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Stability and exponential decay for the compressible viscous non-resistive MHD system

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Abstract

How to construct global solutions of the compressible viscous magnetohydrodynamic (MHD) equations without magnetic diffusion even with small initial data in $\mathbb{R}^3$ or $\mathbb{T}^3$ is still an extremely challenging open problem. The difficulty comes from the lack of magnetic diffusion and the fact that solutions to inviscid equations generally grow in time. Motivated by this open problem, the present paper focuses on a special case of this MHD system in $\mathbb{T}^3$ when the magnetic field is vertical. We establish the global existence and uniqueness of smooth solutions to this system near a steady-state solution. In addition, the solution is shown to be stable and decay exponentially in time. The proof discovers and makes use of the smoothing and stabilizing effect of the steady magnetic field on the perturbations.

Keywords: global solutions, non-resistive compressible MHD, decay rates

Mathematics subject classification: 35Q35, 35A01, 35A02, 76W05

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1. Introduction and the main result

1.1. Model and synopsis of result

In this paper, we are concerned with the large time regularity of the compressible viscous non-resistive magnetohydrodynamic (MHD) equations, which model the motion of electrically conducting fluids in the presence of a magnetic field. It can be written as the following system,

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0, \\ \rho(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mu\Delta\mathbf{v} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{v} + \nabla P = (\nabla\times\mathbf{B})\times\mathbf{B}, \\ \partial_t\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0, \\ \operatorname{div}\mathbf{B} = 0. \end{cases} \tag{1.1}$$

Here the unknowns ρ, $\mathbf{v}$, $\mathbf{B}$, are the density of the fluid, the velocity field, and the magnetic field, respectively. The thermal pressure $P(\rho)$ is assumed to follow a polytropic $\gamma-$law, $P(\rho) = A\rho^\gamma$ for some $A > 0$ and $\gamma \geqslant 1$. The parameters μ and λ are shear viscosity and volume viscosity coefficients, respectively, which satisfy the standard strong parabolicity assumption,

$$\mu > 0 \quad \text{and} \quad \nu \overset{\text{def}}{=} \lambda + 2\mu > 0.$$

The compressible MHD equations can be derived from the isentropic Navier–Stokes-Maxwell system by taking the zero dielectric constant limit [14]. Although the small data global well-posedness on the 2D compressible MHD equations without magnetic diffusion has been successfully settled, this same problem on the 3D counterpart appears to be inaccessible at this moment.

This paper focuses on a very special $2\frac{1}{2}$-D compressible MHD system. The motion of fluids takes place in the plane while the magnetic field acts on fluids only in the vertical direction, namely

$$\mathbf{v} = \left(\mathbf{v}^1(t,x_1,x_2), \mathbf{v}^2(t,x_1,x_2), 0\right) \overset{\text{def}}{=} (\mathbf{u}, 0),$$
$$\rho \overset{\text{def}}{=} \rho(t,x_1,x_2), \quad \mathbf{B} \overset{\text{def}}{=} (0,0,m(t,x_1,x_2)).$$

Then (1.1) is reduced to

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P + \dfrac{1}{2}\nabla m^2 = 0, \\ \partial_t m + \operatorname{div}(m\mathbf{u}) = 0. \end{cases} \tag{1.2}$$

We are interested in the initial boundary value problem for the system (1.2) in torus $\mathbb{T}^2 = (0,d_1)\times(0,d_2)$ with the initial condition

$$(\rho, \mathbf{u}, m)(x,0) = (\rho_0, \mathbf{u}_0, m_0)(x), \quad x \in \mathbb{T}^2,$$

and the periodic boundary condition

$$(\rho,\mathbf{u},m)(x+d,t)=(\rho,\mathbf{u},m)(x,t), \quad t\geqslant 0, \quad d=(d_1,d_2).$$

To overcome the difficulties arising from the non-dissipation on ρ and m, we will rewrite system (1.2). On the basis of the state equations, we reformulate the system (1.2) in terms of variables P, $\mathbf{u}$ and m as

$$\begin{cases}\partial_t P+\operatorname{div}(P\mathbf{u})+(\gamma-1)P\operatorname{div}\mathbf{u}=0,\\ \rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}+\nabla P+\dfrac{1}{2}\nabla m^2=0,\\ \partial_t m+\operatorname{div}(m\mathbf{u})=0,\\ (P,\mathbf{u},m)|_{t=0}=(P_0,\mathbf{u}_0,m_0)\overset{\text{def}}{=}(A\rho_0^\gamma,\mathbf{u}_0,m_0).\end{cases}\tag{1.3}$$

For notational convenience, we write

$$\int_{\mathbb{T}^2}A\rho_0^\gamma\,\mathrm{d}x\overset{\text{def}}{=}\bar{P}\quad\text{and}\quad\int_{\mathbb{T}^2}m_0\,\mathrm{d}x\overset{\text{def}}{=}\bar{m}.$$

Clearly $\bar{P}$ is a positive constant. $\bar{m}$ is non-zero but not required to be positive.

1.2. Main result

Now we can state our main result in the following theorem.

Theorem 1.1. *Assume that the initial data satisfy* $(P_0-\bar{P},m_0-\bar{m})\in H^3(\mathbb{T}^2)$ *and* $\mathbf{u}_0\in H^3(\mathbb{T}^2)$*, with*

$$c_0\leqslant\rho_0\leqslant c_0^{-1}\quad\textit{and}\quad\int_{\mathbb{T}^2}\rho_0\mathbf{u}_0\,\mathrm{d}x=0\tag{1.4}$$

for some constant $c_0>0$*. Then, there exists a small constant* $\varepsilon>0$ *such that, if*

$$\|(P_0-\bar{P},m_0-\bar{m})\|_{H^3}+\|\mathbf{u}_0\|_{H^3}\leqslant\varepsilon,$$

the system (1.3) admits a unique global solution $(P-\bar{P},\mathbf{u},m-\bar{m})$ *such that*

$$(P-\bar{P},m-\bar{m})\in C([0,\infty);H^3),\quad \mathbf{u}\in C([0,\infty);H^3)\cap L^2(\mathbb{R}^+;H^4).$$

Moreover, for any $t\geqslant 0$*, there holds*

$$\|\mathbf{u}(t)\|_{H^3}\leqslant C_1e^{-C_1t},\tag{1.5}$$

for some pure constant $C_1>0$*.*

Remark 1.1. Theorem 1.1 appears to be the first such stability and decay result on this particular MHD system in bounded domains. Our method exploits the hidden wave structure, which provides the desired smoothing and stabilizing effect.

Remark 1.2. Our result does not need any assumption that the vertical magnetic field m is non-negative. What is really crucial here is that the average $\bar{m}$ is not zero. When the average $\bar{m}$ is not zero, the equations of the velocity $\mathbf{u}$ and the perturbation of m, namely $b = m - \bar{m}$ form a wave structure. As we shall see later,

$$\begin{cases} \partial_t \mathbf{u} + \bar{m}\nabla b = \cdots, \\ \partial_t b + \bar{m}\mathrm{div}\mathbf{u} = \cdots, \end{cases} \tag{1.6}$$

where, for simplicity, we have used dots to denote other terms. By taking one more time derivative in (1.6) and making substitutions, we easily see the wave structure in div$\mathbf{u}$ and b,

$$\begin{cases} \partial_{tt}\mathrm{div}\mathbf{u} - \bar{m}^2\Delta\mathrm{div}\mathbf{u} = \cdots, \\ \partial_{tt} b - \bar{m}^2\Delta b = \cdots. \end{cases} \tag{1.7}$$

(1.7) also reveals that the sign of $\bar{m}$ makes no difference. This wave structure allows to extract the designed dissipative effect on b.

When the average $\bar{m}$ is zero, the wave structure disappears and the zero-averaged magnetic field no longer stabilizes the perturbation b. Then the Sobolev norms of b could grow in time and it becomes impossible to establish the desired stability and decay.

1.3. Difficulties and scheme of the proof

Now, let us explain the difficulty and our idea. Due to the lack of the dissipations on both the density and the magnetic field, the stability and large-time behavior problem concerned here is very difficult.

We first remark that stability problem concerned here can not be converted to the compressible Navier–Stokes stability problem. Although adding the equations of ρ and m may lead to a density-like equation for $\rho + m$, but the pressure term can not be rewritten as a power law. Furthermore, $\rho + m$ may not necessarily be positive since m does not have a sign. Therefore the classical compressible Navier–Stokes approach can not be used to solve our stability problem.

To make up for the missing regularization, we consider a small perturbation of the equilibrium $\bar{P}$, $\bar{m}$ for the density and the magnetic field, respectively. In the framework of the perturbation, the local well-posedness of (1.3) can be shown via a procedure that is now standard (see, e.g. [12]). The focus of the proof is on the global bound of $(P - \bar{P}, \mathbf{u}, m - \bar{m})$ in $H^3(\mathbb{T}^2)$. We use the bootstrapping argument and start by making the ansatz that

$$\sup_{t\in[0,T]} \left(\|P - \bar{P}\|_{H^3} + \|\mathbf{u}\|_{H^3} + \|m - \bar{m}\|_{H^3}\right) \leqslant \delta,$$

for suitably chosen $0 < \delta < 1$. The main efforts are devoted to proving that, if the initial norm is taken to be sufficiently small, namely

$$\|P_0 - \bar{P}\|_{H^3} + \|\mathbf{u}_0\|_{H^3} + \|m_0 - \bar{m}\|_{H^3} \leqslant \varepsilon$$

with sufficiently small $\varepsilon > 0$, then

$$\sup_{t\in[0,T]} \left(\|P - \bar{P}\|_{H^3} + \|\mathbf{u}\|_{H^3} + \|m - \bar{m}\|_{H^3}\right) \leqslant \frac{\delta}{2}. \tag{1.8}$$

It is not trivial to prove (1.8). Now, let us explain our main idea. Without loss of generality, we take $\bar{P} = \bar{m} = 1$ in the paper. The starting point is to write the term $\frac{1}{2}\nabla(m-1)^2$ as a new variable rather than the nonlinear term and define

$$p \stackrel{\text{def}}{=} P - 1, \quad b \stackrel{\text{def}}{=} m - 1.$$

Then, we can rewrite (1.3) into the following form

$$\begin{cases} \partial_t p + \gamma \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla p + \gamma p \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla p + \dfrac{1}{2} \nabla m^2 = \text{Nonlinear terms}, \\ \partial_t b + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla b + b \operatorname{div} \mathbf{u} = 0, \\ (p, \mathbf{u}, b)|_{t=0} = (p_0, \mathbf{u}_0, b_0). \end{cases} \tag{1.9}$$

By the standard energy method, we can show that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|(p, \mathbf{u}, b)\|_{H^3}^2 + \mu \|\nabla \mathbf{u}\|_{H^3}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}\|_{H^3}^2 \\ &\qquad \leqslant C\left(\|(p, \mathbf{u}, b)\|_{H^3} + \|(\nabla p, \nabla \mathbf{u}, \nabla b)\|_{H^3}^2\right) \|(p, \mathbf{u}, b)\|_{H^3}^2, \end{aligned} \tag{1.10}$$

from which we can see that (1.10) does not close under small initial data unless some norms of p, b such as $\|p\|_{H^3}^2$ and $\|b\|_{H^3}^2$ occurs on the left. In order to capture the dissipation arising from the complicated coupling between p and b , our idea is to introduce the new pressure φ as

$$\varphi \stackrel{\text{def}}{=} P - 1 + \frac{1}{2} m^2 - \frac{1}{2}. \tag{1.11}$$

After an elementary calculation, we find that the new variable $(\varphi, \mathbf{u})$ satisfies the following equations

$$\begin{cases} \partial_t \varphi + (\gamma + 1) \operatorname{div} \mathbf{u} = \text{Nonlinear terms}, \\ \partial_t \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla \varphi = \text{Nonlinear terms}. \end{cases} \tag{1.12}$$

Especially, the linearized system of (1.12) has the same structure as the compressible Navier–Stokes equations. Hence, by exploiting delicate energy analysis, we can capture the damping effect of φ and smoothing effect of $\mathbf{u}$ in (1.12).

Although we have obtained the damping effect of φ, another difficulty to prove (1.8) is that we still cannot get any damping effect of p, or b, respectively. So, the energy estimate like (1.10) is invalid to our bootstrap argument. We need to make a more dedicated energy estimate as follows

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|(p, \mathbf{u}, b)\|_{H^3}^2 - \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^2} \frac{p}{1+p} \left(\nabla^3 p\right)^2 dx + \mu \|\nabla \mathbf{u}\|_{H^3}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}\|_{H^3}^2 \\ &\quad \leqslant C\left(\left(1 + \|p\|_{H^3}^2\right) \|\mathbf{u}\|_{H^3} + \|\mathbf{u}\|_{H^3}^2 + \|\varphi\|_{H^3}^2\right) \|(p, \mathbf{u}, b)\|_{H^3}^2. \end{aligned} \tag{1.13}$$

Compared to (1.10), the advantage of the refined energy estimates (1.13) is that the time integral of $(1 + \|p(t)\|_{H^3}^2) \|\mathbf{u}(t)\|_{H^3} + \|\mathbf{u}(t)\|_{H^3}^2 + \|\varphi(t)\|_{H^3}^2$ in front of $\|(p, \mathbf{u}, b)\|_{H^3}^2$ is time integrable. This is because the damping effect and smoothing effect on $\varphi, \mathbf{u}$ in (1.12).

In the following, we explain some difficulty about deriving the refined energy estimate (1.13). Due to the lack of damping effect or smoothing effect of p, b, we can only use

$L_t^\infty(H^s)$ norm of p,b in taking energy estimates. Thus, we have to be very careful to deal with nonlinear terms involved in p,b

$$\begin{aligned}&\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p))\cdot\nabla^s p\,\mathrm{d}x,\qquad \int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b)\cdot\nabla^s b\,\mathrm{d}x,\qquad \int_{\mathbb{T}^2}\nabla^s(p\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x,\\&\int_{\mathbb{T}^2}\nabla^s(b\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x,\qquad \frac{1}{2}\int_{\mathbb{T}^2}\nabla^s\nabla b^2\cdot\nabla^s\mathbf{u}\,\mathrm{d}x.\end{aligned}\tag{1.14}$$

For the first two terms

$$\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p))\cdot\nabla^s p\,\mathrm{d}x,\quad\text{and}\quad\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b)\cdot\nabla^s b\,\mathrm{d}x,$$

we mainly use the commutator argument to transform the derivative from ∇p or ∇b to $\mathbf{u}$, see (3.16)–(3.19) for more details. For the last two terms

$$\int_{\mathbb{T}^2}\nabla^s\left(b\mathrm{div}\,\mathbf{u}\right)\cdot\nabla^s p\,\mathrm{d}x,\quad\text{and}\quad\frac{1}{2}\int_{\mathbb{T}^2}\nabla^s\nabla b^2\cdot\nabla^s\mathbf{u}\,\mathrm{d}x,$$

we combine them together and also use the commutator argument, see (3.20) for more detail. However, we cannot use the same strategy as $\int_{\mathbb{T}^2}\nabla^s(b\mathrm{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x$ to deal with the term $\int_{\mathbb{T}^2}\nabla^s(p\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x$ other than the pressure $P(\rho)=\frac{1}{2}\rho^2$. For more general $\gamma-$law, i.e. $P(\rho)=A\rho^\gamma$ for some $A>0$ and $\gamma\geqslant 1$, we need to make some new idea to overcome this difficulty. In fact, we first use the commutator argument to divide this term into two terms

$$\int_{\mathbb{T}^2}\nabla^s\left(p\mathrm{div}\,\mathbf{u}\right)\cdot\nabla^s p\,\mathrm{d}x=\int_{\mathbb{T}^2}[\nabla^s,p]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x+\int_{\mathbb{T}^2}p\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x.\tag{1.15}$$

The term involved in the commutator is easy to control. For the last term in (1.15), the trouble only arise when we deal with the highest order derivative, i.e. $\int_{\mathbb{T}^2}p\nabla^3\mathrm{div}\,\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x$. With the help of the Hölder inequality, this term will be bounded by

$$\begin{aligned}\left|\int_{\mathbb{T}^2}p\nabla^3\mathrm{div}\,\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x\right|&\leqslant C\|p\|_{L^\infty}\left\|\nabla^3\mathrm{div}\,\mathbf{u}\right\|_{L^2}\left\|\nabla^3 p\right\|_{L^2}\\&\leqslant C\|p\|_{H^2}\|\mathrm{div}\,\mathbf{u}\|_{H^3}\|p\|_{H^3}\\&\leqslant C\|p\|_{H^3}^2\|\mathrm{div}\,\mathbf{u}\|_{H^3}.\end{aligned}\tag{1.16}$$

To control the term $\|\mathrm{div}\,\mathbf{u}\|_{H^3}$, we have to use the smoothing effect coming from the velocity equation to absorb this term to the left which will lead to the following inequality

$$\|p\|_{H^3}^2\|\mathrm{div}\,\mathbf{u}\|_{H^3}\leqslant\varepsilon\|\mathrm{div}\,\mathbf{u}\|_{H^3}^2+C\|p\|_{H^3}^4.\tag{1.17}$$

When we use the continuity argument to close the energy estimates, (1.17) implies that we have to ensure that the time integral of $\|p\|_{H^3}^2$ is time integrable, this is a disaster due to the lack of the dissipation of the equation of p. To overcome the difficulty, we shall make full use of the density equation of (3.3) to write

$$\mathrm{div}\,\mathbf{u}=-\frac{\partial_t p+\mathbf{u}\cdot\nabla p}{\gamma\left(p+1\right)},$$

from which we can get

$$-\int_{\mathbb{T}^2} p\nabla^3 \mathrm{div}\,\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x = \frac{1}{\gamma}\int_{\mathbb{T}^2} p\nabla^3\left(\frac{\partial_t p+\mathbf{u}\cdot\nabla p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x$$
$$= \frac{1}{\gamma}\int_{\mathbb{T}^2} p\nabla^3\left(\frac{\partial_t p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x + \int_{\mathbb{T}^2} p\nabla^3\left(\frac{\mathbf{u}\cdot\nabla p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x. \tag{1.18}$$

Now, the degree of nonlinearity on the right hand side of (1.18) is higher than the left, which implies that it is much easier to close the energy estimates in the framework of small initial data. Combining with the previous steps, we then obtain a self-contained energy estimates.

The last step is to establish (1.8) and close the bootstrapping argument. The energy inequality obtained in the previous step, together with interpolation inequalities, allow us to show that

$$\|(\varphi,\mathbf{u})(t)\|_{H^3}\leqslant Ce^{-ct}$$

when $\delta>0$ is taken to be sufficiently small. In particular, the time integral of

$$\left(1+\|p(t)\|_{H^3}^2\right)\|\mathbf{u}(t)\|_{H^3}+\|\mathbf{u}(t)\|_{H^3}^2+\|\varphi(t)\|_{H^3}^2$$

in (3.82) is finite,

$$\int_0^\infty\left(\left(1+\|p(\tau)\|_{H^3}^2\right)\|\mathbf{u}(\tau)\|_{H^3}+\|\mathbf{u}(\tau)\|_{H^3}^2+\|\varphi(\tau)\|_{H^3}^2\right)\mathrm{d}\tau\leqslant C<\infty.$$

Grönwall's inequality then yields

$$\|(p,\mathbf{u},\tau)(t)\|_{H^3}\leqslant C\|(p_0,\mathbf{u}_0,\tau_0)\|_{H^3}.$$

Taking the initial norm to be sufficiently small, we achieve (1.8).

1.4. Organization of the paper

The rest of this paper is structured as follows. In section 2, we recall several functional inequalities to be used in the proof of theorem 1.1. Section 3 proves theorem 1.1. The long proof is accomplished in five subsections. Section 3.1 explains how to prove the local well-posedness and initiates the bootstrapping argument. Section 3.2 provides the energy estimates for $(P-\bar{P},\mathbf{u},m-\bar{m})$. Section 3.3 establishes the energy estimates for $(\varphi,\mathbf{u})$. Section 3.4 explores the dissipation $(\varphi,\mathbf{u})$. Section 3.5 proves the decay estimate and then closes the bootstrapping argument.

Finally, we briefly recall some known related results about the compressible MHD equations.

1.5. Recall some known results

When the effect of the magnetic field is omitted, i.e. $\mathbf{B}=\mathbf{0}$, (1.1) reduces to the isentropic compressible Navier–Stokes system, which has been extensively studied by many researchers, we refer to [1, 9, 30] and the references therein.

Due to its physical importance and mathematical challenge, the mathematical study of the compressible MHD equations has attracted considerable attention. When adding Laplacian term $-\Delta\mathbf{B}$ to the magnetic equation, (1.1) becomes the compressible viscous resistive MHD system. There has been a profuse literature devoted to the compressible viscous resistive MHD system concerning the global existence of weak solutions [4, 7, 8], local and global well-posedness of strong solutions with vacuum [6, 15, 34] and large time behaviour [14, 26]. Next we review some previous results on the case of our consideration, namely the compressible viscous non-resistive MHD system (1.1). Owing to the lack of dissipation mechanism for the magnetic field, the mathematical analysis of (1.1) becomes more delicate and relatively fewer results are available. For weak solutions, Li and Sun [17] obtained the existence and large-time behavior of global weak solutions in 1D case. In 2D case, they [18, 19] also proved the global existence of weak solutions for both isentropic and non-isentropic cases. Later on, the global result of [18] was extended by Liu and Zhang [22] to the density-dependent viscosity coefficient and non-monotone pressure law. For strong solutions, Jiang and Zhang [10] proved the global well-posedness of strong solutions for large data and studied the non-resistive limit in 1D case. Wu and Wu [27] established the global well-posedness of small strong solutions in $\mathbb{R}^2$ by using the systematic approach. Similar results on $\mathbb{T}^2$ were obtained by Wu and Zhu [29]. Zhong [33] constructed the local strong solutions with possible initial vacuum but without any Cho–Choe–Kim type compatibility conditions in $\mathbb{R}^2$. The global existence of smooth solutions on the horizontally infinite flat layer $\Omega = \mathbb{R}^2 \times (0,1)$ for the isentropic and non-isentropic cases was proved by Tan-Wang [24] and Li [16], respectively. Recently, Wu and Zhai [28] proved the global well-posedness of strong solutions on $\mathbb{T}^3$ under the assumptions that the initial data is close enough to an equilibrium state, see [20] for an improvement of [28] for the compressible viscous non-isentropic MHD flows without magnetic diffusion. When $\rho = \text{constant}$, (1.1) becomes the incompressible viscous non-resistive MHD system, we refer to [21, 23, 31, 32] and the references therein for related results.

However, as far as we known, the global well-posedness problem on the compressible viscous non-resistive MHD system in $\mathbb{R}^3$ remains an challenging open problem, even for the small data. As an attempt to solve this problem, by exploiting the Fourier theory and delicate energy method, the authors in this paper [3] considered a special $2\frac{1}{2}$-D compressible non-resistive MHD equations and proved the global existence of strong solutions with small initial data in the critical Besov spaces. Furthermore, we obtained the solution's optimal decay rate when the initial data is further assumed to be in a Besov space of negative index.

2. Preliminaries

First, we describe the notations we shall use in this paper.

Notations : let A, B be two operators, we denote $[A,B] = AB - BA$, the commutator between A and B. Throughout the paper, $C > 0$ stands for a generic 'constant'. For X a Banach space and I an interval of $\mathbb{R}$, for any $f, g, h \in X$, we agree that $\|(f,g,h)\|_X \stackrel{\text{def}}{=} \|f\|_X + \|g\|_X + \|h\|_X$ and denote by $C(I;X)$ the set of continuous functions on I with values in X.

Next, we present several functional inequalities in the proof of our main result. We first recall a weighted Poincaré inequality first established by Desvillettes and Villani in [2].

Lemma 2.1. *Let Ω be a bounded connected Lipschitz domain and $\bar{\varrho}$ be a positive constant. There exists a positive constant C, depending on Ω and $\bar{\varrho}$, such that for any nonnegative function ϱ satisfying*

$$\int_\Omega \varrho \mathrm{d}x = 1, \quad \varrho \leqslant \bar{\varrho},$$

and any $\mathbf{u} \in H^1(\Omega)$, *there holds*

$$\int_\Omega \varrho \left(\mathbf{u} - \int_\Omega \rho \mathbf{u}\, \mathrm{d}x\right)^2 \mathrm{d}x \leqslant C\|\nabla \mathbf{u}\|_{L^2}^2. \tag{2.1}$$

In order to remove the weight function ϱ in (2.1) without resorting to the lower bound of ϱ, we need another variant of Poincaré inequality (see lemma 3.2 in [5]).

Lemma 2.2. *Let* Ω *be a bounded connected Lipschitz domain in* $\mathbb{R}^3$ *and* $p > 1$ *be a constant. Given positive constants* M_0 *and* E_0*, there is a constant* $C = C(E_0, M_0)$ *such that for any non-negative function* ϱ *satisfying*

$$M_0 \leqslant \int_\Omega \varrho \mathrm{d}x \quad \text{and} \quad \int_\Omega \varrho^p \mathrm{d}x \leqslant E_0,$$

and for any $\mathbf{u} \in H^1(\Omega)$, *there holds*

$$\|\mathbf{u}\|_{L^2}^2 \leqslant C\left[\|\nabla \mathbf{u}\|_{L^2}^2 + \left(\int_\Omega \varrho |\mathbf{u}|\, dx\right)^2\right].$$

Lemma 2.3 ([11]). *Let* $s \geqslant 0$ *and* $f, g \in H^s(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. *Then*

$$\|fg\|_{H^s} \leqslant C\left(\|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}\right). \tag{2.2}$$

Lemma 2.4 ([11]). *Let* $s > 0$. *Then there exists a constant* C *such that, for any* $f \in H^s(\mathbb{T}^2) \cap W^{1,\infty}(\mathbb{T}^2)$, $g \in H^{s-1}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$, *there holds*

$$\|[\nabla^s, f\cdot\nabla]g\|_{L^2} \leqslant C\left(\|\nabla f\|_{L^\infty}\|\nabla^s g\|_{L^2} + \|\nabla^s f\|_{L^2}\|\nabla g\|_{L^\infty}\right).$$

Lemma 2.5 ([25]). *Let* $s > 0$ *and* $f \in H^s(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. *Assume that* F *is a smooth function on* $\mathbb{R}$ *with* $F(0) = 0$. *Then we have*

$$\|F(f)\|_{H^s} \leqslant C\left(1 + \|f\|_{L^\infty}\right)^{[s]+1}\|f\|_{H^s},$$

where the constant C *depends on* $\sup_{k\leqslant[s]+2, t\leqslant\|f\|_{L^\infty}}\|F^{(k)}(t)\|_{L^\infty}$.

3. The proof of theorem 1.1

This section is devoted to proving theorem 1.1. The proof is long and is thus divided into several subsections for the sake of clarity.

3.1. Local well-posedness

Given the initial data $(P_0 - \bar{P}, \mathbf{u}_0, m_0 - \bar{m}) \in H^3(\mathbb{T}^2)$, the local well-posedness of (1.3) could be proven by using the standard energy method (see, e.g. [12]). Thus, we may assume that there exists $T > 0$ such that the system (1.3) has a unique solution $(P - \bar{P}, \mathbf{u}, m - \bar{m}) \in C([0,T]; H^3)$. Moreover,

$$\frac{1}{2}c_0 \leqslant \rho(t,x) \leqslant 2c_0^{-1}, \quad \text{for any} t \in [0,T]. \tag{3.1}$$

We use the bootstrapping argument to show that this local solution can be extended into a global one. The goal is to derive a global *a priori* upper bound. To initiate the bootstrapping argument, we make the ansatz that

$$\sup_{t\in[0,T]} \left(\|P-\bar{P}\|_{H^3}+\|\mathbf{u}\|_{H^3}+\|m-\bar{m}\|_{H^3}\right)\leqslant\delta, \tag{3.2}$$

where $0<\delta<1$ obeys requirements to be specified later. In the following subsections we prove that, if the initial norm is taken to be sufficiently small, namely

$$\|P_0-\bar{P}\|_{H^3}+\|\mathbf{u}_0\|_{H^3}+\|m_0-\bar{m}\|_{H^3}\leqslant\varepsilon,$$

with sufficiently small $\varepsilon>0$, then

$$\sup_{t\in[0,T]} \left(\|P-\bar{P}\|_{H^3}+\|\mathbf{u}\|_{H^3}+\|m-\bar{m}\|_{H^3}\right)\leqslant\frac{\delta}{2}.$$

The bootstrapping argument then leads to the desired global bound.

3.2. *Energy estimates for* $(P-\bar{P},\mathbf{u},m-\bar{m})$

We first show the energy estimate which contains the bound for $\mathbf{u}$ only. Without loss of generality, we let $\bar{P}=\bar{m}=1$, and define

$$p\overset{\text{def}}{=}P-1,\quad a\overset{\text{def}}{=}\rho-1,\quad b\overset{\text{def}}{=}m-1.$$

Then, system (1.3) is equivalent to the following system:

$$\begin{cases}\partial_t p+\gamma\operatorname{div}\mathbf{u}+\mathbf{u}\cdot\nabla p+\gamma p\operatorname{div}\mathbf{u}=0,\\ \partial_t\mathbf{u}-\operatorname{div}\left(\bar{\mu}(\rho)\nabla\mathbf{u}\right)-\nabla\left(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\right)+\nabla\varphi=f_1,\\ \partial_t b+\operatorname{div}\mathbf{u}+\mathbf{u}\cdot\nabla b+b\operatorname{div}\mathbf{u}=0,\\ (p,\mathbf{u},b)|_{t=0}=(p_0,\mathbf{u}_0,b_0),\end{cases} \tag{3.3}$$

with

$$\varphi\overset{\text{def}}{=}P-1+\frac{1}{2}m^2-\frac{1}{2},\quad \bar{\mu}(\rho)\overset{\text{def}}{=}\frac{\mu}{\rho},\quad \bar{\lambda}(\rho)\overset{\text{def}}{=}\frac{\lambda+\mu}{\rho},\quad I(a)\overset{\text{def}}{=}\frac{a}{1+a},$$

and

$$f_1\overset{\text{def}}{=}-\mathbf{u}\cdot\nabla\mathbf{u}+I(a)\nabla\varphi+\mu\left(\nabla I(a)\right)\nabla\mathbf{u}+(\lambda+\mu)\left(\nabla I(a)\right)\operatorname{div}\mathbf{u}.$$

We comment that the velocity equation written in (3.3) is slightly different from $(1.3)_2$. This avoids the appearance of the bad term $-I(a)(\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u})$.

In this subsection, we shall prove the following crucial lemma.

Lemma 3.1. *Let* $(p,\mathbf{u},b)\in C([0,T];H^3)$ *be a solution to the system (3.3). There holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(p,\mathbf{u},b)\|_{H^3}^2-\frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3 p\right)^2\mathrm{d}x+\mu\|\nabla\mathbf{u}\|_{H^3}^2+(\lambda+\mu)\|div\,\mathbf{u}\|_{H^3}^2\\ &\quad\leqslant C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2+C\left(\|\mathbf{u}\|_{H^3}+\|\mathbf{u}\|_{H^3}^2+\|\varphi\|_{H^3}^2\right)\|(p,\mathbf{u},b)\|_{H^3}^2.\end{aligned} \tag{3.4}$$

Proof. Let $s=0,1,2,3$. Applying operator ∇^s to the equations of (3.3) and then taking L^2 inner product with $(\frac{1}{\gamma}\nabla^s p,\nabla^s\mathbf{u},\nabla^s b)$ yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\frac{1}{\gamma}\nabla^s p,\nabla^s\mathbf{u},\nabla^s b\right)\right\|_{L^2}^2-\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\nabla\left(\bar{\lambda}(\rho)\,\mathrm{div}\,\mathbf{u}\right)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\\
&\quad=-\frac{1}{\gamma}\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p+\gamma p\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\nabla\varphi\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b+b\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x. \qquad (3.5)
\end{aligned}$$

Due to

$$\nabla\varphi=\nabla p+\nabla b+b\nabla b$$

and the cancellations

$$\begin{aligned}
&\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s\nabla p\cdot\nabla^s\mathbf{u}\,\mathrm{d}x=0,\\
&\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x=0,
\end{aligned}$$

we can further rewrite (3.5) into

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\frac{1}{\gamma}\nabla^s p,\nabla^s\mathbf{u},\nabla^s b\right)\right\|_{L^2}^2-\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\nabla\left(\bar{\lambda}(\rho)\,\mathrm{div}\,\mathbf{u}\right)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\\
&\quad=-\frac{1}{\gamma}\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p+\gamma p\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s(b\nabla b)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\\
&\qquad-\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b+b\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x. \qquad (3.6)
\end{aligned}$$

By using the commutator argument, the second term on the left-hand side can be written as

$$\begin{aligned}
-\int_{\mathbb{T}^2}\nabla^s\mathrm{div}\,(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x&=\int_{\mathbb{T}^2}\nabla^s(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x\\
&=\int_{\mathbb{T}^2}\bar{\mu}(\rho)\nabla\nabla^s\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x+\int_{\mathbb{T}^2}[\nabla^s,\bar{\mu}(\rho)]\nabla\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x. \qquad (3.7)
\end{aligned}$$

It follows from (3.1), for any $t\in[0,T]$, that

$$\int_{\mathbb{T}^2}\bar{\mu}(\rho)\nabla\nabla^s\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x\geqslant c_0^{-1}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2. \qquad (3.8)$$

For the last term in (3.7), we can further rewrite this term into

$$\begin{aligned}
\int_{\mathbb{T}^2}[\nabla^s,\bar{\mu}(\rho)]\nabla\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x&=\int_{\mathbb{T}^2}[\nabla^s,\bar{\mu}(\rho)-\mu+\mu]\nabla\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x\\
&=-\int_{\mathbb{T}^2}[\nabla^s,\mu I(a)]\nabla\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,\mathrm{d}x. \qquad (3.9)
\end{aligned}$$

Bounding nonlinear terms involving composition functions in (3.9) is more elaborate. Throughout we make the assumption that

$$\sup_{t\in\mathbb{R}^+,x\in\mathbb{T}^2} |a(t,x)| \leqslant \frac{1}{2} \tag{3.10}$$

which will enable us to use freely the composition estimate stated in lemma 2.5. Note that as $H^3(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, condition (3.10) will be ensured by the fact that the constructed solution about a has small norm. It them follows from lemma 2.5 that the following composition estimate holds,

$$\|I(a)\|_{H^s} \leqslant C\|a\|_{H^s}, \quad \text{for any} s > 0. \tag{3.11}$$

Moreover, in view of $a = (p+1)^{\frac{1}{\gamma}} - 1$, we can use lemma 2.5 again to deduce that

$$\|a\|_{H^3}^2 \leqslant C\|p\|_{H^3}^2. \tag{3.12}$$

Then, with the aid of lemmas 2.4, 2.5 and (3.11), we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} [\nabla^s, \mu I(a)]\nabla\mathbf{u}\cdot\nabla\nabla^s\mathbf{u}\,dx\right| \\
&\quad\leqslant C\|\nabla\nabla^s\mathbf{u}\|_{L^2}\left(\|\nabla I(a)\|_{L^\infty}\|\nabla^s\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s I(a)\|_{L^2}\right) \\
&\quad\leqslant \frac{c_0^{-1}}{2}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\left(\|\nabla a\|_{L^\infty}^2\|\nabla^s\mathbf{u}\|_{L^2}^2 + \|a\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2\right) \\
&\quad\leqslant \frac{c_0^{-1}}{2}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\|p\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2.
\end{aligned}\tag{3.13}$$

Inserting (3.9) and (3.13) in (3.7) leads to

$$-\int_{\mathbb{T}^2}\nabla^s\text{div}\,(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x \geqslant \frac{c_0^{-1}}{2}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 - C\|p\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2. \tag{3.14}$$

The third term on the left-hand side of (3.6) can be dealt with similarly. Hence, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\frac{1}{\gamma}\nabla^s p, \nabla^s\mathbf{u}, \nabla^s b\right)\right\|_{L^2}^2 + c_0^{-1}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 + c_0^{-1}(\lambda+\mu)\|\nabla^s\text{div}\mathbf{u}\|_{L^2}^2 \\
&\quad\leqslant C\|p\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2 - \frac{1}{\gamma}\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p)\cdot\nabla^s p\,\mathrm{d}x - \int_{\mathbb{T}^2}\nabla^s(p\,\text{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x \\
&\qquad - \int_{\mathbb{T}^2}\nabla^s(b\nabla b)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x - \int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b + b\,\text{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x + \int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x.
\end{aligned}\tag{3.15}$$

To bound the nonlinear terms in (3.15), we first use commutator's estimates to write

$$\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p)\cdot\nabla^s p\,\mathrm{d}x = \int_{\mathbb{T}^2}(\nabla^s(\mathbf{u}\cdot\nabla p) - \mathbf{u}\cdot\nabla\nabla^s p)\cdot\nabla^s p\,\mathrm{d}x + \int_{\mathbb{T}^2}\mathbf{u}\cdot\nabla\nabla^s p\cdot\nabla^s p\,\mathrm{d}x. \tag{3.16}$$

It follows from lemma 2.4 that

$$\left|\int_{\mathbb{T}^2}(\nabla^s(\mathbf{u}\cdot\nabla p)-\mathbf{u}\cdot\nabla\nabla^s p)\cdot\nabla^s p\,\mathrm{d}x\right| \leqslant C\|[\nabla^s,\mathbf{u}\cdot\nabla]p\|_{L^2}\|\nabla^s p\|_{L^2}$$
$$\leqslant C(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s p\|_{L^2}+\|\nabla^s\mathbf{u}\|_{L^2}\|\nabla p\|_{L^\infty})\|\nabla^s p\|_{L^2}$$
$$\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.17}$$

By integration by parts, we have

$$\left|\int_{\mathbb{T}^2}\mathbf{u}\cdot\nabla\nabla^s p\cdot\nabla^s p\,dx\right|\leqslant C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s p\|_{L^2}^2\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2$$

from which and (3.17), we get

$$\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla p)\cdot\nabla^s p\,\mathrm{d}x\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.18}$$

Similarly, there hold

$$\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b)\cdot\nabla^s b\,\mathrm{d}x\leqslant C\|\mathbf{u}\|_{H^3}\|b\|_{H^3}^2, \tag{3.19}$$

and

$$\int_{\mathbb{T}^2}\nabla^s(b\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s(b\nabla b)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x$$
$$=\int_{\mathbb{T}^2}[\nabla^s,b]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}[\nabla^s,b]\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x$$
$$+\int_{\mathbb{T}^2}b\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}b\nabla^s\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x$$
$$=\int_{\mathbb{T}^2}[\nabla^s,b]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}[\nabla^s,b]\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x+\int_{\mathbb{T}^2}b\mathrm{div}\,(\nabla^s b\nabla^s\mathbf{u})\,\mathrm{d}x$$
$$=\int_{\mathbb{T}^2}[\nabla^s,b]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}[\nabla^s,b]\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s b\nabla^s\mathbf{u}\cdot\nabla b\,\mathrm{d}x. \tag{3.20}$$

In view of lemma 2.4, we have

$$\int_{\mathbb{T}^2}[\nabla^s,b]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s b\,\mathrm{d}x\leqslant C\|[\nabla^s,b]\,\mathrm{div}\,\mathbf{u}\|_{L^2}\|\nabla^s b\|_{L^2}$$
$$\leqslant C(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s b\|_{L^2}+\|\nabla^s\mathbf{u}\|_{L^2}\|\nabla b\|_{L^\infty})\|\nabla^s b\|_{L^2}$$
$$\leqslant C\|\mathbf{u}\|_{H^3}\|b\|_{H^3}^2 \tag{3.21}$$

and

$$\int_{\mathbb{T}^2}[\nabla^s,b]\nabla b\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\leqslant C\|[\nabla^s,b]\nabla b\|_{L^2}\|\nabla^s\mathbf{u}\|_{L^2}$$
$$\leqslant C\|\nabla b\|_{L^\infty}\|\nabla^s b\|_{L^2}\|\nabla^s\mathbf{u}\|_{L^2}$$
$$\leqslant C\|\mathbf{u}\|_{H^3}\|b\|_{H^3}^2. \tag{3.22}$$

For the last term in (3.20), it is direct to see that

$$\begin{aligned}\left|-\int_{\mathbb{T}^2}\nabla^s b\nabla^s\mathbf{u}\cdot\nabla b\,\mathrm{d}x\right| &\leqslant C\|\nabla^s b\|_{L^2}\|\nabla^s\mathbf{u}\|_{L^2}\|\nabla b\|_{L^\infty}\\ &\leqslant C\|\mathbf{u}\|_{H^3}\|b\|_{H^3}^2.\end{aligned}\tag{3.23}$$

Hence, combining with (3.19)–(3.23), we obtain

$$\left|\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla b+b\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s b\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s(b\nabla b)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\right|\leqslant C\|\mathbf{u}\|_{H^3}\|b\|_{H^3}^2.\tag{3.24}$$

Next, we have to bound the most difficult term

$$-\int_{\mathbb{T}^2}\nabla^s(p\,\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x.$$

We first use the commutator to rewrite this term into

$$-\int_{\mathbb{T}^2}\nabla^s(p\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x=-\int_{\mathbb{T}^2}[\nabla^s,p]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x-\int_{\mathbb{T}^2}p\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x.\tag{3.25}$$

The first term on the right hand side of (3.25) is easily controlled from lemma 2.4 that

$$\begin{aligned}\left|-\int_{\mathbb{T}^2}[\nabla^s,p]\,\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x\right| &\leqslant C\|[\nabla^s,p]\,\mathrm{div}\,\mathbf{u}\|_{L^2}\|\nabla^s p\|_{L^2}\\ &\leqslant C(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s p\|_{L^2}+\|\nabla^s\mathbf{u}\|_{L^2}\|\nabla p\|_{L^\infty})\|\nabla^s p\|_{L^2}\\ &\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2.\end{aligned}\tag{3.26}$$

Then, we deal with the last term on the right hand side of (3.25). In fact, for $s=0,1,2$, we can bound this term directly as follows

$$\begin{aligned}\left|-\int_{\mathbb{T}^2}p\nabla^s\mathrm{div}\,\mathbf{u}\cdot\nabla^s p\,\mathrm{d}x\right| &\leqslant C\|p\|_{L^\infty}\|\nabla^s\mathrm{div}\,\mathbf{u}\|_{L^2}\|\nabla^s p\|_{L^2}\\ &\leqslant C\|p\|_{H^2}\|\mathrm{div}\,\mathbf{u}\|_{H^2}\|p\|_{H^2}\\ &\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}.\end{aligned}\tag{3.27}$$

However, for the highest regularity $s=3$, the same strategy as (3.27) is invalid. Otherwise, we get by a similar derivation of (3.27) that

$$\left|-\int_{\mathbb{T}^2}p\nabla^3\mathrm{div}\,\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x\right|\leqslant C\|p\|_{L^\infty}\left\|\nabla^3\mathrm{div}\,\mathbf{u}\right\|_{L^2}\left\|\nabla^3 p\right\|_{L^2}\leqslant C\|p\|_{H^3}^2\|\mathrm{div}\,\mathbf{u}\|_{H^3}.\tag{3.28}$$

Moreover, to control the term $\|\mathrm{div}\,\mathbf{u}\|_{H^3}$, we have to use the smoothing effect coming from the velocity equation to absorb this term to the left which will lead to the following inequality

$$\|p\|_{H^3}^2\|\mathrm{div}\,\mathbf{u}\|_{H^3}\leqslant\varepsilon\|\mathrm{div}\,\mathbf{u}\|_{H^3}^2+C\|p\|_{H^3}^4.\tag{3.29}$$

When we use the continuity argument to close the energy estimates, (3.29) implies that we have to ensure that the time integral of $\|p\|_{H^3}^2$ is time integrable, this appears to be impossible due to the lack of the dissipation of the equation of p. To overcome the difficulty, we deduce from the first equation of (3.3) that

$$\operatorname{div}\mathbf{u} = -\frac{\partial_t p + \mathbf{u}\cdot\nabla p}{\gamma(p+1)},$$

from which we have

$$\begin{aligned}
-\int_{\mathbb{T}^2} p\nabla^3 \operatorname{div}\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x &= \frac{1}{\gamma}\int_{\mathbb{T}^2} p\nabla^3\left(\frac{\partial_t p + \mathbf{u}\cdot\nabla p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x\\
&= \frac{1}{\gamma}\int_{\mathbb{T}^2} p\nabla^3\left(\frac{\partial_t p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x + \int_{\mathbb{T}^2} p\nabla^3\left(\frac{\mathbf{u}\cdot\nabla p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x\\
&= D_1 + D_2. \qquad (3.30)
\end{aligned}$$

For the first term D_1 we have

$$\begin{aligned}
D_1 &= \frac{1}{\gamma}\int_{\mathbb{T}^2} p\nabla^3\left(\frac{\partial_t p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x\\
&= \frac{1}{\gamma}\int_{\mathbb{T}^2}\frac{p}{1+p}\nabla^3(\partial_t p)\cdot\nabla^3 p\,\mathrm{d}x + \frac{1}{\gamma}\int_{\mathbb{T}^2} p\sum_{\ell=0}^{2} C_3^\ell\nabla^\ell\partial_t p\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x\\
&= \frac{1}{2\gamma}\int_{\mathbb{T}^2}\frac{p}{1+p}\partial_t\left(\nabla^3 p\right)^2\mathrm{d}x + \frac{1}{\gamma}\int_{\mathbb{T}^2} p\sum_{\ell=0}^{2} C_3^\ell\nabla^\ell\partial_t p\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x\\
&= \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3 p\right)^2\mathrm{d}x - \frac{1}{2\gamma}\int_{\mathbb{T}^2}\frac{1}{(1+p)^2}\partial_t p\left(\nabla^3 p\right)^2\mathrm{d}x\\
&\quad + \frac{1}{\gamma}\int_{\mathbb{T}^2} p\sum_{\ell=0}^{2} C_3^\ell\nabla^\ell\partial_t p\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x. \qquad (3.31)
\end{aligned}$$

Using the first equation of (3.3), we can bound the second term on the right hand side of (3.31) as

$$\begin{aligned}
-\frac{1}{2\gamma}\int_{\mathbb{T}^2}\frac{1}{(1+p)^2}\partial_t p\left(\nabla^3 p\right)^2\mathrm{d}x &= \frac{1}{2\gamma}\int_{\mathbb{T}^2}\frac{1}{(1+p)^2}\left(\mathbf{u}\cdot\nabla p + \gamma p\operatorname{div}\mathbf{u} + \gamma\operatorname{div}\mathbf{u}\right)\left(\nabla^3 p\right)^2\mathrm{d}x\\
&\leqslant C\left((1+\|p\|_{L^\infty})\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla p\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}\right)\left\|\nabla^3 p\right\|_{L^2}^2\\
&\leqslant C\left(1+\|p\|_{H^3}\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \qquad (3.32)
\end{aligned}$$

By the Hölder inequality, the last term in (3.31) can be controlled as

$$\frac{1}{\gamma}\int_{\mathbb{T}^2} p\sum_{\ell=0}^{2} C_3^\ell\nabla^\ell\partial_t p\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x \leqslant C\left\|p\sum_{\ell=0}^{2}\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\right\|_{L^\infty}\|\partial_t p\|_{H^2}\left\|\nabla^3 p\right\|_{L^2}. \qquad (3.33)$$

Recall that

$$\left\|p\sum_{\ell=0}^{2}\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\right\|_{L^\infty} \leqslant C\|p\|_{H^3}^2 \qquad (3.34)$$

and

$$\|\partial_t p\|_{H^2} \leqslant C \|\mathbf{u}\cdot\nabla p + \gamma p \operatorname{div}\mathbf{u} + \gamma \operatorname{div}\mathbf{u}\|_{H^2} \leqslant C\left(\|\mathbf{u}\|_{H^3} + \|\mathbf{u}\|_{H^3}\|p\|_{H^3}\right). \tag{3.35}$$

Hence, we have

$$\frac{1}{\gamma}\int_{\mathbb{T}^2} p \sum_{\ell=0}^{2} C_3^\ell \nabla^\ell \partial_t p \nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x \leqslant C\left(1+\|p\|_{H^3}\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^3. \tag{3.36}$$

Combining (3.32) with (3.36), we get

$$D_1 \leqslant \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3 p\right)^2\mathrm{d}x + C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.37}$$

For the term D_2, we infer that

$$\begin{aligned}
D_2 &= \int_{\mathbb{T}^2} p\nabla^3\left(\frac{\mathbf{u}\cdot\nabla p}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x \\
&= \int_{\mathbb{T}^2}\frac{p}{1+p}\nabla^3\left(\mathbf{u}\cdot\nabla p\right)\cdot\nabla^3 p\,\mathrm{d}x + \int_{\mathbb{T}^2} p\sum_{\ell=0}^{2} C_3^\ell\nabla^\ell\left(\mathbf{u}\cdot\nabla p\right)\nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot\nabla^3 p\,\mathrm{d}x \\
&= D_{2,1} + D_{2,2}.
\end{aligned}$$

We can use the commutator to rewrite $D_{2,1}$ into

$$D_{2,1} = \int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3\left(\mathbf{u}\cdot\nabla p\right) - \mathbf{u}\cdot\nabla\nabla^3 p\right)\cdot\nabla^3 p\,\mathrm{d}x + \int_{\mathbb{T}^2}\frac{p}{1+p}\mathbf{u}\cdot\nabla\nabla^3 p\cdot\nabla^3 p\,\mathrm{d}x. \tag{3.38}$$

Thanks to lemma 2.4, we get

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3\left(\mathbf{u}\cdot\nabla p\right) - \mathbf{u}\cdot\nabla\nabla^3 p\right)\cdot\nabla^3 p\,\mathrm{d}x\right| \\
&\quad\leqslant C\left\|\frac{p}{1+p}\right\|_{L^\infty}\left\|\left[\nabla^3,\mathbf{u}\cdot\nabla\right]p\right\|_{L^2}\left\|\nabla^3 p\right\|_{L^2} \\
&\quad\leqslant C\left(\|\nabla\mathbf{u}\|_{L^\infty}\left\|\nabla^3 p\right\|_{L^2} + \left\|\nabla^3\mathbf{u}\right\|_{L^2}\|\nabla p\|_{L^\infty}\right)\left\|\nabla^3 p\right\|_{L^2} \\
&\quad\leqslant C\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2.
\end{aligned} \tag{3.39}$$

By using the integration by parts, we have

$$\begin{aligned}
\left|\int_{\mathbb{T}^2}\frac{p}{1+p}\mathbf{u}\cdot\nabla\nabla^3 p\cdot\nabla^3 p\,\mathrm{d}x\right| &\leqslant C\left\|\operatorname{div}\left(\frac{p\mathbf{u}}{1+p}\right)\right\|_{L^\infty}\left\|\nabla^3 p\right\|_{L^2}^2 \\
&\leqslant C\left(\|\nabla\mathbf{u}\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}\|\nabla p\|_{L^\infty}\right)\left\|\nabla^3 p\right\|_{L^2}^2 \\
&\leqslant C\left(1+\|p\|_{H^3}\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2
\end{aligned}$$

from which and (3.39) we get

$$D_{2,1} \leqslant C\left(1+\|p\|_{H^3}\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.40}$$

Thanks to the Hölder inequality and (3.34), we can get

$$
\begin{aligned}
D_{2,2} &= \int_{\mathbb{T}^2} p \sum_{\ell=0}^{2} C_3^\ell \nabla^\ell (\mathbf{u}\cdot\nabla p) \nabla^{3-\ell}\left(\frac{1}{1+p}\right)\cdot \nabla^3 p \,\mathrm{d}x \\
&\leqslant C \left\| p \sum_{\ell=0}^{2} \nabla^{3-\ell}\left(\frac{1}{1+p}\right)\right\|_{L^\infty} \|\mathbf{u}\cdot\nabla p\|_{H^2} \left\|\nabla^3 p\right\|_{L^2} \\
&\leqslant C \|\mathbf{u}\|_{H^3} \|p\|_{H^3}^4
\end{aligned}
$$

which combines with (3.40) implies that

$$
D_2 \leqslant C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.41}
$$

Inserting (3.37) and (3.41) into (3.30) leads to

$$
-\int_{\mathbb{T}^2} p\nabla^3 \mathrm{div}\,\mathbf{u}\cdot\nabla^3 p\,\mathrm{d}x \leqslant \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3 p\right)^2 \mathrm{d}x + C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.42}
$$

Consequently, taking the estimates (3.26), (3.27) and (3.42) into (3.25), we get

$$
-\int_{\mathbb{T}^2}\nabla^s (p\mathrm{div}\,\mathbf{u})\cdot\nabla^s p\,\mathrm{d}x \leqslant \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^\ell p\right)^2\mathrm{d}x + C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2. \tag{3.43}
$$

In the following, we have to bound the terms in f_1 of (3.15). To do this, we write

$$
\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x = A_1+A_2+A_3+A_4 \tag{3.44}
$$

with

$$
A_1 \stackrel{\text{def}}{=} -\int_{\mathbb{T}^2}\nabla^s(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x, \qquad A_2 \stackrel{\text{def}}{=} \int_{\mathbb{T}^2}\nabla^s(I(a)\nabla\varphi)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x,
$$

$$
A_3 \stackrel{\text{def}}{=} \int_{\mathbb{T}^2}\nabla^s(\mu(\nabla I(a))\nabla\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x, \qquad A_4 \stackrel{\text{def}}{=} \int_{\mathbb{T}^2}\nabla^s((\lambda+\mu)(\nabla I(a))\mathrm{div}\,\mathbf{u})\cdot\nabla^s\mathbf{u}\,\mathrm{d}x.
$$

We now estimate A_1, A_2, A_3, A_4 one by one. The term A_1 can be bounded the same as (3.17)

$$
|A_1| \leqslant C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^s\mathbf{u}\|_{L^2}^2.
$$

For $s=0$, we can bound the rest terms A_2, A_3, A_4 directly as

$$
\begin{aligned}
|A_2|+|A_3|+|A_4| &\leqslant C\|I(a)\|_{L^\infty}(\|\nabla\varphi\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2})\|\mathbf{u}\|_{L^2} \\
&\leqslant C\|a\|_{H^3}(\|\varphi\|_{H^3}+\|\mathbf{u}\|_{H^3})\|\mathbf{u}\|_{H^3} \\
&\leqslant \frac{c_0^{-1}\mu}{16}\|\mathbf{u}\|_{H^3}^2 + C\|p\|_{H^3}^2\left(\|\varphi\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2\right).
\end{aligned} \tag{3.45}
$$

Due to $\int_{\mathbb{T}^2}\rho\mathbf{u}\,dx = 0$, one can deduce from lemma 2.1 that

$$
\|(\sqrt{\rho}\mathbf{u})(t)\|_{L^2}^2 \leqslant C\|\nabla\mathbf{u}(t)\|_{L^2}^2,
$$

which combines with lemma 2.2 further implies that

$$\|\mathbf{u}(t)\|_{L^2}^2 \leqslant C\|\nabla\mathbf{u}(t)\|_{L^2}^2. \tag{3.46}$$

Hence, we can infer from (3.45) that

$$|A_2|+|A_3|+|A_4| \leqslant \frac{c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|p\|_{H^3}^2\left(\|\varphi\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2\right).$$

For $s=1,2,3$, by lemma 2.3 and (3.11), we have

$$\begin{aligned}|A_2| &\leqslant C\left(\|\nabla\varphi\|_{L^\infty}\|I(a)\|_{H^{s-1}}+\|\nabla\varphi\|_{H^{s-1}}\|I(a)\|_{L^\infty}\right)\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}\\ &\leqslant \frac{c_0^{-1}\mu}{16}\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\|a\|_{L^\infty}^2\|\varphi\|_{H^s}^2 + C\|a\|_{H^s}^2\|\varphi\|_{H^3}^2\\ &\leqslant \frac{c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|p\|_{H^3}^2\|\varphi\|_{H^3}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|A_3|+|A_4| &\leqslant C\left(\|\nabla I(a)\|_{L^\infty}\|\nabla^s\mathbf{u}\|_{L^2}+\|\nabla I(a)\|_{H^{s-1}}\|\nabla\mathbf{u}\|_{L^\infty}\right)\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}\\ &\leqslant \frac{c_0^{-1}\mu}{16}\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\left(\|\nabla a\|_{L^\infty}^2\|\nabla^s\mathbf{u}\|_{L^2}^2+\|a\|_{H^s}^2\|\mathbf{u}\|_{H^3}^2\right)\\ &\leqslant \frac{c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|p\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2\end{aligned}$$

Inserting the bounds for A_1 through A_4 into (3.44), we get

$$\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x \leqslant \frac{c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|\mathbf{u}\|_{H^3}\|\mathbf{u}\|_{H^3}^2 + C\left(\|\varphi\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2\right)\|p\|_{H^3}^2. \tag{3.47}$$

Finally, inserting (3.18), (3.24) and (3.47) into (3.15) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(p,\mathbf{u},b)\|_{H^3}^2 - \frac{1}{2\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^\ell p\right)^2\mathrm{d}x + \mu\|\nabla\mathbf{u}\|_{H^3}^2 + (\lambda+\mu)\|\mathrm{div}\,\mathbf{u}\|_{H^3}^2\\ &\quad\leqslant C\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2 + C\left(\|\mathbf{u}\|_{H^3}+\|\mathbf{u}\|_{H^3}^2+\|\varphi\|_{H^3}^2\right)\|(p,\mathbf{u},b)\|_{H^3}^2.\end{aligned}$$

This finishes the proof of lemma 3.1.

□

3.3. Energy estimates for $(\varphi,\mathbf{u})$

Lemma 3.2. *Let $(p,\mathbf{u},b)\in C([0,T];H^3)$ be a solution to the system (3.3). Then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{u})\|_{H^3}^2 + \frac{c_0^{-1}\mu}{2}\|\nabla\mathbf{u}\|_{H^3}^2 + (\lambda+\mu)\|div\,\mathbf{u}\|_{H^3}^2\\ &\quad\leqslant C\left(\|\mathbf{u}\|_{H^3}+\|(p,\varphi)\|_{H^3}^2+\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2.\end{aligned} \tag{3.48}$$

Proof. We start with the L^2 estimate. To break some technical barrier, it is convenient to rewrite system (3.3) in terms of variables φ and $\mathbf{u}$. Precisely, one has

$$\begin{cases} \partial_t \mathbf{u} - \operatorname{div}\left(\bar{\mu}(\rho)\nabla\mathbf{u}\right) - \nabla\left(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\right) + \nabla\varphi = f_1, \\ \partial_t\varphi + (\gamma+1)\operatorname{div}\mathbf{u} = f_2, \\ (\mathbf{u},\varphi)|_{t=0} = (\mathbf{u}_0,\varphi_0), \end{cases} \tag{3.49}$$

with

$$\begin{aligned} f_1 &\stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{u} + I(a)\nabla\varphi + \mu\left(\nabla I(a)\right)\nabla\mathbf{u} + (\lambda+\mu)\left(\nabla I(a)\right)\operatorname{div}\mathbf{u}, \\ f_2 &\stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\varphi - \gamma\varphi\operatorname{div}\mathbf{u} + \frac{2-\gamma}{2}\left(b^2+2b\right)\operatorname{div}\mathbf{u}. \end{aligned}$$

Taking inner product with $\mathbf{u}$ for the first equation of (3.49), $\frac{\varphi}{\gamma+1}$ for the second equation of (3.49), respectively, then adding up the result together, we then obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}\|_{L^2}^2 + \frac{1}{\gamma+1}\|\varphi\|_{L^2}^2\right) - \int_{\mathbb{T}^2}\operatorname{div}\left(\bar{\mu}(\rho)\nabla\mathbf{u}\right)\cdot\mathbf{u}\,\mathrm{d}x - \int_{\mathbb{T}^2}\nabla\left(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\right)\cdot\mathbf{u}\,\mathrm{d}x \\ &\quad= \int_{\mathbb{T}^2} f_1\cdot\mathbf{u}\,\mathrm{d}x + \frac{1}{\gamma+1}\int_{\mathbb{T}^2} f_2\cdot\varphi\,\mathrm{d}x, \end{aligned} \tag{3.50}$$

where we have used the following cancellations

$$\int_{\mathbb{T}^2}\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x + \int_{\mathbb{T}^2}\operatorname{div}\mathbf{u}\cdot\varphi\,\mathrm{d}x = 0.$$

For the last two terms on the left hand side of (3.50), we get by integration by parts and (1.4) that

$$-\int_{\mathbb{T}^2}\operatorname{div}\left(\bar{\mu}(\rho)\nabla\mathbf{u}\right)\cdot\mathbf{u}\,\mathrm{d}x = \int_{\mathbb{T}^2}\bar{\mu}(\rho)\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,\mathrm{d}x \geqslant c_0^{-1}\mu\|\nabla\mathbf{u}\|_{L^2}^2, \tag{3.51}$$

$$-\int_{\mathbb{T}^2}\nabla\left(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\right)\cdot\mathbf{u}\,dx = \int_{\mathbb{T}^2}\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\cdot\operatorname{div}\mathbf{u}\,dx \geqslant c_0^{-1}(\lambda+\mu)\|\operatorname{div}\mathbf{u}\|_{L^2}^2. \tag{3.52}$$

Next, we shall estimate each term on the right hand side of (3.50). First, it follows from integration by parts and the Hölder inequality that

$$\begin{aligned} \left|\int_{\mathbb{T}^2} f_2\cdot\varphi\,\mathrm{d}x\right| &\leqslant \frac{3c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{L^2}^2 + C\left(\|\nabla\mathbf{u}\|_{L^\infty} + \|b\|_{L^\infty} + \|b\|_{L^\infty}^2\right)\|\varphi\|_{L^2}^2 \\ &\leqslant \frac{3c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{L^2}^2 + C\left(\|(\mathbf{u},b)\|_{H^3} + \|b\|_{H^3}^2\right)\|\varphi\|_{H^3}^2. \end{aligned} \tag{3.53}$$

Thanks to lemma 2.5, the Hölder inequality and the Young inequality, we have

$$\left|\int_{\mathbb{T}^2}\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x\right|\leqslant C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{L^2}^2,$$

$$\begin{aligned}\left|\int_{\mathbb{T}^2}I(a)\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x\right|&\leqslant C\|I(a)\|_{L^\infty}\|\nabla\varphi\|_{L^2}\|\mathbf{u}\|_{L^2}\\&\leqslant\frac{c_0^{-1}\mu}{8}\|\mathbf{u}\|_{L^2}^2+C\|a\|_{L^\infty}^2\|\varphi\|_{H^1}^2,\\&\leqslant\frac{c_0^{-1}\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2+C\|a\|_{L^\infty}^2\|\varphi\|_{H^1}^2,\end{aligned}$$

$$\left|\int_{\mathbb{T}^2}\left((\nabla I(a))\nabla\mathbf{u}+(\nabla I(a))\operatorname{div}\mathbf{u}\right)\cdot\mathbf{u}\,\mathrm{d}x\right|\leqslant\frac{c_0^{-1}\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla a\|_{L^\infty}^2\|\mathbf{u}\|_{L^2}^2$$

from which we get

$$\left|\int_{\mathbb{T}^2}f_1\cdot\mathbf{u}\,dx\right|\leqslant\frac{3c_0^{-1}\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2+C\|a\|_{H^3}^2\|(\mathbf{u},\varphi)\|_{H^3}^2.\tag{3.54}$$

Inserting (3.53), (3.54) into (3.50) and using (3.51), (3.52), we arrive at a basic energy inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{u})\|_{L^2}^2+\frac{5c_0^{-1}\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2+(\lambda+\mu)\|\operatorname{div}\mathbf{u}\|_{L^2}^2\\&\quad\leqslant C\left(\|(\mathbf{u},b)\|_{H^3}+\|(p,b)\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2.\end{aligned}\tag{3.55}$$

Next, we are concerned with the higher energy estimates. Applying ∇^s with $s=1,2,3$ to (3.49) and then taking L^2 inner product with $(\nabla^s\mathbf{u},\frac{1}{\gamma+1}\nabla^s\varphi)$ yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\nabla^s\mathbf{u},\frac{1}{\gamma+1}\nabla^s\varphi\right)\right\|_{L^2}^2-\int_{\mathbb{T}^2}\nabla^s\operatorname{div}\left(\bar{\mu}(\rho)\nabla\mathbf{u}\right)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x-\int_{\mathbb{T}^2}\nabla^s\nabla\left(\bar{\lambda}(\rho)\operatorname{div}\mathbf{u}\right)\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\\&\quad=\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x+\frac{1}{\gamma+1}\int_{\mathbb{T}^2}\nabla^s f_2\cdot\nabla^s\varphi\,\mathrm{d}x.\end{aligned}\tag{3.56}$$

The last two terms on the left hand side of (3.56) can be dealt from (3.14) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\nabla^s\mathbf{u},\frac{1}{\gamma+1}\nabla^s\varphi\right)\right\|_{L^2}^2+c_0^{-1}\mu\left\|\nabla^{s+1}\mathbf{u}\right\|_{L^2}^2+c_0^{-1}(\lambda+\mu)\|\nabla^s\operatorname{div}\mathbf{u}\|_{L^2}^2\\&\quad\leqslant C\|p\|_{H^3}^2\|\mathbf{u}\|_{H^3}^2+C\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x+C\int_{\mathbb{T}^2}\nabla^s f_2\cdot\nabla^s\varphi\,\mathrm{d}x.\end{aligned}\tag{3.57}$$

We now estimate successively terms on the right hand side of (3.57). We first get by a similar derivation of (3.47) that

$$\int_{\mathbb{T}^2}\nabla^s f_1\cdot\nabla^s\mathbf{u}\,\mathrm{d}x\leqslant\frac{c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2+C\|\mathbf{u}\|_{H^3}\|\mathbf{u}\|_{H^3}^2+C\left(\|\varphi\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2\right)\|p\|_{H^3}^2.\tag{3.58}$$

For the first term in f_2, we get by a similar derivation of (3.18) that

$$\int_{\mathbb{T}^2} \nabla^s (\mathbf{u} \cdot \nabla\varphi) \cdot \nabla^s \varphi \, \mathrm{d}x \leqslant \|\mathbf{u}\|_{H^3} \|\varphi\|_{H^3}^2 . \tag{3.59}$$

For the second term in f_2, it follows from lemma 2.3 that

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla^s (\varphi \mathrm{div}\,\mathbf{u}) \cdot \nabla^s \varphi \, \mathrm{d}x &\leqslant C (\|\mathrm{div}\,\mathbf{u}\|_{L^\infty} \|\varphi\|_{H^s} + \|\mathrm{div}\,\mathbf{u}\|_{H^s} \|\varphi\|_{L^\infty}) \|\nabla^s \varphi\|_{L^2} \\ &\leqslant \frac{c_0^{-1}\mu}{16} \|\nabla \mathbf{u}\|_{H^3}^2 + C \left(\|\mathbf{u}\|_{H^3} + \|\varphi\|_{H^3}^2\right) \|\varphi\|_{H^3}^2 . \end{aligned} \tag{3.60}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla^s (b \mathrm{div}\,\mathbf{u}) \cdot \nabla^s \varphi \, \mathrm{d}x &\leqslant C (\|\mathrm{div}\,\mathbf{u}\|_{L^\infty} \|b\|_{H^s} + \|\mathrm{div}\,\mathbf{u}\|_{H^s} \|b\|_{L^\infty}) \|\nabla^s \varphi\|_{L^2} \\ &\leqslant \frac{c_0^{-1}\mu}{16} \|\nabla \mathbf{u}\|_{H^3}^2 + C \|b\|_{H^3}^2 \|\varphi\|_{H^3}^2 , \end{aligned} \tag{3.61}$$

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla^s \left(b^2 \mathrm{div}\,\mathbf{u}\right) \cdot \nabla^s \varphi \, \mathrm{d}x &\leqslant C \left(\|\mathrm{div}\,\mathbf{u}\|_{L^\infty} \|b^2\|_{H^s} + \|\mathrm{div}\,\mathbf{u}\|_{H^s} \|b^2\|_{L^\infty}\right) \|\nabla^s \varphi\|_{L^2} \\ &\leqslant \frac{c_0^{-1}\mu}{16} \|\nabla \mathbf{u}\|_{H^3}^2 + C \left(1 + \|b\|_{H^3}^2\right) \|b\|_{H^3}^2 \|\varphi\|_{H^3}^2 . \end{aligned} \tag{3.62}$$

Collecting (3.59)–(3.62), we can get

$$\int_{\mathbb{T}^2} \nabla^s f_2 \cdot \nabla^s \varphi \, \mathrm{d}x \leqslant \frac{3c_0^{-1}\mu}{16} \|\nabla \mathbf{u}\|_{H^3}^2 + C \left(\|\mathbf{u}\|_{H^3} + \|\varphi\|_{H^3}^2 + \left(1 + \|b\|_{H^3}^2\right) \|b\|_{H^3}^2\right) \|(\varphi, \mathbf{u})\|_{H^3}^2 . \tag{3.63}$$

Plugging (3.58) and (3.63) into (3.57) and combining with (3.55), we arrive at the desired estimate (3.48). This completes the proof of lemma 3.2. □

3.4. Dissipation estimates for $(\varphi, \mathbf{u})$

Next, we find the hidden dissipation of the unknown good function φ.

Lemma 3.3. *Let* $(p, \mathbf{u}, b) \in C([0, T]; H^3)$ *be a solution to the system (3.3), there holds*

$$\begin{aligned}&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|(\varphi, \mathbf{u}, \mathbf{G})\|_{H^3}^2 + \frac{\gamma + 1}{4\nu} \|\varphi\|_{H^3}^2 + \mu \|\nabla \mathbf{u}\|_{H^3}^2 + \frac{\nu}{4} \|\nabla \mathbf{G}\|_{H^3}^2 \\ &\quad \leqslant C \left(\|(p, \mathbf{u})\|_{H^3}^2 + \|\mathbf{u}\|_{H^3} + \left(1 + \|b\|_{H^3}^2\right) \|b\|_{H^3}^2\right) \left(\|\varphi\|_{H^3}^2 + \|\nabla \mathbf{u}\|_{H^3}^2\right) . \end{aligned} \tag{3.64}$$

Proof. Define

$$f_3 \stackrel{\mathrm{def}}{=} -\mathbf{u} \cdot \nabla \mathbf{u} + I(a) \nabla \varphi - I(a) (\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \mathrm{div}\,\mathbf{u}) ,$$

we can transform (3.49) into the following form

$$\begin{cases} \partial_t\varphi + (\gamma+1)\,\mathrm{div}\,\mathbf{u} = f_2, \\ \partial_t\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} + \nabla\varphi = f_3. \end{cases} \tag{3.65}$$

Denote

$$\mathbf{G} \overset{\mathrm{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{\nu}\Delta^{-1}\nabla\varphi. \tag{3.66}$$

Then, we find out that $\mathbf{G}$ satisfies

$$\partial_t\mathbf{G} - \nu\Delta\mathbf{G} = \frac{\gamma+1}{\nu}\mathbb{Q}\mathbf{u} + \mathbb{Q}f_3 + \frac{1}{\nu}\Delta^{-1}\nabla f_2.$$

For $s = 0,1,2,3$, applying ∇^s to the above equation, and taking the L^2-inner product with $\nabla^s\mathbf{G}$ give

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\mathbf{G}\|_{L^2}^2 + \nu\left\|\nabla^{s+1}\mathbf{G}\right\|_{L^2}^2 \\ &\quad\leqslant C\int_{\mathbb{T}^2}\nabla^s\mathbb{Q}\mathbf{u}\cdot\nabla^s\mathbf{G}\,\mathrm{d}x + C\int_{\mathbb{T}^2}\nabla^{s-1}f_2\cdot\nabla^s\mathbf{G}\,\mathrm{d}x + C\int_{\mathbb{T}^2}\nabla^s\mathbb{Q}f_3\cdot\nabla^s\mathbf{G}\,\mathrm{d}x. \end{aligned} \tag{3.67}$$

For $s = 0$, we get by the Young inequality and the Poincaré inequality that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{G}\|_{L^2}^2 + \nu\|\nabla\mathbf{G}\|_{L^2}^2 \\ &\quad\leqslant C\int_{\mathbb{T}^2}\mathbb{Q}\mathbf{u}\cdot\mathbf{G}\,\mathrm{d}x + C\int_{\mathbb{T}^2}\Delta^{-1}\nabla f_2\cdot\mathbf{G}\,\mathrm{d}x + C\int_{\mathbb{T}^2}\mathbb{Q}f_3\cdot\mathbf{G}\,\mathrm{d}x \\ &\quad\leqslant \frac{\nu}{2}\|\mathbf{G}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2 + C\left\|\Delta^{-1}\nabla f_2\right\|_{L^2}^2 + C\|f_3\|_{L^2}^2 \\ &\quad\leqslant \frac{\nu}{2}\|\nabla\mathbf{G}\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|f_2\|_{H^1}^2 + C\|f_3\|_{H^2}^2, \end{aligned}$$

which implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{G}\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\mathbf{G}\|_{L^2}^2 \leqslant C\|\mathbf{u}\|_{H^2}^2 + C\|f_2\|_{H^1}^2 + C\|f_3\|_{H^2}^2. \tag{3.68}$$

For $s = 1,2,3$, with the aid of the Young inequality, we deduce from (3.67) that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\mathbf{G}\|_{L^2}^2 + \frac{\nu}{2}\left\|\nabla^{s+1}\mathbf{G}\right\|_{L^2}^2 &\leqslant C\left(\left\|\nabla^{s-1}\mathbb{Q}\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^{s-2}f_2\right\|_{L^2}^2 + \left\|\nabla^{s-1}f_3\right\|_{L^2}^2\right) \\ &\leqslant C\|\mathbf{u}\|_{H^2}^2 + C\|f_2\|_{H^1}^2 + C\|f_3\|_{H^2}^2. \end{aligned} \tag{3.69}$$

Combining (3.68) with (3.69) and using (3.46), we get for $s = 0,1,2,3$ that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\mathbf{G}\|_{L^2}^2 + \frac{\nu}{2}\left\|\nabla^{s+1}\mathbf{G}\right\|_{L^2}^2 &\leqslant C\left(\left\|\nabla^{s-1}\mathbb{Q}\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^{s-2}f_2\right\|_{L^2}^2 + \left\|\nabla^{s-1}f_3\right\|_{L^2}^2\right) \\ &\leqslant C\|\nabla\mathbf{u}\|_{H^3}^2 + C\|f_2\|_{H^1}^2 + C\|f_3\|_{H^2}^2. \end{aligned} \tag{3.70}$$

By using the fact

$$\mathrm{div}\,\mathbb{Q}\mathbf{u} = \mathrm{div}\,\mathbf{u},$$

we infer from (3.66) and the first equation in (3.49) that φ satisfies a damped transport equation

$$\partial_t\varphi+\frac{\gamma+1}{\nu}\varphi=-(\gamma+1)\operatorname{div}\mathbf{G}+f_2.$$

For the above equation, we get by a similar derivation of (3.70) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\varphi\|_{L^2}^2+\frac{\gamma+1}{\nu}\|\nabla^s\varphi\|_{L^2}^2=-(\gamma+1)\int_{\mathbb{T}^2}\nabla^s\operatorname{div}\mathbf{G}\cdot\nabla^s\varphi\,\mathrm{d}x+\int_{\mathbb{T}^2}\nabla^s f_2\cdot\nabla^s\varphi\,\mathrm{d}x.$$

By the Young inequality, there holds

$$\begin{aligned}\left|-(\gamma+1)\int_{\mathbb{T}^2}\nabla^s\operatorname{div}\mathbf{G}\cdot\nabla^s\varphi\,\mathrm{d}x\right|&\leqslant C\|\nabla^s\operatorname{div}\mathbf{G}\|_{L^2}(\gamma+1)\|\nabla^s\varphi\|_{L^2}\\&\leqslant\frac{\gamma+1}{2\nu}\|\nabla^s\varphi\|_{L^2}^2+C\nu\left\|\nabla^{s+1}\mathbf{G}\right\|_{L^2}^2.\end{aligned}\tag{3.71}$$

The second term on the right hand side of the above equality can be bounded the same as (3.63)

$$\int_{\mathbb{T}^2}\nabla^s f_2\cdot\nabla^s\varphi\,dx\leqslant\frac{3c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2+C\left(\|\mathbf{u}\|_{H^3}+\|\varphi\|_{H^3}^2+\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2.$$

which combines (3.71) implies that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s\varphi\|_{L^2}^2+\frac{\gamma+1}{2\nu}\|\nabla^s\varphi\|_{L^2}^2\leqslant{}&\frac{3c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2+C\nu\left\|\nabla^{s+1}\mathbf{G}\right\|_{L^2}^2\\&+C\left(\|\mathbf{u}\|_{H^3}+\|\varphi\|_{H^3}^2+\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2.\end{aligned}\tag{3.72}$$

Multiplying (3.70) by a suitable large constant and then adding to (3.72) lead to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{G})\|_{H^3}^2+\frac{\gamma+1}{4\nu}\|\varphi\|_{H^3}^2+\frac{\nu}{4}\|\nabla\mathbf{G}\|_{H^3}^2\\&\quad\leqslant\frac{3c_0^{-1}\mu}{16}\|\nabla\mathbf{u}\|_{H^3}^2+C\|\nabla\mathbf{u}\|_{H^3}^2+C\|f_2\|_{H^1}^2+C\|f_3\|_{H^2}^2\\&\qquad+C\left(\|\mathbf{u}\|_{H^3}+\|\varphi\|_{H^3}^2+\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2.\end{aligned}\tag{3.73}$$

Using the fact that $H^2(\mathbb{T}^2)$ is an Banach algebra, the nonlinear terms in f_2,f_3 can be estimated as follows

$$\begin{aligned}\|f_2\|_{H^1}^2&\leqslant C\|\mathbf{u}\cdot\nabla\varphi\|_{H^1}^2+\|\varphi\operatorname{div}\mathbf{u}\|_{H^1}^2+C\|b\operatorname{div}\mathbf{u}\|_{H^1}^2+\left\|b^2\operatorname{div}\mathbf{u}\right\|_{H^1}^2\\&\leqslant C\|\mathbf{u}\|_{H^3}^2\|\varphi\|_{H^3}^2+C\|\varphi\|_{H^3}^2\|\nabla\mathbf{u}\|_{H^3}^2+C\|b\|_{H^3}^2\|\nabla\mathbf{u}\|_{H^3}^2+\|b\|_{H^3}^4\|\nabla\mathbf{u}\|_{H^3}^2\\&\leqslant C\|\mathbf{u}\|_{H^3}^2\|\varphi\|_{H^3}^2+C\left(\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2+\|\varphi\|_{H^3}^2\right)\|\nabla\mathbf{u}\|_{H^3}^2,\end{aligned}\tag{3.74}$$

$$\begin{aligned}\|f_3\|_{H^2}^2 &\leqslant C\|\mathbf{u}\|_{H^2}^2\|\nabla\mathbf{u}\|_{H^2}^2 + C\|a\|_{H^2}^2\|\nabla\varphi\|_{H^2}^2 + C\|a\|_{H^2}^2\|\nabla\mathbf{u}\|_{H^3}^2 \\ &\leqslant C\|(p,\mathbf{u})\|_{H^3}^2\|\nabla\mathbf{u}\|_{H^3}^2 + C\|p\|_{H^3}^2\|\varphi\|_{H^3}^2 . \end{aligned} \tag{3.75}$$

Inserting (3.74) and (3.75) into (3.73) leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{G})\|_{H^3}^2 + \frac{\gamma+1}{4\nu}\|\varphi\|_{H^3}^2 + \frac{\nu}{4}\|\nabla\mathbf{G}\|_{H^3}^2 \\ &\quad\leqslant \left(\frac{3c_0^{-1}\mu}{8}+C\right)\|\nabla\mathbf{u}\|_{H^3}^2 + C\left(\|(p,\mathbf{u},\varphi)\|_{H^3}^2 + \left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|\nabla\mathbf{u}\|_{H^3}^2 \\ &\qquad + C\left(\|\mathbf{u}\|_{H^3} + \|(p,\mathbf{u},\varphi)\|_{H^3}^2 + \left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\|(\varphi,\mathbf{u})\|_{H^3}^2 . \end{aligned} \tag{3.76}$$

Thus, multiplying (3.48) by a suitable large constant and then adding to (3.76), we can finally get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{u},\mathbf{G})\|_{H^3}^2 + \frac{\gamma+1}{4\nu}\|\varphi\|_{H^3}^2 + \mu\|\nabla\mathbf{u}\|_{H^3}^2 + \frac{\nu}{4}\|\nabla\mathbf{G}\|_{H^3}^2 \\ &\quad\leqslant C\left(\|(p,\mathbf{u},\varphi)\|_{H^3}^2 + \|\mathbf{u}\|_{H^3} + \left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2\right)\left(\|\varphi\|_{H^3}^2 + \|\nabla\mathbf{u}\|_{H^3}^2\right), \end{aligned} \tag{3.77}$$

where we have used (3.46) once again. In view of

$$\varphi = p + \frac{1}{2}\left(b^2+2b\right),$$

one has

$$\|\varphi\|_{H^3}^2 \leqslant C\|p\|_{H^3}^2 + C\left(1+\|b\|_{H^3}^2\right)\|b\|_{H^3}^2$$

from which we can arrive at the desired estimate (3.64). Consequently, this completes the proof of lemma 3.3.

□

3.5. *Bootstrap argument*

In this section, we complete the proof of theorem 1.1. Under the assumption of (3.2), we infer from (3.64) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varphi,\mathbf{u},\mathbf{G})\|_{H^3}^2 + \frac{\gamma+1}{4\nu}\|\varphi\|_{H^3}^2 + \mu\|\nabla\mathbf{u}\|_{H^3}^2 + \frac{\nu}{4}\|\nabla\mathbf{G}\|_{H^3}^2 \\ &\quad\leqslant C\delta\left(\delta^3+\delta+1\right)\|\varphi\|_{H^3}^2 + C\delta\left(\delta^3+\delta+1\right)\|\nabla\mathbf{u}\|_{H^3}^2 . \end{aligned} \tag{3.78}$$

Denote

$$\mathscr{E}(t) = \|(\varphi,\mathbf{u},\mathbf{G})\|_{H^3}^2$$

and

$$\mathscr{D}(t) = \frac{\gamma+1}{4\nu}\|\varphi\|_{H^3}^2 + \mu\|\nabla\mathbf{u}\|_{H^3}^2 + \frac{\nu}{4}\|\nabla\mathbf{G}\|_{H^3}^2 .$$

Then, choosing δ small enough in (3.78) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)+\frac{1}{2}\mathscr{D}(t)\leqslant 0. \tag{3.79}$$

It's straightforward to check that

$$\mathscr{E}(t)\leqslant C\mathscr{D}(t)$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)+c\mathscr{E}(t)\leqslant 0.$$

Solving this inequality yields

$$\mathscr{E}(t)\leqslant Ce^{-ct}. \tag{3.80}$$

Hence, we get

$$\int_0^t\left(\|\varphi(\tau)\|_{H^3}+\|\mathbf{u}(\tau)\|_{H^3}+\|\mathbf{u}(\tau)\|_{H^3}^2\right)\mathrm{d}\tau\leqslant C. \tag{3.81}$$

Due to $c_0\leqslant\rho\leqslant c_0^{-1}$, we have

$$\tilde{c}_0\leqslant\frac{1}{1+p}\leqslant(\tilde{c}_0)^{-1}.$$

Hence, there holds

$$\frac{1}{2}\|p\|_{H^3}^2-\frac{1}{2\gamma}\int_{\mathbb{T}^2}\frac{p}{1+p}\left(\nabla^3 p\right)^2\mathrm{d}x\geqslant C\|p\|_{H^3}^2$$

from which and the lemma 3.1, we have

$$\begin{aligned}\|(p,\mathbf{u},b)\|_{H^3}^2\leqslant{}&\|(p_0,\mathbf{u}_0,b_0)\|_{H^3}^2+C\int_0^t\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}\|p\|_{H^3}^2\,\mathrm{d}\tau\\&+C\int_0^t\left(\|\mathbf{u}\|_{H^3}+\|\mathbf{u}\|_{H^3}^2+\|\varphi\|_{H^3}^2\right)\|(p,\mathbf{u},b)\|_{H^3}^2\,\mathrm{d}\tau.\end{aligned} \tag{3.82}$$

Thus, exploiting the Grönwall inequality to (3.82) and using (3.81) imply that

$$\begin{aligned}\|(p,\mathbf{u},b)\|_{H^3}^2&\leqslant C\|(p_0,\mathbf{u}_0,b_0)\|_{H^3}^2\exp\left\{C\int_0^t\left(\left(1+\|p\|_{H^3}^2\right)\|\mathbf{u}\|_{H^3}+\|\mathbf{u}\|_{H^3}^2+\|\varphi\|_{H^3}^2\right)\mathrm{d}\tau\right\}\\&\leqslant C\varepsilon^2.\end{aligned}$$

Taking ε small enough so that $C\varepsilon\leqslant\delta/2$, we deduce from a continuity argument that the local solution can be extended as a global one in time. This completes the proof of our main theorem.

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] Danchin R 2000 Global existence in critical spaces for compressible Navier–Stokes equations *Invent. Math.* **141** 579–614
[2] Desvillettes L and Villani C 2005 On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation *Invent. Math.* **159** 245–316
[3] Dong B, Wu J and Zhai X 2023 Global small solutions to a special $2\frac{1}{2}$-D compressible viscous non-resistive MHD system *J. Nonlinear Sci.* **33** 21
[4] Ducomet B and Feireisl E 2006 The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars *Commun. Math. Phys.* **266** 595–629
[5] Feireisl E 2004 *Dynamics of Viscous Compressible Fluids* (Oxford University Press)
[6] Hong G, Hou X, Peng H and Zhu C 2017 Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum *SIAM J. Math. Anal.* **49** 2409–41
[7] Hu X and Wang D 2008 Global solutions to the three-dimensional full compressible magnetohydrodynamic flows *Commun. Math. Phys.* **283** 255–84
[8] Hu X and Wang D 2010 Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows *Arch. Ration. Mech. Anal.* **197** 203–38
[9] Huang X, Li J and Xin Z 2012 Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations *Commun. Pure Appl. Math.* **65** 549–85
[10] Jiang S and Zhang J 2017 On the non-resistive limit and the magnetic boundary-layer for one-dimensional compressible magnetohydrodynamics *Nonlinearity* **30** 3587–612
[11] Kato T 1990 Liapunov functions and monotonicity in the Euler and Navier–Stokes equations *Functional-Analytic Methods for Partial Differential Equations* (*Lecture Notes in Mathematics* vol 1450) (Springer)
[12] Kawashima S 1984 System of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics *PhD Thesis* Kyoto University
[13] Kawashima S and Shizuta Y 1986 Magnetohydrodynamic approximation of the complete equations for an electromagnetic fluid *Tsukuba J. Math.* **10** 131–49
[14] Li F and Yu H 2011 Optimal decay rate of classical solutions to the compressible magnetohydrodynamic equations *Proc. R. Soc. Edinburgh* A **141** 109–26
[15] Li H, Xu X and Zhang J 2013 Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum *SIAM J. Math. Anal.* **45** 1356–87

[16] Li Y 2022 Global well-posedness for the three-dimensional full compressible viscous non-resistive MHD system *J. Math. Fluid Mech.* **24** 28
[17] Li Y and Sun Y 2019 Global weak solutions and long time behavior for 1D compressible MHD equations without resistivity *J. Math. Phys.* **60** 071511
[18] Li Y and Sun Y 2019 Global weak solutions to a two-dimensional compressible MHD equations of viscous non-resistive fluids *J. Differ. Equ.* **267** 3827–51
[19] Li Y and Sun Y 2021 On global-in-time weak solutions to a two-dimensional full compressible nonresistive MHD system *SIAM J. Math. Anal.* **53** 4142–77
[20] Li Y, Xu H and Zhai X 2023 Global smooth solutions to the 3D compressible viscous non-isentropic magnetohydrodynamic flows without magnetic diffusion *J. Geom. Anal.* **33** 246
[21] Lin F, Xu L and Zhang P 2015 Global small solutions of 2-D incompressible MHD system *J. Differ. Equ.* **259** 5440–85
[22] Liu Y and Zhang T 2021 Global weak solutions to a 2D compressible non-resistivity MHD system with non-monotone pressure law and nonconstant viscosity *J. Math. Anal. Appl.* **502** 125244
[23] Ren X, Wu J, Xiang Z and Zhang Z 2014 Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion *J. Funct. Anal.* **267** 503–41
[24] Tan Z and Wang Y 2018 Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems *SIAM J. Math. Anal.* **50** 1432–70
[25] Triebel H 1983 *Theory of Function Spaces (Monographs in Mathematics)* (Birkhäuser Verlag)
[26] Wu G, Zhang Y and Zou W 2021 Optimal time-decay rates for the 3D compressible magnetohydrodynamic flows with discontinuous initial data and large oscillations *J. Lond. Math. Soc.* **103** 817–45
[27] Wu J and Wu Y 2017 Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion *Adv. Math.* **310** 759–888
[28] Wu J and Zhai X 2023 Global small solutions to the 3D compressible viscous non-resistive MHD system *Math. Models Methods Appl. Sci.* **33** 2629–56
[29] Wu J and Zhu Y 2022 Global well-posedness for 2D non-resistive compressible MHD system in periodic domain *J. Funct. Anal.* **283** 49
[30] Xin Z and Yan W 2013 On blowup of classical solutions to the compressible Navier–Stokes equations *Commun. Math. Phys.* **321** 529–41
[31] Xu L and Zhang P 2015 Global small solutions to three-dimensional incompressible magnetohydrodynamical system *SIAM J. Math. Anal.* **47** 26–65
[32] Zhang T 2016 Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field *J. Differ. Equ.* **260** 5450–80
[33] Zhong X 2020 On local strong solutions to the 2D Cauchy problem of the compressible non-resistive magnetohydrodynamic equations with vacuum *J. Dyn. Differ. Equ.* **32** 505–26
[34] Zhu S 2015 On classical solutions of the compressible magnetohydrodynamic equations with vacuum *SIAM J. Math. Anal.* **47** 2722–53