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Stability for a System of the 2D Incompressible MHD Equations with Fractional Dissipation

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Abstract. Several fundamental problems on the 2D magnetohydrodynamic (MHD) equations with only magnetic diffusion (no velocity dissipation) remain open, especialy in the case when the spatial domain is the whole space $\mathbb{R}^2$. This paper establishes that, near a background magnetic field, any fractional dissipation in one direction in the velocity equation would allow us to establish the global existence and stability for perturbations near the background. The magnetic diffusion here is not required to be given by the standard Laplacian operator but any general fractional Laplacian with positive power.

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1. Introduction

We start with the background information. The magnetohydrodynamic (MHD) system governs the motion of electrically conducting fluids in a magnetic field such as plasmas, liquid metals and electrolytes, and has a wide range of applications in astrophysics, geophysics, cosmology and engineering (see, e.g., [6,14,45]). The MHD system is a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of the electromagnetism. The coupling and interaction between the magnetic field and the fluid enables the MHD system to model many more phenomena than the Navier–Stokes and the Euler equations.

Mathematically the coupling makes it much more challenging to fully understand the MHD systems, even in the 2D case. One significant example is the 2D resistive MHD equations without the velocity dissipation

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t b + u\cdot\nabla b = \Delta b + b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0)=u_0(x),\ \ b(x,0)=b_0(x), \end{cases} \tag{1.1}$$

where $u=u(x,t)$ denotes the fluid velocity, $b=b(x,t)$ the magnetic field, and $p=p(x,t)$ the pressure. (1.1) arises in the study of magnetic reconnection and magnetic turbulence, in which the fluid viscosity can be ignored while the role of resistivity is important (see [45]).

The fluid velocity u in (1.1) obeys the 2D Euler equation with the Lorentz forcing term $b\cdot\nabla b$. Even though the regularity problem on the 2D Euler is well understood, many fundamental problems on the MHD system (1.1) remain open. Among them is the global well-posedness problem for general large initial data. There are substantial developments on this problem (see, e.g., [9–12,16–18,22,29–31,37,38,40,55,59,60,62,64,65]). We now know that the global regularity problem on (1.1) is actually critical. This can be understood from two aspects. If the Laplacian dissipation Δb in (1.1) is replaced by the hyper-dissipation $-(-\Delta)^\beta b$ with any $\beta>1$, then the resulting system is globally well-posed [12,31]. If

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we keep Δb in (1.1) but add fractional dissipation $-(-\Delta)^{\alpha}u$ or even logarithmic dissipation $-\log(2-\Delta)u$, then the slightly dissipated MHD system always possesses a unique global classical solution [22,60,64]. These results illustrate the criticality of the dissipation Δb. The global regularity problem on (1.1) has also been attempted from the uniqueness of weak solutions. (1.1) always possesses a global H^1-weak solution (see, e.g., [11,37]). However, the uniqueness of the H^1-weak solutions remains an open problem.

When the initial datum (u_0, b_0) is assumed to be small, the only global existence and regularity result on (1.1) is for the periodic domain $\mathbb{T}^2$. Assuming the mean of b_0 is zero on $\mathbb{T}^2$, Wei and Zhang [53] were able to obtain the global solution in the Sobolev setting $H^4(\mathbb{T}^2)$. Later Ye and Yin [63] reduced the regularity assumption to the Sobolev space $H^s(\mathbb{T}^2)$ with $s > 2$. We remark that the Sobolev norms of the solutions obtained in [53] and [63] do not admit uniform upper bounds. Therefore, the stability problem (1.1) near the trivial solution remains open even in the periodic setting.

When the spatial domain is the whole space $\mathbb{R}^2$, (1.1) with even small initial data is not known to be globally well-posed, let alone the stability. The Poincare type inequalities in the periodic setting are no longer valid in $\mathbb{R}^2$. Therefore the small data global well-posedness problem on (1.1) in $\mathbb{R}^2$ is widely open.

Motivated by the observed physical phenomenon that the magnetic field can stabilize the electrically conducting fluids (see, e.g., [1–3,14,23,24,34]), substantial investigations have been developed on the global well-posedness and stability problems on the MHD equations near a background magnetic field (see, e.g., [4,5,7,8,13,15,19–21,25–28,32,33,35,36,39,41,43,44,46–48,50,52–54,56–58,66–68]). For the periodic setting $\mathbb{T}^2$ and under suitable symmetry assumptions, Zhou and Zhu [68] obtained the stability of (1.1) near a background magnetic field. Very recently Lin, Suo and Wu solved the stability problem when the spatial domain is $\mathbb{T} \times \mathbb{R}$ [49].

When the spatial domain is $\mathbb{R}^2$, the stability problem on (1.1) near a background magnetic field remains open. This work is partially motivated by the intention to understand this difficult problem. We will show that, near any background magnetic field, adding fractional dissipation in only one direction, say $\nu\partial_2^{2\alpha}u$ with $\alpha > 0$ to the velocity equation in (1.1), would allow us to obtain the global well-posedness and stability. Furthermore, we can replace the magnetic diffusion Δb by any fractional Laplacian dissipation $-(-\Delta)^{\beta}b$ with $\beta > 0$. This result suggests that the stability problem on (1.1) near a background magnetic field is critical. Therefore, this paper examines the following 2D incompressible fractional MHD system,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu\partial_2^{2\alpha}u + B \cdot \nabla B, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t B + u \cdot \nabla B + \eta(-\Delta)^{\beta}B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\nu > 0$ and $\eta > 0$ are real parameters. The fractional partial operator $\partial_2^{2\alpha}$ and the fractional Laplacian operator $(-\Delta)^{\beta}$ are defined by their Fourier transforms,

$$\widehat{\partial_2^{2\alpha}} f(\xi) = \xi_2^{2\alpha}\widehat{f}(\xi), \qquad \widehat{(-\Delta)^{\beta}} f(\xi) = |\xi|^{2\beta}\widehat{f}(\xi).$$

Clearly, (1.2) admits a special class of steady-state solutions represented by the background magnetic field. Attention is focused on the steady-state solution

$$u^{(0)}(x) = (0,0), \quad B^{(0)}(x) = e_1 = (1,0).$$

The perturbation (u, b) around this steady solution with $b = B - e_1$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu\partial_2^{2\alpha}u + b \cdot \nabla b + \partial_1 b, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b + u \cdot \nabla b + \eta(-\Delta)^{\beta}b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\ b(x,0) = b_0(x). \end{cases} \tag{1.3}$$

We establish that any small initial perturbation (u_0, b_0) in $H^3(\mathbb{R}^2)$ leads to a unique global solution (u, b) to (1.3), which remains small and comparable to the size of the initial perturbation for all time.

Theorem 1.1. *Let $\eta,\ \nu > 0$ and $0 < \alpha, \beta \le 1$. Consider (1.3) with the initial data $(u_0, b_0) \in H^3(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon,$$

then (1.3) has a unique global classical solution (u, b) satisfying, for any $t > 0$,

$$\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \left(\|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\Lambda^\beta b\|_{H^3}^2\right)\, d\tau \le C\,\varepsilon^2$$

for some universal constant $C > 0$.

We outline the key points in the proofs of Theorem 1.1. The main difficulty in the proof of Theorem 1.1 is due to the lack of the horizontal dissipation in the velocity equation. In fact, the 2D Navier–Stokes equations with dissipation in just one direction

$$\partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_2^{2\alpha} u, \quad x \in \mathbb{R}^2,\ t > 0$$

is not known to be stable near the trivial solution even when $\alpha = 1$. Certainly we need to fully exploit the stabilizing effect of the magnetic field. The coupling of u and b in (1.3) leads to the wave structure. This can be explained as follows. Applying the Helmholtz-Leray projection operator

$$\mathbb{P} := I - \nabla \Delta^{-1} \nabla \cdot$$

to the velocity equation in (1.3), we eliminate the pressure to obtain

$$\partial_t u = \nu \partial_2^{2\alpha} u + \partial_1 b + N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u + b \cdot \nabla b). \tag{1.4}$$

By separating the linear terms from the nonlinear ones in (1.3), the equation of b can be written as

$$\partial_t b = -\eta(-\Delta)^\beta b + \partial_1 u + N_2, \qquad N_2 = -u \cdot \nabla b + b \cdot \nabla u. \tag{1.5}$$

Thus, (1.3) can be written as

$$\begin{cases} \partial_t u = \nu \partial_2^{2\alpha} u + \partial_1 b + N_1, \\ \partial_t b = -\eta(-\Delta)^\beta b + \partial_1 u + N_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x),\ b(x, 0) = b_0(x). \end{cases}$$

Differentiating (1.4)and (1.5) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt} u - (\nu \partial_2^{2\alpha} - \eta(-\Delta)^\beta)\partial_t u - (\partial_{11} u + \eta\nu \partial_2^{2\alpha}(-\Delta)^\beta u) = N_3, \\ \partial_{tt} b - (\nu \partial_2^{2\alpha} - \eta(-\Delta)^\beta)\partial_t b - (\partial_{11} b + \eta\nu \partial_2^{2\alpha}(-\Delta)^\beta b) = N_4, \end{cases} \tag{1.6}$$

where N_3 and N_4 are given by

$$N_3 = (\partial_t + \eta(-\Delta)^\beta) N_1 + \partial_1 N_2, \qquad N_4 = (\partial_t - \nu \partial_2^{2\alpha}) N_2 + \partial_1 N_1.$$

Both u and b are found to satisfy nonhomogeneous wave type equations with exactly the same linear parts. Moreover, (1.6) exhibits much more regularization than its original counterpart in (1.3). In particular, the term $\partial_{11} u$ in (1.6) provides the desired stabilizing effect on the velocity field. The appearance of this term is originated from the background magnetic field in the x_1-direction.

However, this smoothing and stabilizing effect in x_1 direction is not as strong as what the standard dissipation term provides. An explicit computation reveals that this smoothing effect is actually one-derivative-order lower. This is a weak type regularization. One way to take advantage of this weak stabilizing effect is to design a suitable energy functional, which consists of two pieces. Since we are seeking solutions in H^3, the first piece is defined in terms of the H^3-norm of (u, b) together with the time-integral part from the dissipation terms. More precisely, $E_1(t)$ is defined to be

$$E_1(t) := \sup_{0 \le \tau \le t} (\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2) + 2\nu \int_0^t \|\partial_2^\alpha u(\tau)\|_{H^3}^2 d\tau + 2\eta \int_0^t \|\Lambda^\beta b(\tau)\|_{H^3}^2 d\tau.$$

The second piece takes into account of the weak dissipation, namely

$$E_2(t) := \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 \, d\tau.$$

We combine these two pieces to define the energy functional

$$E(t) = E_1(t) + E_2(t).$$

Our main efforts are devoted to proving that, for any $t > 0$,

$$E(t) \leq E(0) + CE(t)^{3/2}. \tag{1.7}$$

Once we have (1.7), the bootstrapping argument (see, e.g., [51]) implies that if

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon \quad \text{or} \quad E(0) \leq \varepsilon^2,$$

then there exists a constant $C > 0$ such that

$$E(t) \leq C\varepsilon^2, \quad \forall\, t \geq 0.$$

In order to prove (1.7), we need to estimate $E_1(t)$ and $E_2(t)$. These are achieved in the following two propositions.

Proposition 1.2. *For a positive constant $C > 0$ (depending on ν and η), we have*

$$E_1(t) \leq E(0) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t).$$

Proposition 1.3. *For a positive constant $C > 0$ (depending on ν and η), we have*

$$E_2(t) \leq CE_1(0) + CE_1(t) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t).$$

The smoothing and stabilizing effect reflected in the wave structure (1.6) is used in the proof of Proposition 1.3. The term $-\partial_{11} u$ in the wave equation allows us to gain the time integrability of $\|\partial_1 u\|_{L^2}^2$. A quick way to extract from the wave equation this desired dissipative effect is through energy estimates and a mixed L^2-inner product. When we take the time derivative of the inner product $(b, \partial_1 u)$, we have

$$\begin{aligned}
\frac{d}{dt}(b, \partial_1 u) &= (\partial_t b, \partial_1 u) + (b, \partial_t \partial_1 u) \\
&= (\partial_1 u, \partial_1 u) + (b, \partial_{11} b) + \text{other terms} \\
&= \|\partial_1 u\|_{L^2}^2 + (b, \partial_{11} b) + \text{other terms.}
\end{aligned}$$

Since $(\partial_1 u, \partial_1 u) = -(u, \partial_{11} u)$, the process above is pretty much like energy estimates with the term $\partial_{11} u$ in the wave equation. There are more elaborated ways to use the wave structure. When we want to obtain decay rates, we would need to solve the linear wave equation and represent the nonlinear equation in an integral form.

By the equation for the magnetic field

$$\partial_1 u = \partial_t b + u \cdot \nabla b + \eta(-\Delta)^{\beta} b - b \cdot \nabla u,$$

we can convert the estimate of $E_2 := \int_0^t \|\partial_1 u\|_{H^2}^2$ into bounding the terms on the right-hand side,

$$\begin{aligned}\int_0^t \|\partial_1 u\|_{H^2}^2 \, d\tau = & \int_0^t \int \partial_1 u \cdot \partial_t b \, dx d\tau + \int_0^t \int \partial_1 u \cdot u \cdot \nabla b \, dx d\tau \\ & + \eta \int_0^t \int \partial_1 u \cdot (-\Delta)^\beta b \, dx d\tau - \int_0^t \int \partial_1 u \cdot b \cdot \nabla u \, dx d\tau \\ & + \int_0^t \int \nabla \partial_1 u \cdot \partial_t \nabla b \, dx d\tau + \int_0^t \int \partial_1 \nabla u \cdot \nabla (u \cdot \nabla b) \, dx d\tau \\ & + \eta \int_0^t \int \partial_1 \nabla u \cdot \nabla (-\Delta)^\beta b \, dx d\tau - \int_0^t \int \partial_1 \nabla u \cdot \nabla (b \cdot \nabla u) \, dx d\tau \\ & + \int_0^t \int \Delta \partial_1 u \cdot \partial_t \Delta b \, dx d\tau + \int_0^t \int \partial_1 \Delta u \cdot \Delta (u \cdot \nabla b) \, dx d\tau \\ & + \eta \int_0^t \int \partial_1 \Delta u \cdot \Delta (-\Delta)^\beta b \, dx d\tau - \int_0^t \int \partial_1 \Delta u \cdot \Delta (b \cdot \nabla u) \, dx d\tau. \end{aligned}$$

More details are presented in Sect. 4.

We remark that, if we replace $\partial_2^{2\alpha} u$ by $\partial_1^{2\alpha} u$, we won't be able to establish the stability. The background magnetic field $(1, 0)$ can only create a stabilizing and smoothing effect in the x_1-direction. As a consequence, the velocity equation would only have dissipation in the x_1 direction. It is a well-known open problem whether the 2D Navier–Stokes with dissipation in one direction in $\mathbb{R}^2$ is stable in the Sobolev setting.

The rest of the paper is organized as follows. Assuming (1.7), we prove Theorem 1.1 in Sect. 2. Proposition 1.2 and Proposition 1.3 are proven in Sects. 3 and 4, respectively.

2. Proof of Theorem 1.1

Assume Propositions 1.2 and 1.3. This section proves Theorem 1.1. This is a consequence of applying the bootstrapping argument.

Proof of Theorem 1.1. First of all, the local-in-time existence and uniqueness result can be established following a similar procedure as the one for the Navier–Stokes and the Euler equations (see Pages 96–112 of the book by Majda and Bertozzi [42]). This procedure includes several standard steps such as the existence of solutions to regularized equations, uniform bounds, application of the Aubin–Lions Lemma, convergence to the original equation and regularity estimates. Even in the case of no dissipation, the solution can be shown to be continuous in time. A detailed implementation of this procedure on the Navier–Stokes and the Euler equations can be found in [42]. Therefore, for some $T > 0$, we have a local solution (u, b) to (1.3) with

$$(u, b) \in C([0, T); H^3(\mathbb{R}^3)).$$

It then suffices to obtain a global uniform bound on (u, b).

By Propositions 1.2 and 1.3, we have

$$E_1 \leq E(0) + C_1 E_1^{\frac{3}{2}}(t) + C_2 E_2^{\frac{3}{2}}(t), \tag{2.1}$$

$$E_2 \leq C_3 E(0) + C_4 E_1(t) + C_5 E_1^{\frac{3}{2}}(t) + C_6 E_2^{\frac{3}{2}}(t), \tag{2.2}$$

where C_1 through C_6 are positive constants depending on ν and η. Adding (2.1) to a suitable multiple of (2.2) yields

$$E_1 + \frac{1}{2C_4} E_2 \le E(0) + C_1 E_1^{\frac{3}{2}}(t) + C_2\, E_2^{\frac{3}{2}}(t) + \frac{C_3}{2C_4} E(0) + \frac{1}{2} E_1(t) + \frac{C_5}{2C_4} E_1^{\frac{3}{2}}(t) + \frac{C_6}{2C_4} E_2^{\frac{3}{2}}(t),$$

or

$$\frac{1}{2}E_1 + \frac{1}{2C_4} E_2 \le \left(1 + \frac{C_3}{2C_4}\right) E(0) + \left(C_1 + \frac{C_5}{2C_4}\right) E_1^{\frac{3}{2}}(t) + \left(C_2 + \frac{C_6}{2C_4}\right) E_2^{\frac{3}{2}}(t).$$

Therefore, $E(t) := E_1(t) + E_2(t)$ satisfies

$$E(t) \le \widetilde{C}_1 E(0) + \widetilde{C}_2 E^{\frac{3}{2}}(t) \tag{2.3}$$

for two constants $\widetilde{C}_1$ and $\widetilde{C}_2$. An application of the bootstrapping argument to (2.3) leads to the desired upper bound in Theorem 1.1. Indeed, let (u_0, b_0) to be sufficiently small such that $E(0)$ satisfies

$$E(0) = \|(u_0, b_0)\|_{H^3}^2 \le \frac{1}{16\widetilde{C}_1\, \widetilde{C}_2^2} := \varepsilon^2. \tag{2.4}$$

We make the ansatz that, for $t > 0$,

$$E(t) \le \frac{1}{4\widetilde{C}_2^2}. \tag{2.5}$$

It then follows from (2.3) that

$$E(t) \le \widetilde{C}_1 E(0) + \widetilde{C}_2 \frac{1}{2\widetilde{C}_2} E(t) \quad \text{or} \quad E(t) \le 2\widetilde{C}_1 E(0).$$

By (2.4), for all $t > 0$,

$$E(t) \le 2\widetilde{C}_1 \varepsilon^2 = \frac{1}{8\widetilde{C}_2^2},$$

which is just half of the bound in the ansatz (2.5). The bootstrapping argument then asserts that this bound actually holds for all $t > 0$. Thus, we obtain the desired global uniform on $\|(u(t), b(t))\|_{H^3}$. This completes the proof of Theorem 1.1. □

3. Proof of Propostion 1.2

This section is devoted to the proof of Propostion 1.2. We need several basic tool lemmas. The first lemma presents an 1D Sobolev inequality involving fractional derivatives. This 1D inequality is at the core of many higher dimensional anisotropic Sobolev inequalities. The proof of this lemma can be found in [61].

Lemma 3.1. *Assume that f is in $L^q(\mathbb{R})$,*

$$\|f\|_{L^q(\mathbb{R})} \le C \|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}(\frac{1}{2}-\frac{1}{q})} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}(\frac{1}{2}-\frac{1}{q})},$$

where $2 \le q \le \infty$ and $\frac{1}{s}(\frac{1}{2} - \frac{1}{q}) < 1$.

The second lemma is an anisotropic upper bound for a triple product. This inequality is a useful tool in dealing with partial differential equations with anisotropic dissipation and allows us to selectively put directional derivatives on the components of a triple product. It is stated and proven in [11].

Lemma 3.2. *There exists a constant $C > 0$ such that, if f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} |fgh|\, dx \le C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

The next two lemmas recalls two standard Sobolev type inequalities in the 2D case.

Lemma 3.3. *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1 f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_2 f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_{12} f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}.$$

Consequently,

$$\|f\|_{L^\infty} \leq C\|f\|^{\frac{1}{2}}_{H^1}\|\partial_1 f\|^{\frac{1}{2}}_{H^1}, \quad \|f\|_{L^\infty} \leq C\|f\|^{\frac{1}{2}}_{H^1}\|\partial_2 f\|^{\frac{1}{2}}_{H^1}.$$

Lemma 3.4. *Assume that $f \in L^q(\mathbb{R}^2)$ with $2 < q < \infty$. Then*

$$\|f\|_{L^q(\mathbb{R}^2)} \leq C\|f\|^{\frac{2}{q}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{1-\frac{2}{q}}_{L^2(\mathbb{R}^2)}.$$

We are ready to prove Proposition 1.2.

Proof of Proposition 1.2. Due to the equivalence of the norm $\|(u,b)\|_{H^3}$ with the norm $\|(u,b)\|_{L^2} + \|(u,b)\|_{\mathring{H}^3}$, it suffices to bound the L^2 and the homogeneous $\mathring{H}^3$ norms of (u,b). The L^2-bound follows directly from a simple energy estimate and the divergence free condition on u and b,

$$\|u(t)\|^2_{L^2} + \|b(t)\|^2_{L^2} + 2\nu\int_0^t \|\partial_2^\alpha u\|^2_{L^2}\, d\tau + 2\eta\int_0^t \|\Lambda^\beta b\|^2_{L^2}\, d\tau = \|u_0\|^2_{L^2} + \|b_0\|^2_{L^2}.$$

To estimate the homogeneous norm $\|(u,b)\|_{\mathring{H}^3}$, we use the fact that the two norms $\|u\|^2_{\dot{H}^3(\mathbb{R}^2)}$ and $\|\partial_1^3 u\|^2_{L^2(\mathbb{R}^2)} + \|\partial_2^3 u\|^2_{L^2(\mathbb{R}^2)}$ are equivalent in the sense that

$$\|\partial_1^3 u\|^2_{L^2(\mathbb{R}^2)} + \|\partial_2^3 u\|^2_{L^2(\mathbb{R}^2)} \leq \|u\|^2_{\dot{H}^3(\mathbb{R}^2)} \leq 4\left(\|\partial_1^3 u\|^2_{L^2(\mathbb{R}^2)} + \|\partial_2^3 u\|^2_{L^2(\mathbb{R}^2)}\right).$$

The inequalities above can be easily shown. Let $\widehat{u}(\xi)$ denote the Fourier transform of u, namely, for any $\xi \in \mathbb{R}^2$,

$$\widehat{u}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} u(x) dx.$$

By Plancherel's theorem and the basic inequality

$$(\xi_1^2 + \xi_2^2)^3 \leq 4(\xi_1^6 + \xi_2^6),$$

we have

$$\begin{aligned}\|u\|^2_{\dot{H}^3(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\xi|^6 |\widehat{u}(\xi)|^2\, d\xi = \int_{\mathbb{R}^2} (\xi_1^2 + \xi_2^2)^3 |\widehat{u}(\xi)|^2\, d\xi \\ &\leq 4\int_{\mathbb{R}^2} (\xi_1^6 + \xi_2^6)|\widehat{u}(\xi)|^2\, d\xi = 4\left(\|\partial_1^3 u\|^2_{L^2(\mathbb{R}^2)} + \|\partial_2^3 u\|^2_{L^2(\mathbb{R}^2)}\right).\end{aligned}$$

We apply $\partial_i^3 (i = 1,2)$ to (1.3) and then dot with $(\partial_i^3 u, \partial_i^3 b)$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2(\|\partial_i^3 u\|^2_{L^2} + \|\partial_i^3 b\|^2_{L^2}) + \sum_{i=1}^2 \nu\|\partial_i^3\partial_2^\alpha u\|^2_{L^2} + \sum_{i=1}^2 \eta\|\partial_i^3\Lambda^\beta b\|^2_{L^2} \\ &:= J + K + L + M + N,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned} J &= \sum_{i=1}^{2} \int \partial_i^3 \partial_1 b \cdot \partial_i^3 u + \partial_i^3 \partial_1 u \cdot \partial_i^3 b \ dx, \\ K &= -\sum_{i=1}^{2} \int \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \ dx, \\ L &= \sum_{i=1}^{2} \int (\partial_i^3 (b \cdot \nabla b) - b \cdot \nabla \partial_i^3 b) \cdot \partial_i^3 u \ dx, \\ M &= -\sum_{i=1}^{2} \int \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \ dx, \\ N &= \sum_{i=1}^{2} \int (\partial_i^3 (b \cdot \nabla u) - b \cdot \nabla \partial_i^3 u) \cdot \partial_i^3 b \ dx. \end{aligned}$$

By integration by parts, $J = 0$. For the term K, we split K into two terms,

$$\begin{aligned} K &= -\int \partial_1^3 (u \cdot \nabla u) \cdot \partial_1^3 u \ dx - \int \partial_2^3 (u \cdot \nabla u) \cdot \partial_2^3 u \ dx, \\ &= K_1 + K_2. \end{aligned}$$

We first estimate K_1.

$$\begin{aligned} K_1 &= -\int \partial_1^3 (u \cdot \nabla u) \cdot \partial_1^3 u \ dx \\ &= -\sum_{k=1}^{3} C_3^k \int \partial_1^k u \cdot \partial_1^{3-k} \nabla u \cdot \partial_1^3 u \ dx - \int u \cdot \partial_1^3 \nabla u \cdot \partial_1^3 u \ dx \\ &= K_{1,1} + K_{1,2}, \end{aligned}$$

where $C_3^k = \frac{3!}{k!(3-k)!}$ is the binomial coefficient. By Hölder's inequality, Lemmas 3.3 and 3.4,

$$\begin{aligned} K_{1,1} &= -3 \int \partial_1 u \cdot \partial_1^2 \nabla u \cdot \partial_1^3 u \ dx - 3 \int \partial_1^2 u \cdot \partial_1 \nabla u \cdot \partial_1^3 u \ dx - \int \partial_1^3 u \cdot \nabla u \cdot \partial_1^3 u \ dx \\ &\le C \|\partial_1 u\|_{L^\infty} \|\partial_1^2 \nabla u\|_{L^2} \|\partial_1^3 u\|_{L^2} + C \|\partial_1^2 u\|_{L^4} \|\partial_1 \nabla u\|_{L^4} \|\partial_1^3 u\|_{L^2} \\ &\quad + C \|\nabla u\|_{L^\infty} \|\partial_1^3 u\|_{L^2}^2 \\ &\le C \|u\|_{H^3} \|\partial_1 u\|_{H^2}^2 + C \|\partial_1^2 u\|_{L^2}^{1/2} \|\partial_1^2 \nabla u\|_{L^2}^{1/2} \|\partial_1 \nabla u\|_{L^2}^{1/2} \|\partial_1 \nabla^2 u\|_{L^2}^{1/2} \|\partial_1^3 u\|_{L^2} \\ &\le C \|u\|_{H^3} \|\partial_1 u\|_{H^2}^2. \end{aligned}$$

By integration by parts and the divergence free condition,

$$K_{1,2} = -\int u \cdot \partial_1^3 \nabla u \cdot \partial_1^3 u \ dx = -\frac{1}{2} \int u \cdot \nabla (\partial_1^3 u)^2 \ dx = 0.$$

We can rewrite K_2,

$$\begin{aligned} K_2 &= -\int \partial_2^3 (u \cdot \nabla u) \cdot \partial_2^3 u \ dx \\ &= -\sum_{k=1}^{3} C_3^k \int \partial_2^k u \cdot \partial_2^{3-k} \nabla u \cdot \partial_2^3 u \ dx - \int u \cdot \partial_2^3 \nabla u \cdot \partial_2^3 u \ dx \\ &= K_{2,1} + K_{2,2}. \end{aligned}$$

By integration by parts and the divergence-free condition, $K_{2,2}=0$. By Hölder's inequality, Lemmas 3.3 and 3.4,

$$\begin{aligned}K_{2,1} &= -3\int \partial_2 u\cdot\partial_2^2\nabla u\cdot\partial_2^3 u\,dx - 3\int \partial_2^2 u\cdot\partial_2\nabla u\cdot\partial_2^3 u\,dx - \int \partial_2^3 u\cdot\nabla u\cdot\partial_2^3 u\,dx\\
&\le C\|\partial_2 u\|_{L^\infty}\|\partial_2^2\nabla u\|_{L^2}\|\partial_2^3 u\|_{L^2} + C\|\partial_2^2 u\|_{L^4}\|\partial_2\nabla u\|_{L^4}\|\partial_2^3 u\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^\infty}\|\partial_2^3 u\|_{L^2}^2\\
&\le C\|u\|_{H^3}\|\partial_2 u\|_{H^2}^2 + C\|\partial_2^2 u\|_{L^2}^{1/2}\|\partial_2^2\nabla u\|_{L^2}^{1/2}\|\partial_2\nabla u\|_{L^2}^{1/2}\|\partial_2\nabla^2 u\|_{L^2}^{1/2}\|\partial_2^3 u\|_{L^2}\\
&\le C\|u\|_{H^3}\|\partial_2 u\|_{H^2}^2 \le C\|u\|_{H^3}\|\partial_2^\alpha u\|_{H^3}^2.\end{aligned}$$

Therefore,

$$K \le C\|u\|_{H^3}(\|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2). \tag{3.2}$$

To bound L, we decompose it into two parts,

$$\begin{aligned}L &= \sum_{i=1}^{2}\left(\int \partial_i^3(b\cdot\nabla b)\cdot\partial_i^3 u\,dx - \int b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,dx\right)\\
&= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k\int \partial_i^k b\cdot\partial_i^{3-k}\nabla b\cdot\partial_i^3 u\,dx\\
&= \sum_{k=1}^{3} C_3^k\int \partial_1^k b\cdot\partial_1^{3-k}\nabla b\cdot\partial_1^3 u\,dx + \sum_{k=1}^{3} C_3^k\int \partial_2^k b\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 u\,dx\\
&= L_1 + L_2.\end{aligned}$$

By Hölder's inequality, Young's inequality, Lemmas 3.3 and refE:Sobolev,

$$\begin{aligned}L_1 &= 3\int \partial_1 b\cdot\partial_1^2\nabla b\cdot\partial_1^3 u\,dx + 3\int \partial_1^2 b\cdot\partial_1\nabla b\cdot\partial_1^3 u\,dx + \int \partial_1^3 b\cdot\nabla b\cdot\partial_1^3 u\,dx\\
&\le C\|\partial_1 b\|_{L^\infty}\|\partial_1^2\nabla b\|_{L^2}\|\partial_1^3 u\|_{L^2} + C\|\partial_1^2 b\|_{L^4}\|\partial_1\nabla b\|_{L^4}\|\partial_1^3 u\|_{L^2}\\
&\quad + C\|\nabla b\|_{L^\infty}\|\partial_1^3 b\|_{L^2}\|\partial_1^3 u\|_{L^2}\\
&\le C\|\partial_1 b\|_{H^2}\|b\|_{H^3}\|\partial_1 u\|_{H^2} + C\|\partial_1^2 b\|_{L^2}^{1/2}\|\partial_1^2\nabla b\|_{L^2}^{1/2}\|\partial_1\nabla b\|_{L^2}^{1/2}\|\partial_1\nabla^2 b\|_{L^2}^{1/2}\|\partial_1^3 u\|_{L^2}\\
&\quad + C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_1 u\|_{H^2}\\
&\le C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_1 u\|_{H^2} \le C\|b\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned}$$

Similarly,

$$\begin{aligned}L_2 &= 3\int \partial_2 b\cdot\partial_2^2\nabla b\cdot\partial_2^3 u\,dx + 3\int \partial_2^2 b\cdot\partial_2\nabla b\cdot\partial_2^3 u\,dx + \int \partial_2^3 b\cdot\nabla b\cdot\partial_2^3 u\,dx\\
&\le C\|\partial_2 b\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 u\|_{L^2} + C\|\partial_2^2 b\|_{L^4}\|\partial_2\nabla b\|_{L^4}\|\partial_2^3 u\|_{L^2}\\
&\quad + C\|\nabla b\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2}\\
&\le C\|\partial_2 b\|_{H^2}\|b\|_{H^3}\|\partial_2 u\|_{H^2} + C\|\partial_2^2 b\|_{L^2}^{1/2}\|\partial_2^2\nabla b\|_{L^2}^{1/2}\|\partial_2\nabla b\|_{L^2}^{1/2}\|\partial_2\nabla^2 b\|_{L^2}^{1/2}\|\partial_2^3 u\|_{L^2}\\
&\quad + C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_2 u\|_{H^2}\\
&\le C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3} \le C\|b\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^2}^3).\end{aligned}$$

Hence

$$L \le C\|b\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2). \tag{3.3}$$

Then we consider M,

$$\begin{aligned} M &= -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\;dx,\\ &= -\int \partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\;dx - \int \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\;dx,\\ &= M_1 + M_2. \end{aligned}$$

We can rewrite

$$\begin{aligned} M_1 &= -\sum_{k=1}^{3} C_3^k \int \partial_1^k u\cdot\partial_1^{3-k}\nabla b\cdot\partial_1^3 b\;dx - \int u\cdot\partial_1^3\nabla b\cdot\partial_1^3 b\;dx\\ &= M_{1,1} + M_{1,2}. \end{aligned}$$

By Hölder's inequality, Lemmas 3.3 and 3.4,

$$\begin{aligned} M_{1,1} &= -3\int \partial_1 u\cdot\partial_1^2\nabla b\cdot\partial_1^3 b\;dx - 3\int \partial_1^2 u\cdot\partial_1\nabla b\cdot\partial_1^3 b\;dx - \int \partial_1^3 u\cdot\nabla b\cdot\partial_1^3 b\;dx\\ &\le C\|\partial_1 u\|_{L^\infty}\|\partial_1^2\nabla b\|_{L^2}\|\partial_1^3 b\|_{L^2} + C\|\partial_1^2 u\|_{L^4}\|\partial_1\nabla b\|_{L^4}\|\partial_1^3 b\|_{L^2}\\ &\quad + C\|\partial_1^3 u\|_{L^2}\|\partial_1^3 b\|_{L^2}\|\nabla b\|_{L^\infty}\\ &\le C\|\partial_1 b\|_{H^2}^2\|u\|_{H^3} + C\|\partial_1^2 u\|_{L^2}^{1/2}\|\partial_1^2\nabla u\|_{L^2}^{1/2}\|\partial_1\nabla b\|_{L^2}^{1/2}\|\partial_1\nabla^2 b\|_{L^2}^{1/2}\|\partial_1^3 b\|_{L^2}\\ &\quad + C\|\partial_1 b\|_{H^2}\|\nabla b\|_{H^2}\|u\|_{H^3}\\ &\le C\|\partial_1 b\|_{H^2}^2\|u\|_{H^3} + C\|\partial_1 b\|_{H^2}\|\nabla b\|_{H^2}\|u\|_{H^3} \le C\|\Lambda^\beta b\|_{H^3}^2\|u\|_{H^3}. \end{aligned}$$

By integration by parts and the divergence-free condition,

$$M_{1,2} = -\int u\cdot\partial_1^3\nabla b\cdot\partial_1^3 b\;dx = -\frac{1}{2}\int u\cdot\nabla(\partial_1^3 b)^2\;dx = 0.$$

To estimate M_2, we split it into four terms,

$$\begin{aligned} M_2 &= -\int \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\;dx,\\ &= -\sum_{k=1}^{3} C_3^k \int \partial_2^k u\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 b\;dx - \int u\cdot\partial_2^3\nabla b\cdot\partial_2^3 b\;dx\\ &= M_{2,1} + M_{2,2}. \end{aligned}$$

$M_{2,2} = 0$, due to the divergence free condition $\nabla\cdot u = 0$. By Hölder's inequality, Lemmas 3.3, 3.4 and Young's inequality,

$$\begin{aligned} M_{2,1} &= -3\int \partial_2 u\cdot\partial_2^2\nabla b\cdot\partial_2^3 b\;dx - 3\int \partial_2^2 u\cdot\partial_2\nabla b\cdot\partial_2^3 b\;dx - \int \partial_2^3 u\cdot\nabla b\cdot\partial_2^3 b\;dx\\ &\le C\|\partial_2 u\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 b\|_{L^2} + C\|\partial_2^2 u\|_{L^4}\|\partial_2\nabla b\|_{L^4}\|\partial_2^3 b\|_{L^2}\\ &\quad + C\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2}\|\nabla b\|_{L^\infty}\\ &\le C\|\partial_2 b\|_{H^2}^2\|u\|_{H^3} + C\|\partial_2^2 u\|_{L^2}^{1/2}\|\partial_2^2\nabla u\|_{L^2}^{1/2}\|\partial_2\nabla b\|_{L^2}^{1/2}\|\partial_2\nabla^2 b\|_{L^2}^{1/2}\|\partial_2^3 b\|_{L^2}\\ &\quad + C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3}\\ &\le C\|\partial_2 b\|_{H^2}^2\|u\|_{H^3} + +C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3}\\ &\le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2). \end{aligned}$$

Combining the estimates for M_1 and M_2, we obtain

$$M \le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2). \tag{3.4}$$

Now we estimate the term N,

$$
\begin{aligned}
N &= \sum_{i=1}^{2}\left(\int \partial_i^3(b\cdot\nabla u)\cdot\partial_i^3 b\,dx - \int b\cdot\nabla\partial_i^3 u\cdot\partial_i^3 b\,dx\right)\\
&= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k\int \partial_i^k b\cdot\partial_i^{3-k}\nabla u\cdot\partial_i^3 b\,dx\\
&= \sum_{k=1}^{3} C_3^k\int \partial_1^k b\cdot\partial_1^{3-k}\nabla u\cdot\partial_1^3 b\,dx + \sum_{k=1}^{3} C_3^k\int \partial_2^k b\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 b\,dx\\
&= N_1 + N_2.
\end{aligned}
$$

By Hölder's inequality, Lemmas 3.3 and 3.4,

$$
\begin{aligned}
N_1 &= 3\int \partial_1 b\cdot\partial_1^2\nabla u\cdot\partial_1^3 b\,dx + 3\int \partial_1^2 b\cdot\partial_1\nabla u\cdot\partial_1^3 b\,dx + \int \partial_1^3 b\cdot\nabla u\cdot\partial_1^3 b\,dx\\
&\le C\|\partial_1 b\|_{L^\infty}\|\partial_1^2\nabla u\|_{L^2}\|\partial_1^3 b\|_{L^2} + C\|\partial_1^2 b\|_{L^4}\|\partial_1\nabla u\|_{L^4}\|\partial_1^3 b\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^\infty}\|\partial_1^3 b\|_{L^2}^2\\
&\le C\|\partial_1 b\|_{H^2}^2\|u\|_{H^3} + C\|\partial_1^2 b\|_{L^2}^{1/2}\|\partial_1^2\nabla b\|_{L^2}^{1/2}\|\partial_1\nabla u\|_{L^2}^{1/2}\|\partial_1\nabla^2 u\|_{L^2}^{1/2}\|\partial_1^3 b\|_{L^2}\\
&\le C\|\partial_1 b\|_{H^2}^2\|u\|_{H^3} \le C\|\Lambda^\beta b\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}
$$

Similarly,

$$
N_2 \le C\|\partial_2 b\|_{H^2}^2\|u\|_{H^3} \le C\|\Lambda^\beta b\|_{H^3}^2\|u\|_{H^3}.
$$

Combining the estimates of N_1 and N_2, we have

$$
N \le C\|u\|_{H^3}\|\Lambda^\beta b\|_{H^3}^2. \tag{3.5}
$$

Combining the bounds in (3.2),(3.3), (3.4) and (3.5) above leads to

$$
J+K+L+M+N \le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
$$

Inserting the uppder bound for $J+K+L+M+N$ in (3.1) and integrating in time, we get

$$
\begin{aligned}
&\|(u,b)\|_{H^3}^2 + 2\nu\int_0^t \|\partial_2^\alpha u(\tau)\|_{H^3}^2 d\tau + 2\eta\int_0^t \|\Lambda^\beta b(\tau)\|_{H^3}^2 d\tau\\
&\quad\le \|(u_0,b_0)\|_{H^3}^2 + C\int_0^t \|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2)\,d\tau\\
&\quad\le \|(u_0,b_0)\|_{H^3}^2 + C\sup_{0\le\tau\le t}\|(u,b)\|_{H^3}\int_0^t (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2)\,d\tau\\
&\quad + C\sup_{0\le\tau\le t}\|(u,b)\|_{H^3}\int_0^t \|\partial_1 u\|_{H^2}^2\,d\tau\\
&\quad\le E(0) + CE_1^{\frac{1}{2}}(t)E_2(t) + CE_1^{\frac{3}{2}}(t) \le E(0) + CE_1^{\frac{3}{2}}(t) + CE_2^{\frac{3}{2}}(t).
\end{aligned}
$$

This completes the proof of Proposition 1.2. □

4. Proof of Proposition 1.3

This section proves Proposition 1.3. As explained in the introduction, the velocity equation does not involve dissipation in the horizontal direction. The time integral upper bound defined in $E_2(t)$ doesn't follow from the velocity equation. The proof makes use of the coupling and interaction of u and b, as can be seen from the approach of the proof.

Proof of Proposition 1.3. To bound $E_2(t)$, we make use of the relationship between u and b via the equation of the magnetic field

$$\partial_1 u = \partial_t b + u\cdot\nabla b + \eta(-\Delta)^\beta b - b\cdot\nabla u. \tag{4.1}$$

Multiplying (4.1) with $\partial_1 u$ in L^2 and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\|\partial_1 u\|_{L^2}^2 &= \int \partial_1 u\cdot\partial_t b\, dx + \int \partial_1 u\cdot u\cdot\nabla b\, dx\\ &\quad + \eta\int \partial_1 u\cdot(-\Delta)^\beta b\, dx - \int \partial_1 u\cdot b\cdot\nabla u\, dx\\ &:= O_1 + O_2 + O_3 + O_4.\end{aligned}$$

By replacing $\partial_t u$ by the velocity equation in (1.3), we obtain

$$\begin{aligned}O_1 &= \frac{d}{dt}\int \partial_1 u\cdot b\, dx - \int b\cdot\partial_1(\nu\partial_2^{2\alpha}u + b\cdot\nabla b + \partial_1 b - u\cdot\nabla u - \nabla P)\, dx\\ &= O_{1,1} + O_{1,2} + O_{1,3} + O_{1,4} + O_{1,5} + O_{1,6}.\end{aligned}$$

$O_{1,6} = 0$ due to the divergence-free condition $\nabla\cdot b = 0$. By Hölder's inequality, Young's inequality and integration by parts,

$$\begin{aligned}O_{1,2} &= -\nu\int b\cdot\partial_1\partial_2^{2\alpha}u\, dx = \nu\int \partial_1 b\cdot\partial_2^{2\alpha}u\, dx\\ &\le C\|\partial_1 b\|_{L^2}\|\partial_2^\alpha u\|_{H^1} \le C\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3} \le C(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2).\end{aligned}$$

Similarly, we can bound $O_{1,3}$ and $O_{1,4}$.

$$\begin{aligned}O_{1,3} &= -\int b\cdot\partial_1(b\cdot\nabla b)\, dx = \int \partial_1 b\cdot(b\cdot\nabla b)\, dx\\ &\le C\|b\|_{L^\infty}\|\partial_1 b\|_{L^2}\|\nabla b\|_{L^2} \le C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}^2,\end{aligned}$$

and

$$\begin{aligned}O_{1,4} &= -\int b\cdot\partial_1^2 b\, dx = \int \partial_1 b\cdot\partial_1 b\, dx\\ &\le C\|\partial_1 b\|_{L^2}^2 \le C\|\Lambda^\beta b\|_{H^3}^2.\end{aligned}$$

By integration by parts, Young's inequality and Lemma 3.2,

$$\begin{aligned}O_{1,5} &= \int b\cdot\partial_1(u\cdot\nabla u)\, dx = -\int \partial_1 b\cdot(u\cdot\nabla u)\, dx\\ &\le C\|\partial_1 b\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\\ &\le C\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac12}\|u\|_{H^3}\\ &\le C\|u\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned}$$

In a similar manner, by Lemma 3.2 and Young's inequality,

$$\begin{aligned}O_2 &= \int \partial_1 u\cdot u\cdot\nabla b\, dx\\ &\le C\|\partial_1 u\|_{L^2}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\\ &\le C\|\Lambda^\beta b\|_{H^3}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac32}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\\ &\le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).\end{aligned}$$

By Hölder's inequality and Young's inequality,

$$O_3 = \eta\int \partial_1 u\cdot(-\Delta)^\beta b\, dx \le C\|\partial_1 u\|_{L^2}\|\Lambda^\beta b\|_{H^1} \le \frac18\|\partial_1 u\|_{H^2}^2 + C\|\Lambda^\beta b\|_{H^3}^2.$$

By Lemma 3.2 and Young's inequality,

$$\begin{aligned}
O_4 &= -\int \partial_1 u \cdot b \cdot \nabla u \, dx \\
&\le C\|\partial_1 u\|_{L^2}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_2 b\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_1 u\|_{H^2}^{\frac{3}{2}}\|\Lambda^\beta b\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}\|u\|_{H^3}^{\frac{1}{2}} \\
&\le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

This completes the L^2-estimate of $\partial_1 u$.

We turn to the H^1-norm. Applying ∇ to (4.1) and multiplying it by $\nabla\partial_1 u$ in L^2, we have

$$\begin{aligned}
\|\nabla\partial_1 u\|_{L^2}^2 &= \int \nabla\partial_1 u \cdot \partial_t \nabla b \, dx + \int \nabla\partial_1 u \cdot \nabla(u\cdot\nabla b) \, dx \\
&\quad + \eta\int \nabla\partial_1 u \cdot \nabla(-\Delta)^\beta b \, dx - \int \nabla\partial_1 u \cdot \nabla(b\cdot\nabla u)\, dx \\
&:= P_1 + P_2 + P_3 + P_4.
\end{aligned}$$

To bound P_1, we use the integration by parts and the velocity equation in (1.3) to obtain

$$\begin{aligned}
P_1 &= \frac{d}{dt}\int \nabla\partial_1 u\cdot\nabla b\, dx - \int \nabla b\cdot\nabla\partial_1(\nu\partial_2^{2\alpha}u + b\cdot\nabla b + \partial_1 b - u\cdot\nabla u - \nabla P)\, dx \\
&= P_{1,1} + P_{1,2} + P_{1,3} + P_{1,4} + P_{1,5} + P_{1,6}.
\end{aligned}$$

$P_{1,6} = 0$ due to the divergence free condition $\nabla\cdot b = 0$. To bound $P_{1,2}$, we use integration by parts, Hölder's inequality and Young's inequality,

$$\begin{aligned}
P_{1,2} &= -\nu\int \nabla b\cdot\nabla\partial_1\partial_2^{2\alpha}u\, dx = \nu\int \partial_1\nabla b\cdot\partial_2^{2\alpha}\nabla u\, dx \\
&\le C\|\partial_1 b\|_{H^1}\|\partial_2^\alpha u\|_{H^2} \le C\|\Lambda^\beta b\|_{H^3}\|\partial_2^\alpha u\|_{H^3} \\
&\le C(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2).
\end{aligned}$$

Based on Lemma 3.2 and integration by parts, we have

$$\begin{aligned}
P_{1,3} &= -\int \nabla b\cdot\partial_1\nabla(b\cdot\nabla b)\, dx = \int \partial_1\nabla b\cdot\nabla(b\cdot\nabla b)\, dx \\
&= \int \partial_1\nabla b\cdot\nabla b\cdot\nabla b\, dx + \int \partial_1\nabla b\cdot b\cdot\nabla^2 b\, dx \\
&\le C\|\partial_1\nabla b\|_{L^2}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}} \\
&\le C\|b\|_{H^3}\|\Lambda^\beta b\|_{H^3}^2,
\end{aligned}$$

and by integration by parts and Hölder's inequality,

$$\begin{aligned}
P_{1,4} &= -\int \nabla b\cdot\nabla\partial_1^2 b\, dx = \int \partial_1\nabla b\cdot\partial_1\nabla b\, dx \\
&\le C\|\partial_1 b\|_{H^1}^2 \le C\|\Lambda^\beta b\|_{H^3}^2.
\end{aligned}$$

By integration by parts, Young's inequality and Lemma 3.2,

$$\begin{aligned}
P_{1,5} &= \int \nabla b \cdot \partial_1 \nabla (u \cdot \nabla u)\, dx = -\int \partial_1 \nabla b \cdot \nabla (u \cdot \nabla u)\, dx \\
&= -\int \partial_1 \nabla b \cdot \nabla u \cdot \nabla u\, dx - \int \partial_1 \nabla b \cdot u \cdot \nabla^2 u\, dx \\
&\le C\|\partial_1 \nabla b\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1 \nabla b\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\Lambda^\beta b\|_{H^3} \|\partial_2^\alpha u\|_{H^3}^{\frac{1}{2}} \|\partial_1 u\|_{H^2}^{\frac{1}{2}} \|u\|_{H^3} \\
&\le C\|u\|_{H^3} (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

For P_2, by Lemma 3.2 and Young's inequality,

$$\begin{aligned}
P_2 &= \int \nabla \partial_1 u \cdot \nabla (u \cdot \nabla b)\, dx \\
&= \int \nabla \partial_1 u \cdot \nabla u \cdot \nabla b\, dx + \int \nabla \partial_1 u \cdot u \cdot \nabla^2 b\, dx \\
&\le C\|\partial_1 \nabla u\|_{L^2} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1 \nabla u\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_1 u\|^{\frac{1}{2}} \|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla^2 b\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\Lambda^\beta b\|_{H^3}^{\frac{1}{2}} \|\partial_1 u\|_{H^2}^{\frac{3}{2}} \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \\
&\le C\|(u,b)\|_{H^3} (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

Similarly, P_3 and P_4 can be estimated by Hölder's inequality and Young's inequality,

$$P_3 = \eta \int \partial_1 \nabla u \cdot \nabla (-\Delta)^\beta b\, dx \le C\|\partial_1 \nabla u\|_{L^2} \|\Lambda^\beta b\|_{H^2} \le \frac{1}{8}\|\partial_1 u\|_{H^2}^2 + C\|\Lambda^\beta b\|_{H^3}^2.$$

By Lemma 3.2, integration by parts and Young's inequality,

$$\begin{aligned}
P_4 &= -\int \partial_1 \nabla u \cdot \nabla (b \cdot \nabla u)\, dx = \int \partial_1 \Delta u \cdot b \cdot \nabla u\, dx \\
&\le C\|\partial_1 \Delta u\|_{L^2} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_1 u\|_{H^2}^{\frac{3}{2}} \|\Lambda^\beta b\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|u\|_{H^3}^{\frac{1}{2}} \\
&\le C\|(u,b)\|_{H^3} (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

This completes the H^1-estimate of $\partial_1 u$.

Next we proceed to bound the H^2-norm. Applying Δ to (4.1) and multiplying it by $\Delta \partial_1 u$ in L^2, we have

$$\begin{aligned}
\|\Delta \partial_1 u\|_{L^2}^2 &= \int \Delta \partial_1 u \cdot \partial_t \Delta b\, dx + \int \Delta \partial_1 u \cdot \Delta (u \cdot \nabla b)\, dx \\
&\quad + \eta \int \Delta \partial_1 u \cdot \Delta (-\Delta)^\beta b\, dx - \int \Delta \partial_1 u \cdot \Delta (b \cdot \nabla u)\, dx \\
&:= Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

By integration by parts and the velocity equation in (1.3),

$$\begin{aligned}
Q_1 &= \frac{d}{dt} \int \Delta \partial_1 u \cdot \Delta b\, dx - \int \Delta b \cdot \Delta \partial_1 (\nu \partial_2^{2\alpha} u + b \cdot \nabla b + \partial_1 b - u \cdot \nabla u - \nabla P)\, dx \\
&= Q_{1,1} + Q_{1,2} + Q_{1,3} + Q_{1,4} + Q_{1,5} + Q_{1,6}.
\end{aligned}$$

$Q_{1,6} = 0$ because of the divergence-free condition $\nabla \cdot b = 0$. We use integration by parts, Hölder's inequality and Young's inequality,

$$\begin{aligned} Q_{1,2} &= -\nu \int \Delta b \cdot \Delta \partial_1 \partial_2^{2\alpha} u \, dx = \nu \int \partial_1 \Delta b \cdot \partial_2^{2\alpha} \Delta u \, dx \\ &\leq C\|\partial_1 b\|_{H^2} \|\partial_2^\alpha u\|_{H^3} \leq C\|\Lambda^\beta b\|_{H^3} \|\partial_2^\alpha u\|_{H^3} \\ &\leq C(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2). \end{aligned}$$

By integration by parts and Lemma 3.2,

$$\begin{aligned} Q_{1,3} &= -\int \Delta b \cdot \partial_1 \Delta (b \cdot \nabla b) \, dx = \int \partial_1 \Delta b \cdot \Delta (b \cdot \nabla b) \, dx \\ &= \int \partial_1 \Delta b \cdot \Delta b \cdot \nabla b \, dx + 2 \int \partial_1 \Delta b \cdot \nabla b \cdot \nabla^2 b \, dx + \int \partial_1 \Delta b \cdot b \cdot \nabla \Delta b \, dx \\ &= \int \partial_1 \Delta b \cdot \Delta b \cdot \nabla b \, dx + 2 \int \partial_1 \Delta b \cdot \nabla b \cdot \nabla^2 b \, dx \\ &\quad + \int \partial_1 \Delta b \cdot b_1 \partial_1 \Delta b \, dx + \int \partial_1 \Delta b \cdot b_2 \partial_2 \Delta b \, dx \\ &\leq C\|\partial_1 \Delta b\|_{L^2} \|\Delta b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \Delta b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_1 \Delta b\|_{L^2} \|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla^2 b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|b_1\|_{L^\infty} \|\partial_1 \Delta b\|_{L^2}^2 + C\|b_2\|_{L^\infty} \|\partial_1 \Delta b\|_{L^2} \|\partial_2 \Delta b\|_{L^2} \\ &\leq C\|b\|_{H^3} \|\Lambda^\beta b\|_{H^3}^2. \end{aligned}$$

By integration by parts and Hölder's inequality,

$$\begin{aligned} Q_{1,4} &= -\int \Delta b \cdot \Delta \partial_1^2 b \, dx = \int \partial_1 \Delta b \cdot \partial_1 \Delta b \, dx \\ &\leq C\|\partial_1 b\|_{H^2}^2 \leq C\|\Lambda^\beta b\|_{H^3}^2. \end{aligned}$$

We use Young's inequality, integration by parts, Lemma 3.2 and Lemma 3.3 to obtain,

$$\begin{aligned} Q_{1,5} &= \int \Delta b \cdot \partial_1 \Delta (u \cdot \nabla u) \, dx = -\int \partial_1 \Delta b \cdot \Delta (u \cdot \nabla u) \, dx \\ &= -\int \partial_1 \Delta b \cdot \Delta u \cdot \nabla u \, dx - 2 \int \partial_1 \Delta b \cdot \nabla u \cdot \nabla^2 u \, dx - \int \partial_1 \Delta b \cdot u \cdot \nabla \Delta u \, dx \\ &= -\int \partial_1 \Delta b \cdot \Delta u \cdot \nabla u \, dx - 2 \int \partial_1 \Delta b \cdot \nabla u \cdot \nabla^2 u \, dx \\ &\quad - \int \partial_1 \Delta b \cdot u_1 \partial_1 \Delta u \, dx - \int \partial_1 \Delta b \cdot u_2 \partial_2 \Delta u \, dx \\ &\leq C\|\partial_1 \Delta b\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta u\|^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_1 \Delta b\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|u_1\|_{L^\infty} \|\partial_1 \Delta b\|_{L^2} \|\partial_1 \Delta u\|_{L^2} + C\|u_2\|_{L^\infty} \|\partial_1 \Delta b\|_{L^2} \|\partial_2 \Delta u\|_{L^2} \\ &\leq C\|\Lambda^\beta b\|_{H^3} \|\partial_2^\alpha u\|_{H^3}^{\frac{1}{2}} \|\partial_1 u\|_{H^2}^{\frac{1}{2}} \|u\|_{H^3} + C\|\Lambda^\beta b\|_{H^3} \|\partial_1 u\|_{H^2} \|u_1\|_{H^3} \\ &\quad + C\|\Lambda^\beta b\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^2} \|u_2\|_{L^2}^{\frac{1}{4}} \|\partial_1 u_2\|_{H^1}^{\frac{1}{4}} \|\partial_2 u_2\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 u_2\|_{L^2}^{\frac{1}{4}} \\ &\leq C\|\Lambda^\beta b\|_{H^3} \|\partial_2^\alpha u\|_{H^3}^{\frac{1}{2}} \|\partial_1 u\|_{H^2}^{\frac{1}{2}} \|u\|_{H^3} + C\|\Lambda^\beta b\|_{H^3} \|\partial_1 u\|_{H^2} \|u\|_{H^3} \\ &\quad + C\|\Lambda^\beta b\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} \|\partial_2^\alpha u\|_{H^2}^{\frac{3}{2}} \|u\|_{H^3}^{\frac{1}{2}} \\ &\leq C\|(u,b)\|_{H^3} (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_2^\alpha u\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2). \end{aligned}$$

For Q_2, by Hölder's inequality, Lemmas 3.2, 3.3 and Young's inequality,

$$\begin{aligned}
Q_2 &= \int \Delta\partial_1 u \cdot \Delta(u\cdot\nabla b)\ dx \\
&= \int \Delta\partial_1 u\cdot\Delta u\cdot\nabla b\ dx + 2\int \Delta\partial_1 u\cdot\nabla u\cdot\nabla^2 b\ dx + \int \Delta\partial_1 u\cdot u\cdot\nabla\Delta b\ dx \\
&= \int \Delta\partial_1 u\cdot\Delta u\cdot\nabla b\ dx + 2\int \Delta\partial_1 u\cdot\nabla u\cdot\nabla^2 b\ dx \\
&\quad + \int \Delta\partial_1 u\cdot u_1\partial_1\Delta b\ dx + \int \Delta\partial_1 u\cdot u_2\partial_2\Delta b\ dx \\
&\le C\|\partial_1\Delta u\|_{L^2}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}^{\frac12}\|\partial_1\Delta u\|_{L^2}^{\frac12} \\
&\quad + C\|\partial_1\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\partial_2\nabla^2 b\|_{L^2}^{\frac12} \\
&\quad + C\|u_1\|_{L^\infty}\|\partial_1\Delta u\|_{L^2}\|\partial_1\Delta b\|_{L^2} + C\|u_2\|_{L^\infty}\|\partial_1\Delta u\|_{L^2}\|\partial_2\Delta b\|_{L^2} \\
&\le C\|\Lambda^\beta b\|_{H^3}^{\frac12}\|\partial_1 u\|_{H^2}^{\frac32}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12} + C\|u\|_{H^3}\|\partial_1 u\|_{H^2}\|\Lambda^\beta b\|_{H^3} \\
&\le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2).
\end{aligned}$$

Similarly, Q_3 can be estimated by Hölder's inequality and Young's inequality,

$$Q_3 = \eta\int \partial_1\Delta u\cdot\Delta(-\Delta)^\beta b\ dx \le C\|\partial_1\Delta u\|_{L^2}\|\Lambda^\beta b\|_{H^3} \le \frac18\|\partial_1 u\|_{H^2}^2 + C\|\Lambda^\beta b\|_{H^3}^2.$$

By Lemmas 3.2, 3.3, integration by parts and Young's inequality,

$$\begin{aligned}
Q_4 &= -\int \partial_1\Delta u\cdot\Delta(b\cdot\nabla u)\ dx \\
&= \int \partial_1\Delta u\cdot\Delta b\cdot\nabla u\ dx + 2\int \Delta\partial_1 u\cdot\nabla b\cdot\nabla^2 u\ dx + \int \Delta\partial_1 u\cdot b\cdot\nabla\Delta u\ dx \\
&= \int \partial_1\Delta u\cdot\Delta b\cdot\nabla u\ dx + 2\int \Delta\partial_1 u\cdot\nabla b\cdot\nabla^2 u\ dx + \int \Delta\partial_1 u\cdot b_1\partial_1\Delta u\ dx \\
&\quad + \int \Delta\partial_1 u\cdot b_2\partial_2\Delta u\ dx \\
&\le C\|\partial_1\Delta u\|_{L^2}\|\Delta b\|_{L^2}^{\frac12}\|\partial_2\Delta b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12} \\
&\quad + C\|\partial_1\Delta u\|_{L^2}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\partial_1\nabla^2 u\|_{L^2}^{\frac12} \\
&\quad + C\|b_1\|_{L^\infty}\|\partial_1\Delta u\|_{L^2}^2 + C\|b_2\|_{L^\infty}\|\partial_1\Delta u\|_{L^2}\|\partial_2\Delta u\|_{L^2} \\
&\le C\|\partial_1 u\|_{H^2}^{\frac32}\|\Lambda^\beta b\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\|u\|_{H^3}^{\frac12} + C\|b\|_{H^3}\|\partial_1 u\|_{H^2}^2 \\
&\quad + C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_2^\alpha u\|_{H^3} \\
&\le C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2).
\end{aligned}$$

Now we combine all the estimates for O_1 to O_4, P_1 to P_4 and Q_1 to Q_4 to get

$$\begin{aligned}
\frac12\|\partial_1 u\|_{H^2}^2 \le{}& \frac{d}{dt}\int \partial_1 u\cdot b\ dx + \frac{d}{dt}\int \nabla\partial_1 u\cdot\nabla b\ dx + \frac{d}{dt}\int \Delta\partial_1 u\cdot\Delta b\ dx \\
&+ C(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2) \\
&+ C\|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2).
\end{aligned} \tag{4.2}$$

We integrate (4.2) over $[0,t]$ to obtain

$$\begin{aligned}\int_0^t \|\partial_1 u\|_{H^2}^2\, d\tau &\le 2\int \partial_1 u\cdot b\, dx - 2\int \partial_1 u(x,0)\cdot b(x,0)\, dx + 2\int \nabla\partial_1 u\cdot\nabla b\, dx\\ &\quad - 2\int \nabla\partial_1 u(x,0)\cdot\nabla b(x,0)\, dx + 2\int \Delta\partial_1 u\cdot\Delta b\, dx\\ &\quad - 2\int \Delta\partial_1 u(x,0)\cdot\Delta b(x,0)\, dx + C\int_0^t (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2)\, d\tau\\ &\quad + C\int_0^t \|(u,b)\|_{H^3}(\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2)\, d\tau\\ &\le C\|(u,b)\|_{H^3}^2 + C\|(u_0,b_0)\|_{H^3}^2 + C\int_0^t (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2)\, d\tau\\ &\quad + \sup_{0\le\tau\le t}\|(u,b)\|_{H^3}\int_0^t (\|\Lambda^\beta b\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2^\alpha u\|_{H^3}^2)\, d\tau\\ &\le CE_1(0) + CE_1(t) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}.\end{aligned}$$

Thus it implies that

$$E_2(t) \le CE_1(0) + CE_1(t) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}.$$

This completes the proof of Proposition 1.3. □

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Declarations

Conflict of interest On behalf of all the authors, the corresponding author states that there is no conflict of interest.

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