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Stability and optimal decay for the 3D magnetohydrodynamic equations with only horizontal dissipation

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Abstract. This paper develops an effective approach to establishing the optimal decay estimates on solutions of the 3D anisotropic magnetohydrodynamic (MHD) equations with only horizontal dissipation. As our first step, we prove the global existence and stability of solutions to the aforementioned MHD system emanating from any initial data with small H^1-norm. Due to the lack of dissipation in the vertical direction, the large-time behavior does not follow from the classical approaches. The analysis of the nonlinear terms are much more difficult than in the case of full dissipation. In particular, we need to represent the MHD equations in an integral form, exploit cancellations and other properties such as the incompressibility in order to control terms involving vertical derivatives.

1. Introduction

Stability and large-time behavior are among the most essential properties of partial differential equations (PDEs) modeling incompressible fluids. This paper intends to understand these crucial properties for the following 3D anisotropic magnetohydrodynamic (MHD) system with only horizontal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \Delta_h u + b \cdot \nabla b, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t b + u \cdot \nabla b = \eta \Delta_h b + b \cdot \nabla u, \\ \nabla \cdot u = 0,\ \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\ b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where u and b represent the fluid velocity and the magnetic field, respectively, $P = p + \frac{1}{2}|b|^2$ denotes the total pressure, and $\nu > 0$ and $\eta > 0$ are the viscosity and magnetic diffusivity, respectively. Here, $\Delta_h = \partial_1^2 + \partial_2^2$ denotes the horizontal Laplacian.

Anisotropic dissipation arises in the modeling of various fluids and geophysical fluids such as in the Prandtl equation as well as in the study of turbulent flows in Ekman layer [31]. Anisotropic magnetic diffusion is relevant in the study of several astrophysical phenomenon such as the modeling of magnetic reconnection (see, e.g., [32,33]).

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The main goal of this paper is to provide optimal upper bounds on the decay rates of solutions (u, b) to (1.1). In addition, we will also reveal some remarkable large-time behavior of (u, b) such as the accelerated decay rates for the vertical components u_3 and b_3 of u and b, respectively.

The lack of full dissipation in (1.1) prevents us from applying the powerful tools designed for PDEs with full dissipation. The Fourier splitting method of Schonbek has been very successful in understanding the large-time behavior of various fully dissipative PDEs modeling fluids (see, e.g., [1,36,37]). But unfortunately, this method does not appear to apply to the anisotropic dissipation case.

This paper presents a new approach that can effectively extract the large-time behavior of solutions to (1.1). Inspired by a recent work of Ji et al. [19] on the 3D anisotropic Navier–Stokes equations, the approach and techniques of this paper are not merely a parallel extension from the Navier–Stokes to the MHD equations. This paper offers several improvements. For example, the smallness requirement on the initial data in this paper is imposed only on the H^1-norm instead of higher regularity norm as in [19]. Furthermore, this paper reveals some unusual decay properties of (u, b) to (1.1). The third components u_3 and b_3 of u and b, respectively, actually decay faster than the corresponding horizontal ones in the Sobolev setting. This phenomenon was first remarkably observed by Xu and Zhang in the Besov setting [47]. It reflects the enhanced dissipation in the vertical components due to their special evolution structures of u_3 and b_3. We are able to recover this property in the Sobolev setting with no elaborated conditions on the initial data.

The main result established in this paper is summarized in the following theorem.

Theorem 1.1. *Let $k \geq 1$ be an integer. Assume $(u_0, b_0) \in H^k(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, there exists a constant $\varepsilon > 0$ such that, if*

$$\|u_0\|_{H^1(\mathbb{R}^3)} + \|b_0\|_{H^1(\mathbb{R}^3)} \leq \varepsilon, \tag{1.2}$$

then system (1.1) has a unique global solution (u, b) satisfying

$$(u, b) \in L^\infty(0, \infty; H^k(\mathbb{R}^3)), \ \nabla_h u, \nabla_h b \in L^2(0, \infty; H^k(\mathbb{R}^3))$$

and, for any $t > 0$,

$$\begin{aligned} &\|u(t)\|_{H^k}^2 + \|b(t)\|_{H^k}^2 + \int_0^t (\|\nabla_h u(\tau)\|_{H^k}^2 + \|\nabla_h b(\tau)\|_{H^k}^2) d\tau \\ &\leq C(\|u_0\|_{H^k}^2 + \|b_0\|_{H^k}^2), \end{aligned}$$

where $C > 0$ is a constant proportional to the initial norm $\|u_0\|_{H^k}^2 + \|b_0\|_{H^k}^2$.

Furthermore, if $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $s \geq 3$ satisfies, for $\frac{1}{2} < \sigma < 1$,

$$\Lambda_h^{-\sigma} u_0, \ \Lambda_h^{-\sigma} b_0, \ \Lambda_h^{-\sigma} \partial_3 u_0, \ \Lambda_h^{-\sigma} \partial_3 b_0, \ \Lambda_h^{-\sigma} \Lambda_3^{-\frac{\sigma}{2}} u_0, \ \Lambda_h^{-\sigma} \Lambda_3^{-\frac{\sigma}{2}} b_0 \in L^2(\mathbb{R}^3), \tag{1.3}$$

then the global solution (u, b) *of* (1.1) *satisfies*

$$\begin{aligned}
&\|u(t)\|_{H^1(\mathbb{R}^3)} + \|b(t)\|_{H^1(\mathbb{R}^3)} \le C\varepsilon. \\
&\|u(t)\|_{H^s(\mathbb{R}^3)} + \|b(t)\|_{H^s(\mathbb{R}^3)} \le C. \\
&\|\Lambda_h^{-\sigma} u(t)\|_{L^2(\mathbb{R}^3)} + \|\Lambda_h^{-\sigma} b(t)\|_{L^2(\mathbb{R}^3)} \\
&\quad + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2(\mathbb{R}^3)} + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2(\mathbb{R}^3)} \le C.
\end{aligned}$$

$$\|u(t)\|_{L^2(\mathbb{R}^3)} + \|b(t)\|_{L^2(\mathbb{R}^3)} + \|\partial_3 u(t)\|_{L^2(\mathbb{R}^3)} + \|\partial_3 b(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{\sigma}{2}}. \tag{1.4}$$

$$\|\nabla_h u(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla_h b(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1+\sigma}{2}}. \tag{1.5}$$

$$\|u_3(t)\|_{L^2} + \|b_3(t)\|_{L^2} \le C(1+t)^{-\frac{3\sigma}{4}}. \tag{1.6}$$

$$\|\nabla_h u_3(t)\|_{L^2} + \|\nabla_h b_3(t)\|_{L^2} \le C(1+t)^{-(\frac{3\sigma}{4}+\frac{1}{2})}. \tag{1.7}$$

Theorem 1.1 reflects the enhanced regularity and decay rates in the vertical components of u and b. The decay rate for the L^2-norm of the vertical components u_3 and b_3 in (1.6) is higher than that for (u, b) in (1.4). Similarly the rate for $\nabla_h u_3$ and $\nabla_h b_3$ in (1.7) is also higher than that for $\nabla_h u$ and $\nabla_h b$ in (1.5). In addition, the rates for (u, b) in (1.4) and (1.5) are the same as those for the anisotropic heat equation

$$\begin{cases} \partial_t F = \Delta_h F, \quad x \in \mathbb{R}^3,\ t > 0, \\ F(x, 0) = F_0(x) \end{cases}$$

with F_0 satisfying similar assumptions as those for (u_0, b_0). Consequently, the rates obtained in Theorem 1.1 are optimal.

Well-posedness and stability problems on the MHD systems have recently attracted considerable interests, and significant progress has been made (see, e.g., [3–18,20–24,26–28,34,35,39–42,44,45,48–52,54–58]). The references listed here are by no means exhaustive. We shall not attempt to detail these results, but instead describe two closely related work on the 3D MHD equations with anisotropic dissipation. Wu and Zhu [46] investigated the 3D MHD equations with horizontal dissipation and vertical magnetic diffusion and were able to establish the global well-posedness and stability near a background magnetic field. In a preprint submitted for publication [29], Lin, Wu and Zhu considered the 3D MHD equations with velocity dissipation in only one direction and magnetic diffusion in two directions. They showed that any perturbation near a suitable background magnetic field is globally stable. When the dissipation is only in one direction, the velocity nonlinearity does not appear to admit a suitable upper bound when the spatial domain is $\mathbb{R}^3$. By exploiting the symmetric structures in the vorticity formulation to encounter the derivative loss problem as well as the stabilizing effect of the background magnetic field, [29] was able to solve this difficult well-posedness and stability problem. The large-time behavior of the global solutions obtained in [29] and [46] remains open. It is hoped that the method developed in this paper will help solve the large-time behavior problem on the MHD systems considered in [29] and [46].

We outline the main steps in the proof of Theorem 1.1. The proof is naturally divided into two main parts: the stability part and the part for the decay estimates. Assuming that the initial H^1-norm is small, we show that the H^1-norm of the solution is uniformly bounded by the initial H^1-norm. Through an inductive process of controlling the H^k-norm via the H^{k-1}-norm, we further show that any H^k-norm is bounded uniformly and proportional to the initial H^k-norm. We remark that the initial H^k-norm with $k \geq 2$ is not assumed to be small. The decay estimates are shown via the bootstrapping argument (see [38, p.21]). We assume the smallness of the initial H^1-norm as well as the negative Sobolev setting, namely (1.2) and (1.3). In particular, we have

$$\gamma_0 := \|\Lambda_h^{-\sigma} u_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b_0\|_{L^2}^2 < \infty.$$

We note that γ_0 is not assumed to be small. Let (u, b) be the corresponding solution. We make the ansatz that, for $t \in [0, T]$ with $T > 0$,

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2}^2 \leq 3\gamma_0.$$

The initial time interval $[0, T]$ is guaranteed by the local well-posedness. Our main efforts are then devoted to proving the improved inequality, for all $t \in [0, T]$,

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2}^2 \leq 2\gamma_0. \quad (1.8)$$

Then, the bootstrapping argument implies $T = \infty$ and that (1.8) actually holds for any $t < \infty$. The proof of (1.8) requires considerable efforts and is divided into five steps. The first step computes the decay rate for $\nabla u(t)$ and $\nabla b(t)$, while the second step estimates $\|\nabla_h u(t)\|_{L^2}$ and $\|\nabla_h b(t)\|_{L^2}$. The third step reveals enhanced dissipation and higher decay rates for the vertical components u_3 and b_3 and their horizontal derivatives $\nabla_h u_3$ and $\nabla_h b_3$. The fourth step obtains upper bounds on $\|\Lambda^{-\sigma} u\|_{L^2}^2 + \|\Lambda^{-\sigma} b\|_{L^2}^2$ and $\|\Lambda^{-\sigma} \partial_3 u\|_{L^2}^2 + \|\Lambda^{-\sigma} \partial_3 b\|_{L^2}^2$ in terms of the derivatives of u and b. The final step invokes the decay rates for the derivatives from the first three steps to establish (1.8).

The rest of this paper is divided into two sections. Section 2 proves the stability part of Theorem 1.1, while Sect. 3 presents the decay estimates in Theorem 1.1, as outlined in the previous paragraph.

2. Proof of the stability part in Theorem 1.1

We split the proof of Theorem 1.1 into two main parts. The first part establishes the stability result and is presented in this section. The second part verifies the decay rates and will be given in the subsequent section.

First, we state two lemmas to be used in the proof. The first lemma provides an upper bound for the L^p-norm of a one-dimensional function, which serves as a basic ingredient for anisotropic upper bounds. A proof can be found in [53].

Lemma 2.1. *Let* $2 \le p \le \infty$. *Let* $s > \frac{1}{2} - \frac{1}{p}$. *Then, there exists a constant* $C = C(p, s)$ *such that, for any 1D functions* $f \in H^s(\mathbb{R})$,

$$\|f\|_{L^p(\mathbb{R})} \le C \|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

In particular, if $p = \infty$ *and* $s = 1$, *then any* $f = f(x_3) \in H^1(\mathbb{R})$ *satisfies*

$$\|f\|_{L^\infty(\mathbb{R})} \le C \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

The second lemma provides an anisotropic upper bound for the integral of a triple product. It is a very powerful tool in dealing with anisotropic equations. A simple proof of this lemma can be found in [46].

Lemma 2.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\int_{\mathbb{R}^3} |fgh| dx \lesssim \|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$
$$\int_{\mathbb{R}^3} |fgh| dx \lesssim \|f\|_{L^2}^{\frac{1}{4}} \|\partial_1 f\|_{L^2}^{\frac{1}{4}} \|\partial_2 f\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2}^{\frac{1}{4}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_3 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}.$$

With these two lemmas at our disposal, we are ready to prove the stability part of Theorem 1.1.

Proof of the stability result in Theorem 1.1. Since the local well-posedness of (1.1) in H^k with any $k \ge 1$ follows from a standard approach such as Friedrichs' method (see, e.g., [2,30]), this proof focuses on the global *a priori* bounds.

Taking the inner product of (u, b) with the first two equations, integrating by parts and using $\nabla \cdot u = \nabla \cdot b = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \nu \|\nabla_h u\|_{L^2}^2 + \eta \|\nabla_h b\|_{L^2}^2 = 0. \tag{2.1}$$

Integrating in time yields, for $c_0 = \min\{\nu, \eta\}$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + c_0 \int_0^t (\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2) d\tau$$
$$\le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

Applying $\partial_i (i = 1, 2, 3)$ to the first two equations of (1.1), dotting the results by $\partial_i u$ and $\partial_i b$, respectively, integrating over $\mathbb{R}^3$ and adding them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 (\|\partial_i u\|_{L^2}^2 + \|\partial_i b\|_{L^2}^2) + \sum_{i=1}^3 (\nu \|\partial_i \nabla_h u\|_{L^2}^2 + \eta \|\partial_i \nabla_h b\|_{L^2}^2) \\
&= -\sum_{i=1}^3 \int \partial_i (u \cdot \nabla u) \cdot \partial_i u dx + \sum_{i=1}^3 \int \partial_i (b \cdot \nabla b) \cdot \partial_i u dx \\
&\quad - \sum_{i=1}^3 \int \partial_i (u \cdot \nabla b) \cdot \partial_i b dx + \sum_{i=1}^3 \int \partial_i (b \cdot \nabla u) \cdot \partial_i b dx \\
&:= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

Using $\nabla\cdot u=0$, we have

$$
\begin{aligned}
A_1 &= -\sum_{i=1}^{3}\int \partial_i(u\cdot\nabla u)\cdot\partial_i u dx = -\sum_{i=1}^{3}\int \partial_i u\cdot\nabla u\cdot\partial_i u dx\\
&= -\sum_{i=1}^{3}\int \partial_i u_h\cdot\nabla_h u\cdot\partial_i u dx - \sum_{i=1}^{3}\int \partial_i u_3\partial_3 u\cdot\partial_i u dx\\
&:= A_{11}+A_{12}.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
A_{11} &\le C\sum_{i=1}^{3}\|\partial_i u_h\|_{L^2}^{\frac12}\|\partial_1\partial_i u_h\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\nabla_h u\|_{L^2}^{\frac12}\|\partial_i u\|_{L^2}^{\frac12}\|\partial_2\partial_i u\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^1}\|\nabla_h u\|_{H^1}^2.
\end{aligned}
$$

To bound A_{12}, we further divide it into two parts and then apply Lemma 2.1 to obtain

$$
\begin{aligned}
A_{12} &= -\sum_{i=1}^{3}\int \partial_i u_3\partial_3 u\cdot\partial_i u dx\\
&= -\sum_{i=1}^{2}\int \partial_i u_3\partial_3 u\cdot\partial_i u dx - \int \partial_3 u_3\partial_3 u\cdot\partial_3 u dx\\
&= -\sum_{i=1}^{2}\int \partial_i u_3\partial_3 u\cdot\partial_i u dx + \int \nabla_h\cdot u_h\partial_3 u\cdot\partial_3 u dx\\
&\le C\|\nabla_h u_3\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}^{\frac12}\\
&\quad + C\|\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^1}\|\nabla_h u\|_{H^1}^2.
\end{aligned}
$$

Therefore,

$$
A_1 \le C\|u\|_{H^1}\|\nabla_h u\|_{H^1}^2.
$$

Similarly,

$$
\begin{aligned}
A_3 &= \sum_{i=1}^{3}\int \partial_i(u\cdot\nabla b)\cdot\partial_i b\, dx = \sum_{i=1}^{3}\int \partial_i u\cdot\nabla b\cdot\partial_i b\, dx\\
&= \sum_{i=1}^{3}\int \partial_i u_h\cdot\nabla_h b\cdot\partial_i b\, dx + \sum_{i=1}^{3}\int \partial_i u_3\partial_3 b\cdot\partial_i b\, dx\\
&\le C\sum_{i=1}^{3}\|\partial_i u_h\|_{L^2}^{\frac12}\|\partial_1\partial_i u_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_3\nabla_h b\|_{L^2}^{\frac12}\|\partial_i b\|_{L^2}^{\frac12}\|\partial_2\partial_i b\|_{L^2}^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{i=1}^{3}\|\partial_i u_3\|_{L^2}^{\frac12}\|\partial_3\partial_i u_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12}\|\partial_i b\|_{L^2}^{\frac12}\|\partial_2\partial_i b\|_{L^2}^{\frac12}\\
&\le C(\|u\|_{H^1}+\|b\|_{H^1})(\|\nabla_h u\|_{H^1}^2+\|\nabla_h b\|_{H^1}^2).
\end{aligned}
$$

Due to

$$
\int b\cdot\nabla\partial_i b\cdot\partial_i u\,dx+\int b\cdot\nabla\partial_i u\cdot\partial_i b\,dx=0,
$$

we have

$$
\begin{aligned}
A_2+A_4&=\sum_{i=1}^{3}\int\partial_i(b\cdot\nabla b)\cdot\partial_i u\,dx+\sum_{i=1}^{3}\int\partial_i(b\cdot\nabla u)\partial_i b\,dx\\
&=\sum_{i=1}^{3}\int\partial_i b\cdot\nabla b\cdot\partial_i u dx+\sum_{i=1}^{3}\int\partial_i b\cdot\nabla u\cdot\partial_i b\,dx\\
&=\sum_{i=1}^{3}\int\partial_i b_h\cdot\nabla_h b\cdot\partial_i u\,dx+\sum_{i=1}^{3}\int\partial_i b_3\partial_3 b\cdot\partial_i u\,dx\\
&\quad+\sum_{i=1}^{3}\int\partial_i b_h\cdot\nabla_h u\cdot\partial_i b dx+\sum_{i=1}^{3}\int\partial_i b_3\partial_3 u\cdot\partial_i b dx\\
&\le C\sum_{i=1}^{3}\|\partial_i b_h\|_{L^2}^{\frac12}\|\partial_1\partial_i b_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_3\nabla_h b\|_{L^2}^{\frac12}\|\partial_i u\|_{L^2}^{\frac12}\|\partial_2\partial_i u\|_{L^2}^{\frac12}\\
&\quad+C\sum_{i=1}^{3}\|\partial_i b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i b_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12}\|\partial_i u\|_{L^2}^{\frac12}\|\partial_2\partial_i u\|_{L^2}^{\frac12}\\
&\quad+\sum_{i=1}^{3}\|\partial_i b_h\|_{L^2}^{\frac12}\|\partial_1\partial_i b_h\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\nabla_h u\|_{L^2}^{\frac12}\|\partial_i b\|_{L^2}^{\frac12}\|\partial_2\partial_i b\|_{L^2}^{\frac12}\\
&\quad+C\sum_{i=1}^{3}\|\partial_i b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i b_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\|\partial_i b\|_{L^2}^{\frac12}\|\partial_2\partial_i b\|_{L^2}^{\frac12}\\
&\le C(\|u\|_{H^1}+\|b\|_{H^1})(\|\nabla_h u\|_{H^1}^2+\|\nabla_h b\|_{H^1}^2).
\end{aligned}
$$

Combining the estimates above, we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\|u\|_{\dot H^1}^2+\|b\|_{\dot H^1}^2)+\nu\|\nabla_h u\|_{\dot H^1}^2+\eta\|\nabla_h b\|_{\dot H^1}^2\\
&\quad\le C(\|u\|_{H^1}+\|b\|_{H^1})(\|\nabla_h u\|_{H^1}^2+\|\nabla_h b\|_{H^1}^2).
\end{aligned}
\tag{2.2}
$$

Adding (2.2) and (2.1) up yields

$$
\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^1}^2+\|b\|_{H^1}^2)+2\nu\|\nabla_h u\|_{H^1}^2+2\eta\|\nabla_h b\|_{H^1}^2\\
&\quad\le C(\|u\|_{H^1}+\|b\|_{H^1})(\|\nabla_h u\|_{H^1}^2+\|\nabla_h b\|_{H^1}^2).
\end{aligned}
$$

Integrating in time, and choosing ε in (1.2) small enough such that

$$
\|u_0\|_{H^1}+\|b_0\|_{H^1}\le C^{-1}c_0,\quad c_0=\min\{\nu,\eta\},
$$

we obtain

$$\begin{aligned}&\|u(t)\|_{H^1}^2+\|b(t)\|_{H^1}^2+c_0\int_0^t\|\nabla_h u(\tau)\|_{H^1}^2+2\eta\|\nabla_h b(\tau)\|_{H^1}^2d\tau\\&\quad\le\|u_0\|_{H^1}^2+\|b_0\|_{H^1}^2.\end{aligned}$$

Next, we prove by induction on k that

$$\begin{aligned}&\|u(t)\|_{H^k}^2+\|b(t)\|_{H^k}^2+\int_0^t(\|\nabla_h u(\tau)\|_{H^k}^2+\|\nabla_h b(\tau)\|_{H^k}^2)d\tau\\&\quad\le C(\|u_0\|_{H^k}^2+\|b_0\|_{H^k}^2),\end{aligned}\tag{2.3}$$

where $C(\|u_0\|_{H^k}^2+\|b_0\|_{H^k}^2)$ is a constant depending on the initial norm $\|(u_0,b_0)\|_{H^k}$ only. Clearly, (2.3) holds for $k=1$. Assume that for any integer $k\ge 2$, we have

$$\begin{aligned}&\|u(t)\|_{H^{k-1}}^2+\|b(t)\|_{H^{k-1}}^2+\int_0^t(\|\nabla_h u(\tau)\|_{H^{k-1}}^2+\|\nabla_h b(\tau)\|_{H^{k-1}}^2)d\tau\\&\quad\le C(\|u_0\|_{H^{k-1}}^2+\|b_0\|_{H^{k-1}}^2).\end{aligned}\tag{2.4}$$

Applying $\partial_i^k(i=1,2,3)$ to the first two equations of (1.1), dotting the results by $\partial_i^k u$ and $\partial_i^k b$, respectively, integrating over $\mathbb{R}^3$ and adding them up, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3(\|\partial_i^k u\|_{L^2}^2+\|\partial_i^k b\|_{L^2}^2)+\sum_{i=1}^3(\nu\|\partial_i^k\nabla_h u\|_{L^2}^2+\eta\|\partial_i^k\nabla_h b\|_{L^2}^2)\\&\quad=-\sum_{i=1}^3\int\partial_i^k(u\cdot\nabla u)\cdot\partial_i^k u dx+\sum_{i=1}^3\int\partial_i^k(b\cdot\nabla b)\cdot\partial_i^k u dx\\&\qquad-\sum_{i=1}^3\int\partial_i^k(u\cdot\nabla b)\partial_i^k b dx+\sum_{i=1}^3\int\partial_i^k(b\cdot\nabla u)\cdot\partial_i^k b dx\\&\quad:=K_1+K_2+K_3+K_4.\end{aligned}$$

Set $C_k^j=\frac{k!}{j!(k-j)!}$. By $\nabla\cdot u=0$,

$$\begin{aligned}K_1&=-\sum_{i=1}^3\int\partial_i^k(u\cdot\nabla u)\cdot\partial_i^k u dx\\&=-\sum_{i=1}^3\sum_{j=1}^k C_k^j\int\partial_i^j u\cdot\nabla\partial_i^{k-j}u\cdot\partial_i^k u dx\\&=-\sum_{i=1}^3\sum_{j=1}^k C_k^j\int\partial_i^j u_h\cdot\nabla_h\partial_i^{k-j}u\cdot\partial_i^k u dx-\sum_{i=1}^3\sum_{j=1}^k C_k^j\int\partial_i^j u_3\partial_3\partial_i^{k-j}u\cdot\partial_i^k u dx\\&:=K_{11}+K_{12}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
K_{11} &= -\sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_h \cdot \nabla_h \partial_i^{k-j} u \cdot \partial_i^k u dx \\
&\le C \sum_{i=1}^{3}\sum_{j=1}^{k} \|\partial_i^j u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i^j u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_i^{k-j} u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h \partial_i^{k-j} u\|_{L^2}^{\frac{1}{2}} \|\partial_i^k u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^k u\|_{L^2}^{\frac{1}{2}} \\
&\le C \|u\|_{H^k} \|\nabla_h u\|_{H^{k-1}} \|\nabla_h u\|_{H^k} \\
&\le \frac{c_0}{16} \|\nabla_h u\|_{H^k}^2 + C \|u\|_{H^k}^2 \|\nabla_h u\|_{H^{k-1}}^2
\end{aligned}$$

and

$$\begin{aligned}
K_{12} &= -\sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_3 \partial_3 \partial_i^{k-j} u \cdot \partial_i^k u dx \\
&\le C \sum_{i=1}^{3}\sum_{j=1}^{k} \|\partial_i^j u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^j u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^{k-j} u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \partial_i^{k-j} u\|_{L^2}^{\frac{1}{2}} \|\partial_i^k u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^k u\|_{L^2}^{\frac{1}{2}} \\
&\le C \|u\|_{H^k} \|\nabla_h u\|_{H^{k-1}} \|\nabla_h u\|_{H^k} \\
&\le \frac{c_0}{16} \|\nabla_h u\|_{H^k}^2 + C \|u\|_{H^k}^2 \|\nabla_h u\|_{H^{k-1}}^2,
\end{aligned}$$

where we have used the fact, due to $\nabla \cdot u = 0$,

$$\begin{aligned}
\sum_{i=1}^{3} \|\partial_i^j u_3\|_{L^2} &\le C(\|\nabla_h^j u_3\|_{L^2} + \|\partial_3^j u_3\|_{L^2}) \\
&\le C(\|\nabla_h^j u_3\|_{L^2} + \|\partial_3^{j-1} \nabla_h \cdot u_h\|_{L^2}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_3 &= \sum_{i=1}^{3} \int \partial_i^k (u \cdot \nabla b) \cdot \partial_i^k b \, dx \\
&= \sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u \cdot \nabla \partial_i^{k-j} b \cdot \partial_i^k b \, dx \\
&= \sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_h \cdot \nabla_h \partial_i^{k-j} b \cdot \partial_i^k b \, dx + \sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_3 \partial_3 \partial_i^{k-j} b \cdot \partial_i^k b \, dx \\
&\le C \sum_{i,j=1}^{3} \|\partial_i^j u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i^j u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_i^{k-j} b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h \partial_i^{k-j} b\|_{L^2}^{\frac{1}{2}} \|\partial_i^k b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^k b\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \sum_{i,j=1}^{3} \|\partial_i^j u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^j u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^{k-j} b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \partial_i^{k-j} b\|_{L^2}^{\frac{1}{2}} \|\partial_i^k b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^k b\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{c_0}{16} \|\nabla_h u\|_{H^k}^2 + \frac{c_0}{16} \|\nabla_h b\|_{H^k}^2 + C(\|u\|_{H^k}^2 + \|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2 + \|\nabla_h b\|_{H^{k-1}}^2).
\end{aligned}$$

Due to

$$\int b\cdot\nabla\partial_i^k b\cdot\partial_i^k u\,dx+\int b\cdot\nabla\partial_i^k u\cdot\partial_i^k b\,dx=0,$$

we have

$$\begin{aligned}
&K_2+K_4\\
&=\sum_{i=1}^{3}\int\partial_i^k(b\cdot\nabla b)\cdot\partial_i^k u\,dx+\sum_{i=1}^{3}\int\partial_i^k(b\cdot\nabla u)\cdot\partial_i^k b\,dx\\
&=\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b\cdot\nabla\partial_i^{k-j}b\cdot\partial_i^k udx+\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b\cdot\nabla\partial_i^{k-j}u\cdot\partial_i^k b\,dx\\
&=\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b_h\cdot\nabla_h\partial_i^{k-j}b\cdot\partial_i^k udx+\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b_3\partial_3\partial_i^{k-j}b\cdot\partial_i^k udx\\
&\quad+\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b_h\cdot\nabla_h\partial_i^{k-j}u\cdot\partial_i^k b\,dx+\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j b_3\partial_3\partial_i^{k-j}u\cdot\partial_i^k b\,dx\\
&\le C\sum_{i,j=1}^{3}\|\partial_i^j b_h\|_{L^2}^{\frac12}\|\partial_1\partial_i^j b_h\|_{L^2}^{\frac12}\|\nabla_h\partial_i^{k-j}b\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_i^{k-j}b\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_2\partial_i^k u\|_{L^2}^{\frac12}\\
&\quad+C\sum_{i,j=1}^{3}\|\partial_i^j b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i^j b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i^{k-j}b\|_{L^2}^{\frac12}\|\partial_1\partial_3\partial_i^{k-j}b\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_2\partial_i^k u\|_{L^2}^{\frac12}\\
&\quad+\sum_{i,j=1}^{3}\|\partial_i^j b_h\|_{L^2}^{\frac12}\|\partial_1\partial_i^j b_h\|_{L^2}^{\frac12}\|\nabla_h\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_i^k b\|_{L^2}^{\frac12}\|\partial_2\partial_i^k b\|_{L^2}^{\frac12}\\
&\quad+C\sum_{i,j=1}^{3}\|\partial_i^j b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i^j b_3\|_{L^2}^{\frac12}\|\partial_3\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_1\partial_3\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_i^k b\|_{L^2}^{\frac12}\|\partial_2\partial_i^k b\|_{L^2}^{\frac12}\\
&\le\frac{c_0}{16}\|\nabla_h u\|_{H^k}^2+\frac{c_0}{16}\|\nabla_h b\|_{H^k}^2+C(\|u\|_{H^k}^2+\|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2+\|\nabla_h b\|_{H^{k-1}}^2).
\end{aligned}$$

Combining the estimates above, we derive that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{\dot H^k}^2+\|b\|_{\dot H^k}^2)+c_0(\|\nabla_h u\|_{\dot H^k}^2+\|\nabla_h b\|_{\dot H^k}^2)\\
&\quad\le C(\|u\|_{H^k}^2+\|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2+\|\nabla_h b\|_{H^{k-1}}^2).
\end{aligned}\tag{2.5}$$

Adding (2.5) to (2.1) gives

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^k}^2+\|b\|_{H^k}^2)+c_0(\|\nabla_h u\|_{H^k}^2+\|\nabla_h b\|_{H^k}^2)\\
&\quad\le C(\|u\|_{H^k}^2+\|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2+\|\nabla_h b\|_{H^{k-1}}^2).
\end{aligned}$$

By Gronwall's inequality and (2.4),

$$\begin{aligned}
&\|u\|_{H^k}^2+\|b\|_{H^k}^2+\int_0^t(\|\nabla_h u(\tau)\|_{H^k}^2+\|\nabla_h b(\tau)\|_{H^k}^2)d\tau\\
&\quad\le(\|u_0\|_{H^k}^2+\|b_0\|_{H^k}^2)e^{C\int_0^t(\|\nabla_h u(\tau)\|_{H^{k-1}}^2+\|\nabla_h b(\tau)\|_{H^{k-1}}^2)d\tau}\\
&\quad\le(\|u_0\|_{H^k}^2+\|b_0\|_{H^k}^2)e^{C(\|u_0\|_{H^{k-1}}^2+\|b_0\|_{H^{k-1}}^2)}\\
&\quad\le C(\|u_0\|_{H^k}^2+\|b_0\|_{H^k}^2).
\end{aligned}$$

We have thus established (2.3). This finishes the proof of the stability part. □

3. Decay estimates

This section establishes the decay estimates in Theorem 1.1. First, we state two lemmas to be used in the proof.

We need two elementary facts. The first fact, stated in Lemma 3.1, is Minkowski's inequality. It is an extremely useful tool that allows us to estimate the Lebesgue norm with larger index first followed by the Lebesgue norm with a smaller index. The following version is taken from [2, p.4] and a more general statement can be found in [25, p.47].

Lemma 3.1. *Let (X_1,μ_1) and (X_2,μ_2) be two measure spaces. Let f be a nonnegative measurable function over $X_1\times X_2$. For all $1\le p\le q\le\infty$, we have*

$$\big\|\|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)}\big\|_{L^q(X_2,\mu_2)}\le\big\|\|f(x_1,\cdot)\|_{L^q(X_2,\mu_2)}\big\|_{L^p(X_1,\mu_1)}.$$

In particular, for a nonnegative measurable function f over $\mathbb{R}^m\times\mathbb{R}^n$ and for $1\le p\le q\le\infty$,

$$\big\|\|f\|_{L^p(\mathbb{R}^m)}\big\|_{L^q(\mathbb{R}^n)}\le\big\|\|f\|_{L^q(\mathbb{R}^n)}\big\|_{L^p(\mathbb{R}^m)}.$$

The second fact provides an exact L^p-L^q decay estimate for the generalized heat operator associated with a fractional Laplacian. The following lemma and its proof can be found in [43].

Lemma 3.2. *Let $\sigma\ge0$, $\alpha>0$, $\nu>0$, $1\le p\le q\le\infty$. Then,*

$$\|\Lambda^\sigma e^{-\nu(-\Delta)^\alpha t}f\|_{L^q(\mathbb{R}^d)}\le C\,t^{-\frac{\sigma}{2\alpha}-\frac{d}{2\alpha}\left(\frac1p-\frac1q\right)}\|f\|_{L^p(\mathbb{R}^d)}.$$

We are now ready to prove the decay estimates in Theorem 1.1.

Proof of the decay estimates in Theorem 1.1. The framework of the proof is the bootstrapping argument. The H^1-norm of the initial data (u_0,b_0) is assumed to be small, namely

$$\|u_0\|_{H^1}+\|b_0\|_{H^1}\le\varepsilon$$

for some sufficiently small $\varepsilon > 0$. Due to the condition (1.3) on (u_0, b_0), we write

$$\gamma_0 := \|\Lambda_h^{-\sigma} u_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u_0\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b_0\|_{L^2}^2 < \infty. \quad (3.1)$$

We note that γ_0 is not assumed to be small. Let (u, b) be the corresponding solution. We make the ansatz that, for $t \in [0, T]$ with $T > 0$,

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2}^2 \leq 3\gamma_0. \quad (3.2)$$

We remark that the initial time interval $[0, T]$ is guaranteed by the local well-posedness. Our main efforts are then devoted to proving the improved inequality, for all $t \in [0, T]$,

$$\|\Lambda_h^{-\sigma} u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2}^2 \leq 2\gamma_0, \quad (3.3)$$

Then, the bootstrapping argument then implies $T = \infty$ and that (3.3) actually holds for any $t < \infty$.

The rest of proof is devoted to showing (3.3). The proof is very long and thus divided into five steps for the sake of clarity.

Step 1. Decay rate for ∇u and ∇b. More precisely, we show that

$$\begin{aligned} &\|u(t)\|_{L^2} + \|b(t)\|_{L^2} + \|\partial_3 u(t)\|_{L^2} + \|\partial_3 b(t)\|_{L^2} \\ &\quad + \|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \leq C(1+t)^{-\frac{\sigma}{2}}. \end{aligned} \quad (3.4)$$

Applying ∂_3 to the first two equations in (1.1), and then taking the inner product with $(\partial_3 u, \partial_3 b)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_3 u\|_{L^2}^2 + \|\partial_3 b\|_{L^2}^2) + \nu\|\nabla_h \partial_3 u\|_{L^2}^2 + \eta\|\nabla_h \partial_3 b\|_{L^2}^2 \\ &= -\int \partial_3(u\cdot\nabla u)\cdot\partial_3 u\,dx + \int \partial_3(b\cdot\nabla b)\cdot\partial_3 u\,dx \\ &\quad - \int \partial_3(u\cdot\nabla b)\cdot\partial_3 b\,dx + \int \partial_3(b\cdot\nabla u)\cdot\partial_3 b\,dx \\ &:= M_1 + M_2 + M_3 + M_4. \end{aligned}$$

We now bound M_1 through M_4. By $\nabla\cdot u = 0$,

$$\begin{aligned} M_1 &= \int \partial_3(u\cdot\nabla u)\cdot\partial_3 u\,dx = \int \partial_3 u\cdot\nabla u\cdot\partial_3 u\,dx \\ &= \int \partial_3 u_h\cdot\nabla_h u\cdot\partial_3 u\,dx + \int \partial_3 u_3\partial_3 u\cdot\partial_3 u\,dx \\ &:= M_{11} + M_{12}. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} M_{11} &= \int \partial_3 u_h\cdot\nabla_h u\cdot\partial_3 u\,dx \\ &\leq C\|\partial_3 u_h\|_{L^2}^{\frac12}\|\partial_1\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\nabla_h u\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12} \\ &\leq C\|\partial_3 u\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac12}\|\nabla_h\partial_3 u\|_{L^2}^{\frac32} \\ &\leq C\|\partial_3 u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\partial_3 u\|_{L^2}^2). \end{aligned}$$

By $\nabla \cdot u = 0$,

$$\begin{aligned} M_{12} &= \int \partial_3 u_3 \partial_3 u \cdot \partial_3 u \, dx = -\int \nabla_h \cdot u_h \partial_3 u \cdot \partial_3 u \, dx \\ &\leq C \|\nabla_h u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\partial_3 u\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{3}{2}} \\ &\leq C \|\partial_3 u\|_{L^2} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2). \end{aligned}$$

Thus,

$$M_1 \leq C \|\partial_3 u\|_{L^2} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2).$$

Similarly,

$$\begin{aligned} M_3 &= \int \partial_3 (u \cdot \nabla b) \cdot \partial_3 b \, dx = \int \partial_3 u \cdot \nabla b \cdot \partial_3 b \, dx \\ &= \int \partial_3 u_h \cdot \nabla_h b \cdot \partial_3 b \, dx + \int \partial_3 u_3 \partial_3 b \cdot \partial_3 b \, dx \\ &\leq C \|\partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 b\|_{L^2} \\ &\quad + C \|\partial_3 b\|_{L^2} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 b\|_{L^2} \\ &\leq C (\|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} + \|\partial_3 b\|_{L^2}) \\ &\quad \times (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2 + \|\nabla_h \partial_3 b\|_{L^2}^2). \end{aligned}$$

Due to

$$\int b \cdot \nabla \partial_3 b \cdot \partial_3 u \, dx + \int b \cdot \nabla \partial_3 u \cdot \partial_3 b \, dx = 0,$$

we get

$$\begin{aligned} M_2 + M_4 &= \int \partial_3 (b \cdot \nabla b) \cdot \partial_3 u \, dx + \int \partial_3 (b \cdot \nabla u) \cdot \partial_3 b \, dx \\ &= \int \partial_3 b \cdot \nabla b \cdot \partial_3 u \, dx + \int \partial_3 b \cdot \nabla u \cdot \partial_3 b \, dx \\ &\leq C \|\partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 b\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+C\|\partial_3 b\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 b\|_{L^2}\\
\le\ & C(\|\partial_3 u\|_{L^2}+\|\partial_3 b\|_{L^2})\\
&\times(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2).
\end{aligned}
$$

Combining the estimates above yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)+\nu\|\nabla_h\partial_3 u\|_{L^2}^2+\eta\|\nabla_h\partial_3 b\|_{L^2}^2\\
&\quad\le C(\|\partial_3 u\|_{L^2}+\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}+\|\partial_3 b\|_{L^2})\\
&\qquad\times(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2).
\end{aligned}
$$

Adding this to (2.1), together with the Young inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)\\
&\quad+\nu\|\nabla_h u\|_{L^2}^2+\eta\|\nabla_h b\|_{L^2}^2+\nu\|\nabla_h\partial_3 u\|_{L^2}^2+\eta\|\nabla_h\partial_3 b\|_{L^2}^2\\
&\le C(\|\partial_3 u\|_{L^2}+\|\partial_3 b\|_{L^2})(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2).
\end{aligned}
$$

Then, for sufficiently small ε,

$$
\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)\\
&\quad+c_0(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2)\\
&\le 0,
\end{aligned}
\tag{3.5}
$$

where $c_0=\min\{\nu,\eta\}$. Applying the Gagliardo-Nirenberg inequality, together with (3.2), we obtain

$$
\begin{aligned}
\|u\|_{L^2}&=\|\|u\|_{L_h^2}\|_{L_{x_3}^2}\\
&\le C\|\|\Lambda_h^{-\sigma}u\|_{L_h^2}^{\frac{1}{1+\sigma}}\|\nabla_h u\|_{L_h^2}^{\frac{\sigma}{1+\sigma}}\|_{L_{x_3}^2}\\
&\le C\|\Lambda_h^{-\sigma}u\|_{L^2}^{\frac{1}{1+\sigma}}\|\nabla_h u\|_{L^2}^{\frac{\sigma}{1+\sigma}}\\
&\le C\|\nabla_h u\|_{L^2}^{\frac{\sigma}{1+\sigma}}.
\end{aligned}
\tag{3.6}
$$

Similarly, we have

$$
\begin{aligned}
&\|b\|_{L^2}\le C\|\nabla_h b\|_{L^2}^{\frac{\sigma}{1+\sigma}},\\
&\|\partial_3 u\|_{L^2}\le C\|\nabla_h\partial_3 u\|_{L^2}^{\frac{\sigma}{1+\sigma}},\\
&\|\partial_3 b\|_{L^2}\le C\|\nabla_h\partial_3 b\|_{L^2}^{\frac{\sigma}{1+\sigma}}.
\end{aligned}
\tag{3.7}
$$

Inserting these estimates in (3.5), we obtain, for a positive constant $C_0>0$,

$$
\frac{d}{dt}(\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)
$$

$$
\begin{aligned}
&+ C_0(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 + \|\partial_3 b\|_{L^2}^2)^{\frac{1+\sigma}{\sigma}} \\
&\le 0.
\end{aligned}
$$

Integrating in time yields

$$
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\partial_3 u(t)\|_{L^2}^2 + \|\partial_3 b(t)\|_{L^2}^2 \le C(1+t)^{-\sigma}. \tag{3.8}
$$

Applying ∇_h to the first two equations in (1.1), and dotting with $(\nabla_h u, \nabla_h b)$ yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + \nu\|\nabla_h^2 u\|_{L^2}^2 + \eta\|\nabla_h^2 b\|_{L^2}^2 \\
&\quad = -\int \nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,dx + \int \nabla_h(b\cdot\nabla b)\cdot\nabla_h u\,dx \\
&\qquad - \int \nabla_h(u\cdot\nabla b)\cdot\nabla_h b\,dx + \int \nabla_h(b\cdot\nabla u)\cdot\nabla_h b\,dx \\
&\quad := I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

By $\nabla\cdot u = 0$,

$$
\begin{aligned}
I_1 &= \int \nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,dx = \int \nabla_h u\cdot\nabla u\cdot\nabla_h u\,dx \\
&= \int \nabla_h u_h\cdot\nabla_h u\cdot\nabla_h u\,dx + \int \nabla_h u_3\cdot\partial_3 u\cdot\nabla_h u\,dx \\
&:= I_{11} + I_{12}.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
I_{11} &= \int \nabla_h u_h\cdot\nabla_h u\cdot\nabla_h u\,dx \\
&\le C\|\nabla_h u_h\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla_h u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla_h u\|_{L^2}^{\frac{3}{2}}\|\nabla_h^2 u_h\|_{L^2}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla_h u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\partial_3\nabla_h u\|_{L^2}^2 + \|\nabla_h^2 u\|_{L^2}^2).
\end{aligned}
$$

By $\nabla\cdot u = 0$ and Lemma 2.1,

$$
\begin{aligned}
I_{12} &= \int \nabla_h u_3\cdot\partial_3 u\cdot\nabla_h u\,dx \\
&\le C\|\nabla_h u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla_h u\|_{L^2}\|\nabla_h^2 u_h\|_{L^2}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}(\|\nabla_h u\|_{L^2}^2 + \|\partial_1\partial_3 u\|_{L^2}^2 + \|\nabla_h^2 u\|_{L^2}^2).
\end{aligned}
$$

Thus,

$$
I_1 \le C\|\nabla u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\partial_3 u\|_{L^2}^2 + \|\nabla_h^2 u\|_{L^2}^2).
$$

Similarly,

$$\begin{aligned}
I_3 &= \int \nabla_h(u\cdot\nabla b)\cdot\nabla_h b\,dx = \int \nabla_h u\cdot\nabla b\cdot\nabla_h b\,dx\\
&= \int \nabla_h u_h\cdot\nabla_h b\cdot\nabla_h b\,dx + \int \nabla_h u_3\cdot\partial_3 b\cdot\nabla_h b\,dx\\
&\le C\|\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_1\nabla_h b\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_2\nabla_h b\|_{L^2}^{\frac12}\\
&\quad + C\|\nabla_h u_3\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_2\nabla_h b\|_{L^2}^{\frac12}\\
&\le C(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})\\
&\quad \times(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2+\|\nabla_h^2 b\|_{L^2}^2).
\end{aligned}$$

Due to

$$\int b\cdot\nabla\nabla_h b\cdot\nabla_h u\,dx + \int b\cdot\nabla\nabla_h u\cdot\nabla_h b\,dx = 0,$$

we infer that

$$\begin{aligned}
&I_2+I_4\\
&\quad= \int \nabla_h(b\cdot\nabla b)\cdot\nabla_h u\,dx + \int \nabla_h(b\cdot\nabla u)\cdot\nabla_h b\,dx\\
&\quad= \int \nabla_h b\cdot\nabla b\cdot\nabla_h u\,dx + \int \nabla_h b\cdot\nabla u\cdot\nabla_h b\,dx\\
&\quad\le C\|\nabla_h b_h\|_{L^2}^{\frac12}\|\partial_1\nabla_h b_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_2\nabla_h b\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\nabla_h u\|_{L^2}^{\frac12}\\
&\qquad + C\|\nabla_h b_3\|_{L^2}^{\frac12}\|\partial_3\nabla_h b_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_1\partial_3 b\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}^{\frac12}\\
&\qquad + C\|\nabla_h b_3\|_{L^2}^{\frac12}\|\partial_3\nabla_h b_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\partial_2\nabla_h b\|_{L^2}^{\frac12}\\
&\quad\le C\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}^{\frac12}\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3 b\|_{L^2}\\
&\qquad + C\|\partial_3 b\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac12}\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3 b\|_{L^2}\\
&\quad\le C(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})\\
&\qquad \times(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2+\|\nabla_h^2 b\|_{L^2}^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\nu\|\nabla_h^2 u\|_{L^2}^2+\eta\|\nabla_h^2 b\|_{L^2}^2\\
&\quad\le C(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})\\
&\qquad \times(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2+\|\nabla_h^2 b\|_{L^2}^2).
\end{aligned}$$

Adding this to (3.5) and using $\|u\|_{H^1}+\|b\|_{H^1}\le C\varepsilon$ with $\varepsilon<\frac{c_0}{C}$, we have

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2+\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)\\
&\qquad + c_0(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 b\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2+\|\nabla_h^2 b\|_{L^2}^2)\\
&\quad\le 0,
\end{aligned} \tag{3.9}$$

where $c_0 = \min\{\nu, \eta\}$. Applying the Gagliardo-Nirenberg inequality, together with (3.2), we obtain

$$\begin{aligned}
\|\nabla_h u\|_{L^2} &= \|\|\nabla_h u\|_{L^2_h}\|_{L^2_{x_3}} \\
&\leq C\|\|\Lambda_h^{-\sigma} u\|_{L^2_h}^{\frac{1}{2+\sigma}} \|\nabla_h^2 u\|_{L^2_h}^{\frac{1+\sigma}{2+\sigma}}\|_{L^2_{x_3}} \\
&\leq C\|\Lambda_h^{-\sigma} u\|_{L^2}^{\frac{1}{2+\sigma}} \|\nabla_h^2 u\|_{L^2}^{\frac{1+\sigma}{2+\sigma}} \\
&\leq C\|\nabla_h^2 u\|_{L^2}^{\frac{1+\sigma}{2+\sigma}} \\
&\leq C\|\nabla_h^2 u\|_{L^2}^{\frac{\sigma}{1+\sigma}},
\end{aligned} \tag{3.10}$$

where we have used the fact $\frac{1+\sigma}{2+\sigma} > \frac{\sigma}{1+\sigma}$ and $\|\nabla_h^2 u\|_{L^2} \leq C$ in the last inequality. Inserting (3.10) with (3.6)–(3.7) in (3.9), we find, for a positive constant $C_1 > 0$,

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 + \|\partial_3 b\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) \\
&\quad + C_1(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 + \|\partial_3 b\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2)^{\frac{1+\sigma}{\sigma}} \\
&\leq 0.
\end{aligned}$$

Integrating in time gives

$$\|\nabla_h u(t)\|_{L^2}^2 + \|\nabla_h b(t)\|_{L^2}^2 \leq C(1+t)^{-\sigma}. \tag{3.11}$$

We have thus obtained (3.4).

Step 2. Improved decay rates for $\|\nabla_h u(t)\|_{L^2}$ and $\|\nabla_h b(t)\|_{L^2}$. More precisely, we show, for $\frac{1}{2} < \sigma < 1$,

$$\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \leq C(1+t)^{-\frac{1+\sigma}{2}}. \tag{3.12}$$

To this end, we rewrite the first equation in (1.1) in the integral form

$$u(x,t) = e^{\nu\Delta_h t}u_0 + \int_0^t e^{\nu\Delta_h(t-\tau)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(\tau)d\tau, \tag{3.13}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denotes the Leray projection onto divergence-free vector fields. Applying ∇_h to (3.13) yields

$$\nabla_h u(x,t) = \nabla_h e^{\nu\Delta_h t}u_0 + \int_0^t \nabla_h e^{\nu\Delta_h(t-\tau)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(\tau)d\tau.$$

Taking the L^2 norm, we obtain

$$\begin{aligned}\|\nabla_h u(t)\|_{L^2} &\le \|\nabla_h e^{\nu\Delta_h t}u_0\|_{L^2} + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u)(\tau)\|_{L^2}d\tau \\ &\le \|\nabla_h e^{\nu\Delta_h t}u_0\|_{L^2} + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b)(\tau)\|_{L^2}d\tau \\ &\quad + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}d\tau \\ &:= L_1 + L_2 + L_3.\end{aligned} \tag{3.14}$$

By Lemma 3.2,

$$\begin{aligned}L_1 = \|\nabla_h e^{\nu\Delta_h t}u_0\|_{L^2} &\le \|\|\nabla_h e^{\nu\Delta_h t}u_0\|_{L_h^2}\|_{L_{x_3}^2} \\ &\le C(1+t)^{-\frac{1+\sigma}{2}}\|(\|\Lambda_h^{-\sigma}u_0\|_{L_h^2} + \|u_0\|_{L_h^2})\|_{L_{x_3}^2} \\ &\le C(1+t)^{-\frac{1+\sigma}{2}}(\|\Lambda_h^{-\sigma}u_0\|_{L^2} + \|u_0\|_{L^2}) \\ &\le C(1+t)^{-\frac{1+\sigma}{2}}.\end{aligned}$$

For $\sigma < \delta < 1$,

$$\begin{aligned}L_2 &= \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b)(\tau)\|_{L^2}d\tau \\ &\le \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b_h\cdot\nabla_h b)(\tau)\|_{L^2} \\ &\quad + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b_3\partial_3 b)(\tau)\|_{L^2}d\tau \\ &= \int_0^t \|\|\nabla_h e^{\nu\Delta_h(t-\tau)}(b_h\cdot\nabla_h b)(\tau)\|_{L_h^2}\|_{L_{x_3}^2}d\tau \\ &\quad + \int_0^t \|\|\nabla_h e^{\nu\Delta_h(t-\tau)}(b_3\partial_3 b)(\tau)\|_{L_h^2}\|_{L_{x_3}^2}d\tau \\ &\le C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}}\|\|b_h\cdot\nabla_h b(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L_{x_3}^2}d\tau \\ &\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}}\|\|b_3\partial_3 b(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L_{x_3}^2}d\tau.\end{aligned}$$

To proceed, we need to provide suitable bounds for

$$L_{21} := \|\|b_h\cdot\nabla_h b(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L_{x_3}^2} \quad \text{and} \quad L_{22} := \|\|b_3\partial_3 b(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L_{x_3}^2}.$$

We provide a detailed estimate of L_{21} and L_{22}. By Lemma 2.1,

$$L_{21} = \|\|b_h\cdot\nabla_h b\|_{L_h^{\frac{2}{1+\delta}}}\|_{L_{x_3}^2}$$

$$
\begin{aligned}
&\le C\| \|b_h\|_{L_h^{\frac{2}{\delta}}} \|\nabla_h b\|_{L_h^2} \|_{L_{x_3}^2} \\
&\le C\| \|b_h\|_{L_h^{\frac{2}{\delta}}} \|_{L_{x_3}^\infty} \|\nabla_h b\|_{L^2} \\
&\le C\| \|b_h\|_{L_{x_3}^\infty} \|_{L_h^{\frac{2}{\delta}}} \|\nabla_h b\|_{L^2} \\
&\le C\| \|b_h\|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 b_h\|_{L_{x_3}^2}^{\frac{1}{2}} \|_{L_h^{\frac{2}{\delta}}} \|\nabla_h b\|_{L^2} \\
&\le \| \|b_h\|_{L_{x_3}^2}^{\frac{1}{2}} \|_{L_h^{\frac{4}{2\delta-1}}} \| \|\partial_3 b_h\|_{L_{x_3}^2}^{\frac{1}{2}} \|_{L_h^4} \|\nabla_h b\|_{L^2} \\
&\le \| \|b_h\|_{L_h^{\frac{2}{2\delta-1}}} \|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2} \\
&\le C\| \|b_h\|_{L_h^2}^{2\delta-1} \|\nabla_h b_h\|_{L_h^2}^{2-2\delta} \|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2} \\
&\le C\|b_h\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b_h\|_{L^2}^{1-\delta} \|\partial_3 b_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2} \\
&\le C\|b\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b\|_{L^2}^{2-\delta} \|\partial_3 b\|_{L^2}^{\frac{1}{2}}.
\end{aligned} \tag{3.15}
$$

Similarly,

$$
\begin{aligned}
L_{22} &= \| \|b_3 \partial_3 b\|_{L_h^{\frac{2}{1+\delta}}} \|_{L_{x_3}^2} \\
&\le C\|b_3\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b_3\|_{L^2}^{1-\delta} \|\partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2} \\
&\le C\|b_3\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b_3\|_{L^2}^{1-\delta} \|\nabla_h b_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2} \\
&\le C\|b\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{3}{2}-\delta} \|\partial_3 b\|_{L^2}.
\end{aligned}
$$

Incorporating these upper bounds, we obtain

$$
\begin{aligned}
L_2 \le\ & C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b\|_{L^2}^{2-\delta} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} d\tau \\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{3}{2}-\delta} \|\partial_3 b\|_{L^2} d\tau.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_3 &= \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2} d\tau \\
&\le \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u_h\cdot\nabla_h u)(\tau)\|_{L^2} + \int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u_3\partial_3 u)(\tau)\|_{L^2} d\tau \\
&= \int_0^t \| \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u_h\cdot\nabla_h u)(\tau)\|_{L_h^2} \|_{L_{x_3}^2} d\tau \\
&\quad + \int_0^t \| \|\nabla_h e^{\nu\Delta_h(t-\tau)}(u_3\partial_3 u)(\tau)\|_{L_h^2} \|_{L_{x_3}^2} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|\|u_h\cdot\nabla_h u(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L^2_{x_3}} d\tau\\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|\|u_3\partial_3 u(\tau)\|_{L_h^{\frac{2}{1+\delta}}}\|_{L^2_{x_3}} d\tau\\
&\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u_h\|_{L^2}^{\delta-\frac12}\|\nabla_h u_h\|_{L^2}^{1-\delta}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2} d\tau\\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u_3\|_{L^2}^{\delta-\frac12}\|\nabla_h u_3\|_{L^2}^{1-\delta}\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2} d\tau\\
&\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{2-\delta}\|\partial_3 u\|_{L^2}^{\frac12} d\tau\\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{\frac32-\delta}\|\partial_3 u\|_{L^2} d\tau.
\end{aligned}
$$

Inserting these estimates in (3.14) leads to

$$
\begin{aligned}
\|\nabla_h u(t)\|_{L^2} \leq C(1+t)^{-\frac{1+\delta}{2}} &+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac12}\|\nabla_h b\|_{L^2}^{2-\delta}\|\partial_3 b\|_{L^2}^{\frac12} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac12}\|\nabla_h b\|_{L^2}^{\frac32-\delta}\|\partial_3 b\|_{L^2} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{2-\delta}\|\partial_3 u\|_{L^2}^{\frac12} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{\frac32-\delta}\|\partial_3 u\|_{L^2} d\tau.
\end{aligned}
\tag{3.16}
$$

Now, we turn to bound $\nabla_h b$. We rewrite the second equation in (1.1) in the integral form

$$
b(x,t) = e^{\nu\Delta_h t} b_0 + \int_0^t e^{\nu\Delta_h (t-\tau)}(b\cdot\nabla u - u\cdot\nabla b)(\tau) d\tau. \tag{3.17}
$$

Applying ∇_h to (3.17) yields

$$
\nabla_h b(x,t) = \nabla_h e^{\nu\Delta_h t} b_0 + \int_0^t \nabla_h e^{\nu\Delta_h (t-\tau)}(b\cdot\nabla u - u\cdot\nabla b)(\tau) d\tau.
$$

As in (3.16), we have

$$
\begin{aligned}
\|\nabla_h b(t)\|_{L^2} \leq C(1+t)^{-\frac{1+\delta}{2}} &+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac12}\|\nabla_h b\|_{L^2}^{1-\delta}\|\partial_3 b\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|b\|_{L^2}^{\delta-\frac12}\|\nabla_h b\|_{L^2}^{\frac32-\delta}\|\partial_3 u\|_{L^2} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{1-\delta}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2} d\tau\\
&+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u\|_{L^2}^{\delta-\frac12}\|\nabla_h u\|_{L^2}^{\frac32-\delta}\|\partial_3 b\|_{L^2} d\tau.
\end{aligned}
\tag{3.18}
$$

Adding (3.16) and (3.18) gives

$$\begin{aligned}
&\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \le C(1+t)^{-\frac{1+\delta}{2}} \\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} (\|u\|_{L^2} + \|b\|_{L^2})^{\delta-\frac{1}{2}} (\|\nabla_h u\|_{L^2} + \|\nabla_h b\|_{L^2})^{2-\delta} \\
&\quad \times (\|\partial_3 u\|_{L^2} + \|\partial_3 b\|_{L^2})^{\frac{1}{2}} d\tau \\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} (\|u\|_{L^2} + \|b\|_{L^2})^{\delta-\frac{1}{2}} (\|\nabla_h u\|_{L^2} + \|\nabla_h b\|_{L^2})^{\frac{3}{2}-\delta} \\
&\quad \times (\|\partial_3 u\|_{L^2} + \|\partial_3 b\|_{L^2}) d\tau. \qquad (3.19)
\end{aligned}$$

Invoking (3.8) and (3.11) implies, for $\sigma < \delta < 1$,

$$\begin{aligned}
&\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} (1+\tau)^{-\frac{\sigma}{2}(\delta-\frac{1}{2})} (1+\tau)^{-\frac{\sigma}{2}(2-\delta)} (1+\tau)^{-\frac{\sigma}{4}} d\tau \\
&\quad + C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} (1+\tau)^{-\frac{\sigma}{2}(\delta-\frac{1}{2})} (1+\tau)^{-\frac{\sigma}{2}(\frac{3}{2}-\delta)} (1+\tau)^{-\frac{\sigma}{2}} d\tau \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} + C(1+t)^{-(\frac{\delta}{2}+\sigma-\frac{1}{2})} \\
&\le C(1+t)^{-(\frac{\delta}{2}+\sigma-\frac{1}{2})}.
\end{aligned} \qquad (3.20)$$

To improve the decay rate, we implement an iterative procedure. For notational convenience, we set

$$\alpha_0 = \frac{\delta}{2} + \sigma - \frac{1}{2},$$

Inserting (3.20) in (3.19) and using (3.8), we derive that

$$\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \le C(1+t)^{-\frac{1+\sigma}{2}} + C(1+t)^{-\min\{\alpha_1, \frac{1+\sigma}{2}\}},$$

where

$$\alpha_1 = \alpha_0 + \left(\alpha_0 - \frac{\sigma}{2}\right)\left(\frac{3}{2} - \delta\right).$$

Repeating this procedure n times leads to

$$\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \le C(1+t)^{-\frac{1+\sigma}{2}} + C(1+t)^{-\min\{\alpha_n, \frac{1+\sigma}{2}\}}, \qquad (3.21)$$

where

$$\alpha_n = \alpha_0 + \left(\alpha_{n-1} - \frac{\sigma}{2}\right)\left(\frac{3}{2} - \delta\right).$$

We claim that by choosing $n > 1$ sufficiently large and $\delta > \sigma$ close to σ,

$$\alpha_n > \frac{1+\sigma}{2}.$$

In fact, by the iterative formula,

$$\alpha_n = \alpha_0 + \left(\alpha_0 - \frac{\sigma}{2}\right)\left(\frac{3}{2} - \delta + \left(\frac{3}{2} - \delta\right)^2 + \cdots + + \left(\frac{3}{2} - \delta\right)^n\right)$$
$$= \frac{\delta}{2} + \sigma - \frac{1}{2} + \left(\frac{\delta}{2} + \frac{\sigma}{2} - \frac{1}{2}\right)\left(\frac{3}{2} - \delta\right)\frac{1 - \left(\frac{3}{2} - \delta\right)^n}{\delta - \frac{1}{2}}.$$

Since $\frac{1}{2} < \sigma < \delta < 1$, we have

$$0 < \frac{3}{2} - \delta < 1,$$

Therefore, as $n \to \infty$,

$$\alpha_n \to \alpha(\delta)$$

with

$$\alpha(\delta) = \frac{\delta}{2} + \sigma - \frac{1}{2} + \left(\frac{\delta}{2} + \frac{\sigma}{2} - \frac{1}{2}\right)\left(\frac{3}{2} - \delta\right)\left(\delta - \frac{1}{2}\right)^{-1}.$$

Note that

$$\alpha(\sigma) = 1 + \frac{\sigma}{2}.$$

Thus, if δ is close to σ, then $\alpha(\delta)$ would be close to $\alpha(\sigma)$ and $\alpha(\delta) > \frac{1+\sigma}{2}$. Therefore, for sufficiently large n, $\alpha_n > \frac{1+\sigma}{2}$. Then, (3.21) implies

$$\|\nabla_h u(t)\|_{L^2} + \|\nabla_h b(t)\|_{L^2} \leq C(1+t)^{-\frac{1+\sigma}{2}},$$

which is (3.12).

Step 3. Enhanced dissipation for the vertical components. We show in this step that

$$\|u_3(t)\|_{L^2} + \|b_3(t)\|_{L^2} \leq C(1+t)^{-\frac{3\sigma}{4}}, \tag{3.22}$$

$$\|\nabla_h u_3(t)\|_{L^2} + \|\nabla_h b_3(t)\|_{L^2} \leq C(1+t)^{-(\frac{3\sigma}{4}+\frac{1}{2})}. \tag{3.23}$$

To this end, we rewrite the equation of u_3 in (1.1) in the integral form

$$u_3(x,t) = e^{\nu\Delta_h t}u_{03} + \int_0^t e^{\nu\Delta_h(t-\tau)}(b \cdot \nabla b_3 - u \cdot \nabla u_3 - \partial_3 P)(\tau)d\tau. \tag{3.24}$$

Multiplying (3.24) by u_3 and integrating over $\mathbb{R}^3$, we have

$$\|u_3(t)\|_{L^2}^2 \leq \int e^{\nu\Delta_h t}u_{03} \cdot u_3(t)dx$$
$$+ \int_0^t \int e^{\nu\Delta_h(t-\tau)}(b \cdot \nabla b_3 - u \cdot \nabla u_3 - \partial_3 P)(\tau) \cdot u_3(t)dxd\tau.$$

By the Young inequality,

$$\begin{aligned}\|u_3(t)\|_{L^2}^2 \ \le\ & \|e^{\nu\Delta_h t}u_{03}\|_{L^2}^2 + 2\int_0^t\int e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\cdot u_3(t)dxd\tau\\ & +2\int_0^t\int e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u_3)(\tau)\cdot u_3(t)dxd\tau\\ & -2\int_0^t\int e^{\nu\Delta_h(t-\tau)}\partial_3 P(\tau)\cdot u_3(t)dxd\tau\\ := & F_1+F_2+F_3+F_4. \end{aligned}\tag{3.25}$$

We estimate F_1 through F_4. By Plancherel's theorem,

$$\begin{aligned} F_1 &= \|e^{-\nu|\xi_h|t}\hat{u}_{03}\|_{L^2}^2\\ &= \int_{\mathbb{R}^3} e^{-2\nu|\xi_h|^2 t}|\hat{u}_{03}(\xi)|^2 d\xi\\ &= \int_{|\xi_3|\le|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\hat{u}_{03}(\xi)|^2 d\xi + \int_{|\xi_3|>|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\hat{u}_{03}(\xi)|^2 d\xi\\ &= \int_{|\xi_3|\le|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\xi_h|^{2\sigma}|\xi_3|^{\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\hat{u}_{03}(\xi)|^2 d\xi\\ &\quad + \int_{|\xi_3|>|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\xi_h|^{2\sigma+2}|\xi_3|^{\sigma}|\xi_3|^{-2}|\xi_h|^{-2\sigma-2}|\xi_3|^{-\sigma}|\xi_3\hat{u}_{03}(\xi)|^2 d\xi\\ &\le \int_{|\xi_3|\le|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\hat{u}_{03}(\xi)|^2 d\xi\\ &\quad + \int_{|\xi_3|>|\xi_h|} e^{-2\nu|\xi_h|^2 t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\hat{u}_{0h}(\xi)|^2 d\xi\\ &\le C(1+t)^{-\frac{3\sigma}{2}}\int_{\mathbb{R}^3}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\hat{u}_{03}(\xi)|^2 d\xi\\ &\quad + C(1+t)^{-\frac{3\sigma}{2}}\int_{\mathbb{R}^3}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\hat{u}_{0h}(\xi)|^2 d\xi\\ &\le C(1+t)^{-\frac{3\sigma}{2}}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}}\hat{u}_0\|_{L^2}^2, \end{aligned}$$

where we have used the divergence-free condition $\xi_3\hat{u}_{03} = -\xi_h\cdot\hat{u}_{0\,h}$ and the fact $e^{-2\nu\xi_h^2 t}(|\xi_h|^2 t)^{\frac{3\sigma}{2}} \le C$. By $\nabla\cdot u = \nabla\cdot b = 0$, Hölder's inequality and (3.12),

$$\begin{aligned} F_2 &= 2\int_0^t\int e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\,u_3(t)dxd\tau\\ &= 2\int_0^t\int e^{\nu\Delta_h(t-\tau)}\nabla_h\cdot(b_h b_3)(\tau)\,u_3(t)dxd\tau\\ &\quad + 2\int_0^t\int e^{\nu\Delta_h(t-\tau)}\partial_3(b_3 b_3)(\tau)\,u_3(t)dxd\tau\\ &= -2\int_0^t\int e^{\nu\Delta_h(t-\tau)}(b_h b_3)(\tau)\cdot\nabla_h u_3(t)dxd\tau \end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t\int e^{\nu\Delta_h(t-\tau)}(b_3b_3)(\tau)\,\partial_3u_3(t)dxd\tau\\
\le\,& 2\int_0^t \|e^{\nu\Delta_h(t-\tau)}(b_hb_3)(\tau)\|_{L^2}d\tau\|\nabla_h u_3(t)\|_{L^2}\\
&+2\int_0^t \|e^{\nu\Delta_h(t-\tau)}(b_3b_3)(\tau)\|_{L^2}d\tau\|\nabla_h u_h(t)\|_{L^2}\\
\le\,& C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t \|e^{\nu\Delta_h(t-\tau)}(b_hb_3)(\tau)\|_{L^2}d\tau\\
&+C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t \|e^{\nu\Delta_h(t-\tau)}(b_3b_3)(\tau)\|_{L^2}d\tau\\
:=\,& F_{21}+F_{22}.
\end{aligned}
$$

By (3.4) and (3.12),

$$
\begin{aligned}
\int_0^t \|e^{\nu\Delta_h(t-\tau)}(b_hb_3)(\tau)\|_{L^2}d\tau &\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\|b_hb_3\|_{L^1_h}\|_{L^2_{x_3}}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|\|b_h\|_{L^2_h}\|_{L^2_{x_3}}\|\|b_3\|_{L^2_h}\|_{L^\infty_{x_3}}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_h\|_{L^2}\|\|b_3\|_{L^\infty_{x_3}}\|_{L^2_h}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_h\|_{L^2}\|\|b_3\|^{\frac{1}{2}}_{L^2_{x_3}}\|\partial_3b_3\|^{\frac{1}{2}}_{L^2_{x_3}}\|_{L^2_h}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_h\|_{L^2}\|b_3\|^{\frac{1}{2}}_{L^2}\|\partial_3b_3\|^{\frac{1}{2}}_{L^2}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_h\|_{L^2}\|b_3\|^{\frac{1}{2}}_{L^2}\|\nabla_hb_h\|^{\frac{1}{2}}_{L^2}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}\|b_h\|_{L^2}\|b_3\|^{\frac{1}{2}}_{L^2}\|\nabla_hb_h\|^{\frac{1}{2}}_{L^2}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3\sigma}{4}}(1+\tau)^{-\frac{1+\sigma}{4}}d\tau\\
&\le \begin{cases} C(1+t)^{-(\sigma-\frac{1}{4})}, & if\ \frac{1}{2}<\sigma<\frac{3}{4},\\ C(1+t)^{-\frac{1}{2}}\ln(1+t), & if\ \sigma=\frac{3}{4},\\ C(1+t)^{-\frac{1}{2}}, & if\ \sigma>\frac{3}{4},\end{cases}\\
&\le C(1+t)^{-\frac{\sigma}{2}}.
\end{aligned}
$$

Therefore,

$$
F_{21}\le C(1+t)^{-\frac{1+2\sigma}{2}}\le C(1+t)^{-\frac{3\sigma}{2}}.
$$

Similarly,

$$
F_{22}\le C(1+t)^{-\frac{3\sigma}{2}}.
$$

Thus,

$$
F_2\le C(1+t)^{-\frac{3\sigma}{2}}.
$$

F_3 has the same bound as F_2, namely

$$F_3 \leq C(1+t)^{-\frac{3\sigma}{2}}.$$

Since

$$\partial_3 P = \partial_3 \Delta^{-1} \nabla \cdot (b \cdot \nabla b - u \cdot \nabla u),$$

we have

$$\begin{aligned}
F_4 &= -2 \int_0^t \int e^{\nu \Delta_h (t-\tau)} \partial_3 P(\tau)\, u_3(t) dx d\tau \\
&= -2 \int_0^t \int e^{\nu \Delta_h (t-\tau)} \partial_3 \Delta^{-1} \nabla \cdot (b \cdot \nabla b)(\tau)\, u_3(t) dx d\tau \\
&\quad + 2 \int_0^t \int e^{\nu \Delta_h (t-\tau)} \partial_3 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)(\tau) \cdot u_3(t) dx d\tau \\
&= 2 \int_0^t \int e^{\nu \Delta_h (t-\tau)} \nabla \Delta^{-1} \cdot (b \cdot \nabla b)(\tau)\, \partial_3 u_3(t) dx d\tau \\
&\quad - 2 \int_0^t \int e^{\nu \Delta_h (t-\tau)} \nabla \Delta^{-1} \cdot (u \cdot \nabla u)(\tau)\, \partial_3 u_3(t) dx d\tau \\
&:= F_{41} + F_{42}.
\end{aligned}$$

By $\nabla \cdot b = 0$,

$$\begin{aligned}
\nabla \cdot (b \cdot \nabla b) &= \sum_{i=1}^{3} \sum_{j=1}^{3} \partial_j \partial_i (b_i b_j) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{2} \partial_j \partial_i (b_i b_j) + \sum_{i=1}^{3} \partial_3 \partial_i (b_i b_3).
\end{aligned}$$

Then,

$$\begin{aligned}
F_{41} &\leq 2 \int_0^t \|e^{\nu \Delta_h (t-\tau)} \sum_{i=1}^{3} \sum_{j=1}^{2} \partial_j \partial_i \Delta^{-1} (b_i b_j)(\tau)\|_{L^2} \|\partial_3 u_3(t)\|_{L^2} d\tau \\
&\quad + 2 \int_0^t \|e^{\nu \Delta_h (t-\tau)} \sum_{i=1}^{3} \partial_3 \partial_i \Delta^{-1} (b_i b_3)(\tau)\|_{L^2} \|\partial_3 u_3(t)\|_{L^2} d\tau \\
&\leq C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t \|e^{\nu \Delta_h (t-\tau)} \sum_{i=1}^{3} \sum_{j=1}^{2} \partial_j \partial_i \Delta^{-1} (b_i b_j)(\tau)\|_{L^2} d\tau \\
&\quad + C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t \|e^{\nu \Delta_h (t-\tau)} \sum_{i=1}^{3} \partial_3 \partial_i \Delta^{-1} (b_i b_3)(\tau)\|_{L^2} d\tau \\
&:= F_{411} + F_{412}.
\end{aligned}$$

By Planchrel's theorem, together with (3.4) and (3.12), then for $2\sigma - \frac{1}{2} < \gamma < \frac{3}{2}$,

$$
\begin{aligned}
&\int_0^t \|e^{-\nu|\xi_h|^2(t-\tau)}|\xi|^{-1}|\xi_h|\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(b_ib_j)}(\tau)\|_{L^2}d\tau \\
&\le C\int_0^t \|\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi|^{-1}|\xi_h|\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(b_ib_j)}(\tau)\|_{L^2_{\xi_3}}\|_{L^2_h}d\tau \\
&\le C\int_0^t \|\||\xi|^{-1}\|_{L^2_{\xi_3}}\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi_h|\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(b_ib_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau \\
&\le C\int_0^t \|\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi_h|^{\frac{1}{2}}\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(b_ib_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\||\xi_h|^{-(\gamma-\frac{1}{2})}\|\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(b_ib_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|\|\sum_{i=1}^{3}\sum_{j=1}^{2}\Lambda_h^{-(\gamma-\frac{1}{2})}(b_ib_j)(\tau)\|_{L^2_h}\|_{L^1_{x_3}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|\|\sum_{i=1}^{3}\sum_{j=1}^{2}(b_ib_j)(\tau)\|_{L_h^{\frac{4}{1+2\gamma}}}\|_{L^1_{x_3}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|\|b(\tau)\|^2_{L_h^{\frac{8}{1+2\gamma}}}\|_{L^1_{x_3}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|\|b(\tau)\|_{L^2_h}^{\frac{1+2\gamma}{4}}\|\nabla_h b(\tau)\|_{L^2_h}^{\frac{3-2\gamma}{4}}\|^2_{L^2_{x_3}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|b(\tau)\|_{L^2}^{\frac{1+2\gamma}{2}}\|\nabla_h b(\tau)\|_{L^2}^{\frac{3-2\gamma}{2}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}}(1+\tau)^{-\frac{\sigma(1+2\gamma)}{4}}(1+\tau)^{-\frac{(1+\sigma)(3-2\gamma)}{4}}d\tau \\
&\le C(1+t)^{-(\sigma-\frac{1}{4})}.
\end{aligned}
$$

Here, we have used the simple fact

$$\||\xi|^{-1}\|_{L^2_{\xi_3}} = |\xi_h|^{-\frac{1}{2}}.$$

Thus,

$$F_{411} \le C(1+t)^{-(\frac{3\sigma}{2}+\frac{1}{4})} \le C(1+t)^{-\frac{3\sigma}{2}}.$$

By (3.4) and (3.12),

$$\begin{aligned}
&\int_0^t \|e^{\nu\Delta_h(t-\tau)} \frac{\sum_{i=1}^3 \partial_3\partial_i(b_i b_3)}{\Delta}(\tau)\|_{L^2} d\tau \\
&\quad \le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|\|bb_3(\tau)\|_{L_h^{\frac{2}{1+\gamma}}}\|_{L^2_{x_3}} d\tau \\
&\quad \le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|b_3\|_{L^2}^{\gamma-\frac{1}{2}} \|\nabla_h b_3\|_{L^2}^{1-\gamma} \|\partial_3 b_3\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2} d\tau \\
&\quad \le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|b\|_{L^2}^{\gamma+\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{3}{2}-\gamma} d\tau \\
&\quad \le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} (1+t)^{-\frac{\sigma}{2}(\gamma+\frac{1}{2})} (1+t)^{-\frac{(1+\sigma)(\frac{3}{2}-\gamma)}{2}} d\tau \\
&\quad \le \begin{cases} C(1+t)^{-(\sigma-\frac{1}{4})}, & if\ \sigma < \frac{\gamma}{2}+\frac{1}{4}, \\ C(1+t)^{-\frac{\gamma}{2}}\ln(1+t), & if\ \sigma = \frac{\gamma}{2}+\frac{1}{4}, \\ C(1+t)^{-\frac{\gamma}{2}}, & if\ \sigma > \frac{\gamma}{2}+\frac{1}{4}, \end{cases}
\end{aligned}$$

for any $\frac{1}{2} < \gamma < 1$. By choosing γ near 1, we obtain

$$F_{412} \le C(1+t)^{-(\frac{3\sigma}{2}+\frac{1}{4})} + C(1+t)^{-\frac{1+\sigma+\gamma}{2}} \ln(1+t) \le C(1+t)^{-\frac{3\sigma}{2}}.$$

Therefore,

$$F_4 \le C(1+t)^{-\frac{3\sigma}{2}}.$$

Inserting the bounds for F_1 through F_4 in (3.25), we obtain

$$\|u_3(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3\sigma}{2}}.$$

To bound $\|b_3(t)\|_{L^2}$, we rewrite the equation of b_3 in (1.1) in the integral form

$$b_3(x,t) = e^{\nu\Delta_h t} b_{03} + \int_0^t e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla u_3 - u\cdot\nabla b_3)(\tau)d\tau.$$

This equation is similar as (3.24). It is simpler than (3.24) since it does not have the pressure term. Therefore, a similar process leads to

$$\|b_3(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3\sigma}{2}}.$$

We have thus obtained (3.22).

Now, we turn to (3.23). Applying ∇_h to (3.24) yields

$$\nabla_h u_3(x,t) = \nabla_h e^{\nu\Delta_h t} u_{03} + \int_0^t \nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3 - u\cdot\nabla u_3 - \partial_3 P)(\tau)d\tau,$$

Taking the L^2 norm, we obtain

$$\begin{aligned}
\|\nabla_h u_3(t)\|_{L^2}^2 &= \int \nabla_h e^{\nu\Delta_h t} u_{03}\cdot\nabla_h u_3(t)dx \\
&\quad + \int_0^t\int \nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3 - u\cdot\nabla u_3 - \partial_3 P)(\tau)\cdot\nabla_h u_3(t)dxd\tau.
\end{aligned}$$

By Young's and Hölder's inequalities,

$$\begin{aligned}
\|\nabla_h u_3(t)\|_{L^2}^2 \le \|\nabla_h e^{\nu\Delta_h t}u_{03}\|_{L^2}^2 &+ 2\int_0^t\int \nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\cdot\nabla_h u_3(t)dxd\tau\\
&- 2\int_0^t\int \nabla_h e^{\nu\Delta_h(t-\tau)}(u\cdot\nabla u_3)(\tau)\cdot\nabla_h u_3(t)dxd\tau\\
&- 2\int_0^t\int \nabla_h e^{\nu\Delta_h(t-\tau)}\partial_3 P(\tau)\cdot\nabla_h u_3(t)dxd\tau\\
&:= B_1+B_2+B_3+B_4.
\end{aligned} \tag{3.26}$$

As in the estimate of F_1,

$$\begin{aligned}
B_1 &= \|\nabla_h e^{\nu\Delta_h t}u_{03}\|_{L^2}^2\\
&\le C(1+t)^{-1}\|e^{\nu\Delta_h t}u_{03}\|_{L^2}^2\\
&\le C(1+t)^{-1}\|e^{-\nu\xi_h t}\hat{u}_{03}\|_{L^2}^2\\
&\le C(1+t)^{-\frac{3\sigma}{2}-1}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}-\frac14}\hat{u}_0\|_{L^2}^2\\
&\le C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}$$

Similar estimates as those for L_2 and F_{21} above, together with (3.8) and (3.11), yield

$$\begin{aligned}
B_2 &= 2\int_0^t\int \nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\cdot\nabla_h u_3(t)dxd\tau\\
&\le 2\int_0^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\|_{L^2}d\tau\|\nabla_h u_3(t)\|_{L^2}\\
&= 2\int_0^{\frac t2} \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\|_{L^2}d\tau\|\nabla_h u_3(t)\|_{L^2}\\
&\quad + 2\int_{\frac t2}^t \|\nabla_h e^{\nu\Delta_h(t-\tau)}(b\cdot\nabla b_3)(\tau)\|_{L^2}d\tau\|\nabla_h u_3(t)\|_{L^2}\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac t2}(t-\tau)^{-1}\|b_h\|_{L^2}^{\frac12}\|\partial_3 b_h\|_{L^2}^{\frac12}\|\nabla_h b_3\|_{L^2}d\tau\\
&\quad + C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac t2}(t-\tau)^{-1}\|b_3\|_{L^2}^{\frac12}\|\partial_3 b_3\|_{L^2}^{\frac12}\|\partial_3 b_3\|_{L^2}d\tau\\
&\quad + C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac t2}^t(t-\tau)^{-\frac{1+\sigma}{2}}\|b_h\|_{L^2}^{\sigma-\frac12}\|\nabla_h b_h\|_{L^2}^{1-\sigma}\|\partial_3 b_h\|_{L^2}^{\frac12}\|\nabla_h b_3\|_{L^2}d\tau\\
&\quad + C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac t2}^t(t-\tau)^{-\frac{1+\sigma}{2}}\|b_3\|_{L^2}^{\sigma-\frac12}\|\nabla_h b_3\|_{L^2}^{1-\sigma}\|\partial_3 b_3\|_{L^2}^{\frac12}\|\partial_3 b_3\|_{L^2}d\tau\\
&\le C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}$$

Similarly,

$$B_3 \le C(1+t)^{-(\frac{3\sigma}{2}+1)}.$$

To estimate B_4, we replace $P = -\Delta^{-1}\nabla \cdot (u \cdot \nabla u)$ and divide it into four parts,

$$\begin{aligned}
B_4 &= -2\int_0^t \int \nabla_h e^{\nu\Delta_h(t-\tau)} \partial_3 P(\tau) \cdot \nabla_h u_3(t) dx d\tau \\
&= -2\int_0^t \int \Delta_h e^{\nu\Delta_h(t-\tau)} P(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&= 2\int_0^t \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (u \cdot \nabla u)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&\quad - 2\int_0^t \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (b \cdot \nabla b)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&= 2\int_0^{\frac{t}{2}} \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (u \cdot \nabla u)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&\quad + 2\int_{\frac{t}{2}}^t \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (u \cdot \nabla u)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&\quad - 2\int_0^{\frac{t}{2}} \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (b \cdot \nabla b)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&\quad - 2\int_{\frac{t}{2}}^t \int \Delta_h e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (b \cdot \nabla b)(\tau)\, \nabla_h \cdot u_h(t) dx d\tau \\
&:= B_{41} + B_{42} + B_{43} + B_{44}.
\end{aligned} \tag{3.27}$$

To estimate B_{41}, we further distinguish the horizontal derivatives from the vertical ones to write

$$\begin{aligned}
B_{41} &= 2\int_0^{\frac{t}{2}} \int \nabla_h^2 e^{\nu\Delta_h(t-\tau)} \Delta^{-1}\nabla \cdot (u \cdot \nabla u)(\tau) \cdot \nabla_h u_h(t) dx d\tau \\
&= 2\int_0^{\frac{t}{2}} \int \nabla_h^2 e^{\nu\Delta_h(t-\tau)} \sum_{i=1}^3 \sum_{j=1}^2 \partial_j \partial_i \Delta^{-1}(u_i u_j)(\tau) \cdot \nabla_h u_h(t) dx d\tau \\
&\quad + 2\int_0^{\frac{t}{2}} \int \nabla_h^2 e^{\nu\Delta_h(t-\tau)} \sum_{i=1}^2 \partial_3 \partial_i \Delta^{-1}(u_i u_3)(\tau) \cdot \nabla_h u_h(t) dx d\tau \\
&\quad + 2\int_0^{\frac{t}{2}} \int \nabla_h^2 e^{\nu\Delta_h(t-\tau)} \partial_3 \partial_3 \Delta^{-1}(u_3 u_3)(\tau) \cdot \nabla_h u_h(t) dx d\tau \\
&:= B_{411} + B_{412} + B_{413}.
\end{aligned}$$

As in the proof of F_{411}, we have, for $2\sigma - \frac{1}{2} < \gamma < \frac{3}{2}$,

$$\begin{aligned}
&B_{411} + B_{412} \\
&\quad \le 2\|\nabla_h u_h(t)\|_{L^2} \int_0^{\frac{t}{2}} \|\Delta_h e^{\nu\Delta_h(t-\tau)} \sum_{i=1}^3 \sum_{j=1}^2 \partial_j \partial_i \Delta^{-1}(u_i u_j)(\tau)\|_{L^2} d\tau \\
&\qquad + 2\|\nabla_h u_h(t)\|_{L^2} \int_0^{\frac{t}{2}} \|\Delta_h e^{\nu\Delta_h(t-\tau)} \sum_{i=1}^2 \partial_3 \partial_i \Delta^{-1}(u_i u_3)(\tau)\|_{L^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi|^{-1}|\xi_h|^3\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(u_iu_j)}(\tau)\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}\|\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi|^{-1}|\xi_h|^3\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(u_iu_j)}(\tau)\|_{L^2_{\xi_3}}\|_{L^2_h}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}\|\||\xi|^{-1}\|_{L^2_{\xi_3}}\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi_h|^3\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(u_iu_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}\|\|e^{-\nu|\xi_h|^2(t-\tau)}|\xi_h|^{\frac{5}{2}}\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(u_iu_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-(1+\frac{\gamma}{2})}\|\|\xi_h|^{-(\gamma-\frac{1}{2})}\|\sum_{i=1}^{3}\sum_{j=1}^{2}\widehat{(u_iu_j)}(\tau)\|_{L^\infty_{\xi_3}}\|_{L^2_h}d\tau\\
&\le C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})}.
\end{aligned}
$$

By Lemma 3.2,

$$
\begin{aligned}
B_{413}&=2\int_0^{\frac{t}{2}}\int\Delta_h e^{\nu\Delta_h(t-\tau)}\partial_3\partial_3\Delta^{-1}(u_3u_3)(\tau)\cdot\nabla_h\cdot u_h(t)dxd\tau\\
&\le 2\|\nabla_h u_h(t)\|_{L^2}\int_0^{\frac{t}{2}}\|\Delta_h e^{\nu\Delta_h(t-\tau)}\partial_3\partial_3\Delta^{-1}(u_3u_3)(\tau)\|_{L^2}d\tau\\
&\le 2\|\nabla_h u_h(t)\|_{L^2}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{2}}\|\|u_3u_3\|_{L^1_h}\|_{L^2_{x_3}}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{2}}\|u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3u_3\|_{L^2}^{\frac{1}{2}}\|u_3\|_{L^2}d\tau\\
&\le C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})}.
\end{aligned}
$$

Therefore,

$$
B_{41}\le C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})}.
$$

Similarly,

$$
\begin{aligned}
B_{43}&\le C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})}.\\
B_{42}&=2\int_{\frac{t}{2}}^{t}\int\Delta_h e^{\nu\Delta_h(t-\tau)}\Delta^{-1}\nabla\cdot(u\cdot\nabla u)(\tau)\cdot\nabla_h\cdot u_h(t)dxd\tau\\
&\le 2\|\nabla_h u_3(t)\|_{L^2}\int_{\frac{t}{2}}^{t}\|\Delta_h e^{\nu\Delta_h(t-\tau)}\Delta^{-1}\nabla\cdot(u\cdot\nabla u)(\tau)\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}\|\|\nabla_h(u\otimes u)(\tau)\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L^2_{x_3}}\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}d\tau
\end{aligned}
$$

$$\leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.$$

Similarly,

$$B_{44} \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.$$

Inserting the estimates above in (3.27), we obtain

$$\begin{aligned} B_4 &\leq C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})} + C(1+t)^{-(\frac{3\sigma}{2}+1)} \\ &\leq C(1+t)^{-(\frac{3\sigma}{2}+1)}. \end{aligned}$$

Substituting the bounds of B_1, B_2, B_3 and B_4 into (3.26), we have

$$\|\nabla_h u_3(t)\|_{L^2}^2 \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.$$

Similarly, we have

$$\|\nabla_h b_3(t)\|_{L^2}^2 \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.$$

This completes the proof of (3.23).

Step 4. Estimates of $\|\Lambda_h^{-\sigma} u\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b\|_{L^2}^2$. More precisely, we show that (u, b) obeys, for $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned} &\frac{d}{dt}(\|\Lambda_h^{-\sigma} u\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b\|_{L^2}^2) \\ &\quad\leq C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{2-\sigma}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\Lambda_h^{-\sigma} u\|_{L^2} \\ &\qquad + C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 b\|_{L^2}\|\Lambda_h^{-\sigma} u\|_{L^2} \\ &\qquad + C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\Lambda_h^{-\sigma} u\|_{L^2} \\ &\qquad + C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 u\|_{L^2}\|\Lambda_h^{-\sigma} u\|_{L^2} \\ &\qquad + C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h b\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2} \\ &\qquad + C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 b\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2} \\ &\qquad + C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{1-\sigma}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2} \\ &\qquad + C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 u\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\frac{d}{dt}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}^2 \\ &\quad\leq C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h b\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1} b\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\ &\qquad + C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{2-\sigma-\frac{1}{s}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\ &\qquad + C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1} b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\ &\qquad + C\|\partial_3 u\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h u\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1} u\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma-\frac{1}{s}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}\\
&+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1} u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}
\end{aligned}\tag{3.29}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}^2\\
&\le C\|\partial_3 u\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h u\|_{L^2}^{\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h b\|_{L^2}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1}u\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}\\
&\quad+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{s-1}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}\\
&\quad+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\nabla_h b\|_{L^2}^{\frac{s-1}{s}}\|\partial_3^{s+1} b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}\\
&\quad+C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h b\|_{L^2}^{\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h u\|_{L^2}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}\\
&\quad+C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{1-\sigma}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{s-1}{s}}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}\\
&\quad+C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{3}{2}-\sigma}\|\nabla_h u\|_{L^2}^{\frac{s-1}{s}}\|\partial_3^{s+1} u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 b\|_{L^2}.
\end{aligned}\tag{3.30}
$$

We first prove (3.28). Applying $\Lambda_h^{-\sigma}$ to the first two equations of (1.1), and taking the L^2-inner products with $\Lambda_h^{-\sigma}u$ and $\Lambda_h^{-\sigma}b$, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda_h^{-\sigma}u\|_{L^2}^2+\|\Lambda_h^{-\sigma}b\|_{L^2}^2)+\nu\|\Lambda_h^{1-\sigma}u\|_{L^2}^2+\eta\|\Lambda_h^{1-\sigma}b\|_{L^2}^2\\
&\quad=\int\Lambda_h^{-\sigma}(b\cdot\nabla b)\cdot\Lambda_h^{-\sigma}u\,dx-\int\Lambda_h^{-\sigma}(u\cdot\nabla u)\cdot\Lambda_h^{-\sigma}u\,dx\\
&\qquad-\int\Lambda_h^{-\sigma}(u\cdot\nabla b)\cdot\Lambda_h^{-\sigma}b\,dx+\int\Lambda_h^{-\sigma}(b\cdot\nabla u)\cdot\Lambda_h^{-\sigma}b\,dx\\
&\quad:=J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.31}
$$

Using Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$
\begin{aligned}
J_1&=\int\Lambda_h^{-\sigma}(b\cdot\nabla b)\cdot\Lambda_h^{-\sigma}u\\
&\le\|\Lambda_h^{-\sigma}(b\cdot\nabla b)\|_{L^2}\|\Lambda_h^{-\sigma}u\|_{L^2}\\
&=\|\|\Lambda_h^{-\sigma}(b\cdot\nabla b)\|_{L_h^2}\|_{L_{x_3}^2}\|\Lambda_h^{-\sigma}u\|_{L^2}\\
&\le C(\|\|b_h\cdot\nabla_h b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L_{x_3}^2}+\|\|b_3\partial_3 b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L_{x_3}^2})\|\Lambda_h^{-\sigma}u\|_{L^2}\\
&:=C(J_{11}+J_{12})\|\Lambda_h^{-\sigma}u\|_{L^2}.
\end{aligned}
$$

As in the estimate of (3.15), we have, for $\frac{1}{2}<\sigma<1$,

$$
\begin{aligned}
J_{11}&=\|\|b_h\cdot\nabla_h b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L_{x_3}^2}\\
&\le C\|\|b_h\|_{L_h^{\frac{2}{\sigma}}}\|\nabla_h b\|_{L_h^2}\|_{L_{x_3}^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\| \|b_h\|_{L_h^{\frac{2}{\sigma}}} \|_{L^\infty_{x_3}} \|\nabla_h b\|_{L^2}\\
&\le C\| \|b_h\|_{L^\infty_{x_3}} \|_{L_h^{\frac{2}{\sigma}}} \|\nabla_h b\|_{L^2}\\
&\le C\| \|b_h\|_{L^2_{x_3}}^{\frac12} \|\partial_3 b_h\|_{L^2_{x_3}}^{\frac12} \|_{L_h^{\frac{2}{\sigma}}} \|\nabla_h b\|_{L^2}\\
&\le \| \|b_h\|_{L^2_{x_3}}^{\frac12} \|_{L_h^{\frac{4}{2\sigma-1}}} \| \|\partial_3 b_h\|_{L^2_{x_3}}^{\frac12} \|_{L_h^4} \|\nabla_h b\|_{L^2}\\
&\le \| \|b_h\|_{L_h^{\frac{2}{2\sigma-1}}} \|_{L^2_{x_3}}^{\frac12} \|\partial_3 b_h\|_{L^2}^{\frac12} \|\nabla_h b\|_{L^2}\\
&\le C\| \|b_h\|_{L_h^2}^{2\sigma-1} \|\nabla_h b\|_{L_h^2}^{2-2\sigma} \|_{L^2_{x_3}}^{\frac12} \|\partial_3 b_h\|_{L^2}^{\frac12} \|\nabla_h b\|_{L^2}\\
&\le C\|b_h\|_{L^2}^{\sigma-\frac12} \|\nabla_h b_h\|_{L^2}^{1-\sigma} \|\partial_3 b_h\|_{L^2}^{\frac12} \|\nabla_h b\|_{L^2}\\
&\le C\|b\|_{L^2}^{\sigma-\frac12} \|\nabla_h b\|_{L^2}^{2-\sigma} \|\partial_3 b\|_{L^2}^{\frac12}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_{12} &= \| \|b_3 \partial_3 b\|_{L_h^{\frac{2}{1+\sigma}}} \|_{L^2_{x_3}}\\
&\le C\|b_3\|_{L^2}^{\sigma-\frac12} \|\nabla_h b_3\|_{L^2}^{1-\sigma} \|\partial_3 b_3\|_{L^2}^{\frac12} \|\partial_3 b\|_{L^2}\\
&\le C\|b_3\|_{L^2}^{\sigma-\frac12} \|\nabla_h b_3\|_{L^2}^{1-\sigma} \|\nabla_h b_h\|_{L^2}^{\frac12} \|\partial_3 b\|_{L^2}\\
&\le C\|b\|_{L^2}^{\sigma-\frac12} \|\nabla_h b\|_{L^2}^{\frac32-\sigma} \|\partial_3 b\|_{L^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J_1 \le\ & C\|b\|_{L^2}^{\sigma-\frac12} \|\nabla_h b\|_{L^2}^{1-\sigma} \|\partial_3 b\|_{L^2}^{\frac12} \|\nabla_h b\|_{L^2} \|\Lambda_h^{-\sigma} u\|_{L^2}\\
&+ C\|b\|_{L^2}^{\sigma-\frac12} \|\nabla_h b\|_{L^2}^{\frac32-\sigma} \|\partial_3 b\|_{L^2} \|\Lambda_h^{-\sigma} u\|_{L^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_2 &= -\int \Lambda_h^{-\sigma}(u\cdot\nabla u)\cdot \Lambda_h^{-\sigma} u\\
&\le C\|u\|_{L^2}^{\sigma-\frac12} \|\nabla_h u\|_{L^2}^{2-\sigma} \|\partial_3 u\|_{L^2}^{\frac12} \|\Lambda_h^{-\sigma} u\|_{L^2}\\
&\quad + C\|u\|_{L^2}^{\sigma-\frac12} \|\nabla_h u\|_{L^2}^{\frac32-\sigma} \|\partial_3 u\|_{L^2} \|\Lambda_h^{-\sigma} u\|_{L^2}.
\end{aligned}
$$

J_3 can be similarly bounded as J_1,

$$
\begin{aligned}
J_3 &= -\int \Lambda_h^{-\sigma}(u\cdot\nabla b)\cdot \Lambda_h^{-\sigma} b\\
&\le \|\Lambda_h^{-\sigma}(u\cdot\nabla b)\|_{L^2} \|\Lambda_h^{-\sigma} b\|_{L^2}\\
&= \| \|\Lambda_h^{-\sigma}(u\cdot\nabla b)\|_{L_h^2} \|_{L^2_{x_3}} \|\Lambda_h^{-\sigma} b\|_{L^2}
\end{aligned}
$$

$$\begin{aligned}
&\le C(\| \|u_h\cdot\nabla_h b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L^2_{x_3}}+\|\|u_3\partial_3 b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L^2_{x_3}})\|\Lambda_h^{-\sigma}b\|_{L^2}\\
&:= C(J_{31}+J_{32})\|\Lambda_h^{-\sigma}b\|_{L^2}.
\end{aligned}$$

J_{31} and J_{32} are bounded as follows.

$$\begin{aligned}
J_{31} &= \|\|u_h\cdot\nabla_h b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L^2_{x_3}}\\
&\le C\|\|u_h\|_{L_h^{\frac{2}{\sigma}}}\|\nabla_h b\|_{L_h^2}\|_{L^2_{x_3}}\\
&\le C\|\|u_h\|_{L_h^{\frac{2}{\sigma}}}\|_{L^\infty_{x_3}}\|\nabla_h b\|_{L^2}\\
&\le C\|\|u_h\|_{L^\infty_{x_3}}\|_{L_h^{\frac{2}{\sigma}}}\|\nabla_h b\|_{L^2}\\
&\le C\|\|u_h\|_{L^2_{x_3}}^{\frac12}\|\partial_3 u_h\|_{L^2_{x_3}}^{\frac12}\|_{L_h^{\frac{2}{\sigma}}}\|\nabla_h b\|_{L^2}\\
&\le \|\|u_h\|_{L^2_{x_3}}^{\frac12}\|_{L_h^{\frac{4}{2\sigma-1}}}\|\|\partial_3 u_h\|_{L^2_{x_3}}^{\frac12}\|_{L_h^4}\|\nabla_h b\|_{L^2}\\
&\le \|\|u_h\|_{L_h^{\frac{2}{2\sigma-1}}}\|_{L^2_{x_3}}^{\frac12}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}\\
&\le C\|\|u_h\|_{L_h^2}^{2\sigma-1}\|\nabla_h u\|_{L_h^2}^{2-2\sigma}\|_{L^2_{x_3}}^{\frac12}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}\\
&\le C\|u_h\|_{L^2}^{\sigma-\frac12}\|\nabla_h u_h\|_{L^2}^{1-\sigma}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}\\
&\le C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
J_{32} &= \|\|u_3\partial_3 b\|_{L_h^{\frac{2}{1+\sigma}}}\|_{L^2_{x_3}}\\
&\le C\|u_3\|_{L^2}^{\sigma-\frac12}\|\nabla_h u_3\|_{L^2}^{1-\sigma}\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}\\
&\le C\|u_3\|_{L^2}^{\sigma-\frac12}\|\nabla_h u_3\|_{L^2}^{1-\sigma}\|\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}\\
&\le C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{\frac32-\sigma}\|\partial_3 b\|_{L^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
J_3 \le\ & C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h b\|_{L^2}\|\Lambda_h^{-\sigma}b\|_{L^2}\\
&+C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{\frac32-\sigma}\|\partial_3 b\|_{L^2}\|\Lambda_h^{-\sigma}b\|_{L^2}.
\end{aligned}$$

Similarly,

$$J_4 = -\int \Lambda_h^{-\sigma}(b\cdot\nabla u)\cdot\Lambda_h^{-\sigma}b\,dx$$

$$\leq C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{1-\sigma}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2}$$
$$+ C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 u\|_{L^2}\|\Lambda_h^{-\sigma} b\|_{L^2}.$$

Inserting the bounds for J_1, J_2, J_3 and J_4 in (3.31), we obtain (3.28).

Now we prove (3.29). Applying $\Lambda_h^{-\sigma}\partial_3$ to the first two equations of (1.1), and taking the L^2-inner products with $\Lambda_h^{-\sigma}\partial_3 u$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}^2 + \nu\|\Lambda_h^{1-\sigma}\partial_3 u\|_{L^2}^2 \\
&= \int \Lambda_h^{-\sigma}\partial_3(b\cdot\nabla b)\cdot\Lambda_h^{-\sigma}\partial_3 u\,dx - \int \Lambda_h^{-\sigma}\partial_3(u\cdot\nabla u)\cdot\Lambda_h^{-\sigma}\partial_3 u\,dx \\
&:= N_1 + N_2. \qquad (3.32)
\end{aligned}$$

By Hölder's inequality and the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}
N_1 &= \int \Lambda_h^{-\sigma}\partial_3(b\cdot\nabla b)\cdot\Lambda_h^{-\sigma}\partial_3 u \\
&\leq \|\Lambda_h^{-\sigma}\partial_3(b\cdot\nabla b)\|_{L^2}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\
&= (\|\Lambda_h^{-\sigma}(\partial_3 b\cdot\nabla b)\|_{L^2} + \|\Lambda_h^{-\sigma}(b\cdot\nabla\partial_3 b)\|_{L^2})\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\
&\leq C(\|\Lambda_h^{-\sigma}(\partial_3 b_h\cdot\nabla_h b)\|_{L^2} + \|\Lambda_h^{-\sigma}(\partial_3 b_3\partial_3 b)\|_{L^2} + \|\Lambda_h^{-\sigma}(b_h\cdot\nabla_h\partial_3 b)\|_{L^2} \\
&\quad + \|\Lambda_h^{-\sigma}(b_3\partial_3^2 b)\|_{L^2})\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2} \\
&:= C(N_{11} + N_{12} + N_{13} + N_{14})\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}.
\end{aligned}$$

As in the estimate of J_{11},

$$\begin{aligned}
N_{11} &= \|\Lambda_h^{-\sigma}(\partial_3 b_h\cdot\nabla_h b)\|_{L^2} \\
&\leq C\|\partial_3 b_h\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h\partial_3 b_h\|_{L^2}^{1-\sigma}\|\partial_3^2 b_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h b\|_{L^2}.
\end{aligned}$$

Further invoking the interpolation inequalities,

$$\|\nabla_h\partial_3 b_h\|_{L^2} \leq C\|\nabla_h b_h\|_{L^2}^{\frac{s-1}{s}}\|\nabla_h\partial_3^s b_h\|_{L^2}^{\frac{1}{s}},$$
$$\|\partial_3^2 b_h\|_{L^2} \leq C\|\partial_3 b_h\|_{L^2}^{\frac{s-1}{s}}\|\partial_3^{s+1} b_h\|_{L^2}^{\frac{1}{s}},$$

we obtain

$$\begin{aligned}
N_{11} &\leq C\|\partial_3 b_h\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_h\|_{L^2}^{\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b_h\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3 b_h\|_{L^2}^{\frac{s-1}{2s}}\|\partial_3^{s+1} b_h\|_{L^2}^{\frac{1}{2s}}\|\nabla_h b\|_{L^2} \\
&\leq C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h b\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1} b\|_{L^2}^{\frac{1}{2s}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_{12} &= \|\Lambda_h^{-\sigma}(\partial_3 b_3\partial_3 b)\|_{L^2} \\
&\leq C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h\partial_3 b\|_{L^2}^{1-\sigma}\|\partial_3^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_3 b_3\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b_h\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3 b\|_{L^2}^{\frac{s-1}{2s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{2s}}\|\nabla_h b_h\|_{L^2}\\
&\le C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h b\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{2s}},\\
N_{13} &= \|\Lambda_h^{-\sigma}(b_h\cdot\nabla_h\partial_3 b)\|_{L^2}\\
&\le C\|b_h\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_h\|_{L^2}^{1-\sigma}\|\partial_3 b_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 b\|_{L^2}\\
&\le C\|b_h\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_h\|_{L^2}^{1-\sigma}\|\partial_3 b_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{s-1}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1}{s}}\\
&\le C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{2-\sigma-\frac{1}{s}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1}{s}},\\
N_{14} &= \|\Lambda_h^{-\sigma}(b_3\partial_3^2 b)\|_{L^2}\\
&\le C\|b_3\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_3\|_{L^2}^{1-\sigma}\|\partial_3 b_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 b\|_{L^2}\\
&\le C\|b_3\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_3\|_{L^2}^{1-\sigma}\|\partial_3 b_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 b_3\|_{L^2}^{\frac{s-1}{s}}\|\partial_3^{s+1}b_3\|_{L^2}^{\frac{1}{s}}\\
&\le C\|b_3\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b_3\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1}b_3\|_{L^2}^{\frac{1}{s}}\\
&\le C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{s}}.
\end{aligned}
$$

Incorporating these upper bounds yields

$$
\begin{aligned}
N_1 \le\, & C\|\partial_3 b\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h b\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}\\
&+C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{2-\sigma-\frac{1}{s}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^s b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}\\
&+C\|b\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h b\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1}b\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
N_2 =\, & -\int \Lambda_h^{-\sigma}\partial_3(u\cdot\nabla u)\cdot\Lambda_h^{-\sigma}\partial_3 u\,dx\\
\le\, & C\|\partial_3 u\|_{L^2}^{\sigma-\frac{1}{2s}}\|\nabla_h u\|_{L^2}^{1+\frac{(s-1)(1-\sigma)}{s}}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1-\sigma}{s}}\|\partial_3^{s+1}u\|_{L^2}^{\frac{1}{2s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}\\
&+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma-\frac{1}{s}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^s u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}\\
&+C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{5}{2}-\sigma-\frac{1}{s}}\|\partial_3^{s+1}u\|_{L^2}^{\frac{1}{s}}\|\Lambda_h^{-\sigma}\partial_3 u\|_{L^2}.
\end{aligned}
$$

Inserting the above bounds into (3.32), we obtain (3.29). Since (3.30) can be proven similarly, we omit the details.

Step 5. Completion of the bootstrapping argument. This step finishes the bootstrapping argument and proves (3.3). Integrating (3.28) over $[0, t]$ with $0 < t \le T$, together with (3.4) and (3.12), we obtain

$$\begin{aligned}
&\|\Lambda_h^{-\sigma}u(t)\|_{L^2}^2+\|\Lambda_h^{-\sigma}b(t)\|_{L^2}^2\\
&\quad\le \|\Lambda_h^{-\sigma}u_0\|_{L^2}^2+\|\Lambda_h^{-\sigma}b_0\|_{L^2}^2+C\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}u(\tau)\|_{L^2}^2+\|\Lambda_h^{-\sigma}b(\tau)\|_{L^2}^2)\\
&\qquad\times\epsilon^{\delta_0}\int_0^t\Big((1+\tau)^{-(\frac{\sigma^2}{2}-\delta_0)}(1+\tau)^{-\frac{(1+\sigma)(2-\sigma)}{2}}\\
&\qquad+(1+\tau)^{-(\frac{\sigma^2}{2}+\frac{\sigma}{4}-\delta_0)}(1+\tau)^{-\frac{(1+\sigma)(\frac{3}{2}-\sigma)}{2}}\Big)d\tau\\
&\quad\le \|\Lambda_h^{-\sigma}u_0\|_{L^2}^2+\|\Lambda_h^{-\sigma}b_0\|_{L^2}^2+C\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}u(\tau)\|_{L^2}^2\\
&\qquad+\|\Lambda_h^{-\sigma}b(\tau)\|_{L^2}^2)\epsilon^{\delta_0}\int_0^t(1+\tau)^{-(\frac{3+2\sigma}{4}-\delta_0)}d\tau\\
&\quad\le \|\Lambda_h^{-\sigma}u_0\|_{L^2}^2+\|\Lambda_h^{-\sigma}b_0\|_{L^2}^2+C\epsilon^{\delta_0}\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}u(\tau)\|_{L^2}^2+\|\Lambda_h^{-\sigma}b(\tau)\|_{L^2}^2),
\end{aligned}\tag{3.33}$$

where we have used the fact $\|u(t)\|_{L^2}+\|b(t)\|_{L^2}\le C\varepsilon$, and $\delta_0>0$ is chosen small enough such that $\frac{3+2\sigma}{4}-\delta_0>1$, which is certainly achievable due to the assumption $\frac{1}{2}<\sigma<1$. Integrating (3.29) over $[0,t]$, together with (3.4) and (3.12), we obtain, for $s\ge 3$,

$$\begin{aligned}
&\|\Lambda_h^{-\sigma}\partial_3u(t)\|_{L^2}^2\\
&\quad\le \|\Lambda_h^{-\sigma}\partial_3u_0\|_{L^2}^2+C\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}\partial_3u(\tau)\|_{L^2}^2)\\
&\qquad\times\epsilon^{\delta_0}\int_0^t\Big((1+\tau)^{-(\frac{\sigma^2}{2}-\frac{\sigma}{4s}-\delta_0)}(1+\tau)^{-(\frac{1+\sigma}{2}+\frac{(s-1)(1-\sigma^2)}{s})}\\
&\qquad+(1+\tau)^{-(\frac{\sigma^2}{2}-\delta_0)}(1+\tau)^{-\frac{(1+\sigma)(2-\sigma-\frac{1}{s})}{2}}\\
&\qquad+(1+\tau)^{-(\frac{\sigma^2}{2}-\frac{\sigma}{4}-\delta_0)}(1+\tau)^{-\frac{(1+\sigma)(\frac{5}{2}-\sigma-\frac{1}{s})}{2}}\Big)d\tau\\
&\quad\le \|\Lambda_h^{-\sigma}\partial_3u_0\|_{L^2}^2+C\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}\partial_3u(\tau)\|_{L^2}^2)\\
&\qquad\times\epsilon^{\delta_0}\int_0^t(1+\tau)^{-(\frac{\sigma^2}{2}-\frac{\sigma}{4s}-\delta_0)}(1+\tau)^{-(\frac{1+\sigma}{2}+\frac{(s-1)(1-\sigma^2)}{s})}d\tau\\
&\quad\le \|\Lambda_h^{-\sigma}\partial_3u_0\|_{L^2}^2+C\epsilon^{\delta_0}\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}\partial_3u(\tau)\|_{L^2}^2),
\end{aligned}\tag{3.34}$$

where $\delta_0>0$ is chosen small enough such that $\frac{\sigma^2}{2}-\frac{\sigma}{4s}+\frac{1+\sigma}{2}+\frac{(s-1)(1-\sigma^2)}{s}-\delta_0>1$. Similarly, we have

$$\|\Lambda_h^{-\sigma}\partial_3b(t)\|_{L^2}^2\le\|\Lambda_h^{-\sigma}\partial_3u_0\|_{L^2}^2+C\epsilon^{\delta_0}\sup_{0\le\tau\le t}(\|\Lambda_h^{-\sigma}\partial_3b(\tau)\|_{L^2}^2)\tag{3.35}$$

with $\frac{1}{2} < \sigma < 1$ and $s \geq 3$. Adding (3.33), (3.34) and (3.35), together with (3.1), we obtain

$$\begin{aligned}&\|\Lambda_h^{-\sigma} u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(t)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 b(t)\|_{L^2}^2 \\ &\leq \gamma_0 + C\varepsilon^{\delta_0} \sup_{0\leq\tau\leq t} (\|\Lambda_h^{-\sigma} u(\tau)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} b(\tau)\|_{L^2}^2 + \|\Lambda_h^{-\sigma} \partial_3 u(\tau)\|_{L^2}^2 \\ &\quad + \|\Lambda_h^{-\sigma} \partial_3 b(\tau)\|_{L^2}^2).\end{aligned}$$

By choosing ε sufficiently small such that $C\varepsilon^{\delta_0} < \min\{\frac{1}{3}, \frac{1}{3}\gamma_0\}$, then this inequality, together with the Young inequality, yields (3.3) for all $t \in [0, T]$. Then, the bootstrapping argument implies that $T = \infty$ and (3.3) holds for all $t < \infty$. This completes the proof of Theorem 1.1. □

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