DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **10**, Numbers **1&2**, January & March **2004**

Website: http://AIMsciences.org

pp. **543–556**

REGULARITY RESULTS FOR WEAK SOLUTIONS OF THE 3D MHD EQUATIONS

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Abstract. We study regularity of general and axisymmetric weak solutions of the 3D MHD equations with dissipation and resistance. A general weak solution is shown to be smooth if it satisfies a Serrin condition. The regularity of axisymmetric weak solutions is analyzed through the MHD equations in cylindrical coordinates, whose concrete form is derived here using Gibbs' notion of dyadic product. We establish that it is sufficient to impose conditions on certain components (in cylindrical coordinates) of an axisymmetric weak solution in order for the solution to be regular.

1. **Introduction.** We consider the 3D MHD equations with dissipation and resistance

$$\begin{cases} \partial_t u + u\cdot\nabla u &= -\nabla P + b\cdot\nabla b + \nu\Delta u \\ \partial_t b + u\cdot\nabla b &= b\cdot\nabla u + \eta\Delta b \\ \nabla\cdot u = 0, & \nabla\cdot b = 0, \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^3$, $t\ge 0$, u is the flow velocity, b is the magnetic field, P is the total pressure, ν is the kinematic viscosity and η is the resistivity. For any prescribed initial data $(u_0,b_0)\in L^2(\mathbb{R}^3)$, the MHD equations (1.1) have been shown to possess global L^2 weak solutions ([4], [5]). Because of the smoothing effects of dissipation and resistance, L^2 weak solutions are more regular than in the basic class $L^\infty([0,T];L^2(\mathbb{R}^3))\cap L^2([0,T];H^1(\mathbb{R}^3))$. Several regularity results concerning L^2 weak solutions of the MHD equations have been established. In [7] the H^1-norm of (u,b) is shown to control derivatives of any higher order and as a consequence any possible singularity must occur in the first-order derivative of (u,b). In [2] Caflisch, Klapper and Steele extended the well-known result of Beale, Kato and Majda ([1]) to the 3D MHD equations to draw the conclusion that the finiteness of the $L^1([0,T];L^\infty)$-norms of the vorticity $\omega=\nabla\times u$ and the current density $j=\nabla\times b$ implies global regularity. In [8] possible singularity formation is linked to the development i of special geometric structure of ω and j.

The major results we are about to present in this paper are further developments on the regularity of weak solutions of the 3D MHD equations (1.1). Theorem 2.1 (in Section 2) states that a general weak solution (u,b) actually belongs to $L^\infty([0,T];H^k)$ for any $k\ge 1$ if (u,b) satisfies a Serrin condition over $[0,T]$ ([6]). Section 3 concerns axisymmetric weak solutions of the MHD equations and our study

1991 *Mathematics Subject Classification.* 76W05, 76D03, 76DXX .
Key words and phrases. MHD equations, weak solution, axisymmetry, global regularity.
This research was partially supported by NSF grant DMS 9971926 and by a Ralph E. Powe Junior Faculty Enhancement Award from the Oak Ridge Associated Universities.

of their regularity is motivated by a recent work of Chae and Lee on solutions of the Navier-Stokes equations [3]. We conclude in this section that an axisymmetric weak solution (u, b) is regular if certain components of (u, b) in cylindrical coordinates satisfy a generalized Serrin condition. We will defer the precise statement to Theorem 3.1. To take advantage of the axisymmetry, we use the MHD equations in cylindrical coordinates rather than in Cartesian coordinates. Our experience of having difficulty finding the full MHD equations in cylindrical coordinates impelled us to derive them ourselves and our major tool is Gibbs' notion of dyadic product. This is an effective approach and many tedious calculations are avoided. Equations of both (u, b) and (ω, j) in cylindrical coordinates are presented in the appendix and the results provided there serve as records for future reference.

2. **A general regularity result.**

The regularity result we are about to present in this section is valid for general classical weak solutions of the MHD equations including those with axisymmetry discussed in the next section. We conclude in this section that a weak solution (u, b) is actually smooth if (u, b) satisfies an appropriate assumption. We emphasize that this assumption does not involve derivatives of (u, b) and the precise statement is given in the following theorem.

Theorem 2.1. *Let $T > 0$, $p > 3$ and $q \geq 2$ satisfy $2/q + 3/p = 1$. Assume that $(u_0, b_0) \in L^p \cap H^1$ and (u, b) is a weak solution of the MHD equations (1.1) with initial data (u_0, b_0). If $(u, b) \in L^q([0, T]; L^p)$, then*

(a) $(u, b) \in L^\infty([0, T]; L^p)$,
(b) $(\nabla u, \nabla b) \in L^\infty([0, T]; L^2)$.

It has been shown in [7] that any H^k-norm $(k \geq 1)$ of a weak solution (u, b) is finite as long as its H^1-norm remains bounded. This allows us to draw as a consequence of Theorem 2.1 the following conclusion.

Corollary 2.2. *Let (u, b) be a weak solution of the MHD equations satisfying the assumptions of Theorem 2.1. Then $(u, b) \in L^\infty([0, T]; H^k)$ for any $k \geq 1$.*

Proof of Theorem 2.1 We first remark that the seemingly formal steps in this proof can all be made rigorous by going through an approximating procedure. It is easy to verify that (u, b) satisfies

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} [|u|^p + |b|^p]\,dx + \frac{4(p-1)}{p^2}\left[\nu \int_{\mathbb{R}^3} |\nabla(|u|^{p/2})|^2 dx + \eta \int_{\mathbb{R}^3} |\nabla(|b|^{p/2})|^2 dx\right]$$

$$= \int_{\mathbb{R}^3} (b\cdot\nabla b)(|u|^{p-2}u)\ dx + \int_{\mathbb{R}^3} (b\cdot\nabla u)(|b|^{p-2}b)\,dx - \int_{\mathbb{R}^3} (u\cdot\nabla P)|u|^{p-2} dx \quad (2.1)$$

We now focus our attention on the three terms on the right. For the second term, we have

$$\begin{aligned}
\textstyle\int (b\cdot\nabla u)(|b|^{p-2}b)\,dx &= -\textstyle\int u\cdot\left[b\cdot\nabla(|b|^{p-2}b)\right]\,dx \\
&\le (p-1)\textstyle\int |u||\nabla b||b|^{p-1}\,dx \\
&\le C_\epsilon \textstyle\int |u|^2|b|^p\,dx + \frac{\epsilon\, p^2}{4}\textstyle\int |b|^{p-2}|\nabla b|^2\,dx \\
&= C_\epsilon \textstyle\int |u|^2|b|^p\,dx + \epsilon\textstyle\int |\nabla(|b|^{p/2})|^2 dx \\
&\le C_\epsilon \|u\|_{L^p}^2\ \||b|^{p/2}\|_{L^{2p/(p-2)}}^2 + \epsilon\textstyle\int |\nabla(|b|^{p/2})|^2 dx \\
&\le C_\epsilon \|u\|_{L^p}^2\ \||b|^{p/2}\|_{L^2}^{2-6/p}\,\|\nabla(|b|^{p/2})\|_{L^2}^{6/p} + \epsilon\textstyle\int |\nabla(|b|^{p/2})|^2 dx \\
&\le 2\epsilon\textstyle\int |\nabla(|b|^{p/2})|^2 dx + C_\epsilon\|u\|_{L^p}^{2p/(p-3)}\textstyle\int |b|^p dx,
\end{aligned}$$

where $\epsilon > 0$ is small and C_ϵ is a constant depending on ϵ. For the first term, we integrate by parts to obtain

$$\begin{aligned}
\textstyle\int (b\cdot\nabla b)(|u|^{p-2}u)\ dx &\le \frac{\epsilon\, p^2}{4}\textstyle\int |u|^{p-2}|\nabla u|^2 dx + C_\epsilon\textstyle\int |u|^{p-2}|b|^4 dx \\
&= \epsilon\textstyle\int |\nabla(|u|^{p/2})|^2 dx + C_\epsilon\textstyle\int |u|^{p-2}|b|^4 dx.
\end{aligned}$$

For p_1, q_1 and r_1 satisfying $1/p_1 + 1/q_1 + 1/r_1 = 1$,

$$\int |u|^{p-2}|b|^4 dx = \int |b|^2\,|b|^2\,|u|^{p-2} dx \le C\|b^2\|_{L^{p_1}}\,\|b^2\|_{L^{q_1}}\,\||u|^{p-2}\|_{L^{r_1}}. \tag{2.2}$$

We then choose

$$p_1 = \frac{p}{2}, \quad q_1 = \frac{3p}{p-1}, \quad r_1 = \frac{3p}{2p-5} \tag{2.3}$$

to get

$$\int |u|^{p-2}|b|^4 dx \le C\|b\|_{L^p}^2\ \||b|^{p/2}\|_{L^{12/(p-1)}}^{4/p}\ \||u|^{p/2}\|_{L^{6(p-2)/(2p-5)}}^{2(1-2/p)}.$$

Applying the Gagliardo-Nirenberg inequality and Young's inequality, there obtains

$$\begin{aligned}
C_\epsilon\textstyle\int |u|^{p-2}|b|^4 dx &\le C\,\|b\|_{L^p}^2\ \||b|^{p/2}\|_{L^2}^{1-\frac{3}{p}} \\
&\quad\times\|\nabla(|b|^{p/2})\|_{L^2}^{\frac{7}{p}-1}\ \||u|^{p/2}\|_{L^2}^{1-\frac{3}{p}}\ \|\nabla(|u|^{p/2})\|_{L^2}^{1-\frac{1}{p}} \\
&\le \epsilon\textstyle\int |\nabla(|u|^{p/2})|^2 dx + \epsilon\textstyle\int |\nabla(|b|^{p/2})|^2 dx \\
&\quad + C_\epsilon\|u\|_{L^p}^{\frac{2p}{p-3}}\textstyle\int |u|^p dx + C_\epsilon\|u\|_{L^p}^{\frac{2p}{p-3}}\textstyle\int |b|^p dx
\end{aligned} \tag{2.4}$$

We now deal with the last term on the right hand side of (2.1). Obviously,

$$\begin{aligned}
\int (u\cdot\nabla P)|u|^{p-2}dx &= -(p-2)\int P(u\cdot\nabla u\cdot u)|u|^{p-4}dx \\
&\le \frac{\epsilon p^2}{4}\int |u|^{p-2}|\nabla u|^2 dx + C_\epsilon\int |P|^2|u|^{p-2}dx \\
&= \epsilon\int |\nabla(|u|^{p/2})|^2 dx + C_\epsilon\int |P|^2|u|^{p-2}dx.
\end{aligned} \tag{2.5}$$

The second term in (2.5) is similar in nature to and can be handled as the term in (2.2). In fact, for p_1, q_1 and r_1 defined as in (2.3)

$$\int |P|^2 |u|^{p-2} dx \le C \|P\|_{L^{p_1}} \|P\|_{L^{q_1}} \||u|^{p-1}\|_{L^{r_1}}$$

$$\le C \left(\|u\|_{L^p}^2 + \|b\|_{L^p}^2\right) \left(\||u|^{p/2}\|_{L^{12/(p-1)}}^{4/p} + \||b|^{p/2}\|_{L^{12/(p-1)}}^{4/p}\right) \||u|^{p/2}\|_{L^{6(p-2)/(2p-5)}}^{2(1-2/p)},$$

where we have used the relation $-\Delta P = \partial_i\partial_j(u_i u_j) - \partial_i \partial_j (b_i b_j)$. The estimate of this term can then be completed as in (2.4).

Combining the above estimates, we find that

$$\frac{d}{dt}\int [|u|^p + |b|^p]\, dx + C\int \left[\nu|\nabla(|u|^{p/2})|^2 + \eta|\nabla(|b|^{p/2})|^2\right] dx$$

$$\le C\, \|u\|_{L^p}^{\frac{2p}{p-3}} \int [|u|^p + |b|^p]\, dx$$

Applying Gronwall's inequality then leads to the conclusion stated in (a).

We now prove (b). One easily verifies that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + \nu \int |\Delta u|^2 dx$$

$$= -\int (\partial_{kj}^2 P)(\partial_k u_j) - \int \partial_k u_i\, \partial_i u_j\, \partial_k u_j + \int \partial_k b_i\, \partial_i\, b_j \partial_k u_j$$

$$\frac{1}{2}\frac{d}{dt}\int |\nabla b|^2 dx + \eta \int |\Delta b|^2 dx = -\int \partial_k u_i\, \partial_i b_j\, \partial_k b_j + \int \partial_k b_i\, \partial_i u_j\, \partial_k b_j,$$

where the repeated indices are summed. We now estimate the nonlinear terms. Using the following Gagliardo-Nirenberg inequalities

$$\|\nabla u\|_{L^3} \le C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}},$$

$$\|\nabla u\|_{L^3} \le C \|u\|_{L^p}^{\frac{p}{p+6}} \|\Delta u\|_{L^2}^{\frac{6}{p+6}},$$

we obtain that for $p > 3$

$$\begin{aligned} \left|\int \partial_k u_i \partial_i u_j \partial_k u_j\right| &\le \|\nabla u\|_{L^3}^3 = \|\nabla u\|_{L^3}^{1+\frac{6}{p}} \|\nabla u\|_{L^3}^{2-\frac{6}{p}} \\ &\le C\left[\|u\|_{L^p}^{\frac{p}{p+6}} \|\Delta u\|_{L^2}^{\frac{6}{p+6}}\right]^{1+\frac{6}{p}} \left[\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}\right]^{2-\frac{6}{p}} \\ &= C\|u\|_{L^p} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{1+\frac{3}{p}} \\ &\le \epsilon \|\Delta u\|_{L^2}^2 + C_\epsilon \|u\|_{L^p}^{\frac{2p}{p-3}} \|\nabla u\|_{L^2}^2 \end{aligned}$$

Now we treat the term involving the pressure P. Using the relation

$$-\Delta P = \partial_j u_i\, \partial_i u_j - \partial_j b_i\, \partial_i b_j,$$

we obtain

$$\begin{aligned} \left|\int (\partial_{kj}^2 P)(\partial_k u_j)\right| &\le C\|\partial_{kj}^2 P\|_{L^{\frac{3}{2}}} \|\partial_k u_j\|_{L^3} \\ &\le C\|\Delta P\|_{L^{\frac{3}{2}}} \|\nabla u\|_{L^3} \\ &\le C\|\nabla u\|_{L^3}^3 + C\|\nabla u\|_{L^3} \|\nabla b\|_{L^3}^2. \end{aligned}$$

Other terms can be estimated in a similar fashion. For example,

$$\left|\int \partial_k b_i\, \partial_i u_j\, \partial_k b_j\right| \le C\|\nabla u\|_{L^3}\, \|\nabla b\|_{L^3}^2 \le C\|\nabla u\|_{L^3}^3 + C\|\nabla b\|_{L^3}^3$$

Gathering these estimates, we obtain

$$\frac{d}{dt}\int \left[|\nabla u|^2 + |\nabla b|^2\right] dx + \int \left[\nu|\Delta u|^2 + \eta|\Delta b|^2\right] dx$$
$$\le C\|u\|_{L^p}^{\frac{2p}{p-3}}\, \|\nabla u\|_{L^2}^2 + C\|b\|_{L^p}^{\frac{2p}{p-3}}\, \|\nabla b\|_{L^2}^2.$$

Applying Gronwall's inequality then implies (b). This completes the proof of Theorem 2.1.

3. **Regularity results for axisymmetric weak solutions.** We now turn our attention to weak solutions with axisymmetry. The axisymmetric u and b satisfy a simplified version of (A.5), namely

$$\frac{\partial u_r}{\partial t} + u\cdot\nabla u_r - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + b\cdot\nabla b_r - \frac{b_\theta^2}{r} + \nu\left(\Delta u_r - \frac{u_r}{r^2}\right)$$
$$\frac{\partial u_\theta}{\partial t} + u\cdot\nabla u_\theta + \frac{u_\theta u_r}{r} = b\cdot\nabla b_\theta + \frac{b_\theta b_r}{r} + \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2}\right)$$
$$\frac{\partial u_z}{\partial t} + u\cdot\nabla u_z = -\frac{\partial p}{\partial z} + b\cdot\nabla b_z + \nu\Delta u_z$$
$$\frac{\partial b_r}{\partial t} + u\cdot\nabla b_r = b\cdot\nabla u_r + \eta\left(\Delta b_r - \frac{1}{r^2} b_r\right)$$
$$\frac{\partial b_\theta}{\partial t} + u\cdot\nabla b_\theta + \frac{u_\theta b_r}{r} = b\cdot\nabla u_\theta + \frac{b_\theta u_r}{r} + \eta\left(\Delta b_\theta - \frac{b_\theta}{r^2}\right)$$
$$\frac{\partial b_z}{\partial t} + u\cdot\nabla b_z = b\cdot\nabla u_z + \eta\Delta b_z.$$

Because of the axisymmetry, the differential operators in these equations do not involve partial derivatives with respect to θ. For example, $u\cdot\nabla u_r$ should read $u_r\partial_r u_r + u_z\partial_z u_r$ and the Laplacian operator $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}$.

The equations for the vorticity ω and current density j with axisymmetry can be obtained either by setting partial derivatives with respect to θ equal to zero in the general form (A.6) and (A.7) or by differentiating the equations of u and b above and utilizing the definitions of ω and j, i.e.,

$$\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \frac{1}{r}\partial_r(r u_\theta);$$
$$j_r = -\partial_z b_\theta, \quad j_\theta = \partial_z b_r - \partial_r b_z, \quad j_z = \frac{1}{r}\partial_r(r b_\theta).$$

Either way we find that (ω, j) obeys

$$\frac{\partial \omega_r}{\partial t} + (u_r\partial_r + u_z\partial_z)\omega_r - (\omega_r\partial_r + \omega_z\partial_z)u_r$$
$$= (b_r\partial_r + b_z\partial_z)j_r - (j_r\partial_r + j_z\partial_z)b_r + \nu\left(\Delta - \frac{1}{r^2}\right)\omega_r \tag{3.1}$$

$$\partial_t\omega_\theta + (u_r\partial_r + u_z\partial_z)\omega_\theta - \frac{u_r\omega_\theta}{r}$$
$$= (b_r\partial_r + b_z\partial_z)j_\theta - \frac{b_r j_\theta}{r} + \nu\left(\Delta\omega_\theta - \frac{\omega_\theta}{r^2}\right) + \frac{1}{r}\partial_z(u_\theta^2 - b_\theta^2) \tag{3.2}$$

$$\partial_t\omega_z + (u_r\partial_r + u_z\partial_z)\omega_z = (\omega_r\partial_r + \omega_z\partial_z)u_z$$
$$+(b_r\partial_r + b_z\partial_z)j_z - (j_r\partial_r + j_z\partial_z)b_z + \nu\Delta\omega_z \tag{3.3}$$

$$\frac{\partial j_r}{\partial t} + (u_r\partial_r + u_z\partial_z)j_r = (b_r\partial_r\omega_r + b_z\partial_z\omega_r) + \eta\left(\Delta j_r - \frac{j_r}{r^2}\right)$$
$$+(\partial_z u_r\partial_r b_\theta + \partial_z u_z\partial_z b_\theta) - (\partial_z b_r\partial_r u_\theta + \partial_z b_z\partial_z u_\theta) + \frac{\partial_z(u_\theta b_r - u_r b_\theta)}{r} \tag{3.4}$$

$$\partial_t j_\theta + (u_r\partial_r + u_z\partial_z)j_\theta = (b_r\partial_r + b_z\partial_z)\omega_\theta + \eta\left(\Delta j_\theta - \frac{j_\theta}{r^2}\right)$$
$$+(\partial_z u_r + \partial_r u_z)(\partial_z b_z - \partial_r b_r) + (\partial_r b_z + \partial_z b_r)(\partial_r u_r - \partial_z u_z), \tag{3.5}$$

where $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}$.

Theorem 3.1. *Let $T > 0$, $p > 2$ and $q \geq 2$ satisfy $2/q + 2/p = 1$. Assume that (u_0, b_0) is in $L^2(drdz)$ and (u, b) is axisymmetric weak solution of the MHD equations (1.1) with initial data (u_0, b_0). If u_r, u_θ, b_r and b_θ are in $L^q([0,T]; L^p(drdz))$, then ω_r, ω_θ, j_r and j_θ are in $L^\infty([0,T]; L^2(drdz)) \cap L^2([0,T]; H^1(drdz))$.*

Proof of Theorem 3.1. Multiplying (3.1) by ω_r, (3.4) by j_r, (3.2) by ω_θ, (3.5) by j_θ, summing the results, and integrating with respect to $drdz$ over $[0,\infty) \times (-\infty,\infty)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\int(\omega_r^2 + \omega_\theta^2 + j_r^2 + j_\theta^2)drdz$$
$$+\nu\int\int(|\nabla\omega_r|^2 + |\nabla\omega_\theta|^2)drdz + \eta\int\int(|\nabla j_r|^2 + |\nabla j_\theta|^2)$$
$$+\nu\int\int\left[\frac{(\omega_r)^2}{r^2} + \frac{(\omega_\theta)^2}{r^2}\right]drdz + \eta\int\int\left[\frac{(j_r)^2}{r^2} + \frac{(j_\theta)^2}{r^2}\right]drdz$$
$$= \nu\int\int\frac{1}{r}\left[(\partial_r\omega_r)\omega_r + (\partial_r\omega_\theta)\omega_\theta)\right]drdz$$
$$-\int\int((u_r\partial_r + u_z\partial_z)\omega_r)\omega_r\,drdz$$
$$+\int\int((\omega_r\partial_r + \omega_z\partial_z)u_r)\omega_r drdz$$
$$+\int\int\left[(b_r\partial_r + b_z\partial_z)j_r - (j_r\partial_r + j_z\partial_z)b_r\right]\omega_r\,drdz$$
$$+\int\int(b_r\partial_r j_\theta + b_z\partial_z j_\theta - u_r\partial_r\omega_\theta - u_z\partial_z\omega_\theta)\omega_\theta\,drdz$$
$$+\int\int\frac{\omega_\theta}{r}(u_r\omega_\theta - u_\theta\omega_r + b_\theta j_r - b_r j_\theta)\,drdz$$
$$+\eta\int\int\frac{1}{r}\left[(\partial_r j_r)j_r + (\partial_r j_\theta)j_\theta\right]drdz$$
$$-\int\int((u_r\partial_r + u_z\partial_z)j_r)j_r\,drdz$$
$$+\int\int(b_r\partial_r\omega_r + b_z\partial_z\omega_r)j_r\,drdz$$
$$+\int\int j_r\left[(\partial_z u_r\partial_r b_\theta + \partial_z u_z\partial_z b_\theta) - (\partial_z b_r\partial_r u_\theta + \partial_z b_z\partial_z u_\theta)\right]drdz$$

$$+\int\int \frac{j_r\,\partial_z(u_\theta b_r - u_r b_\theta)}{r}\,drdz$$

$$+\int\int (b_r\partial_r\omega_\theta + b_z\partial_z\omega_\theta - u_r\partial_r j_\theta - u_z\partial_z j_\theta) j_\theta\,drdz$$

$$+\int\int j_\theta\,[(\partial_z u_r + \partial_r u_z)(\partial_z b_z - \partial_r b_r) + (\partial_r b_z + \partial_z b_r)(\partial_r u_r - \partial_z u_z)]\,drdz$$

For notational convenience, we label the terms on the right hand side of the above equation as R_1, R_2,$\cdots$, and R_{13} in the order they appear on the right hand side. To obtain bounds for these terms, we use extensively the Hölder inequality, the Gagliardo-Nirenberg inequality and the Young inequality. We may not explicitly mention the particular inequality used in a certain estimate when it is fairly clear which inequality is utilized. We will also omit the bounds and $drdz$ in the integrals below since it is understood that they are with respect to $drdz$ over $[0,\infty)\times(-\infty,\infty)$. In addition, $\|\cdot\|_{L^p}$ should read $\|\cdot\|_{L^p(drdz)}$.

$$R_1 \le \frac{\nu}{2}\int\int\left[\frac{(\omega_r)^2}{r^2} + \frac{(\omega_\theta)^2}{r^2}\right] + \frac{\nu}{2}\int\int\left[(\partial_r\omega_r)^2 + (\partial_r\omega_\theta)^2\right] \tag{3.6}$$

$$R_7 \le \frac{\eta}{2}\int\int\left[\frac{(j_r)^2}{r^2} + \frac{(j_\theta)^2}{r^2}\right] + \frac{\eta}{2}\int\int\left[(\partial_r j_r)^2 + (\partial_r j_\theta)^2\right] \tag{3.7}$$

Integrating by parts and using $\nabla\cdot u = 0$ and $\nabla\cdot b = 0$, we obtain

$$R_2 = -\int\int \frac{u_r}{r}\omega_r^2 \le \left(\int\int\frac{\omega_r^2}{r^2}\right)^{\frac12}\left(\int\int u_r^2\omega_r^2\right)^{\frac12}$$

$$\le \epsilon\int\int\frac{\omega_r^2}{r^2} + \epsilon\|\nabla\omega_r\|_{L^2}^2 + C_\epsilon\|u_r\|_{L^p}^{\frac{2p}{p-2}}\,\|\omega_r\|_{L^2}^2 \tag{3.8}$$

and similarly

$$R_8 \le \epsilon\int\int\frac{j_r^2}{r^2} + \epsilon\|\nabla j_r\|_{L^2}^2 + C_\epsilon\|u_r\|_{L^p}^{\frac{2p}{p-2}}\,\|j_r\|_{L^2}^2, \tag{3.9}$$

where $\epsilon > 0$ is small and C_ϵ is a constant depending on ϵ. To estimate R_3, we split it into two terms R_{31} and R_{32}. Integrating by parts and recalling $\omega_z = u_\theta/r + \partial_r u_\theta$,

$$R_3 = R_{31} + R_{32} = -2\int\int u_r(\partial_r\omega_r)\omega_r + \int\int\left(\frac{u_\theta}{r} + \partial_r u_\theta\right)\partial_z u_r\,\omega_r$$

and

$$R_{31} \le \epsilon\int\int(\partial_r\omega_r)^2 + \epsilon\|\nabla\omega_r\|_{L^2}^2 + C_\epsilon\|u_r\|_{L^p}^{\frac{2p}{p-2}}\|\omega_r\|_{L^2}^2.$$

We further split R_{32} into R_{321} and R_{322}.

$$R_{32} = \int\int\frac{u_\theta}{r}\,\partial_z u_r\,\omega_r + \int\int\partial_r u_\theta\,\partial_z u_r\,\omega_r = R_{321} + R_{322}$$

and

$$R_{321} \le \epsilon\int\int\frac{\omega_r^2}{r^2} + C_\epsilon\|u_\theta\partial_z u_r\|_{L^2}^2 \le \epsilon\int\int\frac{\omega_r^2}{r^2} + C_\epsilon\|u_\theta\|_{L^p}^2\|\omega_\theta\|_{L^{\frac{2p}{p-2}}}$$

$$\le \epsilon\int\int\frac{\omega_r^2}{r^2} + \epsilon\|\nabla\omega_\theta\|_{L^2}^2 + C_\epsilon\|u_\theta\|_{L^p}^{\frac{2p}{p-2}}\,\|\omega_\theta\|_{L^2}^2.$$

For R_{322}, we integrate by parts and obtain

$$R_{322} = -\int (\partial_r\partial_z u_\theta) u_r \omega_r - \int\int u_r (\partial_r u_\theta)(\partial_z \omega_r)$$
$$= \int\int (\partial_r \omega_r) u_r \omega_r - \int\int u_\theta \partial_r u_r (\partial_z \omega_r) - \int\int u_\theta u_r \partial_r\partial_z \omega_r$$
$$= 2\int\int (\partial_r \omega_r) u_r \omega_r - \int\int u_\theta \partial_r u_r \partial_z \omega_r + \int\int u_\theta \partial_z u_r \partial_r \omega_r .$$

The first term of R_{322} can be estimated as before. In estimating the last two terms of R_{322}, we use the inequalities

$$\|\nabla u_r\|_{L^p} \le C\|\omega_\theta\|_{L^p}, \qquad \|\nabla u_\theta\|_{L^p} \le C\|\omega_z\|_{L^p}$$

valid for any $p > 1$. We obtain

$$R_{322} \le \epsilon \int\int \left[(\partial_r \omega_r)^2 + (\partial_z \omega_r)^2\right] + \epsilon \left[\|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla \omega_r\|_{L^2}^2\right]$$
$$+ C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_r\|_{L^2}^2 + C_\epsilon \|u_\theta\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2$$

Collecting the estimates for R_3, we obtain

$$R_3 \le \epsilon \int\int \left[(\partial_r \omega_r)^2 + (\partial_z \omega_r)^2\right] + \epsilon \left[\|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla \omega_r\|_{L^2}^2\right]$$
$$+ \epsilon \int\int \frac{\omega_r^2}{r^2} + C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_r\|_{L^2}^2 + C_\epsilon \|u_\theta\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2 . \tag{3.10}$$

We combine R_4 and R_9 to eliminate certain terms. Using $\nabla \cdot b = 0$ ($\partial_r b_r + \partial_z b_z + b_r/r = 0$ in cylindrical coordinates) and $j_z = b_\theta/r + \partial_r b_\theta$, we obtain

$$R_4 + R_9 = \int\int \frac{b_r}{r} j_r \omega_r - \int\int j_r (\partial_r b_r) \omega_r - \int\int \left(\frac{b_\theta}{r} + \partial_r b_\theta\right) (\partial_z b_r) \omega_r .$$

Integrating by parts in the second term of $R_4 + R_9$, we can handle the first three terms of $R_4 + R_9$ as before. Recalling that $j_r = -\partial_z b_\theta$ and integrating by parts, the last term of $R_4 + R_9$ becomes

$$-\int\int \partial_r b_\theta \, \partial_z b_r \, \omega_r = \int\int b_\theta \left(\partial_r \partial_z b_r\right) \omega_r + \int\int b_\theta \partial_z b_r \partial_r \omega_r$$
$$= -\int\int b_\theta \partial_r b_r \partial_z \omega_r + \int\int j_r \partial_r b_r \omega_r + \int\int b_\theta \partial_z b_r \partial_r \omega_r$$

which can be handled as the term R_{322}. Thus

$$R_4 + R_9 \le \epsilon \int\int \frac{j_r^2 + \omega_r^2}{r^2} + \epsilon \left(\|\nabla \omega_r\|_{L^2}^2 + \|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_r\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right)$$
$$+ C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \left(\|\omega_r\|_{L^2}^2 + \|j_r\|_{L^2}^2\right) + C_\epsilon \|b_\theta\|_{L^p}^{\frac{2p}{p-2}} \left(\|\omega_\theta\|_{L^2}^2 + \|j_\theta\|_{L^2}^2\right) \tag{3.11}$$

We combine R_5 and R_{12}. Integrating by parts and using $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we obtain

$$R_5 + R_{12} = \int\int \frac{b_r}{r} \omega_\theta \, j_\theta - \frac{1}{2} \int\int \frac{u_r}{r} (\omega_\theta^2 + j_\theta^2),$$

which can be easily treated. In fact,

$$R_5 + R_{12} \le \epsilon \int\int \frac{\omega_\theta^2}{r^2} + \epsilon \left(\|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right) + C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \|j_\theta\|_{L^2}^2$$
$$+ C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \left(\|\omega_\theta\|_{L^2}^2 + \|j_\theta\|_{L^2}^2\right) . \tag{3.12}$$

We now turn to R_6 which can be directly bounded as follows.

$$R_6 \le \epsilon \int\int \frac{\omega_\theta^2}{r^2} + \epsilon \left(\|\nabla \omega_r\|_{L^2}^2 + \|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_r\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right)$$
$$+C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2 + C_\epsilon \|u_\theta\|_{L^p}^{\frac{2p}{p-2}} \|\omega_r\|_{L^2}^2$$
$$+C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \|j_\theta\|_{L^2}^2 + C_\epsilon \|b_\theta\|_{L^p}^{\frac{2p}{p-2}} \|j_r\|_{L^2}^2 \tag{3.13}$$

We split R_{10} into four terms: R_{101}, R_{102}, R_{103} and R_{104}.

$$R_{101} = \int\int j_r(\partial_z u_r)(\partial_r b_\theta) = -\int\int b_\theta \partial_r \partial_z u_r j_r - \int b_\theta \partial_z u_r \partial_r j_r$$
$$= -\int\int (\partial_r u_r) j_r^2 + \int\int b_\theta \partial_r u_r \, \partial_z j_r - \int b_\theta \partial_z u_r \partial_r j_r$$

Using $\nabla \cdot u = 0$,

$$R_{102} = \int\int \left(\partial_r u_r + \frac{u_r}{r}\right) j_r^2$$

We perform similar procedure on R_{103} and R_{104}. Therefore

$$R_{10} \le \epsilon \int\int \frac{j_r^2 + \omega_r^2}{r^2} + \epsilon \left(\|\nabla \omega_r\|_{L^2}^2 + \|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_r\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right)$$
$$+C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|j_r\|_{L^2}^2 + C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_r\|_{L^2}^2$$
$$+C_\epsilon \|u_\theta\|_{L^p}^{\frac{2p}{p-2}} \|j_\theta\|_{L^2}^2 + C_\epsilon \|b_\theta\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2 \tag{3.14}$$

Using $\omega_r = -\partial_z u_\theta$ and $j_r = -\partial_z b_\theta$, we can rewrite R_{11} as follows.

$$R_{11} = \int\int \frac{j_r}{r} \left[-\omega_r b_r + u_\theta(\partial_z b_r) - b_\theta(\partial_z u_r) + u_r j_r\right]$$
$$\le \epsilon \int\int \frac{j_r^2}{r^2} + \epsilon \left(\|\nabla \omega_r\|_{L^2}^2 + \|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_r\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right)$$
$$+C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|j_r\|_{L^2}^2 + C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_r\|_{L^2}^2$$
$$+C_\epsilon \|u_\theta\|_{L^p}^{\frac{2p}{p-2}} \|j_\theta\|_{L^2}^2 + C_\epsilon \|b_\theta\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2 \tag{3.15}$$

We now estimate the last term R_{13}. Using $\nabla \cdot u = 0$, $\nabla \cdot b = 0$, $\omega_\theta = \partial_z u_r - \partial_r u_z$ and $j_\theta = \partial_z b_r - \partial_r b_z$, we have

$$R_{13} = \int\int j_\theta \left[(2\partial_z u_r - \omega_\theta)\left(-2\partial_r b_r - \frac{b_r}{r}\right) + (2\partial_z b_r - j_\theta)\left(2\partial_r u_r + \frac{u_r}{r}\right)\right]$$
$$= 4\int\int j_\theta(\partial_z b_r \, \partial_r u_r - \partial_z u_r \, \partial_r b_r) + 2\int\int \frac{j_\theta}{r}(u_r \partial_z b_r - b_r \partial_z u_r)$$
$$+2\int\int j_\theta \, \omega_\theta \partial_r b_r - 2\int\int j_\theta^2 \partial_r u_r + \int\int \frac{j_\theta}{r}\omega_\theta b_r - \int\int \frac{j_\theta}{r} j_\theta u_r$$

Integration by parts in the first term yields

$$4\int\int b_r \partial_z u_r \partial_r j_\theta - 4\int\int b_r \partial_r u_r \partial_z j_\theta$$

and we then have no difficulty bounding the terms in R_{13}.

$$R_{13} \le \epsilon \int\int \frac{j_\theta^2}{r^2} + \epsilon \left(\|\nabla \omega_\theta\|_{L^2}^2 + \|\nabla j_\theta\|_{L^2}^2\right)$$
$$+C_\epsilon \|u_r\|_{L^p}^{\frac{2p}{p-2}} \|j_\theta\|_{L^2}^2 + C_\epsilon \|b_r\|_{L^p}^{\frac{2p}{p-2}} \|\omega_\theta\|_{L^2}^2 \tag{3.16}$$

Combining the estimates (3.6),(3.7),(3.8), (3.9), (3.10),(3.11), (3.12), (3.13), (3.14), (3.15) and (3.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\int(\omega_r^2+\omega_\theta^2+j_r^2+j_\theta^2)drdz \\
&+\nu\int\int(|\nabla\omega_r|^2+|\nabla\omega_\theta|^2)drdz+\eta\int\int(|\nabla j_r|^2+|\nabla j_\theta|^2) \\
&+\nu\int\int\left[\frac{(\omega_r)^2}{r^2}+\frac{(\omega_\theta)^2}{r^2}\right]drdz+\eta\int\int\left[\frac{(j_r)^2}{r^2}+\frac{(j_\theta)^2}{r^2}\right]drdz \\
&\le C_\epsilon\left(\|u_r\|_{L^p}^{\frac{2p}{p-2}}+\|u_\theta\|_{L^p}^{\frac{2p}{p-2}}+\|b_r\|_{L^p}^{\frac{2p}{p-2}}+\|b_\theta\|_{L^p}^{\frac{2p}{p-2}}\right) \\
&\times\left(\|\omega_r\|_{L^2}^2+\|\omega_\theta\|_{L^2}^2+\|j_r\|_{L^2}^2+\|j_\theta\|_{L^2}^2\right).
\end{aligned}$$

Applying Gronwall's inequality then leads to the conclusion sought. This completes the proof of Theorem 3.1.

Appendix. The MHD equations in cylindrical coordinates. We use the notion of dyadic product introduced by J. W. Gibbs to derive in a uniform fashion the equations for (u, b) and (ω, j) in cylindrical coordinates. This approach of deriving equations in cylindrical coordinates is effective. The full MHD equations in cylindrical coordinates can be rarely found in textbooks and the results provided here serve as records for future reference. To offer a clear presentation, we divide the appendix into four parts.

A.1. **Dyadic product.** We now give Gibbs' definition of the dyadic product of two vectors. Let **i**, **j** and **k** be an orthonormal basis of a coordinate system and **c** and **d** be two vectors with coordinates (c_1, c_2, c_3) and (d_1, d_2, d_3), respectively. The dyadic product of **c** and **d**, denoted **cd**, is given by

$$\begin{aligned}
\mathbf{cd} &= c_1d_1\mathbf{ii}+c_1d_2\mathbf{ij}+c_1d_3\mathbf{ik} \\
&+c_2d_1\mathbf{ji}+c_2d_2\mathbf{jj}+c_2d_3\mathbf{jk} \\
&+c_3d_1\mathbf{ki}+c_3d_1\mathbf{kj}+c_3d_3\mathbf{kk}.
\end{aligned}$$

Such a product is neither a dot nor a cross product and has no geometric significance. But when it operates on a vector or a vector operates on it, it gives a definite geometric quantity: another vector. More precisely, if **f** is dotted to **cd**, then **f**·(**cd**) = (**f**·**c**) **d**. Note that the order of the dyadic product and the operating vector must be preserved. That is, **f**·(**cd**) $\neq$ (**cd**)·**f**.

A.2. ∇**,** $\nabla\cdot$**,** Δ **and** $u\cdot\nabla v$ **in cylindrical coordinates.** In this subsection we derive the expressions of ∇, $\nabla\cdot$, Δ and $u\cdot\nabla v$ in cylindrical coordinates. We will use e_r, e_θ and e_z to denote the standard orthonormal basis of the cylindrical coordinate system. One easily checks by geometric means that

$$\frac{\partial e_r}{\partial r}=\frac{\partial e_\theta}{\partial r}=\frac{\partial e_z}{\partial r}=0, \quad \frac{\partial e_r}{\partial z}=\frac{\partial e_\theta}{\partial z}=\frac{\partial e_z}{\partial z}=0, \tag{A.1}$$

$$\frac{\partial e_r}{\partial\theta}=e_\theta, \quad \frac{\partial e_\theta}{\partial\theta}=-e_r, \quad \frac{\partial e_\theta}{\partial\theta}=0 \tag{A.2}$$

and that the gradient operator ∇ in cylindrical coordinates is given by

$$\nabla = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z}.$$

Let u be a vector field with cylindrical coordinates u_r, u_θ and u_z. Then

$$\begin{aligned}
\nabla \cdot u \quad &= \left(e_r \tfrac{\partial}{\partial r} + e_\theta \tfrac{1}{r}\tfrac{\partial}{\partial \theta} + e_z \tfrac{\partial}{\partial z}\right) \cdot (e_r u_r + e_\theta u_\theta + e_z u_z) \\
&= e_r \cdot \left(e_r \tfrac{\partial u_r}{\partial r} + u_r \tfrac{\partial e_r}{\partial r} + e_\theta \tfrac{\partial u_\theta}{\partial r} + u_\theta \tfrac{\partial e_\theta}{\partial r} + e_z \tfrac{\partial u_z}{\partial r} + u_z \tfrac{\partial e_z}{\partial r}\right) \\
&\quad + e_\theta \tfrac{1}{r} \cdot \left(e_r \tfrac{\partial u_r}{\partial \theta} + u_r \tfrac{\partial e_r}{\partial \theta} + e_\theta \tfrac{\partial u_\theta}{\partial \theta} + u_\theta \tfrac{\partial e_\theta}{\partial \theta} + e_z \tfrac{\partial u_z}{\partial \theta} + u_z \tfrac{\partial e_z}{\partial \theta}\right) \\
&\quad + e_z \cdot \left(e_r \tfrac{\partial u_r}{\partial z} + u_r \tfrac{\partial e_r}{\partial z} + e_\theta \tfrac{\partial u_\theta}{\partial z} + u_\theta \tfrac{\partial e_\theta}{\partial z} + e_z \tfrac{\partial u_z}{\partial z} + u_z \tfrac{\partial e_z}{\partial z}\right)
\end{aligned}$$

Using (A.1) and (A.2), there obtains

$$\nabla \cdot u = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

The definition of dyadic product applies not only to two vectors but also to a vector operator and a vector. We now derive an explicit expression for ∇v in cylindrical coordinates, where v is a vector with cylindrical coordinates v_r, v_θ and v_z. After applying (A.1) and (A.2), we obtain

$$\begin{aligned}
\nabla v \quad &= \left(e_r \tfrac{\partial}{\partial r} + e_\theta \tfrac{1}{r}\tfrac{\partial}{\partial \theta} + e_z \tfrac{\partial}{\partial z}\right)(e_r v_r + e_\theta v_\theta + e_z v_z) \\
&= e_r e_r \tfrac{\partial v_r}{\partial r} + e_r e_\theta \tfrac{\partial v_\theta}{\partial r} + e_r e_z \tfrac{\partial v_z}{\partial r} \\
&\quad + e_\theta e_r \tfrac{1}{r}\tfrac{\partial v_r}{\partial \theta} + e_\theta e_\theta \tfrac{1}{r}\left(v_r + \tfrac{\partial v_\theta}{\partial \theta}\right) - e_\theta e_r \tfrac{v_\theta}{r} + e_\theta e_z \tfrac{1}{r}\tfrac{\partial v_z}{\partial \theta} \\
&\quad + e_z e_r \tfrac{\partial v_r}{\partial z} + e_z e_\theta \tfrac{\partial v_\theta}{\partial z} + e_z e_z \tfrac{\partial v_z}{\partial z}
\end{aligned} \tag{A.3}$$

We now calculate the dot product $u \cdot \nabla v$. Using $e_r \cdot (e_r e_r) = (e_r \cdot e_r) e_r = e_r$, $e_r \cdot (e_r e_\theta) = (e_r \cdot e_r) e_\theta = e_\theta$, etc., we have

$$\begin{aligned}
u \cdot \nabla v \quad &= (e_r u_r + e_\theta u_\theta + e_z u_z) \cdot \Big[e_r e_r \tfrac{\partial v_r}{\partial r} + e_r e_\theta \tfrac{\partial v_\theta}{\partial r} + e_r e_z \tfrac{\partial v_z}{\partial r} \\
&\quad + e_\theta e_r \tfrac{1}{r}\tfrac{\partial v_r}{\partial \theta} + e_\theta e_\theta \tfrac{1}{r}\left(v_r + \tfrac{\partial v_\theta}{\partial \theta}\right) - e_\theta e_r \tfrac{v_\theta}{r} + e_\theta e_z \tfrac{1}{r}\tfrac{\partial v_z}{\partial \theta} \\
&\quad + e_z e_r \tfrac{\partial v_r}{\partial z} + e_z e_\theta \tfrac{\partial v_\theta}{\partial z} + e_z e_z \tfrac{\partial v_z}{\partial z} \Big] \\
&= \left(u_r \tfrac{\partial v_r}{\partial r} + \tfrac{u_\theta}{r}\tfrac{\partial v_r}{\partial \theta} + u_z \tfrac{\partial v_r}{\partial z} - \tfrac{u_\theta v_\theta}{r}\right) e_r \\
&\quad + \left(u_r \tfrac{\partial v_\theta}{\partial r} + \tfrac{u_\theta}{r}\tfrac{\partial v_\theta}{\partial \theta} + u_z \tfrac{\partial v_\theta}{\partial z} + \tfrac{u_\theta v_r}{r}\right) e_\theta \\
&\quad + \left(u_r \tfrac{\partial v_z}{\partial r} + \tfrac{u_\theta}{r}\tfrac{\partial v_z}{\partial \theta} + u_z \tfrac{\partial v_z}{\partial z}\right) e_z \\
&= \left(u \cdot \nabla v_r - \tfrac{u_\theta v_\theta}{r}\right) e_r + \left(u \cdot \nabla v_\theta + \tfrac{u_\theta v_r}{r}\right) e_\theta + (u \cdot \nabla v_z) e_z,
\end{aligned}$$

where we have used $u \cdot \nabla$ to denote $u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$.

Since Δu can be seen as a dot product of ∇ and ∇u, we can derive in a similar fashion an explicit formula for Δu in cylindrical coordinates. In fact,

$$\Delta u = \nabla \cdot (\nabla u) = \left(e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z}\right) \cdot (\nabla u) \tag{A.4}$$

Inserting the gradient formula (A.3) (with v replaced by u) in (A.4), we obtain after some simplification that

$$\begin{aligned}
\Delta u \;&= \left(\tfrac{\partial^2 u_r}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial u_r}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2 u_r}{\partial \theta^2} + \tfrac{\partial^2 u_r}{\partial z^2} - \tfrac{u_r}{r^2} - \tfrac{2}{r^2}\tfrac{\partial u_\theta}{\partial \theta}\right) e_r \\
&\quad + \left(\tfrac{\partial^2 u_\theta}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial u_\theta}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2 u_\theta}{\partial \theta^2} + \tfrac{\partial^2 u_\theta}{\partial z^2} + \tfrac{2}{r^2}\tfrac{\partial u_r}{\partial \theta} - \tfrac{u_\theta}{r^2}\right) e_\theta \\
&\quad + \left(\tfrac{\partial^2 u_z}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial u_z}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2 u_z}{\partial \theta^2} + \tfrac{\partial^2 u_z}{\partial z^2}\right) e_z \\
&= \left(\Delta u_r - \tfrac{u_r}{r^2} - \tfrac{2}{r^2}\tfrac{\partial u_\theta}{\partial \theta}\right) e_r + \left(\Delta u_\theta + \tfrac{2}{r^2}\tfrac{\partial u_r}{\partial \theta} - \tfrac{u_\theta}{r^2}\right) e_\theta + \Delta u_z e_z,
\end{aligned}$$

where Δ denotes $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$.

A.3. **Equations of (u,b) in cylindrical coordinates.** Using the building blocks in the previous subsection, we can now rewrite the equations of (u,b) in cylindrical coordinates as follows.

$$\begin{aligned}
\tfrac{\partial u_r}{\partial t} + u\cdot\nabla u_r - \tfrac{u_\theta^2}{r} \;&=\; -\tfrac{\partial p}{\partial r} + b\cdot\nabla b_r - \tfrac{b_\theta^2}{r} + \nu\left(\Delta u_r - \tfrac{u_r}{r^2} - \tfrac{2}{r^2}\tfrac{\partial u_\theta}{\partial \theta}\right) \\
\tfrac{\partial u_\theta}{\partial t} + u\cdot\nabla u_\theta + \tfrac{u_\theta u_r}{r} \;&=\; -\tfrac{1}{r}\tfrac{\partial p}{\partial \theta} + b\cdot\nabla b_\theta + \tfrac{b_\theta b_r}{r} + \nu\left(\Delta u_\theta - \tfrac{u_\theta}{r^2} + \tfrac{2}{r^2}\tfrac{\partial u_r}{\partial \theta}\right) \\
\tfrac{\partial u_z}{\partial t} + u\cdot\nabla u_z \;&=\; -\tfrac{\partial p}{\partial z} + b\cdot\nabla b_z + \nu\Delta u_z \\
\tfrac{\partial b_r}{\partial t} + u\cdot\nabla b_r \;&=\; b\cdot\nabla u_r + \eta\left(\Delta b_r - \tfrac{1}{r^2} b_r - \tfrac{2}{r^2}\tfrac{\partial b_\theta}{\partial \theta}\right) \\
\tfrac{\partial b_\theta}{\partial t} + u\cdot\nabla b_\theta + \tfrac{u_\theta b_r}{r} \;&=\; b\cdot\nabla u_\theta + \tfrac{b_\theta u_r}{r} + \eta\left(\Delta b_\theta - \tfrac{b_\theta}{r^2} + \tfrac{2}{r^2}\tfrac{\partial b_r}{\partial \theta}\right) \\
\tfrac{\partial b_z}{\partial t} + u\cdot\nabla b_z \;&=\; b\cdot\nabla u_z + \eta\Delta b_z
\end{aligned} \tag{A.5}$$

A.4. $\nabla\times$ **and equations of (ω, j) in cylindrical coordinates.** We now turn our attention to the equations of the vorticity ω and current density j in cylindrical coordinates and we start with the expression of $\nabla \times u$.

$$\begin{aligned}
\nabla\times u \;&= \left(e_r\tfrac{\partial}{\partial r} + e_\theta \tfrac{1}{r}\tfrac{\partial}{\partial \theta} + e_z\tfrac{\partial}{\partial z}\right)\times(e_r u_r + e_\theta u_\theta + e_z u_z) \\
&= e_r\times\tfrac{\partial e_r}{\partial r}u_r + e_r\times e_\theta\tfrac{\partial u_\theta}{\partial r} + e_r\times\tfrac{\partial e_\theta}{\partial r}u_\theta + e_r\times e_z\tfrac{\partial u_z}{\partial r} + e_r\times\tfrac{\partial e_z}{\partial r}u_z \\
&\quad + e_\theta\times e_r\tfrac{1}{r}\tfrac{\partial u_r}{\partial \theta} + e_\theta\times\tfrac{\partial e_r}{\partial \theta}\tfrac{u_r}{r} + e_\theta\times\tfrac{\partial e_\theta}{\partial \theta}\tfrac{u_\theta}{r} + e_\theta\times e_z\tfrac{1}{r}\tfrac{\partial u_z}{\partial \theta} + e_\theta\times\tfrac{\partial e_z}{\partial \theta}\tfrac{1}{r}u_z \\
&\quad + e_z\times e_r\tfrac{\partial u_r}{\partial z} + e_z\times\tfrac{\partial e_r}{\partial z}u_r + e_z\times\tfrac{\partial e_\theta}{\partial z}u_\theta + e_z\times e_\theta\tfrac{\partial u_\theta}{\partial z} + e_z\times\tfrac{\partial e_z}{\partial z}u_z.
\end{aligned}$$

Inserting (A.1) and (A.2) in the above expression, there obtains

$$\nabla\times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)e_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)e_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)e_z$$

We now recall the equations of (ω, j) in Cartesian coordinates for the purpose of comparing with their equations in cylindrical coordinates,

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u + b \cdot \nabla j - j \cdot \nabla b + \nu \Delta \omega$$

$$\partial_t j + \nabla \times (u \cdot \nabla b) = \nabla \times (b \cdot \nabla u) + \eta \Delta j.$$

Taking the curl of (A.5) and using the above expression of $\nabla \times$, we find that ω satisfies the following equations in cylindrical coordinates.

$$\begin{aligned} \frac{\partial \omega_r}{\partial t} + u \cdot \nabla \omega_r &= \omega \cdot \nabla u_r + b \cdot \nabla j_r - j \cdot \nabla b_r \\ &\quad + \nu \left(\Delta \omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2} \frac{\partial \omega_\theta}{\partial \theta} \right) \\ \frac{\partial \omega_\theta}{\partial t} + u \cdot \nabla \omega_\theta + \frac{u_\theta \omega_r}{r} &= \omega \cdot \nabla u_\theta + \frac{\omega_\theta u_r}{r} + b \cdot \nabla j_\theta - j \cdot \nabla b_\theta \\ &\quad + \frac{b_\theta j_r - b_r j_\theta}{r} + \nu \left(\Delta \omega_\theta - \frac{\omega_\theta}{r^2} + \frac{2}{r^2} \frac{\partial \omega_r}{\partial \theta} \right) \\ \frac{\partial \omega_z}{\partial t} + u \cdot \nabla \omega_z &= \omega \cdot \nabla u_z + b \cdot \nabla j_z - j \cdot \nabla b_z + \nu \Delta \omega_z \end{aligned} \tag{A.6}$$

To rewrite the equation for j in cylindrical coordinates, we start with the term $\nabla \times (u \cdot \nabla b)$.

$$\begin{aligned} \nabla \times (u \cdot \nabla b) &= \left(e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z} \right) \times \Big[\left(u \cdot \nabla b_r - \frac{u_\theta b_\theta}{r} \right) e_r \\ &\quad + \left(u \cdot \nabla b_\theta + \frac{u_\theta b_r}{r} \right) e_\theta + (u \cdot \nabla b_z) e_z \Big] \\ &= e_r \left[\frac{1}{r} \frac{\partial}{\partial \theta} (u \cdot \nabla b_z) - \frac{\partial}{\partial z} \left(u \cdot \nabla b_\theta + \frac{u_\theta b_r}{r} \right) \right] \\ &\quad + e_\theta \left[\frac{\partial}{\partial z} \left(u \cdot \nabla b_r - \frac{u_\theta b_\theta}{r} \right) - \frac{\partial}{\partial r} (u \cdot \nabla b_z) \right] \\ &\quad + e_z \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r u \cdot \nabla b_\theta + u_\theta b_r \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left(u \cdot \nabla b_r - \frac{u_\theta b_\theta}{r} \right) \right] \end{aligned}$$

We intend to write $\nabla \times (u \cdot \nabla b)$ as a sum of $u \cdot \nabla j$ and a term to be determined below. Since

$$\begin{aligned} j &= \nabla \times b \\ &= e_r \left[\frac{1}{r} \frac{\partial b_z}{\partial \theta} - \frac{\partial b_\theta}{\partial z} \right] + e_\theta \left[\frac{\partial b_r}{\partial z} - \frac{\partial b_z}{\partial r} \right] + e_z \left[\frac{1}{r} \frac{\partial (r b_\theta)}{\partial r} - \frac{1}{r} \frac{\partial b_r}{\partial \theta} \right]. \\ u \cdot \nabla j &= e_r \left[u \cdot \nabla j_r - \frac{u_\theta j_\theta}{r} \right] + e_\theta \left[u \cdot \nabla j_\theta + \frac{u_\theta j_r}{r} \right] + e_z [u \cdot \nabla j_z], \end{aligned}$$

we obtain after a tedious calculation

$$\begin{aligned} \nabla \times (u \cdot \nabla b) &= u \cdot \nabla j \\ &\quad + e_r \left[\frac{\partial u}{\partial \theta} \cdot \nabla b_z - \frac{\partial u}{\partial z} \cdot \nabla b_\theta + \frac{u_r}{r^2} \frac{\partial b_z}{\partial \theta} - \frac{u_\theta}{r} \frac{\partial b_z}{\partial r} - \frac{b_r}{r} \frac{\partial u_\theta}{\partial z} \right] \\ &\quad + e_\theta \left[\frac{\partial u}{\partial z} \cdot \nabla b_r - \frac{\partial u}{\partial r} \cdot \nabla b_z - \frac{b_\theta}{r} \frac{\partial u_\theta}{\partial z} \right] + e_z \Big[\frac{1}{r} \frac{\partial (r u)}{\partial r} \cdot \nabla b_\theta \\ &\quad - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \cdot \nabla b_r - u \cdot \nabla \left(\frac{b_\theta}{r} \right) + \frac{b_\theta}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial (u_\theta b_r)}{\partial r} - \frac{u_r}{r^2} \frac{\partial b_r}{\partial \theta} \Big]. \end{aligned}$$

One can derive the expression of $\nabla \times (b \cdot \nabla u)$ in cylindrical coordinates in a similar fashion. In cylindrical coordinates the current density equations can then be written as follows.

$$\begin{aligned}
\frac{\partial j_r}{\partial t} + u \cdot \nabla j_r &= b \cdot \nabla \omega_r + \eta \left(\Delta j_r - \frac{j_r}{r^2} - \frac{2}{r^2}\frac{\partial j_\theta}{\partial \theta} \right) + R_r(u,b), \\
\frac{\partial j_\theta}{\partial t} + u \cdot \nabla j_\theta &= b \cdot \nabla \omega_\theta + \eta \left(\Delta j_\theta - \frac{j_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} \right) + R_\theta(u,b), \\
\frac{\partial j_z}{\partial t} + u \cdot \nabla j_z &= b \cdot \nabla \omega_z + \eta \Delta j_z + R_z(u,b),
\end{aligned} \tag{A.7}$$

where $R_r(u,b)$, $R_\theta(u,b)$ and $R_z(u,b)$ are given by

$$\begin{aligned}
R_r(u,b) &= \left[\frac{\partial b}{\partial \theta} \cdot \nabla u_z - \frac{\partial b}{\partial z} \cdot \nabla u_\theta + \frac{b_r}{r^2}\frac{\partial u_z}{\partial \theta} - \frac{b_\theta}{r}\frac{\partial u_r}{\partial z} - \frac{u_r}{r}\frac{\partial b_\theta}{\partial z} \right] \\
&\quad - \left[\frac{\partial u}{\partial \theta} \cdot \nabla b_z - \frac{\partial u}{\partial z} \cdot \nabla b_\theta + \frac{u_r}{r^2}\frac{\partial b_z}{\partial \theta} - \frac{u_\theta}{r}\frac{\partial b_r}{\partial z} - \frac{b_r}{r}\frac{\partial u_\theta}{\partial z} \right], \\
R_\theta(u,b) &= \left[\frac{\partial b}{\partial z} \cdot \nabla u_r - \frac{\partial b}{\partial r} \cdot \nabla u_z - \frac{u_\theta}{r^2}\frac{\partial b_z}{\partial \theta} \right] \\
&\quad - \left[\frac{\partial u}{\partial z} \cdot \nabla b_r - \frac{\partial u}{\partial r} \cdot \nabla b_z - \frac{b_\theta}{r^2}\frac{\partial u_z}{\partial \theta} \right], \\
R_z(u,b) &= \left[\frac{1}{r}\frac{\partial (rb)}{\partial r} \cdot \nabla u_\theta - \frac{1}{r}\frac{\partial b_\theta}{\partial \theta} \cdot \nabla u_r - b \cdot \nabla \left(\frac{u_\theta}{r} \right) \right. \\
&\quad \left. + \frac{u_\theta}{r^2}\frac{\partial b_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial (b_\theta u_r)}{\partial r} - \frac{b_r}{r^2}\frac{\partial u_r}{\partial \theta} \right] - \left[\frac{1}{r}\frac{\partial (ru)}{\partial r} \cdot \nabla b_\theta - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \cdot \nabla b_r \right. \\
&\quad \left. - u \cdot \nabla \left(\frac{b_\theta}{r} \right) + \frac{b_\theta}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial (u_\theta b_r)}{\partial r} - \frac{u_r}{r^2}\frac{\partial b_r}{\partial \theta} \right].
\end{aligned}$$

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Received January 2002; revision October 2002; second revision May 2003.

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