ K_1(\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j u), \\
H_j =& -\frac{\mu_2^2}{\nu}(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u) \\
&- \nu(\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla\cdot\tau) \\
&+ \frac{\mu_2}{2}(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla\cdot\tau) \\
&+ \frac{\mu_2}{2}(\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j u),
\end{aligned}
$$

and

$$
I_j = -(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u).
$$

Lemma 3.3 *Let (u,τ) be a solution of (1.2) on $[0,T)$. Then there exist two constants C_1, C_2 independent of T such that*

$$
\begin{aligned}
&E^l(t) + E^h(t) \le C_1 E_0 \\
&+C_2 \int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|G_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \\
&+|V_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \tau\|_{L^2})\Big)\mathrm{d}t' \\
&+C_2 \int_0^t \Big(\sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|H_j|/(\|\dot{\Delta}_j u\|_{L^2} \\
&+\|\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla\cdot\tau\|_{L^2}) + |I_j|/\|\dot{\Delta}_j u\|_{L^2}\Big) \\
&+\sum_{j>0} 2^{\frac{d}{2}j}|Y_j|/\|\dot{\Delta}_j \tau\|_{L^2}\Big)\mathrm{d}t',
\end{aligned}
$$

where $G_j, H_j, \tilde{I}_j$ are defined as in Lemma 3.2 and

$$
\begin{aligned}
V_j &= -\mu_2(\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j u) - \mu_1(\dot{\Delta}_j(u\cdot\nabla\tau), \dot{\Delta}_j\tau) - \mu_1(\dot{\Delta}_j Q(\tau,\nabla u), \dot{\Delta}_j\tau), \\
Y_j &= -(\dot{\Delta}_j(u\cdot\nabla\tau), \dot{\Delta}_j\tau) - (\dot{\Delta}_j Q(\tau,\nabla u), \dot{\Delta}_j\tau).
\end{aligned}
$$

3.1 Proof of Lemma 3.2

Proof of Lemma 3.2 Without loss of generality, we assume that

$$\|\dot{\Delta}_j u\|_{L^2},\ \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2},\ \|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}\neq 0.$$

Step 1: Estimate of $E_0^l(t)$.

Let $\widetilde{C}_0 = 2^{j_0}$ for some fixed integer j_0 to be chosen in Step 2. By Lemma 2.1, there exist two constants $\widetilde{C}_1$, $\widetilde{C}_2$ such that, for any function f,

$$\widetilde{C}_1 2^{\kappa j}\|\dot{\Delta}_j f\|_{L^p} \le \|\Lambda^{\kappa}\dot{\Delta}_j f\|_{L^p} \le \widetilde{C}_2 2^{\kappa j}\|\dot{\Delta}_j f\|_{L^p}. \tag{3.2}$$

Here $\kappa\in[0,m]$ for a large positive integer m, and $\widetilde{C}_1$ and $\widetilde{C}_1$ are constants depending on m. Applying the operators $\dot{\Delta}_j\mathbb{P}$ and $\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot$ to the first and the second equation of the system (1.2), respectively, we obtain

$$\begin{cases}(\dot{\Delta}_j u)_t + \nu\Lambda^{2\alpha}\dot{\Delta}_j u - \mu_1\Lambda\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau = -\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u),\\ (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)_t + \frac{\mu_2}{2}\Lambda\dot{\Delta}_j u = -\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)).\end{cases} \tag{3.3}$$

Taking the L^2-inner product of the first equation of (3.3) with $\dot{\Delta}_j u$, and of the second with $\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 + \nu\|\Lambda^{\alpha}\dot{\Delta}_j u\|_{L^2}^2 - \mu_1(\Lambda\dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau), \dot{\Delta}_j u)\\ &\quad = -(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\|_{L^2}^2 + \frac{\mu_2}{2}(\Lambda\dot{\Delta}_j u, \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\ &\quad = -(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)).\end{aligned} \tag{3.5}$$

Taking the L^2-inner product of the first equation of (3.3) with $\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau$, applying $\Lambda^{2\alpha-1}$ to the second equation, taking the L^2-inner product with $\dot{\Delta}_j u$, and then summing up both equations, we get

$$\begin{aligned}&\frac{d}{dt}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u) + \nu(\Lambda^{2\alpha}\dot{\Delta}_j u, \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau)\\ &\qquad - \mu_1\|\Lambda^{\alpha-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 + \frac{\mu_2}{2}\|\Lambda^{\alpha}\dot{\Delta}_j u\|_{L^2}^2\\ &\quad = -(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau) -\\ &\qquad (\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j u).\end{aligned} \tag{3.6}$$

For some constant $K_1 > 0$ to be determined later, $\frac{\mu_2}{2}$(3.4)+μ_1(3.5)-K_1(3.6) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-2K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau,\dot\Delta_j u)\Big)\\&\quad+(\frac{\mu_2\nu}{2}-\frac{\mu_2K_1}{2})\|\Lambda^\alpha\dot\Delta_j u\|_{L^2}^2+\mu_1K_1\|\Lambda^{\alpha-1}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\&\quad-\nu K_1(\Lambda^{2\alpha}\dot\Delta_j u,\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau)\\&\quad=G_j.\end{aligned}\tag{3.7}$$

Thanks to $\frac{1}{2}\le\alpha\le1$ and (3.2), we have, for $\epsilon_0,\epsilon_1>0$,

$$\begin{aligned}2K_1|(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau,\dot\Delta_j u)|&\le\frac{1}{\epsilon_0}\|\dot\Delta_j u\|_2^2+\epsilon_0\widetilde{C}_2^2\widetilde{C}_0^{2(2\alpha-1)}K_1^2\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2,\\ \nu K_1|(\Lambda^{2\alpha}\dot\Delta_j u,\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau)|&\le\frac{\nu\widetilde{C}_0^{2(2\alpha-1)}\widetilde{C}_2^2K_1}{2\epsilon_1}\|\Lambda^\alpha\dot\Delta_j u\|_{L^2}^2\\&\quad+\frac{\epsilon_1\nu K_1}{2}\|\Lambda^{\alpha-1}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2.\end{aligned}$$

Choosing $\epsilon_0=\frac{4}{\mu_2},\epsilon_1=\frac{\mu_1}{\nu}$, and K_1 small enough and substituting the second inequality above into (3.7), we derive

$$\begin{aligned}&\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-2K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau,\dot\Delta_j u)\\&\quad\approx\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-2K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau,\dot\Delta_j u)\Big)\\&\quad+(\frac{\mu_2\nu}{2}-\frac{\mu_2K_1}{2}-\frac{\nu^2\widetilde{C}_0^{2(2\alpha-1)}\widetilde{C}_2^2K_1}{2\mu_1})\widetilde{C}_1^2 2^{2\alpha j}\|\dot\Delta_j u\|_{L^2}^2\\&\quad+\frac{\mu_1\widetilde{C}_1^2K_1}{2}2^{2\alpha j}\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\&\quad\le|G_j|.\end{aligned}\tag{3.8}$$

by using (3.2). Choosing a small positive constant K_1 such that

$$(\frac{\mu_2\nu}{2}-\frac{\mu_2K_1}{2}-\frac{\nu^2\widetilde{C}_0^{2(2\alpha-1)}\widetilde{C}_2^2K_1}{2\mu_1})>0,\quad\Big(\mu_1-\frac{4\widetilde{C}_0^{2(2\alpha-1)}\widetilde{C}_2^2K_1^2}{\mu_2}\Big)>0,$$

and dividing both sides of (3.8) by $\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}$, we have

$$\begin{aligned}&\frac{d}{dt}\sqrt{\Big(\frac{\mu_2}{2}\|\dot{\Delta}_j u\|_{L^2}^2 + \mu_1\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - 2K_1(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u)\Big)}\\&\quad + 2^{2\alpha j}\Big(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big)\\&\le C_2|G_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}).\end{aligned}\tag{3.9}$$

Multiplying both sides of (3.9) by $2^{j(\frac{d}{2}+1-2\alpha)}$, summing over $j\le j_0$ (actually we can choose $j_0=0$, see Step 2), and performing a time integration, we have

$$\begin{aligned}E_0^l(t) &\le C_1(\|u_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}})\\&\quad + C_2\int_0^t \sum_{j\le j_0} 2^{j(\frac{d}{2}+1-2\alpha)}|G_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'.\end{aligned}\tag{3.10}$$

Step 2: Estimate of $E_0^h(t)$.

Applying $\Lambda^{2\alpha-1}$ to the second equation of (3.3) and taking the L^2-inner product with $\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 + \frac{\mu_2}{2}(\Lambda^{2\alpha}\dot{\Delta}_j u, \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau)\\&\quad = -(\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau).\end{aligned}\tag{3.11}$$

For some positive constant K_2 to be determined later, ν(3.11)-$\frac{\mu_2}{2}$(3.6)+K_2(3.4) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(K_2\|\dot{\Delta}_j u\|_{L^2}^2 + \nu\|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \mu_2(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u)\Big)\\&\quad + \big(\nu K_2 - \frac{\mu_2^2}{4}\big)\|\Lambda^\alpha\dot{\Delta}_j u\|_{L^2}^2 + \frac{\mu_1\mu_2}{2}\|\Lambda^{\alpha-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \mu_1K_2(\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u)\\&= H_j.\end{aligned}\tag{3.12}$$

For any $\epsilon_0, \epsilon_1 > 0$, we have

$$\begin{aligned}\mu_2|(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u)| &\le \frac{\epsilon_0}{2}\|\dot{\Delta}_j u\|_2^2 + \frac{\mu_2^2}{2\epsilon_0}\|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2,\\ \mu_1K_2|(\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau, \dot{\Delta}_j u)| &\le \frac{\epsilon_1\mu_1K_2}{2}\|\Lambda^{1-\alpha}\dot{\Delta}_j u\|_{L^2}^2 + \frac{\mu_1K_2}{2\epsilon_1}\|\Lambda^{\alpha-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2.\end{aligned}\tag{3.13}$$

Thanks to $j\ge j_0+1$ and $\frac{1}{2}\le\alpha\le 1$, we obtain

$$\frac{\epsilon_1\mu_1K_2}{2}\|\Lambda^{1-\alpha}\dot{\Delta}_j u\|_{L^2}^2 \leqslant \frac{\epsilon_1\mu_1K_2}{2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\alpha)}\|\Lambda^\alpha\dot{\Delta}_j u\|_{L^2}^2.\tag{3.14}$$

Combining (3.12), (3.13), and (3.14), and choosing

$$K_2 = \epsilon_0 = \frac{\mu_2^2}{\nu}, \quad \epsilon_1 = \frac{2K_2}{\mu_2},$$

we have

$$\begin{aligned} &K_2\|\dot\Delta_j u\|_{L^2}^2 + \nu\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \mu_2(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau, \dot\Delta_j u) \\ &\quad\approx \|\dot\Delta_j u\|_{L^2}^2 + \|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(K_2\|\dot\Delta_j u\|_{L^2}^2 + \nu\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \mu_2(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau, \dot\Delta_j u)\Big) \\ &\quad + \big(\frac{3}{4}\mu_2^2 - \frac{\mu_1\mu_2^3}{\nu^2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\alpha)}\big)\|\Lambda^\alpha\dot\Delta_j u\|_{L^2}^2 + \frac{\mu_1\mu_2}{4}\|\Lambda^{\alpha-1}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 \leqslant |H_j|. \end{aligned} \tag{3.16}$$

Thanks to the condition on α in Theorem 1, we then choose j_0 or $\widetilde{C}_0$ to ensure that

$$\big(\frac{3}{4}\mu_2^2 - \frac{\mu_1\mu_2^3}{\nu^2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\alpha)}\big) > 0.$$

Based on Remark 2.2, we can define $j_0 = 0$ without loss of generality. Dividing both sides of (3.16) by $\|\dot\Delta_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}$, we get

$$\begin{aligned} &\frac{d}{dt}\sqrt{K_2\|\dot\Delta_j u\|_{L^2}^2 + \nu\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \mu_2(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau, \dot\Delta_j u)} \\ &\quad + 2^{2(1-\alpha)j}(\|\dot\Delta_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}) \\ &\le C_2|H_j|/(\|\dot\Delta_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}). \end{aligned} \tag{3.17}$$

Multiplying both sides of (3.17) by $2^{j(\frac{d}{2}+1-2\alpha)}$, summing over $j > 0$ and integrating in time, we obtain

$$\begin{aligned} &\sup_t \|u\|^h_{\dot B_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot B_{2,1}^{\frac{d}{2}-1}} + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot B_{2,1}^{\frac{d}{2}+1-2\alpha}} dt' \\ &\quad\lesssim \|u_0\|^h_{\dot B_{2,1}^{\frac{d}{2}+1-2\alpha}} + \|\tau_0\|^h_{\dot B_{2,1}^{\frac{d}{2}}} \\ &\qquad + \int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}|H_j|/(\|\dot\Delta_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}) dt'. \end{aligned} \tag{3.18}$$

Next we show the smoothing effect on u. We obtain from the first equation of (3.3)

$$\frac{d}{dt}\|\dot\Delta_j u\|_{L^2} + 2^{2\alpha j}\|\dot\Delta_j u\|_{L^2} \le C\|\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2} + C|I_j|/\|\dot\Delta_j u\|_{L^2}.$$

Thus, we have

$$\sup_t \|u\|^h_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' \le \|u_0\|^h_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + C\int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} \mathrm{d}t' \\ + C\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)} |I_j|/\|\dot{\Delta}_j u\|_{L^2} \mathrm{d}t'. \tag{3.19}$$

In order to eliminate the term $C\int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t'$ on the right-hand side of the inequality above, calculating (3.18)+η_1(3.19), and choosing η_1 small enough such that $\eta_1 C \le \frac{1}{2}$, we obtain

$$E_0^h(t) \le C_1\big(\|u_0\|^h_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \\ +C_2\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|H_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2}) \\ +|I_j|/\|\dot{\Delta}_j u\|_{L^2}\Big)\mathrm{d}t'. \tag{3.20}$$

Combining (3.10) and (3.20) leads to the desired inequality in Lemma 3.2. This completes the proof. □

3.2 Proof of Lemma 3.3

Proof of Lemma 3.3 Without loss of generality, we assume

$$\|\dot{\Delta}_j u\|_{L^2},\ \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2},\ \|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|_{L^2},\ \|\dot{\Delta}_j\tau\|_{L^2} \ne 0.$$

Step 1: Estimate of $E^l(t)$.

In this step we will estimate $\sup_t \|\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}}$ instead of $\sup_t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}}$. From (1.2), it is easy to obtain the following two equations:

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|^2_{L^2} + \nu\|\Lambda^\alpha\dot{\Delta}_j u\|^2_{L^2} = \mu_1(\dot{\Delta}_j\nabla\cdot\tau, \dot{\Delta}_j u) - (\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j u) \tag{3.21}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\tau\|^2_{L^2} = \mu_2(\dot{\Delta}_j D(u), \dot{\Delta}_j\tau) \\ -(\dot{\Delta}_j(u\cdot\nabla\tau), \dot{\Delta}_j\tau) - (\dot{\Delta}_j Q(\tau,\nabla u), \dot{\Delta}_j\tau), \tag{3.22}$$

where we have used

$$(\dot{\Delta}_j\nabla p, \dot{\Delta}_j u) = 0. \tag{3.23}$$

Thanks to

$$(\dot\Delta_j \nabla\cdot\tau, \dot\Delta_j u) + (\dot\Delta_j D(u), \dot\Delta_j \tau) = 0, \tag{3.24}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \tau\|_{L^2}\Big) \le C_2|V_j|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \tau\|_{L^2}).$$

These inequalities allow us to obtain the following estimates for the low frequencies:

$$\begin{aligned}&\sup_t \|u\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \sup_t \|\tau\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} \le C_1\left(\|u_0\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\right)\\ &+C_2\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}|V_j|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \tau\|_{L^2})\mathrm{d}t'.\end{aligned} \tag{3.25}$$

Combining (3.25) with (3.10), we have

$$\begin{aligned}E^l(t) \le\ & C_1(\|u_0\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|^l_{\dot B^{\frac{d}{2}+1-2\alpha}})\\ &+C_2\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)} \times \Big(|G_j|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\\ &+|V_j|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \tau\|_{L^2})\Big)\mathrm{d}t'.\end{aligned} \tag{3.26}$$

Step 2: Estimate of $E^h(t)$.

In this step we will estimate $\sup_t \|\tau\|^h_{\dot B^{\frac{d}{2}}_{2,1}}$ instead of $\sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot B^{\frac{d}{2}-1}_{2,1}}$. It follows from (3.22) that

$$\frac{d}{dt}\big(\|\dot\Delta_j \tau\|_2\big) \le C2^j\|\dot\Delta_j u\|_2 + C|Y_j|/\|\dot\Delta_j\tau\|_{L^2}.$$

It is easy to obtain the following estimate for high frequencies:

$$\begin{aligned}\sup_t \|\tau\|^h_{\dot B^{\frac{d}{2}}_{2,1}} \le \|\tau_0\|^h_{\dot B^{\frac{d}{2}}_{2,1}} &+ C\int_0^t \|u\|^h_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\ &+ C\int_0^t \sum_{j>0} 2^{\frac{d}{2}j}|Y_j|/\|\dot\Delta_j\tau\|_{L^2}\mathrm{d}t'.\end{aligned} \tag{3.27}$$

To eliminate the term $C\int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} dt'$ on the right side of (3.27), we calculate (3.20)+η_2(3.27), and choose η_2 small enough such that $\eta_2 C \le \frac{1}{2}$ to get

$$E^h(t) \le C_1(\|u_0\|^h_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}) + C_2\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|H_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla\cdot\tau\|_{L^2}) + |I_j|/\|\dot{\Delta}_j u\|_{L^2}\Big) + 2^{\frac{d}{2}j}|Y_j|/\|\dot{\Delta}_j \tau\|_{L^2} dt'. \tag{3.28}$$

Combining this with (3.26) finishes the proof of Lemma 3.3. □

3.3 Proof of Proposition 3.1

Proof of Proposition 3.1 To estimate G_j and H_j, we rewrite them as

$$G_j = G_j^1 + G_j^2 + G_j^3,$$
$$H_j = H_j^1 + H_j^2 + H_j^3,$$

where

$$\begin{aligned}
G_j^1 &= -\frac{\mu_2}{2}(\dot{\Delta}_j \mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u) \\
&\quad -\mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) \\
&\quad +K_1(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot{\Delta}_j u), \\
G_j^2 &= -\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau), \\
G_j^3 &= K_1(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau) + K_1(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot{\Delta}_j u), \\
H_j^1 &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau) \\
&\quad +\frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot{\Delta}_j u) - \frac{\mu_2^2}{\nu}(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u), \\
H_j^2 &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau), \\
H_j^3 &= \frac{\mu_2}{2}(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau) + \frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot{\Delta}_j u).
\end{aligned}$$

Step 1: Estimate for $G_j^1,\ H_j^1,\ V_j,\ Y_j,\ I_j$.

We first deal with the terms in $G_j^1,\ H_j^1,\ V_j,\ Y_j,\ I_j$ excluding those with $Q(\tau,\nabla u)$. Thanks to $\nabla\cdot u = 0$, we have

$$(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u) = (\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j u). \tag{3.29}$$

Making use of Proposition 2.7 and Proposition 2.9, we obtain

$$
\begin{aligned}
&|(\dot\Delta_j(u\cdot\nabla u),\dot\Delta_j u)|\lesssim c_j 2^{-j(\frac{d}{2}+1-2\alpha)}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\|\dot\Delta_j u\|_{L^2},\\
&|(\dot\Delta_j(u\cdot\nabla \tau),\dot\Delta_j \tau)|\lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}(j)}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\|\dot\Delta_j \tau\|_{L^2}.
\end{aligned}
\tag{3.30}
$$

We now estimate those terms with $Q(\tau,\nabla u)$ in $G_j^1,\ H_j^1,\ V_j,\ I_j$, and Y_j. By Lemma 2.1 and Hölder's inequality, for $j\le 0$,

$$
\begin{aligned}
&\big|-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)+K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u),\dot\Delta_j u)\big|\\
&\quad\lesssim(1+2^{(2\alpha-1)j})\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\\
&\quad\lesssim\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}),\\
&\quad\big|-\mu_1(\dot\Delta_j Q(\tau,\nabla u),\dot\Delta_j\tau)\big|\lesssim\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}\|\dot\Delta_j\tau\|_{L^2}.
\end{aligned}
\tag{3.31}
$$

For $j\ge 0$, we have

$$
\begin{aligned}
&\big|-\nu(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau)\big|\\
&\quad+\frac{\mu_2}{2}\big|(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u),\dot\Delta_j u)\big|\\
&\quad\lesssim 2^{(2\alpha-1)j}\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}(\|\dot\Delta_j u\|_{L^2}+\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}),\\
&\big|-(\dot\Delta_j Q(\tau,\nabla u),\dot\Delta_j\tau)\big|\lesssim\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}\|\dot\Delta_j\tau\|_{L^2}.
\end{aligned}
\tag{3.32}
$$

Combining (3.29), (3.30), (3.31), and (3.32), we conclude that

$$
\begin{aligned}
&\int_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\alpha)}\Big(|G_j^1|/(\|\dot\Delta_j u\|_2+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\\
&\quad+|V_j|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\tau\|_{L^2}\Big)\mathrm{d}t'\\
&\quad+\int_0^t\sum_{j>0}\Big(2^{j(\frac{d}{2}+1-2\alpha)}\Big(|H_j^1|/(\|\dot\Delta_j u\|_{L^2}+\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2})+|I_j|/\|\dot\Delta_j u\|_{L^2}\Big)\\
&\quad+2^{\frac{d}{2}j}|Y_j|/\|\dot\Delta_j\tau\|_{L^2}\Big)\mathrm{d}t'\\
&\lesssim\Big(\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}+\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\Big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\
&\quad+\int_0^t\|Q(\tau,\nabla u)\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\mathrm{d}t'\\
&\lesssim\Big(\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}+\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\Big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t',
\end{aligned}
\tag{3.33}
$$

where we have used Remark 2.8 in the second inequality.

Step 2: Estimates for G_j^2, G_j^3.

To estimate G_j^2, we follow Proposition 2.10 to divide it into three parts, namely $G_j^2 = G_j^{2,1} + G_j^{2,2} + G_j^{2,3}$, where

$$\begin{aligned}
G_j^{2,1} &= -\mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau),\\
G_j^{2,2} &= -\mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(\nabla u\cdot\nabla\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau),\\
G_j^{2,3} &= \mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau).
\end{aligned}$$

According to Proposition 2.9,

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}|G_j^{2,1}|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\
&\lesssim \int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)} c_j 2^{-j}2^{-j(\frac{d}{2}-2\alpha)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\mathrm{d}t'\\
&\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.
\end{aligned} \tag{3.34}$$

Next we estimate the terms $G_j^{2,2}$ and $G_j^{2,3}$. Thanks to $\nabla\cdot u = 0$, we have the following two equations:

$$\begin{aligned}
[\nabla u\cdot\nabla\tau]^i &\triangleq \sum_{j,k}\partial_j u^k\partial_k\tau^{i,j} = \sum_{j,k}\partial_k(\partial_j u^k\tau^{i,j}),\\
[\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau]^i &\triangleq \sum_k \partial_i u^k\partial_k\Delta^{-1}\nabla\cdot\nabla\cdot\tau\\
&= \sum_k \partial_k(\partial_i u^k\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau).
\end{aligned} \tag{3.35}$$

By Remark 2.8,

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}\left|G_j^{2,2}+G_j^{2,3}\right|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\
&\lesssim \int_0^t \|\nabla u\cdot\nabla\tau\|^l_{\dot{B}^{\frac{d}{2}-2\alpha}_{2,1}} + \|\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}-2\alpha}_{2,1}}\mathrm{d}t'\\
&\lesssim \int_0^t \|\nabla u\otimes\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}} + \|\nabla u\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\mathrm{d}t'\\
&\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.
\end{aligned} \tag{3.36}$$

Combining (3.34) and (3.36) yields

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)} |G_j^2|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \lesssim E^2(t). \quad (3.37)$$

Now we estimate the term G_j^3. We again follow Proposition 2.10 to rewrite $G_j^3 = G_j^{3,1} + G_j^{3,2} + G_j^{3,3}$, where

$$\begin{aligned} G_j^{3,1} &= K_1\big((\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau) + (\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \dot\Delta_j u)\big),\\ G_j^{3,2} &= K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}(\nabla u\cdot\nabla\tau), \dot\Delta_j u),\\ G_j^{3,3} &= -K_1(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \dot\Delta_j u). \end{aligned}$$

For $j\le 0$ and by Proposition 2.9, we have

$$\begin{aligned} |G_j^{3,1}| &\lesssim c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\Big(2^{-j(\frac{d}{2}+1-2\alpha)}\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}\\ &\quad + 2^{(2\alpha-2)j}2^{-(\frac{d}{2}-2\alpha)j}\|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\|\dot\Delta_j u\|_{L^2}\Big)\\ &\lesssim c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}2^{-j(\frac{d}{2}+2-4\alpha)}\Big(\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}}\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}\\ &\quad + \|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\|\dot\Delta_j u\|_{L^2}\Big). \end{aligned}$$

Thanks to $\frac{1}{2}\le\alpha\le 1$, we have

$$\begin{aligned} &\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}|G_j^{3,1}|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\lesssim \int_0^t \sum_{j\le 0} c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}2^{(2\alpha-1)j}\Big(\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\Big)\\ &\lesssim \big(\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'. \end{aligned} \quad (3.38)$$

As in the estimates for $G_j^{2,2}$ and $G_j^{2,3}$, we have the following estimates for $G_j^{3,2}$ and $G_j^{3,3}$:

$$\begin{aligned} &\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}\big|G_j^{3,2}+G_j^{3,3}\big|/(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\lesssim \sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'. \end{aligned} \quad (3.39)$$

Combining this with (3.38), we obtain

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}|G_j^3|/(\|\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \lesssim G^2(t). \tag{3.40}$$

Step 3: Estimates for H_j^2, H_j^3.

We now deal with H_j^2, which can be rewritten as $H_j^2 = H_j^{2,1} + H_j^{2,2} + H_j^{2,3}$ according to Proposition 2.10, where

$$\begin{aligned} H_j^{2,1} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau),\\ H_j^{2,2} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\tau), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau),\\ H_j^{2,3} &= \nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau). \end{aligned}$$

By Proposition 2.9, we obtain

$$|H_j^{2,1}| \lesssim c_j 2^{(2\alpha-2)j}2^{-j(\frac{d}{2}-1)}\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2}.$$

Through a simple computation, we derive

$$\begin{aligned} &\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}|H_j^{2,1}|/(\|\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t'. \end{aligned} \tag{3.41}$$

Then we estimate $H_j^{2,2}$ and $H_j^{2,3}$. Using Remark 2.8 and (3.35), we obtain

$$\begin{aligned} &\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}\big|H_j^{2,2}+H_j^{2,3}\big|/(\|\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\lesssim \int_0^t \|\nabla u\otimes\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla u\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}}}\mathrm{d}t'\\ &\lesssim \int_0^t \|\nabla u\otimes\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}} + \|\nabla u\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\mathrm{d}t'\\ &\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t'. \end{aligned} \tag{3.42}$$

Combining (3.41) with (3.42), we have

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)}|H_j^2|/(\|\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \lesssim E^2(t). \tag{3.43}$$

Now we consider H_j^3, which can be rewritten as $H_j^3 = H_j^{3,1} + H_j^{3,2} + H_j^{3,3}$ by Proposition 2.10, where

$$H_j^{3,1} = \frac{\mu_2}{2}\big((\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau) + (\Lambda^{2\alpha-2}\dot{\Delta}_j(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \dot{\Delta}_j u)\big),$$
$$H_j^{3,2} = \frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\tau), \dot{\Delta}_j u),$$
$$H_j^{3,3} = -\frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \dot{\Delta}_j u).$$

First we estimate $H_j^{3,1}$. Making use of Proposition 2.9, we obtain

$$\begin{aligned}\big|H_j^{3,1}\big| \lesssim c_j\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\big(&2^{-j(\frac{d}{2}+1-2\alpha)}\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}}\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2}\\ &+ 2^{j(2\alpha-2)}2^{-j(\frac{d}{2}-1)}\|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\|\dot{\Delta}_j u\|_{L^2}\big).\end{aligned}$$

Then, we derive

$$\begin{aligned}&\int_0^t\sum_{j>0}2^{j(\frac{d}{2}+1-2\alpha)}|H_j^{3,1}|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\quad\lesssim \big(\sup_t\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}} + \sup_t\|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\big)\int_0^t\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t'.\end{aligned}\tag{3.44}$$

As in the estimates of $H_j^{2,2}$ and $H_j^{2,3}$, we have

$$\begin{aligned}&\int_0^t\sum_{j>0}2^{j(\frac{d}{2}+1-2\alpha)}\big|H_j^{3,2} + H_j^{3,3}\big|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\ &\lesssim \sup_t\|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t'.\end{aligned}\tag{3.45}$$

Combining (3.44) with (3.45) gives

$$\int_0^t\sum_{j>0}2^{j(\frac{d}{2}+1-2\alpha)}|H_j^3|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \lesssim E^2(t).\tag{3.46}$$

Putting together (3.33), (3.37), (3.40), (3.43), and (3.46) leads to Proposition 3.1. □

4 The Global Existence and the Uniqueness

This section is devoted to the global existence and uniqueness part of Theorem 1.1. The global existence is a consequence of the local existence and the global *a priori* bound. The local existence part can be obtained by modifying those in [15] or [8]. We shall not repeat it here. The global *a priori* bound follows from the *a priori* inequality (3.1) and a bootstrapping argument.

When $\alpha = 1$, the uniqueness can be established by directly performing the L^2-estimate on the equation satisfied by the difference $(\delta u, \delta\tau)$ between two solutions (u_1, τ_1) and (u_2, τ_2). However, when $\frac{1}{2} \le \alpha < 1$, the uniqueness can no longer be achieved by the L^2-estimate. The reason is due to the term $Q(\tau_1, \nabla\delta u)$, which requires the control of $\|\nabla\delta u\|_{L^2}$. But the dissipation in the velocity equation can only absorb $\|\Lambda^\alpha \delta u\|_{L^2}$ for $\frac{1}{2} \le \alpha < 1$. Therefore, direct energy estimate on the difference in L^2 would not work. The new idea here is to make use of the enhanced dissipation as in the proof of the existence part.

The rest of this section is divided into two subsections. The first subsection is devoted to the global *a priori* bound while the second to the uniqueness when $\frac{1}{2} < \alpha < 1$ or $\alpha = \frac{1}{2}$ and $\nu^2 \geqslant C\mu_1\mu_2$.

4.1 The Global Existence

By Proposition 3.1, we have

$$E(t) \le C_1 E_0 + C_2 E^2(t)$$

for some positive constants C_1 and C_2. By a standard bootstrap argument, if $E_0 \leqslant \varepsilon$ for sufficiently small $\varepsilon > 0$, then for all $t > 0$ we have

$$E(t) \leqslant C\varepsilon. \tag{4.1}$$

which, in particular, implies desired global *a priori* bound and thus the global existence of solutions to the system (1.2).

4.2 The Uniqueness

Assume (u_1, τ_1) and (u_2, τ_2) are two solutions of (1.2) with the same initial data. Denote $\delta u = u_1 - u_2$, $\delta\tau = \tau_1 - \tau_2$, $\delta p = p_1 - p_2$, then $(\delta u, \delta\tau)$ satisfies

$$\begin{cases} (\delta u)_t + \nu\Lambda^{2\alpha}\delta u + \nabla\delta p = \mu_1 \nabla\cdot\delta\tau - u_1\cdot\nabla\delta u - \delta u\cdot\nabla u_2, \\ (\delta\tau)_t + u_1\cdot\nabla\delta\tau = \mu_2 D(\delta u) - \delta u\cdot\nabla\tau_2 - Q(\tau_1, \nabla\delta u) - Q(\delta\tau, \nabla u_2), \\ \nabla\cdot\delta u = 0, \\ \delta u(0, x) = 0;\ \ \delta\tau(0, x) = 0. \end{cases} \tag{4.2}$$

As in the derivation of Lemma 3.2 and Lemma 3.3, we have

$$\sup_t \|\delta u\|_{\dot B^{\frac{d}{2}+1-2\alpha, \frac{d}{2}-2\alpha}} + \sup_t \|\delta\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha, \frac{d}{2}-1}} + \int_0^t \|\delta u\|_{\dot B^{\frac{d}{2}+1, \frac{d}{2}}} \mathrm{d}t'$$
$$+ \int_0^t (\|\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|^l_{\dot B^{\frac{d}{2}+1}_{2,1}} + \|\mathbb{P}\nabla\cdot\delta\tau\|^h_{\dot B^{\frac{d}{2}-2\alpha}_{2,1}})\mathrm{d}t'$$

$$
\begin{aligned}
&\lesssim C_2\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|G_j^{'}|/(\|\dot\Delta_j\delta u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\\
&\quad+|V_j^{'}|/(\|\dot\Delta_j\delta u\|_{L^2}+\|\dot\Delta_j\delta\tau\|_{L^2})\Big)\mathrm{d}t'\\
&\quad+C_2\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)}\Big(|H_j^{'}|/(\|\dot\Delta_j\delta u\|_{L^2}+\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\\
&\quad+|I_j^{'}|/\|\dot\Delta_j\delta u\|_{L^2}\Big)+2^{(\frac{d}{2}-1)j}|Y_j^{'}|/\|\dot\Delta_j\delta\tau\|_{L^2}\mathrm{d}t',
\end{aligned}
$$

where $G_j^{'}=G_j^{'1}+G_j^{'2}+G_j^{'3}$ with

$$
\begin{aligned}
G_j^{'1}&=-\frac{\mu_2}{2}((\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u),\dot\Delta_j\delta u)+(\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2),\dot\Delta_j\delta u))\\
&\quad-\mu_1((\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\quad-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau))\\
&\quad+K_1((\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot\Delta_j\delta u)+(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot\Delta_j\delta u)),\\
G_j^{'2}&=-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\quad-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau),\\
G_j^{'3}&=K_1((\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)+(\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau))\\
&\quad+K_1((\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot\Delta_j\delta u)+(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot\Delta_j\delta u)),
\end{aligned}
$$

and $H_j^{'}=H_j^{'1}+H_j^{'2}+H_j^{'3}$ with

$$
\begin{aligned}
H_j^{'1}&=-\nu(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)\\
&\quad-\nu(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)\\
&\quad+\frac{\mu_2}{2}((\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot\Delta_j\delta u)+(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot\Delta_j\delta u))\\
&\quad-\frac{\mu_2^2}{\nu}((\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u),\dot\Delta_j\delta u)+(\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2),\dot\Delta_j\delta u)),\\
H_j^{'2}&=-\nu(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)\\
&\quad-\nu(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau),\\
H_j^{'3}&=\frac{\mu_2}{2}(\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)+\frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot\Delta_j\delta u)\\
&\quad+\frac{\mu_2}{2}(\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2),\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\delta\tau)+\frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot\Delta_j\delta u),
\end{aligned}
$$

and

$$
\begin{aligned}
V_j^{'}&=-\mu_2((\dot\Delta_j(u_1\cdot\nabla\delta u),\dot\Delta_j\delta u)+(\dot\Delta_j(\delta u\cdot\nabla u_2),\dot\Delta_j\delta u))\\
&\quad-\mu_1((\dot\Delta_j(u_1\cdot\nabla\delta\tau),\dot\Delta_j\delta\tau)+(\dot\Delta_j(\delta u\cdot\nabla\tau_2),\dot\Delta_j\delta\tau))\\
&\quad-\mu_1((\dot\Delta_j Q(\tau_1,\nabla\delta u),\dot\Delta_j\delta\tau)+(\dot\Delta_j Q(\delta\tau,\nabla u_2),\dot\Delta_j\delta\tau)),\\
I_j^{'}&=-((\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u),\dot\Delta_j\delta u)+(\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2),\dot\Delta_j\delta u)),
\end{aligned}
$$

$$\begin{aligned}\tilde{Y}_j' = &-((\dot{\Delta}_j(u_1\cdot\nabla\delta\tau), \dot{\Delta}_j\delta\tau) + (\dot{\Delta}_j(\delta u\cdot\nabla\tau_2), \dot{\Delta}_j\delta\tau))\\ &-((\dot{\Delta}_j Q(\tau_1, \nabla\delta u), \dot{\Delta}_j\delta\tau) + (\dot{\Delta}_j Q(\delta\tau, \nabla u_2), \dot{\Delta}_j\delta\tau)).\end{aligned}$$

Based on Proposition 2.7 and Proposition 2.9, we have

$$|(\dot{\Delta}_j(u_1\cdot\nabla\delta u), \dot{\Delta}_j\delta u)| \lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}(j)}\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}\|\dot{\Delta}_j\delta u\|_{L^2}, \tag{4.3}$$

$$|(\dot{\Delta}_j(\delta u\cdot\nabla u_2), \dot{\Delta}_j\delta u)| \lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}(j)}\|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}\|\dot{\Delta}_j\delta u\|_{L^2}, \tag{4.4}$$

$$|(\dot{\Delta}_j(u_1\cdot\nabla\delta\tau), \dot{\Delta}_j\delta\tau)| \lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}(j)}\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}\|\dot{\Delta}_j\delta\tau\|_{L^2}, \tag{4.5}$$

$$|(\dot{\Delta}_j(\delta u\cdot\nabla\tau_2), \dot{\Delta}_j\delta\tau)| \lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}(j)}\|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\nabla\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}\|\dot{\Delta}_j\delta\tau\|_{L^2}. \tag{4.6}$$

For $j \le 0$, we have

$$\begin{aligned}&\big|-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\big|\\ &\quad+K_1\big|(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u), \dot{\Delta}_j\delta u)\big|\\ &\lesssim \|\dot{\Delta}_j Q(\tau_1,\nabla\delta u)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\big|-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\big|\\ &\quad+K_1\big|(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2), \dot{\Delta}_j\delta u)\big|\\ &\lesssim \|\dot{\Delta}_j Q(\delta\tau,\nabla u_2)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\end{aligned} \tag{4.8}$$

$$\big|-\mu_1(\dot{\Delta}_j Q(\tau_1,\nabla\delta u), \dot{\Delta}_j\delta\tau)\big| \lesssim \|\dot{\Delta}_j Q(\tau_1,\nabla\delta u)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2}, \tag{4.9}$$

$$\big|-\mu_1(\dot{\Delta}_j Q(\delta\tau,\nabla u), \dot{\Delta}_j\delta\tau)\big| \lesssim \|\dot{\Delta}_j Q(\delta\tau,\nabla u_2)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2} \tag{4.10}$$

and for $j \ge 0$, we have

$$\begin{aligned}&\big|-\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau)\big|\\ &\quad+\frac{\mu_2}{2}\big|(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u), \dot{\Delta}_j\delta u)\big|\\ &\lesssim 2^{(2\alpha-1)j}\|\dot{\Delta}_j Q(\tau_1,\nabla\delta u)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2} + \|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\big|-\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2), \Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau)\\ &\quad+\frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2), \dot{\Delta}_j\delta u)\big|\end{aligned}$$

$$\lesssim 2^{(2\alpha-1)j}\|\dot{\Delta}_j Q(\delta\tau,\nabla u_2)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}), \quad (4.12)$$
$$\left|-(\dot{\Delta}_j Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta\tau)\right| \lesssim \|\dot{\Delta}_j Q(\tau_1,\nabla\delta u)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2}, \quad (4.13)$$
$$\left|-(\dot{\Delta}_j Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\delta\tau)\right| \lesssim \|\dot{\Delta}_j Q(\delta\tau,\nabla u_2)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2}. \quad (4.14)$$

Combining (4.3) through (4.14), we conclude

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}\Big(|G_j^{'1}|/(\|\dot{\Delta}_j\delta u\|_2+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\\
&\quad+|V_j^{'}|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\delta\tau\|_{L^2}\Big)dt'\\
&\quad+\int_0^t\sum_{j>0}2^{j(\frac{d}{2}-2\alpha)}\Big(|H_j^{'1}|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})+|I^{'}{}_j|/\|\dot{\Delta}_j\delta u\|_{L^2}\Big)\\
&\quad+2^{\frac{d}{2}j}|Y_j^{'}|/\|\dot{\Delta}_j\delta\tau\|_{L^2}dt'\\
&\lesssim \sup_t\big(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}+\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}\big)\int_0^t\|(u_1,u_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}dt'\\
&\quad+\int_0^t\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|\nabla\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}dt'\\
&\quad+\int_0^t\|(Q(\tau_1,\nabla\delta u),Q(\delta\tau,\nabla u_2))\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}dt'\\
&\lesssim \sup_t\big(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}+\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}\big)\int_0^t\|(u_1,u_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}dt'\\
&\quad+\sup_t(\|\tau_1\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}+\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}})\int_0^t\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}dt'.
\end{aligned} \quad (4.15)$$

It is easy to the deal with $G_j^{'2}$ and $G_j^{'3}$ directly by Proposition 2.7,

$$\begin{aligned}
&\int_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\alpha)}|G_j^{'2}|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})dt'\\
&\lesssim\int_0^t(\|u_1\cdot\nabla\delta\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}}+\|\delta u\cdot\nabla\tau_2\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}})dt'\\
&\lesssim\int_0^t(\|u_1\delta\tau\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}}+\|\delta u\tau_2\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\alpha}})dt'\\
&\lesssim\sup_t\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}\int_0^t\|u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}dt'\\
&\quad+t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}})
\end{aligned} \quad (4.16)$$

and

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\alpha)}|G_j^{'3}|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t' \\
&\lesssim \int_0^t (\|u_1\delta u\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}}+\|\delta u\nabla u_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}}+\|u_1\delta\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}}+\|\delta u\tau_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\alpha}_{2,1}})\mathrm{d}t' \\
&\lesssim \sup_t(\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}+\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}})\int_0^t(\|u_1\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}})\mathrm{d}t' \\
&\quad +t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}). \qquad (4.17)
\end{aligned}$$

To estimate $H_j^{'2}$, we rewrite it as $H_j^{'2} = H_j^{'2,1}+H_j^{'2,2}+H_j^{'2,3}+H_j^{'2,4}$ by Proposition 2.10, where

$$\begin{aligned}
H_j^{'2,1} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(u_1\cdot\nabla\mathbb{P}\nabla\cdot\delta\tau),\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau), \\
H_j^{'2,2} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u_1\cdot\nabla\delta\tau),\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau), \\
H_j^{'2,3} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}(\nabla u_1\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\delta\tau),\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau), \\
H_j^{'2,4} &= -\nu(\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\Lambda^{2\alpha-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot\delta\tau). \qquad (4.18)
\end{aligned}$$

Taking advantage of Proposition 2.9, we get

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)}|H_j^{'2,1}|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t' \\
&\lesssim \sup_t\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}\int_0^t\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'. \qquad (4.19)
\end{aligned}$$

From Proposition 2.7, we derive

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)}\big|H_j^{'2,2}+H_j^{'2,3}\big|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t' \\
&\lesssim \int_0^t \|\nabla u_1\otimes\delta\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\nabla u_1\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\delta\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t' \\
&\lesssim \sup_t\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}}\int_0^t\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t' \qquad (4.20)
\end{aligned}$$

and

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)}\big|H_j^{'2,4}\big|/(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \int_0^t \|\delta u \otimes \nabla \tau_2\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t' \lesssim \sup_t \|\tau_2\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t'. \tag{4.21}$$

To estimate $H_j^{'3}$, we rewrite it as $H_j^{'3} = H_j^{'3,1} + H_j^{'3,2} + H_j^{'3,3} + H_j^{'3,4} + H_j^{'3,5}$ by Proposition 2.10, where

$$\begin{aligned}
H_j^{'3,1} &= \frac{\mu_2}{2}\big((\dot{\Delta}_j \mathbb{P}(u_1 \cdot \nabla \delta u), \Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla \cdot \delta\tau) \\
&\quad + (\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}(u_1 \cdot \nabla \mathbb{P}\nabla \cdot \delta\tau), \dot{\Delta}_j \delta u)\big), \\
H_j^{'3,2} &= \frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}(\nabla u_1 \cdot \nabla \delta\tau), \dot{\Delta}_j \delta u), \\
H_j^{'3,3} &= \frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}(\nabla u_1 \cdot \nabla \Delta^{-1}\nabla \cdot \nabla \cdot \delta\tau), \dot{\Delta}_j \delta u), \\
H_j^{'3,4} &= \frac{\mu_2}{2}(\dot{\Delta}_j \mathbb{P}(\delta u \cdot \nabla u_2), \Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla \cdot \delta\tau), \\
H_j^{'3,5} &= \frac{\mu_2}{2}(\Lambda^{2\alpha-2}\dot{\Delta}_j \mathbb{P}\nabla \cdot (\delta u \cdot \nabla \tau_2), \dot{\Delta}_j \delta u).
\end{aligned} \tag{4.22}$$

According to Proposition 2.9,

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)} |H_j^{'3,1}| / (\|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla \cdot \delta\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim \big(\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha, \frac{d}{2}-2\alpha}} + \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha, \frac{d}{2}-1}}\big) \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'.
\end{aligned} \tag{4.23}$$

As in the estimates of $H_j^{'2,2}$, $H_j^{'2,3}$, and $H_j^{'2,4}$, we have

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)} |H_j^{'3,2} + H_j^{'3,3}| / (\|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla \cdot \delta\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha, \frac{d}{2}-1}} \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)} |H_j^{'3,5}| / (\|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla \cdot \delta\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim \sup_t \|\tau_2\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t'.
\end{aligned}$$

Using Proposition 2.7, we have

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-2\alpha)} |H_j'^{3,4}|/(\|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \int_0^t \|\delta u\cdot\nabla u_2\|^h_{\dot{B}^{\frac{d}{2}-2\alpha}_{2,1}}\mathrm{d}t' \lesssim \sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}} \int_0^t \|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.$$

Combining the estimates above, we have

$$\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}} + \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}} + \int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}}}\mathrm{d}t'$$
$$\lesssim \sup_t \left(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}} + \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}\right)\int_0^t (\|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|u_1\|_{\dot{B}^{\frac{d}{2}}_{2,1}})\mathrm{d}t'$$
$$+\sup_t \|(\tau_1,\tau_2)\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t'$$
$$+t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}}).$$

Note that

$$\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t' \lesssim t\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}} + \int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}}}\mathrm{d}t'.$$

Thanks to the uniform a priori estimates, we have $\sup_t \|(\tau_1,\tau_2)\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}} \lesssim \varepsilon$. Then we can choose ε and t small such that

$$\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-2\alpha}} + \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}-1}} \leqslant 0.$$

This proves the uniqueness.

5 Higher Regularity for More Regular Data

This section proves the higher regularity part of Theorem 1.1 when the initial data are more regular. Due to the weak smoothing effect for u, some of the terms such as $Q(\tau,\nabla u)$ cannot be directly bounded. To overcome this difficulty, we make use of the enhanced dissipation revealed in the wave structure (1.4). More precisely, we prove a lemma similar to Lemma 3.2 and Lemma 3.3, and then establish an inequality as the one in Proposition 3.1.

We first state the regularitzation lemma.

Lemma 5.1 *Let* (u,τ) *be a solution to the system* (1.2) *on* $[0,T)$. *Let* $E_0^{'h}$ *and* $E^{'h}$ *be defined as in* (1.15) *and* (1.16), *respectively. Then the following inequality holds*

$$E_0^{'h}(t)+E^{'h}(t)\le C_1E_0^{'}+C_2\int_0^t\sum_{j>0}2^{j(\frac{d}{2}+2-2\alpha)}\Big(H_j/(\|\dot\Delta_ju\|_{L^2}+\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2})+I_j/\|\dot\Delta_ju\|_{L^2}\Big)+2^{(\frac{d}{2}+1)j}Y_j/\|\dot\Delta_j\tau\|_{L^2}\mathrm{d}t',$$

where $E_0^{'}=\|u_0\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}}+\|\tau_0\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}}$, *and* C_1 *and* C_2 *are independent of* T.

The proof of Lemma 5.1 are similar to those for Lemma 3.2 and Lemma 3.3 and thus the details are omitted.

Similar to Proposition 3.1, we have the following a priori estimates.

Proposition 5.2 *Assume that* (1.2) *has a solution* (u,τ) *on* $[0,T)$. *Then, there exist two positive constants* C_1, C_2 *independent of* T *such that*

$$E_0^{'h}(t)+E^{'h}(t)\le C_1E_0^{'}+C_2((E_0^{'h}(t)+E^{'h}(t))E(t)+E^2(t)).\tag{5.1}$$

Proof For $j>0$, it is easy to get

$$\begin{aligned}|(\dot\Delta_j(u\cdot\nabla u),\dot\Delta_ju)|&\lesssim c_j2^{-j(\frac{d}{2}+2-2\alpha)}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}}\|\dot\Delta_ju\|_{L^2},\\|(\dot\Delta_j(u\cdot\nabla\tau),\dot\Delta_j\tau)|&\lesssim c_j2^{-(\frac{d}{2}+1)j}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}}\|\dot\Delta_j\tau\|_{L^2},\end{aligned}\tag{5.2}$$

using Proposition 2.7 and Proposition 2.9. As in the derivation of (3.33), we have

$$\begin{aligned}&\int_0^t\sum_{j>0}2^{j(\frac{d}{2}+2-2\alpha)}\Big(|H_j^1|/(\|\dot\Delta_ju\|_{L^2}+\|\Lambda^{2\alpha-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2})+|I_j|/\|\dot\Delta_ju\|_{L^2}\Big)\\&\quad+2^{(\frac{d}{2}+1)j}|Y_j|/\|\dot\Delta_j\tau\|_{L^2}\mathrm{d}t'\\&\lesssim\Big(\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}}+\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}}\Big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\&\quad+\int_0^t\|Q(\tau,\nabla u)\|^h_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\&\lesssim\Big(\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}}+\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}}\Big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\&\quad+\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}}\int_0^t\|u\|^h_{\dot B^{\frac{d}{2}+2}_{2,1}}\mathrm{d}t'\\&\lesssim(E_0^{'h}(t)+E^{'h}(t))E(t)+E^2(t).\end{aligned}\tag{5.3}$$

By Proposition 2.9,

$$|H_j^{2,1}| \lesssim c_j 2^{(2\alpha-2)j} 2^{-j\frac{d}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\alpha,\frac{d}{2}}} \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2},$$

which implies

$$\begin{aligned}&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+2-2\alpha)} |H_j^{2,1}|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \\ &\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}} \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t'.\end{aligned} \tag{5.4}$$

Taking advantage of Proposition 2.7, we obtain

$$\begin{aligned}&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+2-2\alpha)} \big|H_j^{2,2} + H_j^{2,3}\big|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \\ &\lesssim \int_0^t \|\nabla u \otimes \tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla u \otimes \Delta^{-1}\nabla\cdot\nabla\cdot\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' \\ &\lesssim \big(\sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}} + \sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}}\big) \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' \\ &\quad + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}}} \int_0^t \|u\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \mathrm{d}t' \\ &\lesssim (E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t).\end{aligned} \tag{5.5}$$

Combining (5.4) with (5.5) leads to

$$\begin{aligned}&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+2-2\alpha)} |H_j^2|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \\ &\lesssim (E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t).\end{aligned}$$

By Proposition 2.9,

$$\begin{aligned}\big|H_j^{3,1}\big| \lesssim c_j \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \big(&2^{-j(\frac{d}{2}+2-2\alpha)} \|u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}} \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2} \\ &+ 2^{2\alpha-2} 2^{-j\frac{d}{2}} \|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\alpha,\frac{d}{2}}} \|\dot{\Delta}_j u\|_{L^2}\big).\end{aligned}$$

Then, we have

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+2-2\alpha)} |H_j^{3,1}|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \big(\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+2-2\alpha}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}}\big) \int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' \\ \lesssim (E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t). \tag{5.6}$$

Dealing with $H_j^{3,2}$ and $H_j^{3,3}$ in the same way as we estimate $H_j^{2,2}$ and $H_j^{2,3}$ gives

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1-2\alpha)} \big|H_j^{3,2} + H_j^{3,3}\big| / (\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\ \lesssim (E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t). \tag{5.7}$$

Combining (5.6) and (5.7), we obtain

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+2-2\alpha)} |H_j^3| / (\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{2\alpha-2}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\ \lesssim (E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t). \tag{5.8}$$

Finally, combining (5.3), (5.6), and (5.8) finishes the proof of Proposition 5.2. □

We now prove the desired higher regularity. By Proposition 5.2,

$$E_0^{'h}(t) + E^{'h}(t) \le C_1 E_0^{'} + C_2((E_0^{'h}(t) + E^{'h}(t))E(t) + E^2(t)).$$

Therefore, thanks to the results in Sect. 4.1, we can choose ε small such that $C_2 E(t) \leqslant \frac{1}{2}$ such that

$$E_0^{'h}(t) + E^{'h}(t) \lesssim E_0^{'} + \varepsilon.$$

Combining this and the results in Sect. 4.1, we have

$$\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha.\frac{d}{2}+2-2\alpha}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+1}} + \int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}+2}} \mathrm{d}t' \\ + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}+2-2\alpha}_{2,1}} \mathrm{d}t' \lesssim E_0^{'} + \varepsilon.$$

Similarly, we can choose ε small (depending on s) such that the bound holds

$$\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\alpha.\frac{d}{2}+1+s-2\alpha}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\alpha,\frac{d}{2}+s}} + \int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}+1+s}} \mathrm{d}t' \\ + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + \int_0^t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}+1+s-2\alpha}_{2,1}} \mathrm{d}t' \\ \lesssim \|u\|^h_{\dot{B}^{\frac{d}{2}+1+s-2\alpha}_{2,1}} + \|\tau\|^h_{\dot{B}^{\frac{d}{2}+s}_{2,1}} + \varepsilon.$$

This completes the proof of the higher regularity in Theorem 1.1.

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