COMMUN. MATH. SCI.
Vol. 21, No. 6, pp. 1531–1548

UNIQUENESS OF WEAK SOLUTIONS TO THE BOUSSINESQ EQUATIONS WITH FRACTIONAL DISSIPATION*

RUIHONG JI†, DAN LI‡, AND JIAHONG WU§

Abstract. This paper examines the existence and uniqueness of weak solutions to the d-dimensional Boussinesq equations with fractional dissipation $(-\Delta)^\alpha u$ and fractional thermal diffusion $(-\Delta)^\beta\theta$. The aim is at the uniqueness of weak solutions in the weakest possible inhomogeneous Besov spaces. We establish the local existence and uniqueness in the functional setting $u\in L^\infty(0,T;B^{d/2-2\alpha+1}_{2,1}(\mathbb{R}^d))$ and $\theta\in L^\infty(0,T;B^{d/2}_{2,1}(\mathbb{R}^d))$ when $\alpha>1/4$, $\beta\geq 0$ and $\alpha+2\beta\geq 1$. By decomposing the bilinear term into different frequencies, we are able to obtain a suitable upper bound on the bilinear term, which allows us to close the estimates in the aforementioned Besov spaces.

Keywords. Boussinesq equations; Littlewood-Paley; weak solution; uniqueness.

AMS subject classifications. 35A05; 35Q35; 76D03.

1. Introduction

This paper examines the existence and uniqueness of weak solutions to the d-dimensional incompressible Boussinesq equations with fractional dissipation,

$$\begin{cases} \partial_t u+\nu(-\Delta)^\alpha u=-u\cdot\nabla u-\nabla p+\theta\mathbf{e}_d, \quad x\in\mathbb{R}^d, t>0,\\ \partial_t\theta+\eta(-\Delta)^\beta\theta=-u\cdot\nabla\theta, \quad x\in\mathbb{R}^d, t>0,\\ \nabla\cdot u=0, \quad x\in\mathbb{R}^d, t>0\\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where u, p and θ represent the velocity, the pressure and the temperature, respectively, and $\nu>0$, $\eta>0$, $\alpha\geq 0$ and $\beta\geq 0$ are real parameters. The fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi)=|\xi|^{2\alpha}\widehat{f}(\xi),$$

and

$$\widehat{f}(\xi)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)dx.$$

When $\alpha=\beta=1$, (1.1) reduce to the standard 2D Boussinesq equations with Laplacian dissipation. The standard Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and also play an important role in the study of Raleigh-Bénard convection (see, e.g., [9, 14, 24, 26, 32, 33]).

Although (1.1) with fractional dissipation appears to be a purely mathematical generalization, (1.1) may be physically relevant. Firstly, closely related equations such as the surface quasi-geostrophic equation model important geophysical phenomena (see, e.g., [10, 15, 26]). Secondly, there are geophysical circumstances in which the Boussinesq

*Received: November 16, 2019; Accepted (in revised form): November 18, 2022. Communicated by Shi Jin.

†Geomathematics Key Laboratory of Sichuan Province, Chengdu University of Technology, Chengdu 610059, P.R. China (jiruihong09@cdut.cn).

‡Corresponding author. Department of Mathematics, Sichuan University, Chengdu 610064, P.R. China (dandy0219@hotmail.com).

§Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States (jiahong.wu@okstate.edu).

equations with fractional Laplacian may arise. Flows in the middle atmosphere traveling upward undergo changes due to the change of atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [6,14]). Thirdly, it may be possible to derive the Boussinesq equations with fractional dissipation from the Boltzmann-type equations using suitable rescalings.

The Boussinesq equations have always been of great interest in mathematics. The Boussinesq equations have recently gained renewed interests and there have been substantial developments on the well-posedness problem, especially when the Boussinesq equations involve only partial or fractional dissipation (see, e.g., [1–3,5,7,11,13,16–21, 29,34,35]), there are two different focuses on the well-posedness problem. One is the global existence and regularity of classical solutions while the other is the uniqueness of solutions in a weak functional setting.

Our main result can be stated as follows.

THEOREM 1.1. *Let $d\geq 2$. Consider (1.1) with α and β satisfying*

$$\alpha>\frac{1}{4},\quad \beta\geq 0,\quad \alpha+2\beta\geq 1.$$

Assuming (u_0,θ_0) obeys $\nabla\cdot u_0=\nabla\cdot\theta_0=0$, and

$$u_0\in B^{\frac{d}{2}+1-2\alpha}_{2,1}(\mathbb{R}^d),\quad \theta_0\in B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d).$$

Then, there exist $T>0$ and a unique weak solution (u,θ) of (1.1) on [0,T] satisfying

$$u\in C([0,T];B^{\frac{d}{2}+1-2\alpha}_{2,1}(\mathbb{R}^d))\cap L^1(0,T;B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)), \tag{1.2}$$

and

$$\theta\in C([0,T];B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d))\cap L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1}(\mathbb{R}^d)). \tag{1.3}$$

The topics on the existence and uniqueness of solutions to fluid equations under optimal regularity assumptions have recently attracted considerable interests. One can find several related results on the Boussinesq and the magneto-hydrodynamic (MHD) equations (see, e.g., [8,12,22,23,31]). In particular we make a comparison between the assumptions on the fractional indices and the initial regularity in this paper and those in Jiu, Suo, Wu and Yu on the non-resistive MHD equations [22]. In the case when $\alpha<1$, the assumptions on α and the initial regularity in [22] are

$$0<\alpha<1,\quad u_0,b_0\in B^{\sigma}_{2,\infty}(\mathbb{R}^d)\quad \text{with } \sigma>1+\tfrac{d}{2}-\alpha.$$

In this paper the conditions are

$$\alpha>\frac{1}{4},\quad \beta\geq 0,\quad \alpha+2\beta\geq 1,\quad u_0\in \mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d),\quad \theta_0\in \mathring{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d).$$

Why do we need $\alpha>\frac{1}{4}$ here? Can we allow $\alpha>0$? This is mainly due to our lower regularity assumption on θ_0. If we increase the assumption on θ_0 to

$$\theta_0\in \mathring{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d), \tag{1.4}$$

we can then drop the requirement $\alpha > \frac{1}{4}$. The requirement $\alpha > \frac{1}{4}$ is needed only at one spot, namely in the estimate of J_4 on page 10. For the sake of clarity, we copy the estimate of J_4 here. The term of J_4 is bounded by

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)} J_4 \, d\tau \\
&\le C \int_0^t \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} e^{-C_0 2^{2\alpha j}(t-\tau)} \|\Delta_j \theta^{(n)}\|_{L^2} \, d\tau \\
&\le C \int_0^t \sum_j 2^{\frac{d}{2}j} \|\Delta_j \theta^{(n)}\|_{L^2} e^{-C_0 2^{2\alpha j}(t-\tau)} 2^{(1-2\alpha)j} \, d\tau \\
&\le C \sum_j 2^{(1-4\alpha)j} (1 - e^{-C_1 t 2^{2\alpha j}}) \|\theta^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \\
&\le CM(1-e^{-C_1 T}).
\end{aligned}$$

We need $\alpha > \frac{1}{4}$ in order for the summation to be finite

$$\sum_j 2^{(1-4\alpha)j} < \infty.$$

If we change the regularity setting to (1.4), then we do not need $\alpha > \frac{1}{4}$. In fact, then J_4 can be alternatively bounded by

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)} J_4 \, d\tau \\
&\le C \int_0^t \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} e^{-C_0 2^{2\alpha j}(t-\tau)} \|\Delta_j \theta^{(n)}\|_{L^2} \, d\tau \\
&\le C \int_0^t \sum_j 2^{(\frac{d}{2}+1-\alpha)j} \|\Delta_j \theta^{(n)}\|_{L^2} e^{-C_0 2^{2\alpha j}(t-\tau)} 2^{(-\alpha)j} \, d\tau \\
&\le C \sum_j 2^{(-3\alpha)j} (1 - e^{-C_1 t 2^{2\alpha j}}) \|\theta^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-\alpha}_{2,1})} \\
&\le CM(1-e^{-C_1 T}),
\end{aligned}$$

and $\alpha > 0$ would be sufficient. This explains why we need $\alpha > \frac{1}{4}$ when we make a lower regularity assumption on θ_0, and how we can actually remove it by making more regularity assumption on θ_0.

We explain why the assumptions on the combination of the indices α and β, and the regularity setting of the initial data are optimal. The regularity assumption on u_0, namely $u_0 \in B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)$ is the same as that for the Navier-Stokes with fractional dissipation and is optimal. The assumptions that $\theta_0 \in \mathring{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$ and $\alpha + 2\beta \ge 1$ are needed in order to deal with the term $\int \widetilde{u} \cdot \nabla \theta^{(1)} \cdot \widetilde{\theta} \, dx$ in the proof of the uniqueness result. Therefore the conditions $\theta_0 \in \mathring{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$ and $\alpha + 2\beta \ge 1$ are optimal. Due to many differences between the technical details for the MHD and the Boussinesq equations, we feel that our existence and uniqueness result with optimal regularity assumptions may be useful for further investigations on the MHD equations.

We describe the framework in the proof of Theorem 1.1. The proof is naturally divided into two parts: the existence part and the uniqueness part. The existence of weak solutions is shown by the method of successive approximation. By choosing a suitable functional setup, constructing a successive approximation sequence and proving the uniform boundedness, one can prove that the limit of such sequence is indeed a weak solution of (1.1). The Littlewood-Paley decomposition and Besov space techniques are employed to facilitate the proof of the uniform bounds. The uniqueness of weak solution in the regularity class (1.2) and (1.3) can be established by directly working with the L^2-norm of the difference between any two weak solutions.

The rest of this paper is divided into three sections. Section 2 provides the definition of the Besov spaces and related tools. Section 3 proves the existence part of Theorem 1.1 while Section 4 establishes the uniqueness part of Theorem 1.1.

2. Preparation

This section serves as a preparation. We provide the definition of the Besov spaces and related facts to be used in the subsequent sections. More details can be found in several books and many papers ([4, 25, 27, 28, 30]). In addition, we prove bounds on triple products involving Fourier localized functions to be used extensively in the sections that follow. We start with the partition of unity. Let $B(0,r)$ and $C(0,r_1,r_2)$ denote the standard ball and the annulus, respectively,

$$B(0,r)=\{\xi\in\mathbb{R}^d:|\xi|\le r\},\quad C(0,r_1,r_2)=\{\xi\in\mathbb{R}^d:r_1\le|\xi|\le r_2\}.$$

There are two compactly supported smooth radial functions ϕ and ψ satisfying

$$\begin{aligned}&\operatorname{supp}\phi\subset B(0,\frac{4}{3}),\quad \operatorname{supp}\psi\subset C(0,\frac{3}{4},\frac{8}{3})\\ &\phi(\xi)+\sum_{j\ge 0}\psi(2^{-j}\xi)=1\quad\text{for all}\quad \xi\in\mathbb{R}^d.\end{aligned}\tag{2.1}$$

We use $\widetilde{h}$ and h to denote the inverse Fourier transforms of ϕ and ψ respectively,

$$\widetilde{h}=\mathscr{F}^{-1}\phi,\quad h=\mathscr{F}^{-1}\psi.$$

In addition, for notational convenience, we write $\psi_j(\xi)=\psi(2^{-j}\xi)$. By a simple property of the Fourier transform,

$$h_j(x):=\mathscr{F}^{-1}(\psi_j)(x)=2^{dj}h(2^jx).$$

The inhomogeneous dyadic block operators Δ_j are defined as follows

$$\begin{aligned}&\Delta_j f=0\quad\text{for}\quad j\le -2,\\ &\Delta_{-1}f=\widetilde{h}*f=\int_{\mathbb{R}^d}f(x-y)\widetilde{h}(y)\,dy,\end{aligned}$$

and

$$\Delta_j f=h_j*f=2^{dj}\int_{\mathbb{R}^d}f(x-y)h(2^jy)\,dy\quad\text{for}\quad j\ge 0.$$

The corresponding inhomogeneous low frequency cut-off operator S_j is defined by

$$S_j f=\sum_{k\le j-1}\Delta_k f.$$

For any function f in the usual Schwarz class $\mathcal{S}$, (2.1) implies

$$\hat{f}(\xi) = \phi(\xi)\hat{f}(\xi) + \sum_{j\ge 0}\psi(2^{-j}\xi)\hat{f}(\xi), \tag{2.2}$$

or, in terms of the inhomogeneous dyadic block operators,

$$f = \sum_{j\ge -1}\Delta_j f \quad \text{or} \quad \mathrm{Id} = \sum_{j\ge -1}\Delta_j,$$

where Id denotes the identity operator. More generally, for any F in the space of tempered distributions, denoted $\mathcal{S}'$, (2.2) still holds but in the distributional sense. That is, for $F\in\mathcal{S}'$,

$$F = \sum_{j\ge -1}\Delta_j F \quad \text{or} \quad \mathrm{Id} = \sum_{j\ge -1}\Delta_j \quad \text{in} \quad \mathcal{S}'. \tag{2.3}$$

In fact, one can verify that

$$S_j F := \sum_{k\le j-1}\Delta_k F \to F \quad \text{as} \quad j\to\infty \quad \text{in} \quad \mathcal{S}'.$$

(2.3) is referred to as the Littlewood-Paley decomposition for tempered distributions. In terms of the inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely

$$FG = \sum_{|j-k|\le 2} S_{k-1}F\Delta_k G + \sum_{|j-k|\le 2}\Delta_k F S_{k-1}G + \sum_{k\ge j-1}\Delta_k F\widetilde{\Delta}_k G,$$

where $\widetilde{\Delta}_k = \Delta_{k-1}+\Delta_k+\Delta_{k+1}$. The above result holds due to $\nabla\cdot F = 0$. This is the so-called Bony decomposition. The inhomogeneous Besov space can be defined in terms of Δ_j specified as above.

Definition 2.1. *The inhomogeneous Besov space $B^s_{p,q}$ with $1\le p,q\le\infty$ and $s\in\mathbb{R}$ consists of $f\in\mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Bernstein's inequality is a useful tool on Fourier localized functions and can trade derivatives for integrability. The following lemma provides Bernstein-type inequalities for fractional derivatives.

Lemma 2.1. *Let $\alpha\ge 0$. Let $1\le p,q\le\infty$.*

(1) *If f satisfies*

$$supp\,\widehat{f} \subset \{\xi\in\mathbb{R}^d : |\xi|\le K2^j\},$$

for some integer j and a constant $K>0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*

$$supp\,\widehat{f} \subset \{\xi\in\mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0<K_1\le K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Next, we state and prove bounds for the triple products involving Fourier localized functions. These bounds will be used quite frequently in the proof of Theorem 1.1 in the subsequent section.

LEMMA 2.2. *Let $j\ge 0$ be an integer. Let Δ_j be the inhomogeneous Littlewood-Paley localization operator. Let F be a divergence-free vector field. Then there hold*

$$
\begin{aligned}
|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx| \le C\|\Delta_j H\|_{L^2}\Big(2^j\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k G\|_{L^2}\\
+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m G\|_{L^2}\\
+\sum_{k\ge j-1}2^j 2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Big)
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j G\,dx| \le C\|\Delta_j G\|_{L^2}\Big(\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k G\|_{L^2}\\
+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le j}2^{(1+\frac{d}{2})m}\|\Delta_m G\|_{L^2}\\
+\sum_{k\ge j-1}2^j 2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Big).
\end{aligned}
\tag{2.5}
$$

The proof can be found in [12].

3. The local existence part

This section proves the existence part of Theorem 1.1. The approach is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1.1) in the weak sense.

Proof. **(Proof of existence part of Theorem 1.1.)** We consider a successive approximation sequence $(u^{(n)},\theta^{(n)})$ satisfying

$$
\begin{cases}
u^{(1)}=S_2u_0, \qquad \theta^{(1)}=S_2\theta_0,\\
\partial_t u^{(n+1)}+\nu(-\Delta)^\alpha u^{(n+1)}=\mathbb{P}(-u^{(n)}\cdot\nabla u^{(n+1)})+\theta^{(n)}\mathbf{e}_d, \quad x\in\mathbb{R}^d, t>0\\
\partial_t\theta^{(n+1)}+\eta(-\Delta)^\beta\theta^{(n+1)}=-u^{(n)}\cdot\nabla\theta^{(n+1)}, \quad x\in\mathbb{R}^d, t>0\\
\nabla\cdot u^{(n+1)}=0, \quad x\in\mathbb{R}^d, t>0\\
u^{(n+1)}(x,0)=S_{n+1}u_0, \qquad \theta^{(n+1)}(x,0)=S_{n+1}\theta_0, \quad x\in\mathbb{R}^d,
\end{cases}
\tag{3.1}
$$

where $\mathbb{P}$ is the standard Leray projection. For

$$M=2(\|u_0\|_{B^{\frac{d}{2}+1-2\alpha}_{2,1}}+\|\theta_0\|_{B^{\frac{d}{2}}_{2,1}}),$$

$T>0$ being sufficiently small and $0<\delta<1$ (to be specified later), we set

$$
\begin{aligned}
Y\equiv\{(u,\theta)\,|\,\|u\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})}\le M, \quad \|\theta\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}\le M,\\
\|u\|_{L^1(0,T;B^{\frac{d}{2}+1}_{2,1})}\le\delta, \quad \|\theta\|_{L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1})}\le\delta\}.
\end{aligned}
\tag{3.2}
$$

Our goal is to show that $\{(u^{(n)},\theta^{(n)})\}$ has a subsequence that converges to the weak solution of (1.1). This process consists of three main steps. The first step is to show that $(u^{(n)},\theta^{(n)})$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma while the last step is to show that the limit is indeed a weak solution of (1.1) .

Our main effort is devoted to showing the uniform bound for $(u^{(n)},\theta^{(n)})$ in Y. This is proven by induction. Clearly,

$$\|u^{(1)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})}\le M,\quad \|\theta^{(1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}\le M.$$

If $T>0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1(0,T;B^{\frac{d}{2}+1}_{2,1})}\le T\|S_2u_0\|_{B^{\frac{d}{2}+1}_{2,1}}\le CT\|u_0\|_{B^{\frac{d}{2}+1-2\alpha}_{2,1}}\le\delta,$$

and

$$\|\theta^{(1)}\|_{L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1})}\le T\|S_2\theta_0\|_{B^{\frac{d}{2}+2\beta}_{2,1}}\le CT\|\theta_0\|_{B^{\frac{d}{2}}_{2,1}}\le\delta.$$

Assuming that $(u^{(n)},\theta^{(n)})$ obeys the bounds defined in Y, namely

$$\|u^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})}\le M,\quad \|\theta^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}\le M,$$

and

$$\|u^{(n)}\|_{L^1(0,T;B^{\frac{d}{2}+1}_{2,1})}\le\delta,\quad \|\theta^{(n)}\|_{L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1})}\le\delta.$$

We prove that $(u^{(n+1)},\theta^{(n+1)})$ obeys the same bound for suitably selected $T>0$ and $\delta>0$. For the sake of clarity, the proof of the four bounds is achieved in the following four subsections.

3.1. The estimate of $u^{(n+1)}$ in $B^{1-2\alpha+\frac{d}{2}}_{2,1}(\mathbb{R}^d)$. Let $j\ge 0$ be an integer. Applying Δj to the second equation in (3.1) and then dotting with $\Delta_j u^{(n+1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|^2_{L^2}+\nu\|\Lambda^\alpha\Delta_j u^{(n+1)}\|^2_{L^2}=I_1+I_2,\tag{3.3}$$

where

$$I_1=-\int\Delta_j(u^{(n)}\cdot\nabla u^{(n+1)})\cdot\Delta_j u^{(n+1)}\,dx,$$

and

$$I_2=\int\Delta_j\theta^{(n)}\mathbf{e}_d\cdot\Delta_j u^{(n+1)}\,dx.$$

The dissipative part admits a lower bound

$$\nu\left\|\Lambda^\alpha\Delta_j u^{(n+1)}\right\|^2_{L^2}\ge C_0 2^{2\alpha j}\left\|\Delta_j u^{(n+1)}\right\|^2_{L^2},$$

where $C_0>0$ is a constant. According to Lemma 2.2, I_1 can be bounded by

$$|I_1|\le C\|\Delta_j u^{(n+1)}\|^2_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}$$

$$+C\|\Delta_j u^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}$$
$$+C\|\Delta_j u^{(n+1)}\|_{L^2}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\Delta_k u^{(n)}\|_{L^2}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}.$$

I_2 can be bounded by

$$\begin{aligned}|I_2| &= \int \Delta_j\theta^{(n)}\mathbf{e}_d\cdot\Delta_j u^{(n+1)}\,dx\\ &\le \|\Delta_j\theta^{(n)}\mathbf{e}_d\|_{L^2}\|\Delta_j u^{(n+1)}\|_{L^2}.\end{aligned}$$

Inserting the estimates above in (3.3) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2}+C_0 2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2}\le J_1+J_2+J_3+J_4, \tag{3.4}$$

where

$$\begin{aligned}J_1 &= C\|\Delta_j u^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},\\ J_2 &= C\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2},\\ J_3 &= C2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2},\end{aligned}$$

and

$$J_4=\|\Delta_j\theta^{(n)}\mathbf{e}_d\|_{L^2}.$$

Integrating (3.4) in time yields

$$\begin{aligned}\|\Delta_j u^{(n+1)}(t)\|_{L^2} \le{}& e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\\ &+\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}(J_1+J_2+J_3+J_4)\,d\tau.\end{aligned} \tag{3.5}$$

Multiplying (3.5) by $2^{(1+\frac{d}{2}-2\alpha)j}$ and summing over j, we obtain

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\le\|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}+\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}(J_1+\cdots+J_4)\,d\tau. \tag{3.6}$$

The terms on the right-hand side can be estimated as follows. For any $t\le T$, we have

$$\begin{aligned}&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}J_1\,d\tau\\ &\le C\int_0^t\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\|\Delta_j u^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau\\ &\le C\|u^{(n+1)}\|_{L^\infty(0,t;B^{1+\frac{d}{2}-2\alpha}_{2,1})}\|u^{(n)}\|_{L^1(0,t;B^{1+\frac{d}{2}}_{2,1})}\end{aligned}$$

$$\le C\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}\|u^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})}$$
$$\le C\delta\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}.$$

The term with J_2 is bounded by

$$\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}J_2\,d\tau$$
$$\le C\int_0^t\sum_j 2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j}2^{2\alpha(m-j)}2^{(1+\frac{d}{2}-2\alpha)m}\|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\,d\tau$$
$$\le C\int_0^t\|u^{(n)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}}}\|u^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}\,d\tau$$
$$\le C\delta\|u^{(n+1)}\|_{L^\infty(0,t;B_{2,1}^{1+\frac{d}{2}-2\alpha})}$$
$$\le C\delta\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}.$$

The term with J_3 is bounded by

$$\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}J_3\,d\tau$$
$$=\int_0^t\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\,d\tau$$
$$=C\int_0^t\sum_j\sum_{k\ge j-1}2^{(2+\frac{d}{2}-2\alpha)(j-k)}2^{(1+\frac{d}{2})k}\|\Delta_k u^{(n)}\|_{L^2}2^{(1+\frac{d}{2}-2\alpha)k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\,d\tau$$
$$\le C\int_0^t\|u^{(n)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}}}\|u^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}\,d\tau$$
$$\le C\|u^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})}\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}$$
$$\le C\delta\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}.$$

The term of J_4 is bounded by

$$\begin{aligned}\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}J_4\,d\tau &\le C\int_0^t\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}e^{-C_0 2^{2\alpha j}(t-\tau)}\|\Delta_j\theta^{(n)}\|_{L^2}\,d\tau\\ &\le C\int_0^t\sum_j 2^{\frac{d}{2}j}\|\Delta_j\theta^{(n)}\|_{L^2}e^{-C_0 2^{2\alpha j}(t-\tau)}2^{(1-2\alpha)j}\,d\tau\\ &\le C\sum_j 2^{(1-4\alpha)j}(1-e^{-C_1 t 2^{2\alpha j}})\|\theta^{(n)}\|_{L^\infty(0,T;B_{2,1}^{\frac{d}{2}})}\\ &\le CM(1-e^{-C_1 T}),\end{aligned}$$

where we have used the fact that there exists $C_1\ge 0$ satisfying, for $j\ge 0$

$$\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}\,d\tau\le C2^{(-2\alpha)j}(1-e^{-C_1 T}).$$

Collecting the bounds above and inserting them in (3.6), we obtain, for any $t\le T$,

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\le\|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}+C\delta\|u^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})}+CM(1-e^{C_1T}).\tag{3.7}$$

3.2. The estimate of $\theta^{(n+1)}$ in $B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$. We apply Δ_j to the third equation in (3.1) and then dotting with $\Delta_j\theta^{(n+1)}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta^{(n+1)}\|^2_{L^2}+C_12^{2\beta j}\|\Delta_j\theta^{(n+1)}\|^2_{L^2}\le B_1,\tag{3.8}$$

with

$$B_1=-\int\Delta_j(u^{(n)}\cdot\nabla\theta^{(n+1)})\cdot\Delta_j\theta^{(n+1)}\,dx.$$

By Lemma 2.2, we have

$$\begin{aligned}|B_1|\le{}&C\|\Delta_j\theta^{(n+1)}\|^2_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_mu^{(n)}\|_{L^2}\\&+C\|\Delta_j\theta^{(n+1)}\|_{L^2}\|\Delta_ju^{(n)}\|_{L^2}\sum_{m\le j}2^{(1+\frac{d}{2})m}\|\Delta_m\theta^{(n+1)}\|_{L^2}\\&+C\|\Delta_j\theta^{(n+1)}\|_{L^2}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k\theta^{(n+1)}\|_{L^2}\|\Delta_ku^{(n)}\|_{L^2}.\end{aligned}$$

Inserting the estimate above in (3.8) and eliminating $\|\Delta_j\theta^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j\theta^{(n+1)}\|_{L^2}+C_12^{2\beta j}\|\Delta_j\theta^{(n+1)}\|_{L^2}\le K_1+K_2+K_3,\tag{3.9}$$

$$\begin{aligned}K_1&=C\|\Delta_j\theta^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_mu^{(n)}\|_{L^2},\\K_2&=C\|\Delta_ju^{(n)}\|_{L^2}\sum_{m\le j}2^{(1+\frac{d}{2})m}\|\Delta_m\theta^{(n+1)}\|_{L^2},\end{aligned}$$

and

$$K_3=C2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k\theta^{(n+1)}\|_{L^2}\|\Delta_ku^{(n)}\|_{L^2}.$$

Integrating (3.9) in time yields, for any $t<T$,

$$\|\Delta_j\theta^{(n+1)}(t)\|_{L^2}\le e^{-C_12^{2\beta t}}\|\Delta_j\theta_0^{(n+1)}\|_{L^2}+\int_0^te^{-C_12^{2\beta j}(t-\tau)}(K_1+K_2+K_3)\,d\tau.\tag{3.10}$$

Multiplying (3.10) by $2^{\frac{d}{2}j}$ and summing over j, we obtain

$$\|\theta^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}}\le\|\theta_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}}+\sum_j2^{\frac{d}{2}j}\int_0^te^{-C_12^{2\beta j}(t-\tau)}(K_1+K_2+K_3)\,d\tau.\tag{3.11}$$

By the simple bound

$$e^{-C_1 2^{2\beta j(t-\tau)}} \le 1,$$

the term with K_1 is bounded by

$$\begin{aligned}
&\sum_j 2^{\frac{d}{2}j} \int_0^t K_1\, d\tau \\
&\le \sum_j 2^{\frac{d}{2}j} \int_0^t C\|\Delta_j \theta^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2}\, d\tau \\
&\le C\int_0^t \sum_j 2^{\frac{d}{2}j} \|\Delta_j \theta^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2}\, d\tau \\
&\le C\|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_{2,1})} \\
&\le C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}.
\end{aligned}$$

The estimate for the term with K_2 is similar,

$$\begin{aligned}
\sum_j 2^{\frac{d}{2}j} \int_0^t K_2\, d\tau &\le C\|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_{2,1})} \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \\
&\le C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}.
\end{aligned}$$

The term with K_3 is also similar,

$$\begin{aligned}
&\sum_j 2^{\frac{d}{2}j} \int_0^t K_3\, d\tau \\
&\le C\int_0^t \sum_j \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k \theta^{(n+1)}\|_{L^2} 2^{(1+\frac{d}{2})(j-k)} 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\
&\le C\|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_{2,1})} \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \\
&\le C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}.
\end{aligned}$$

Collecting the estimates and inserting them in (3.11), we have, for any $t \le T$,

$$\|\theta^{(n+1)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \le \|\theta_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}. \tag{3.12}$$

3.3. The estimate of $\|u^{(n+1)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)}$. We multiply (3.5) by $2^{(1+\frac{d}{2})j}$, sum over j and integrate in time to obtain

$$\begin{aligned}
\|u^{(n+1)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_{2,1})} &\le \int_0^T \sum_j 2^{(1+\frac{d}{2})j} e^{-C_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \\
&\quad + \int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} (J_1+J_2+J_3+J_4)\, d\tau\, ds.
\end{aligned} \tag{3.13}$$

We estimate the terms on the right and start with the first term

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} e^{-C_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt$$
$$= \frac{1}{C_0} \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} (1-e^{-C_0 2^{2\alpha j} T}) \|\Delta_j u_0^{(n+1)}\|_{L^2}. \tag{3.14}$$

Since $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}$, it follows from the Dominated Convergence Theorem that

$$\lim_{T\to 0} \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} (1-e^{-C_0 2^{2\alpha j} T}) \|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} e^{-C_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \le \frac{\delta}{4}.$$

Applying Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau\, ds \\
&= C \int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\
&\le C \sum_j 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \int_0^T e^{-C_0 2^{2\alpha j} s}\, ds \\
&\le C(1-e^{-C_2 T}) \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\le C(1-e^{-C_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|u^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \\
&\le C\delta(1-e^{-C_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})},
\end{aligned}$$

where we have used the fact that there exists $C_2 > 0$ satisfying, for $j \ge 0$

$$\int_0^T e^{-C_0 2^{2\alpha j} s}\, ds \le C 2^{-2\alpha j} (1-e^{-C_2 T}). \tag{3.15}$$

We remark that the functional settings here are the inhomogeneous Besov spaces and index j is bounded below. This is the reason why there is $C_2 > 0$ satisfying (3.15). This can not be done for homogeneous Besov spaces. The terms of J_2 and J_3 can be similarly estimated and obey the same bound,

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau\, ds \le C\delta(1-e^{-C_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}$$

and

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau\, ds \le C\delta(1-e^{-C_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}.$$

The term with J_4 is bounded by

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \, ds \\
=&\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} \|\Delta_j \theta^{(n)}\|_{L^2} \, d\tau \, ds \\
\le& C(1-e^{-C_2 T}) \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j \theta^{(n)}\|_{L^2} \, d\tau \\
\le& C(1-e^{-C_2 T}) \|\theta^{(n)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})} \\
\le& C(1-e^{-C_2 T}) \|\theta^{(n)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+2\beta})}.
\end{aligned}$$

Collecting the estimates above leads to

$$\begin{aligned}
\|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})} \le& \frac{\delta}{4} + C\delta(1-e^{-C_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \\
&+ C(1-e^{-C_2 T}) \|\theta^{(n)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+2\beta})}.
\end{aligned} \tag{3.16}$$

3.4. The estimate of $\|\theta^{(n+1)}\|_{L^1\left(0,T;B_{2,1}^{\frac{d}{2}+2\beta}\right)}$. Multiplying (3.11) by $2^{(\frac{d}{2}+2\beta)j}$, summing over j and integrating in time, we have

$$\begin{aligned}
&\|\theta^{(n+1)}(t)\|_{L^1\left(0,T;B_{2,1}^{\frac{d}{2}+2\beta}\right)} \\
\le& \int_0^T \sum_j 2^{(2\beta+\frac{d}{2})j} e^{-C_1 2^{2\beta j} t} \|\Delta_j \theta_0^{(n+1)}\|_{L^2} \, dt \\
&+ \int_0^T \sum_j 2^{(2\beta+\frac{d}{2})j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} (K_1+K_2+K_3) \, d\tau \, ds.
\end{aligned}$$

The terms on the right can be bounded as follows,

$$\begin{aligned}
&\int_0^T \sum_j 2^{(2\beta+\frac{d}{2})j} e^{-C_1 2^{2\beta j} t} \|\Delta_j \theta_0^{(n+1)}\|_{L^2} \, dt \\
=& C_1^{-1} \sum_j 2^{\frac{d}{2}j} (1-e^{-C_1 2^{2\beta j} T}) \|\Delta_j \theta_0^{(n+1)}\|_{L^2}.
\end{aligned}$$

Since $\theta_0 \in B_{2,1}^{\frac{d}{2}}$, it follows from the Dominated Convergence Theorem that

$$\lim_{T\to 0} \sum_j 2^{\frac{d}{2}j} (1-e^{-C_1 2^{2\beta j} T}) \|\Delta_j \theta_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} e^{-C_1 2^{2\beta j} t} \|\Delta_j \theta_0^{(n+1)}\|_{L^2} \, dt \le \frac{\delta}{4}.$$

The terms involving K_1 through K_3 can be bounded as follows

$$
\begin{aligned}
&\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} K_1 \, d\tau \, ds \\
&\le C \int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} \|\Delta_j \theta^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \, ds \\
&\le C \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^T \|\Delta_j \theta^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \int_0^T e^{-C_1 2^{2\beta j} s} \, ds \\
&\le C(1-e^{-C_3 T}) \int_0^T \sum_j 2^{\frac{d}{2}j} \|\Delta_j \theta^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \\
&\le C(1-e^{-C_3 T}) \|\theta^{(n+1)}\|_{L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T;B^{\frac{d}{2}+1}_{2,1}\right)} \\
&\le C\delta(1-e^{-C_3 T}) \|\theta^{(n+1)}\|_{L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)},
\end{aligned}
$$

where $C_3 > 0$ is a constant. The term with K_2 and K_3 admit the same bounds,

$$
\begin{aligned}
&\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} K_2 \, d\tau \, ds \le C\delta(1-e^{-C_3 T}) \|\theta^{(n+1)}\|_{L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)} \\
&\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} K_3 \, d\tau \, ds \le C\delta(1-e^{-C_3 T}) \|\theta^{(n+1)}\|_{L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)}.
\end{aligned}
$$

Collecting the estimates above, we conclude

$$
\|\theta^{(n+1)}\|_{L^1\left(0,T;B^{\frac{d}{2}+2\beta}_{2,1}\right)} \le \frac{\delta}{4} + C\delta(1-e^{-C_3 T}) \|\theta^{(n+1)}\|_{L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)}. \tag{3.17}
$$

In fact, if we choose T and δ satisfying

$$
C\delta \le \frac{1}{4} \quad \text{and} \quad C(1-e^{-C_1 T}) \le \frac{1}{4},
$$

then (3.7) implies

$$
\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le \frac{1}{2}M + \frac{1}{4}\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \frac{1}{4}M.
$$

So

$$
\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le M.
$$

Similary, if $C\delta \le \frac{1}{4}$ and $\|\theta_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \le \frac{1}{2}M$, then (3.12) states

$$
\|\theta^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \le M.
$$

According to (3.16) and (3.17), if we choose T sufficiently small such that

$$
C(1-e^{-C_2 T})M \le \frac{1}{4}, \quad C(1-e^{-C_3 T})M \le \frac{1}{2},
$$

then

$$\|u^{(n+1)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \le \delta, \quad \|\theta^{(n+1)}\|_{L^1\left(0,T;B^{2\beta+\frac{d}{2}}_{2,1}\right)} \le \delta.$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u,\theta)\in Y$ such that a subsequence of $(u^{(n)},\theta^{(n)})$ satisfies

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty\left(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1}\right)\cap L^1\left(0,T;B^{\frac{d}{2}+1}_{2,1}\right),$$

$$\theta^{(n)} \overset{*}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty\left(0,T;B^{\frac{d}{2}}_{2,1}\right)\cap L^1\left(0,T;B^{\frac{d}{2}+2\beta}_{2,1}\right).$$

In order to show that (u,θ) is a weak solution of (1.1), we need to further extract a subsequence which converges strongly to (u,θ). This is done via the Aubin-Lions Lemma. We can show by making use of the equations in (3.1) that $(\partial_t u^{(n)},\partial_t\theta^{(n)})$ is uniformly bounded in

$$\partial_t u^{(n)} \in L^1\left(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1}\right)\cap L^2\left(0,T;B^{\frac{d}{2}+1-3\alpha}_{2,1}\right),$$

$$\partial_t \theta^{(n)} \in L^1\left(0,T;B^{\frac{d}{2}}_{2,1}\right)\cap L^2\left(0,T;B^{\frac{d}{2}-\beta}_{2,1}\right).$$

Since we are in the case of the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of the weakly convergent subsequence, still denoted by $(u^{(n)},\theta^{(n)})$, has the following strongly convergent property,

$$(u^{(n)},\theta^{(n)}) \to (u,\theta) \quad \text{in} \quad L^2(0,T;B^{\gamma}_{2,1}(Q)),$$

where $\frac{d}{2}-\alpha\le\gamma\le\frac{d}{2}$ and $Q\subset\mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u,θ) is indeed a weak solution of (1.1). This process is routine and we omit the details. This completes the proof for the existence part of Theorem 1.1. ◻

4. The uniqueness part

This section proves the uniqueness part of Theorem 1.1.

Proof. **(Proof of the uniqueness part of Theorem 1.1.)** Assuming that $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ are two solutions of (1.1) in the regularity class in (1.2) and (1.3). Their difference $(\widetilde{u},\widetilde{\theta})$ with

$$\widetilde{u}=u^{(2)}-u^{(1)}, \quad \widetilde{\theta}=\theta^{(2)}-\theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t\widetilde{u}+\nu(-\Delta)^\alpha\widetilde{u}=-\mathbb{P}(u^{(2)}\cdot\nabla\widetilde{u}+\widetilde{u}\cdot\nabla u^{(1)})+\widetilde{\theta}, \\ \partial_t\widetilde{\theta}+\eta(-\Delta)^\beta\widetilde{\theta}=-u^{(2)}\cdot\nabla\widetilde{\theta}-\widetilde{u}\cdot\nabla\theta^{(1)}, \\ \nabla\cdot\widetilde{u}=0, \\ \widetilde{u}(x,0)=0, \quad \widetilde{\theta}(x,0)=0. \end{cases} \tag{4.1}$$

We focus on the case when $\alpha>1/4$, $\beta\ge 0$, $\alpha+2\beta\ge 1$. We estimate the difference $(\widetilde{u},\widetilde{\theta})$ in $L^2(\mathbb{R}^d)$. Dotting (4.1) by $(\widetilde{u},\widetilde{\theta})$ and applying the divergence-free condition, we find

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{\theta}(t))\|_{L^2}^2)+\nu\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+\eta\|\Lambda^\beta\widetilde{\theta}\|_{L^2}^2=L_1+L_2+L_3,$$

where

$$L_1=-\int \widetilde{u}\cdot\nabla u^{(1)}\cdot\widetilde{u}\,dx, \tag{4.2}$$

$$L_2=-\int \widetilde{u}\cdot\nabla\theta^{(1)}\cdot\widetilde{\theta}\,dx, \tag{4.3}$$

and

$$L_3=\int \widetilde{u}\cdot\widetilde{\theta}\,dx. \tag{4.4}$$

By Hölder's inequality,

$$|L_1|\le\|\nabla u^{(1)}\|_{L^\infty}\|\widetilde{u}\|_{L^2}^2\le C\|u^{(1)}\|_{B_{2,1}^{\frac{d}{2}+1}}\|\widetilde{u}\|_{L^2}^2.$$

To bound L_2, we set

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{d},\quad \frac{1}{q}=\frac{\alpha}{d}.$$

Applying Hölder's inequality,

$$\begin{aligned}|L_2|&\le\|\widetilde{\theta}\|_{L^2}\|\nabla\theta^{(1)}\|_{L^q}\|\widetilde{u}\|_{L^p}\\ &\le C\|\widetilde{\theta}\|_{L^2}\|\theta^{(1)}\|_{B_{2,1}^{\frac{d}{2}+2\beta}}\|\Lambda^\alpha\widetilde{u}\|_{L^2}\\ &\le\frac{\nu}{2}\|\Lambda^\alpha u\|_{L^2}+C\|\theta^{(1)}\|^2_{B_{2,1}^{\frac{d}{2}+2\beta}}\|\widetilde{\theta}\|_{L^2}^2,\end{aligned}$$

where we have used the inequalities, due to $\frac{\alpha}{2}+\beta\ge\frac{1}{2}$,

$$\|\widetilde{u}\|_{L^p}\le C\|\Lambda^\alpha\widetilde{u}\|_{L^2},$$

and

$$\begin{aligned}\|\nabla\theta^{(1)}\|_{L^q}&\le\sum_{j\ge-1}\|\Delta_j\nabla\theta^{(1)}\|_{L^q}\\ &\le C\sum_{j\ge-1}2^{j+jd(\frac{1}{2}-\frac{1}{q})}\|\Delta_j\theta^{(1)}\|_{L^2}\\ &\le C\sum_{j\ge-1}2^{\frac{d}{2}j+(1-\frac{d}{q})j}\|\Delta_j\theta^{(1)}\|_{L^2}\\ &\le C\sum_{j\ge-1}2^{\frac{d}{2}j+2\beta j}\|\Delta_j\theta^{(1)}\|_{L^2}\\ &\le C\|\theta^{(1)}\|_{B_{2,1}^{\frac{d}{2}+2\beta}}.\end{aligned}$$

Applying Hölder's inequality

$$|L_3|\le\|\widetilde{u}\|_{L^2}\|\widetilde{\theta}\|_{L^2}.$$

Combining these estimates lead to

$$\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+\nu\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+2\eta\|\Lambda^\beta\widetilde{\theta}\|_{L^2}^2$$

$$\leq C\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}}(\|\widetilde{u}\|^2_{L^2}+\|\widetilde{\theta}\|^2_{L^2})+C\|\theta^{(1)}\|^2_{B^{\frac{d}{2}+2\beta}_{2,1}}\|\widetilde{\theta}\|^2_{L^2}. \tag{4.5}$$

Since $(u^{(1)},\theta^{(1)})$ is in the regularity class (1.2) and (1.3), we obtain

$$\int_0^T \|u^{(1)}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}dt<\infty,$$
$$\int_0^T \|\theta^{(1)}(t)\|_{B^{\frac{d}{2}+2\beta}_{2,1}}dt<\infty.$$

Applying Gronwall's inequality to (4.5), we have

$$\|\widetilde{u}\|_{L^2}=\|\widetilde{\theta}\|_{L^2}=0,$$

which leads to the desired uniqueness. □

Acknowledgments. Ji is supported by the National Natural Science Foundation of China (Grants No. 12001065) and Creative Research Groups of the Natural Science Foundation of Sichuan (Grants No. 2023NSFSC1984), Wu is partially supported by NSF grant DMS 2104682 and the AT&T Foundation at Oklahoma State University.

REFERENCES

[1] D. Adhikari, C. Cao, and J. Wu, *The 2D Boussinesq equations with vertical viscosity and vertical diffusivity*, J. Differ. Equ., 249:1078–1088, 2010. 1

[2] D. Adhikari, C. Cao, and J. Wu, *Global regularity results for the 2D Boussinesq equations with vertical dissipation*, J. Differ. Equ., 251:1637–1655, 2011. 1

[3] D. Adhikari, C. Cao, J. Wu, and X. Xu, *Small global solutions to the damped two-dimensional Boussinesq equations*, J. Differ. Equ., 256:3594–3613, 2014. 1

[4] J. Bergh and J. Löfström, *Interpolation Space, An Introduction*, Springer-Verlag, Berlin Heidelberg-New York, 223, 1976. 2

[5] C. Cao and J. Wu, *Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal., 208:985–1004, 2013. 1

[6] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*-II, Geophy. J. R. Astr. Soc., 13:529–539, 1967. 1

[7] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203:497–513, 2006. 1

[8] J. Chemin, D. McCormick, J. Robinson, and J. Rodrigo, *Local existence for the non-resistive MHD equations in Besov spaces*, Adv. Math., 286:1–31, 2016. 1

[9] P. Constantin and C.R. Doering, *Infinite Prandtl number convection*, J. Stat. Phys., 94:159–172, 1999. 1

[10] P. Constantion, A. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar*, Nonlinearity, 7:1495–1533, 1994. 1

[11] P. Constantin and V. Vicol, *Nonlinear maximum princples for dissipative linear nonlocal operators and applications*, Geom. Funct. Anal., 22:1289–1321, 2012. 1

[12] Y. Dai, R. Ji, and J. Wu, *Unique weak solutions of the magnetohydrodynamic equation with fractional dissipation*, Z. Angew. Math. Mech., 100(7):e201900290, 2020. 1, 2

[13] R. Danchin, *Remarks on the lifespan of the solutions to some models of incompressible fluid mechanics*, Proc. Amer. Math. Soc., 141:1979–1993, 2013. 1

[14] A.E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, London, 1982. 1

[15] I. Held, R. Pierrehumbert, S. Garner, and K. Swanson, *Surface quasi-geostrophic dynamic*, J. Fluid Mech., 282:1–20, 1995. 1

[16] T. Hmidi, *On a maximum principle and its application to the logarithmically critical Boussinesq system*, Anal. Partial Differ. Equ., 4:247–284, 2011. 1

[17] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for a Boussinesq Navier-Stokes with critical dissipation*, J. Differ. Equ., 249:2147–2174, 2010. 1

[18] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for Euler-Boussinesq system with critical dissipation*, Commun. Partial Differ. Equ., 36:420–445, 2011. 1

[19] T. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst., 12:1–12, 2005. 1

[20] J. Jia, J. Peng, and K. Li, *On the global well-posedness of a generalized 2D Boussinesq equations*, Nonlinear Differ. Equ. Appl., 22(4):911–945, 2015. 1

[21] Q. Jiu, C. Miao, J. Wu, and Z. Zhang, *The two-dimensional incompressible Boussinesq equations with general critical dissipation*, SIAM J. Math. Anal., 46:3426–3452, 2014. 1

[22] Q. Jiu, X. Suo, J. Wu, and H. Yu, *Unique weak solutions of the non-resistive magnetohydrodynamic equations with fractional dissipation*, Commun. Math. Sci., 18(4):987–1022, 2020. 1

[23] J. Li, W. Tan, and Z. Yin, *Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces*, Adv. Math., 317:786–798, 2017. 1

[24] A.J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics, AMS/CIMS, 9, 2003. 1

[25] C. Miao, J. Wu, and Z. Zhang, *Littlewood-Paley Theory and Its Applications in Partial Differential Equations of Fluid Dynamics*, Science Press (in Chinese), Beijing, China, 2012. 2

[26] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1987. 1

[27] H. Rahouri, J-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011. 2

[28] T. Runst and W. Sickel, *Sobolev Space of Fractional Order, Nemytskij Operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996. 2

[29] A. Stefanov and J. Wu, *A global regularity result for the 2D Boussineq equations with critical dissipation*, J. Anal. Math., 137(1):269–290, 2019. 1

[30] H. Triebel, *Theory of Function Spaces* II, Birkhauser Verlag, 1992. 2

[31] R. Wan, *On the uniqueness for the 2D MHD equations without magnetic diffusion*, Nonlinear Anal. Real World Appl., 30:32–40, 2016. 1

[32] B. Wen, N. Dianati, E. Lunasin, G. Chini, and C. Doering, *New upper bounds and reduced dynamical modeling for Rayleigh-Bénard convection in a fluid saturated porous layer*, Commun. Nonlinear Sci. Numer. Simul., 17:2191–2199, 2012. 1

[33] J. Whitehead and C. Doering, *Internal heating driven convection at infinite Prandtl number*, J. Math. Phys., 52(11):093101, 2011. 1

[34] J. Wu, X. Xu, L. Xue, and Z. Ye, *Regularity results for the 2D Boussinesq equations with critical and supercritical dissipation*, Commun. Math. Sci., 14:1963–1997, 2016. 1

[35] X. Xu, *Global regularity of solutions of 2D Boussinesq equations with fractional diffusion*, Nonlinear Anal., 72:677–681, 2010. 1