Nonlinearity

PAPER

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To cite this article: Jiahong Wu and Kazuo Yamazaki 2024 *Nonlinearity* **37** 115014

IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **37** (2024) 115014 (48pp) https://doi.org/10.1088/1361-6544/ad7f18

Convex integration solution of two-dimensional hyperbolic Navier–Stokes equations*

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Received 23 July 2024
Accepted for publication 24 September 2024
Published 8 October 2024

Recommended by Dr Anna L Mazzucato

Abstract

Hyperbolic Navier–Stokes equations replace the heat operator within the Navier–Stokes equations with a damped wave operator. Due to this second-order temporal derivative term, there exist no known bounded quantities for its solution; consequently, various standard results for the Navier–Stokes equations such as the global existence of a weak solution, that is typically constructed via Galerkin approximation, are absent in the literature. In this manuscript, we employ the technique of convex integration on the two-dimensional hyperbolic Navier–Stokes equations to construct a weak solution with prescribed energy and thereby prove its non-uniqueness. The main difficulty is the second-order temporal derivative term, which is too singular to be estimated as a linear error. One of our novel ideas is to use the time integral of the temporal corrector perturbation of the Navier–Stokes equations as the temporal corrector perturbation for the hyperbolic Navier–Stokes equations.

Keywords: convex integration, energy, fractional Laplacian, hyperbolicity, Navier–Stokes equations

Mathematics Subject Classification numbers: 35A02, 35L70, 35Q30

* This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University where the authors had fruitful discussions. The first author is partially supported by the National Science Foundation (DMS 2104682 and DMS 2309748). The second author is supported by the Simons Foundation (962572, KY).
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1. Introduction and review of previous works

1.1. Motivation from physics and real-world applications

The Navier–Stokes equations is a prominent system of partial differential equations (PDEs) in hydrodynamics that has various applications in real world such as fluid mechanics, aerodynamics, medicine, and even finance. More than half a century ago, Cattaneo [12, 13] and Vernotte [47] proposed replacing the heat operator with a damped wave operator to make the propagation speed of heat transfer finite. This idea was subsequently extended by others such as Carrassi and Morro [11]. More recently, Couland, Hachicha, and Raugel [19] derived a general version of the hyperbolic Navier–Stokes equations by replacing the Fourier law with the law proposed by Cattaneo. The hyperbolic Navier–Stokes equations (1) of our main interest in this manuscript is precisely [4, equation (1.6)] by Brenier, Natalini, and Puel, which can be considered as an approximation of the general version in [19].

One of the most fundamental issues concerning the standard Navier–Stokes equations is the uniqueness of its global-in-time weak solution that has been known to exist since the pioneering works of Leray and Hopf [31, 36]. Although the non-uniqueness of Leray–Hopf weak solution to the three-dimensional (3D) Navier–Stokes equations remains open, recent breakthrough technique of convex integration has advanced our understanding of this challenging problem. The hyperbolic Navier–Stokes equations differ from the standard Navier–Stokes equations by a second-order temporal derivative term and even the global L^2_x-bound remains unknown. The classical approach such as the Galerkin approximation on a torus to construct a global-in-time weak solution strongly relies on such bounded quantities. Hence, setting aside the question of uniqueness, to the best of our knowledge, there is currently no known construction of a weak solution to the hyperbolic Navier–Stokes equations from non-trivial initial data on an arbitrarily large time interval.

Considering such a unique difficulty of the absence of any bounded quantities, we turn to the non-traditional approach, specifically the recent breakthrough technique of convex integration, and for *a priori* fixed arbitrary $T>0$ and prescribed energy, construct a weak solution with the prescribed energy on $[0,T]$ and consequently prove its non-uniqueness. Extending the current convex integration technique to the hyperbolic Navier–Stokes equations with the second-order temporal derivative term requires substantial modifications. With several new novel ideas and optimization over multiple parameters, we achieve this goal. To the best of our knowledge, this is the

- first construction of a non-trivial solution to the hyperbolic Navier–Stokes equations on $[0,T]$ for an arbitrarily large $T>0$ *a priori* fixed,
- first ill-posedness result for the hyperbolic Navier–Stokes equations,
- and first attempt of convex integration on a physically meaningful hyperbolic system of PDEs with a second-order temporal derivative term.

1.2. Review of previous results

We define $\mathbb{N} \triangleq \{1,2,\ldots,\}$, $\mathbb{N}_0 \triangleq \{0\}\cup\mathbb{N}$, $\mathbb{T}^d \triangleq [-\pi,\pi]^d$, and a fractional Laplacian $(-\Delta)^m$ for $m\in\mathbb{R}$ to satisfy $(-\Delta)^m f(x) \triangleq \sum_{k\in\mathbb{Z}^d}|k|^{2m}\hat{f}(k)e^{ik\cdot x}$ for $d\in\mathbb{N}$. Let us denote by $v:\mathbb{R}_+\times\mathbb{T}^d\mapsto\mathbb{R}^d$ the velocity field, $\pi:\mathbb{R}_+\times\mathbb{T}^d\mapsto\mathbb{R}$ the pressure field, and $\eta\geqslant 0$ the kinematic viscosity, so that we may write down the hyperbolic Navier–Stokes equations generalized via a fractional diffusion $\eta(-\Delta)^m v$ as

$$\gamma\partial_{tt}v+\partial_t v+\eta(-\Delta)^m v+(v\cdot\nabla)v+\nabla\pi=0, \tag{1a}$$
$$\nabla\cdot v=0, \tag{1b}$$

where $\gamma\geqslant 0$. The case $\gamma=0$ recovers the generalized Navier–Stokes equations, additionally considering $m=1$ recovers the Navier–Stokes equations, and also taking $\eta=0$ leads to the Euler equations. For simplicity, we assume $\eta=1$ hereafter whenever $\eta>0$. For the Navier–Stokes equations in case $\gamma=0$, taking $L^2(\mathbb{T}^d)$-inner products with v under the assumption of sufficient regularity of the solution leads to the energy identity of

$$\|v(t)\|_{L_x^2}^2+2\int_0^t\|(-\Delta)^{\frac{m}{2}}v\|_{L_x^2}^2\mathrm{d}s=\|v(0)\|_{L_x^2}^2. \tag{2}$$

Based on this fundamental property, there is a rich theory of the Navier–Stokes equations over 90 years of investigations starting from the pioneering work of Leray [36].

In sharp contrast, the energy identity (2) fails in case $\gamma>0$ due to the extra term

$$\gamma\int_0^t\int_{\mathbb{T}^d}\partial_{ss}v\cdot v\mathrm{d}x\mathrm{d}s.$$

Additionally, taking $L^2(\mathbb{T}^d)$-inner products with $\partial_t v$ and summing the resulting equations create a different problem this time due to

$$\int_0^t\int_{\mathbb{T}^d}(v\cdot\nabla)v\cdot\partial_s v\mathrm{d}x\mathrm{d}s.$$

Consequently, there are no known bounded quantities for the solution to the hyperbolic Navier–Stokes equations (1). Very recently, Ji, Li, Tian, and Wu [34] proved that in the spatial domain $\mathbb{R}^d$ for $d\geqslant 2$, the hyperbolic Navier–Stokes equations (1) possess a unique global-in-time mild solution under constraints on γ and initial data, and the solution converges to the solution of the Navier–Stokes equations as $\gamma\searrow 0$ in certain Sobolev norms. Nevertheless, rigorous results for the hyperbolic Navier–Stokes equations in the current literature, in general, are extremely limited due to the lack of bounded quantities disabling one from following the known classical approaches on the Navier–Stokes equations. In turn, this presents a unique opportunity in which a non-classical approach that does not rely on the energy inequality (2) of the Navier–Stokes equations may be applied to the hyperbolic Navier–Stokes equations instead to shed light from a different angle to improve our understanding, and a good candidate for such a technique is the convex integration, that we briefly review next.

Convex integration has its roots in geometry, specifically the famous C^1-isometric embedding theorem of Nash [43]. It has seen rapid developments in the past two decades fueled by the goal of proving Onsager's conjecture [44], the positive direction being that every weak solution $v\in C^\alpha(\mathbb{T}^3)$ to the 3D Euler equations for $\alpha>\frac{1}{3}$ conserves its energy and the negative direction being the existence of a weak solution $v\in C^\alpha(\mathbb{T}^3)$ for $\alpha<\frac{1}{3}$ that fails to conserve its energy. While Constantin *et al* [17], and Eyink [25] in 1994 proved its positive direction, De Lellis and Székelyhidi [21] in 2009, by partially using ideas from [42] by Müller and Šverák, proved the existence of a solution $v\in L_{t,x}^\infty$ to the dD Euler equations for $d\in\mathbb{N}\setminus\{1\}$ with compact support in space and time, extending the previous works of Scheffer [45] and Shnirelman [46] that proved analogous results with regularity in $L_{t,x}^2$ in the 2D case. After further extensions (e.g. [6, 22, 23]), Isett [32] proved the negative direction of Onsager's conjecture in any dimension $d\geqslant 3$. The case $d=2$ was excluded in [32] due to the absence of Mikado flows in the 2D case; nevertheless, Giri and Radu [26] recently settled the 2D case as well via a new approach of Newton-Nash iteration.

Via an introduction of intermittent Beltrami waves, Buckmaster and Vicol [8] proved the non-uniqueness of weak solutions to the 3D Navier–Stokes equations, and it was followed by many more: [5, 14, 38, 40] on the Navier–Stokes equations; [10] on power-law model; [39] on Boussinesq system; [3] on magnetohydrodynamics system; [20, 41] on transport equation; [7, 33] on active scalars. Concerning the non-uniqueness of Leray-Hopf weak solutions to the 3D Navier–Stokes equations which requires higher regularity than weak solutions, Colombo *et al* [16] extended the approach of [6] and proved that Leray–Hopf weak solutions to the generalized Navier–Stokes equations, which is (1) with $\gamma = 0$, are non-unique for $m < \frac{1}{5}$. Subsequently, De Rosa [24] utilized some ideas from [32] and extended [16] up to $m < \frac{1}{3}$. Finally, more recently, Albritton *et al* [1] proved the non-uniqueness of Leray–Hopf weak solutions to the 3D Navier–Stokes equations under some non-zero force.

Despite the seemingly wide applicability, there exist plenty of PDEs to which we do not know how to employ the convex integration technique. In particular, during a workshop 'Criticality and Stochasticity in Quasilinear Fluid Systems' at the American Institute of Mathematics in 2021, one participant suggested an open question of whether one can apply the convex integration technique to dispersive or hyperbolic PDEs such as the wave equation. Some workshop participants attempted but came out empty-handed in terms of concrete results. Nevertheless, in this work we succeed in employing the convex integration technique to the hyperbolic Navier–Stokes equations (1) and thereby construct a solution with prescribed energy that is non-unique on $[0,T]$ for *a priori* fixed arbitrarily large $T > 0$; we present our result formally next in section 2.

2. Statement of our main result

Let us present our main result in theorem 2.1; its style of presentation has some similarities to [10, theorem B]. We define $a \wedge b \triangleq \min\{a,b\}$.

Theorem 2.1. *Fix $m \in (0, \frac{2}{3})$, an arbitrary $T > 0$, as well as any*

$$e \in C^1(\mathbb{R}; [\underline{e}, \infty)) \text{ such that } \|e\|_{C([-2,T])} \leqslant \bar{e} \text{ and } \|e'\|_{C([-2,T])} \leqslant \tilde{e}, \tag{3a}$$

$$\text{where } 4 \leqslant \underline{e} \leqslant \bar{e} < \infty, \tilde{e} \in [0, \infty). \tag{3b}$$

Then there exists a constant $\beta = \beta(m) \in (0,1)$ sufficiently small such that the following holds. There exists a mean-zero function

$$v \in C\left([0,T]; H^{\beta}\left(\mathbb{T}^2\right)\right) \cap C^{\beta}\left([0,T]; L^2\left(\mathbb{T}^2\right)\right) \tag{4}$$

such that

(1) v solves the hyperbolic Navier–Stokes equations (1) distributionally; i.e.

$$\int_0^T \int_{\mathbb{T}^2} v(s,x) \cdot \left(\gamma \partial_{ss}\phi - \partial_s \phi + (-\Delta)^m \phi - (v \cdot \nabla)\phi\right)(s,x)\,dxds = 0,$$
$$\int_{\mathbb{T}^2} v(t) \cdot \nabla \psi = 0 \;\forall\, t \in [0,T],$$

for any divergence-free $\phi \in C^2([0,T] \times \mathbb{T}^2)$ such that both $\phi, \partial_t \phi$ vanish at $t = 0$ and $t = T$, and any $\psi \in C^1(\mathbb{T}^2)$;

(2)

$$\|v(t)\|_{L_x^2}^2 = e(t) \quad \forall\, t \in [0,T]. \tag{5}$$

Additionally, if two such energies e_1 and e_2 obeying the same bounds $\underline{e}, \bar{e}$, and $\tilde{e}$ in (3) coincide on $[0,t]$, then there exist corresponding solutions v_1 and v_2 that also coincide on $[0, t \wedge T]$, implying non-uniqueness of distributional solutions for the hyperbolic Navier–Stokes equations (1).

The lower bound of 4 in (3) is arbitrary and any strictly positive real number suffices in the proof of theorem 2.1.

Remark 2.1. Initially, we attempted to prove the 3D analogue of theorem 2.1 but faced various difficulties unable to close several necessary estimates. Using the generalized intermittent jets in higher spatial dimensions from [37], we investigated to see if the difficulties in the 3D case could be overcome in higher dimensions but we saw that the obstacles still remained. In fact, the difficulties we faced in the 3D case interestingly became worse as the spatial dimension increased. This is very counterintuitive to the theory of convex integration in which in general, the lower dimension poses more difficulties; e.g. recall that Isett [32] proved Onsager's conjecture for all $d \geqslant 3$ but not in case $d = 2$. In any event, this is how we realized that the only path forward for us with our current approach is actually the lower dimensional case, namely when $d = 2$, which finally led to theorem 2.1 after various optimizations over all parameters. We make further comments on this issue in remark 4.5.

Remark 2.2. Our convex integration scheme will specifically utilize the 2D intermittent stationary flows originally introduced by Choffrut *et al* [15] for the 2D Euler equations, subsequently extended by Buckmaster *et al* [7] to the 2D surface quasi-geostrophic equations, by Luo and Qu [38] to the 2D Navier–Stokes equations, and by Yamazaki [49, 51] to the stochastic case implementing the smooth cut-off function "χ introduced in [40, p. 7] (see (198)).

The main result from [38, corollary 1.2] is the construction of a non-trivial weak solution to the 2D Navier–Stokes equations diffused by $(-\Delta)^m$ for all $m \in [0,1)$ that has compact temporal support which implies non-uniqueness because the zero function is a solution to the Navier–Stokes equations starting from zero initial data. A zero function also solves the hyperbolic Navier–Stokes equations (1). Unfortunately, the approach of [38] directly conflicts with one of our new novel ideas to handle the hyperbolic term (see remark 4.1). Consequently, we were not able to extend [38] to the hyperbolic Navier–Stokes equations (1). Because further explanation requires more notations, let us elaborate on this difficulty in remark 4.4.

In [9, theorem 7.1], Buckmaster and Vicol constructed a weak solution $v(t,x)$ to the 3D Navier–Stokes equations such that its kinetic energy at least doubles from time $t = 0$ to $t = 1$: $\|v(1)\|_{L_x^2}^2 > 2\|v(0)\|_{L_x^2}^2$. This implies non-uniqueness because one can take such a solution $v(t,x)$ constructed via convex integration, consider the solution $v(0,x)$ at $t = 0$ as initial data, and employ Galerkin approximation to it to construct another solution $v^{\mathrm{G}}(t,x)$ such that $v^{\mathrm{G}}(0,x) = v(0,x)$ and $\|v^{\mathrm{G}}(1)\|_{L_x^2}^2 \leqslant \|v(0)\|_{L_x^2}^2$. Our first successful result was actually an extension of [9, section 7] to the 2D hyperbolic Navier–Stokes equations. However, in contrast to the Navier–Stokes equations, this result does not allow us to conclude non-uniqueness because even if we take the solution constructed via convex integration at time $t = 0$, we cannot construct another classical solution via Galerkin approximation. Although theorem 2.1 with prescribed energy is stronger in various ways, we leave a sketch of the proof of extension of [9, theorem 7.1] to (1) in appendix D due to its independent mathematical interest.

Finally, Burczak *et al* in [10, theorems A and B] introduced a very nice approach to construct solutions to the power-law model with prescribed energy, which particularly proved to be amenable to the stochastic case (e.g. [29, 48]). We adapt the convex integration scheme of [15, 38] to such a prescribed energy approach from [10] to prove theorem 2.1.

Remark 2.3. We briefly point out an interesting development in the research area of the convex integration technique applied on PDEs forced by random noise of relevance to our manuscript. There are various PDEs forced by random noise that is very rough such as the space-time white noise, and they have been studied in the physics literature for many decades. The lack of smoothness of such a force transmits to the roughness of its solution and the product within the nonlinear term becomes ill-defined according to Bony's estimates that informally states that a product fg is well-defined if and only if $f \in \mathcal{C}_x^{\alpha_1}, g \in \mathcal{C}_x^{\alpha_2}$ for $\sum_{j=1}^{2} \alpha_j > 0$. Such PDEs are called singular stochastic PDEs (SPDEs), and its research direction has experienced significant advances due to the recent breakthrough inventions of the theory of regularity structures by Hairer [28] and the theory of paracontrolled distributions by Gubinelli *et al* [27]. For example, Zhu and Zhu [52] constructed a local-in-time solution to the 3D Navier–Stokes equations forced by space-time white noise using these theories. Yet, even these powerful techniques have limitations: the constructed solutions are local-in-time, and the techniques, in general, apply only to locally subcritical singular SPDEs, which informally require their nonlinear terms to be smoother than the noise (see [28, assumption 8.3] for a precise definition of local subcriticality). Remarkably, Hofmanov *et al* [30] were able to employ the convex integration technique to the 2D surface quasi-geostrophic equations in the locally critical and even supercritical cases; this was the first construction of any solution to any singular SPDE in the locally critical and supercritical cases; not only that, the solutions were global-in-time and non-unique.

In contrast, the hyperbolic Navier–Stokes equations (1) is a physically meaningful system of PDEs with no known bounded quantities, barring any success in applications of the classical Galerkin approximation to construct a global-in-time weak solution. Yet, we were able to construct a non-trivial weak solutions on $[0,T]$ for an arbitrarily large $T > 0$ *a priori* fixed and prove non-uniqueness. The results of [30] and our theorem 2.1 suggest that the technique of convex integration has proven to be not only a breakthrough technique to demonstrate non-uniqueness of weak solutions to various PDEs in hydrodynamics but a new technique to construc solutions for PDEst, although non-unique, when no other means are available.

Remark 2.4. With remark 2.3 in mind, we wish to recall the 2D Kuramoto–Sivashinsky equation that has applications in diverse areas such as the instabilities in laminar flame fronts. It can be informally written as

$$\partial_t u + (u \cdot \nabla) u = -\Delta u - \Delta^2 u \tag{6}$$

solved by $u : \mathbb{R}_+ \times \mathbb{T}^2 \mapsto \mathbb{R}^2$; we refer to [18, 35] and references therein for details. Due to the lack of divergence-free property in sharp contrast to the Navier–Stokes equations, the solution to the 2D Kuramoto–Sivashinsky equation (6) shares the same property as the hyperbolic Navier–Stokes equations (1), namely the absence of any known bounded quantities. Consequently, to the best of our knowledge, global-in-time existence of a solution to the 2D Kuramoto–Sivashinsky equation starting from an arbitrary initial data remains unknown (see

[2, 18]). It would be interesting if some ideas from this manuscript can contribute to this research direction in future.

We summarize some of the novelties and significances of theorem 2.1 and its proof.

- Theorem 2.1 allows us, for the first time, to fix an arbitrary $T > 0$, and prove the existence result, ill-posed type, of a non-trivial weak solution to the hyperbolic Navier–Stokes equations on $[0, T]$.
- To the best of our knowledge, theorem 2.1 presents the first convex integration scheme on a hyperbolic equation with a second-order temporal derivative term ∂_{tt}. In fact, to the best of our knowledge, it is the first time that a convex integration scheme with prescribed energy is applied on a PDE for which we do not even know if it has any bounded quantities at all.
- Within the proof of theorem 2.1, we took a time integral of the temporal corrector of the Navier–Stokes equations to handle the second-order temporal derivative term in (1). (See remark 4.1 for details.)
- There are many parameters such as $r, \mu, \sigma^{-1}, l, b$ and p^*, for all of which we had to discover non-empty intervals and optimize to obtain theorem 2.1 for all $m \in (0, \frac{2}{3})$. (See (31) and the discussion thereafter.) We go through details of such derivations of the parameters in appendices A and B to better explain their optimality.

We also comment that considering that we were able to adapt [10, theorems A and B] and prescribe energy in theorem 2.1, it is very likely that we can adapt the proof of [10, theorem C] and construct a solution to the 2D hyperbolic Navier–Stokes equations (1) with prescribed initial data to give a second proof of non-uniqueness. We choose to leave this to future works.

In what follows, we describe preliminaries and past results in section 3, prove theorem 2.1 in section 4, and leave additional computations in appendix C for completeness. Appendix D consists of a sketch of proof of the extension of [9, theorem 7.1] to the hyperbolic Navier–Stokes equations (1). Hereafter, we consider (1) with $\gamma = 1$ for simplicity; the case $\gamma \in \mathbb{R}_+ \setminus \{1\}$ can be attained with straightforward modifications of the following proof.

3. Preliminaries

We write $A \lesssim_{a,b} B$ to imply the existence of a constant $C = C(a,b) \geqslant 0$ such that $A \leqslant CB$; additionally, we write $A \approx_{a,b} B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \overset{(\cdot)}{\lesssim} B$ to indicate that this inequality is due to an equation $(\cdot)$. Vector components will be indicated by super-indices, and we define $x^{\perp} \triangleq (-x^2, x^1)$. We denote a tensor product by $\otimes$ while the trace-free tensor product by

$$f \mathring{\otimes} g \triangleq \begin{pmatrix} f^1 g^1 - \frac{1}{2} f \cdot g & f^1 g^2 \\ f^2 g^1 & f^2 g^2 - \frac{1}{2} f \cdot g \end{pmatrix}$$

for any $\mathbb{R}^2$-valued maps f and g. We write for $N \in \mathbb{N}_0$ and $p \in [1, \infty]$,

$$\|f\|_{C^N_{t,x}} \triangleq \sum_{0 \leqslant n + |\alpha| \leqslant N} \|\partial_t^n D^\alpha f\|_{C_{t,x}}, \quad \|f\|_{C_t L^p_x} \triangleq \sup_{s \in [0,t]} \|f(s)\|_{L^p_x}. \tag{7}$$

We also define $L^2_\sigma \triangleq \{f \in L^2_x : \nabla \cdot f = 0\}$, reserve $\mathbb{P} \triangleq \mathrm{Id} - \nabla \Delta^{-1} \mathrm{div}$ as the Leray projection operator, and $\mathbb{P}_{\leqslant r}$ to be a Fourier operator with a Fourier symbol of $1_{|\xi| \leqslant r}(\xi)$ for any $r \in [0, \infty)$.

Lemma 3.1 (Geometric lemma from [7, lemma 4.2]). *Let $B_\epsilon(Id)$ denote the ball of symmetric 2×2 matrices, centered at Id of radius $\epsilon>0$. Then there exists $\epsilon_\gamma>0$ with which there exist disjoint finite subsets $\Lambda^+,\Lambda^-\subset\mathbb{S}^1\cap\mathbb{Q}^2$ and smooth positive functions*

$$\gamma_\zeta\in C^\infty\left(B_{\epsilon_\gamma}(Id)\right),\quad \zeta\in\Lambda^\pm,$$

such that

(1) $5\Lambda^\pm\subset\mathbb{Z}^2$,
(2) if $\zeta\in\Lambda^\pm$, *then* $-\zeta\in\Lambda^\pm$ *and* $\gamma_\zeta=\gamma_{-\zeta}$,
(3)

$$R=\frac{1}{2}\sum_{\zeta\in\Lambda^\pm}\left(\gamma_\zeta(R)\right)^2\left(\zeta^\perp\otimes\zeta^\perp\right)\quad\forall\, R\in B_{\epsilon_\gamma}(Id),\tag{8}$$

(4) $|\zeta+\zeta'|\geqslant\frac{1}{2}$ *for all* $\zeta,\zeta'\in\Lambda^\pm$ *such that* $\zeta+\zeta'\neq 0$.

We define $\Lambda\triangleq\Lambda^+\cup\Lambda^-$. For convenience, we fix the following universal constants

$$C_\Lambda\triangleq 2\left[\epsilon_\gamma^{-1}\left(\pi^2+\frac{\epsilon_\gamma}{48}\right)+\frac{5}{8}\right]^{\frac{1}{2}}|\Lambda|\text{ and }M\triangleq C_\Lambda\sup_{\zeta\in\Lambda}\|\gamma_\zeta\|_{C\left(B_{\epsilon_\gamma}(\mathrm{Id})\right)};\tag{9}$$

the reason for this definition of C_Λ is due to (55). Next, we describe some notations and results concerning the 2D intermittent stationary flows introduced in [15] (e.g. [15, lemma 4]) and extended in [38]. For all $\zeta\in\Lambda$ and any frequency parameter $\lambda\in 5\mathbb{N}$, we define b_ζ and its potential ψ_ζ as

$$b_\zeta(x)\triangleq b_{\zeta,\lambda}(x)\triangleq i\zeta^\perp e^{i\lambda\zeta\cdot x},\quad \psi_\zeta(x)\triangleq\psi_{\zeta,\lambda}(x)\triangleq\frac{1}{\lambda}e^{i\lambda\zeta\cdot x}\tag{10}$$

(see [15, equation (14)]). It follows that for all $N\in\mathbb{N}_0$,

$$b_\zeta(x)=\nabla^\perp\psi_\zeta(x),\quad \nabla\cdot b_\zeta(x)=0,\quad \nabla^\perp\cdot b_\zeta(x)=\Delta\psi_\zeta(x)=-\lambda^2\psi_\zeta(x),\tag{11a}$$

$$\overline{b_\zeta}(x)=b_{-\zeta}(x),\ \ \overline{\psi_\zeta}(x)=\psi_{-\zeta}(x),\ \ \|b_\zeta\|_{C_x^N}\overset{(7)}{\leqslant}(N+1)\lambda^N,\ \ \|\psi_\zeta\|_{C_x^N}\overset{(7)}{\leqslant}(N+1)\lambda^{N-1}.\tag{11b}$$

Similarly to [8, equations (3.5*b*), (3.5*c*), and (3.6) on p 111], we consider a 2D Dirichlet kernel for $r\in\mathbb{N}$

$$D_r(x)\triangleq\frac{1}{2r+1}\sum_{k\in\Omega_r}e^{ik\cdot x}\text{ with }\Omega_r\triangleq\left\{k=\begin{pmatrix}k^1 & k^2\end{pmatrix}^T: k^i\in\mathbb{Z}\cap[-r,r]\text{ for } i=1,2\right\},$$

where T denotes a transpose, that satisfies $\|D_r\|_{L_x^p}\lesssim r^{1-\frac{2}{p}}$ for all $p\in(1,\infty]$ and $\|D_r\|_{L_x^2}=2\pi$. The role of r is to parametrize the number of frequencies along edges of the cube Ω_r. We introduce σ such that $\lambda\sigma\in 5\mathbb{N}$ to parametrize the spacing between frequencies, or equivalently such that the resulting rescaled kernel is $(\mathbb{T}/\lambda\sigma)^2$-periodic. In particular, this will be needed in application of lemma 3.4. Lastly, we introduce μ that measures the amount of temporal oscillation in the building blocks. In sum, the parameters we introduced are required to satisfy

$$1\ll r\ll\mu\ll\sigma^{-1}\ll\lambda,\quad r\in\mathbb{N},\quad\text{and}\quad \lambda,\lambda\sigma\in 5\mathbb{N}.\tag{12}$$

Now we define the directed-rescaled Dirichlet kernel by

$$\eta_\zeta(t,x) \triangleq \eta_{\zeta,\lambda,\sigma,r,\mu}(t,x) \triangleq \begin{cases} D_r\left(\lambda\sigma(\zeta\cdot x+\mu t),\lambda\sigma\zeta^\perp\cdot x\right) & \text{if } \zeta\in\Lambda^+, \\ \eta_{-\zeta,\lambda,\sigma,r,\mu}(t,x) & \text{if } \zeta\in\Lambda^-, \end{cases} \tag{13}$$

so that

$$\frac{1}{\mu}\partial_t\eta_\zeta(t,x) = \pm(\zeta\cdot\nabla)\eta_\zeta(t,x)\ \forall\,\zeta\in\Lambda^\pm, \tag{14a}$$

$$\fint_{\mathbb{T}^2}\eta_\zeta^2(t,x)\mathrm{d}x = 1, \quad \text{and} \quad \|\eta_\zeta\|_{L_t^\infty L_x^p} \lesssim r^{1-\frac{2}{p}}\ \forall\,p\in(1,\infty] \tag{14b}$$

(see [8, equations (3.8)–(3.10)]). Finally, we define the intermittent 2D stationary flow as

$$W_\zeta(t,x) \triangleq W_{\zeta,\lambda,\sigma,r,\mu}(t,x) \triangleq \eta_{\zeta,\lambda,\sigma,r,\mu}(t,x)\,b_{\zeta,\lambda}(x) \tag{15}$$

(see [8, equation (3.11)]). We note that Luo and Qu [38, equation (4.15)] called W_ζ 'intermittent 2D stationary flow' because they adapted the 2D stationary flow introduced in [15] to an intermittent form. Similarly to the 3D case in [8] it follows that for all $\zeta,\vartheta\in\Lambda$ (see [38, Equations (4.16)-(4.19)] and also [8, equations (3.13) and (3.14)])

$$\mathbb{P}_{\leqslant 2\lambda}\mathbb{P}_{\geqslant\frac{\lambda}{2}}W_\zeta = W_\zeta, \tag{16a}$$

$$\mathbb{P}_{\leqslant 4\lambda}\mathbb{P}_{\geqslant\frac{\lambda}{5}}\left(W_\zeta\mathring{\otimes}W_\vartheta\right) = W_\zeta\mathring{\otimes}W_\vartheta \quad \text{if } \zeta+\vartheta\neq 0, \tag{16b}$$

$$\mathbb{P}_{\geqslant\frac{\lambda\sigma}{2}}\left(W_\zeta\mathring{\otimes}W_\vartheta\right) = \mathbb{P}_{\neq 0}\left(W_\zeta\mathring{\otimes}W_\vartheta\right), \tag{16c}$$

$$\mathbb{P}_{\neq 0}\eta_\zeta = \mathbb{P}_{\geqslant\frac{\lambda\sigma}{2}}\eta_\zeta. \tag{16d}$$

Lemma 3.2 ([38, lemmas 4.2 and 4.3]; see [8, proposition 3.5]). *Define η_ζ and W_ζ respectively by (13) and (15), and assume (12). Then*

(1) For any $\{a_\zeta\}_{\zeta\in\Lambda}\subset\mathbb{C}$ such that $a_{-\zeta}=\bar{a}_\zeta$, a function $\sum_{\zeta\in\Lambda}a_\zeta W_\zeta$ is $\mathbb{R}$-valued.
(2) for any $p\in(1,\infty]$, k, $N\in\{0,1,2,3\}$,

$$\|\nabla^N\partial_t^k W_\zeta\|_{L_t^\infty L_x^p} \lesssim_{N,k,p} \lambda^N(\lambda\sigma r\mu)^k r^{1-\frac{2}{p}}, \tag{17a}$$

$$\|\nabla^N\partial_t^k \eta_\zeta\|_{L_t^\infty L_x^p} \lesssim_{N,k,p} (\lambda\sigma r)^N(\lambda\sigma r\mu)^k r^{1-\frac{2}{p}}. \tag{17b}$$

Lemma 3.3 ([15, definition 9, lemma 10], also [38, definition 7.1, lemmas 7.2 and 7.3]). *For $f\in C(\mathbb{T}^2)$, set*

$$\mathcal{R}f \triangleq \nabla g + (\nabla g)^T - (\nabla\cdot g)Id, \tag{18}$$

where g satisfies $\Delta g = f - \fint_{\mathbb{T}^2}f\mathrm{d}x$ and $\fint_{\mathbb{T}^2}g\mathrm{d}x = 0$. Then for any $f\in C(\mathbb{T}^2)$ such that $\fint_{\mathbb{T}^2}f\mathrm{d}x=0$, $\mathcal{R}f(x)$ is a trace-free symmetric matrix for all $x\in\mathbb{T}^2$. Moreover, $\nabla\cdot\mathcal{R}f=f$ and $\fint_{\mathbb{T}^2}\mathcal{R}f(x)\mathrm{d}x=0$. When f is not mean-zero, we overload the notation and denote by $\mathcal{R}f\triangleq\mathcal{R}(f-\fint_{\mathbb{T}^2}f\mathrm{d}x)$. Finally, for all $p\in(1,\infty)$, $\|\mathcal{R}\|_{L_x^p\mapsto W_x^{1,p}}\lesssim 1, \|\mathcal{R}\|_{C_x\mapsto C_x}\lesssim 1$, and $\|\mathcal{R}f\|_{L_x^p}\lesssim\|(-\Delta)^{-\frac{1}{2}}f\|_{L_x^p}$.

Lemma 3.4 ([38, lemma 6.2]). *Let $f,g\in C^\infty(\mathbb{T}^2)$ where g is also $(\mathbb{T}/\kappa)^2$-periodic for some $\kappa\in\mathbb{N}$. Then there exists a constant $C\geqslant 0$ such that*

$$\|fg\|_{L_x^2} \leqslant \|f\|_{L_x^2}\|g\|_{L_x^2} + C\kappa^{-\frac{1}{2}}\|f\|_{C_x^1}\|g\|_{L_x^2}. \tag{19}$$

Lemma 3.5 ([38, lemma 7.4]). *For any* $p \in (1,\infty), \lambda \in \mathbb{N}, a \in C^2(\mathbb{T}^2)$, *and* $f \in L^p(\mathbb{T}^2)$,

$$\|(-\Delta)^{-\frac{1}{2}}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geqslant\lambda}f)\|_{L^p_x} \lesssim \lambda^{-1}\|a\|_{C^2_x}\|f\|_{L^p_x}. \tag{20}$$

4. Proof of theorem 2.1

4.1. Proof of theorem 2.1 assuming proposition 4.2

We fix the function e that satisfies (3). We set for $q \in \mathbb{N}_0$,

$$\lambda_q \triangleq a^{b^q}, \qquad \delta_q \triangleq \lambda_1^{2\beta}\lambda_q^{-2\beta}, \tag{21}$$

where

$$a \in 10\mathbb{N}, \quad a \geqslant a_0, \tag{22}$$

$b \in \mathbb{N}$, and $\beta \in (0,1)$ will be selected subsequently. It is useful that $\delta_1 = 1$, e.g. in the proof of proposition 4.1. We set a convention that $\sum_{1\leqslant r\leqslant 0} c_r = 0$ for any $c_r \in \mathbb{R}$. Hereafter, we impose on ourselves

$$3 \leqslant a^{b\beta} \tag{23}$$

without any significant difficulties because we can take a_0 as large and $\beta > 0$ as small as we wish and still maintain this inequality (23). Then, (23) allows us to define

$$t_q \triangleq -2 + \sum_{0\leqslant\iota\leqslant q}\delta_\iota^{\frac{1}{2}} \leqslant -\frac{1}{2} \text{for all } q \in \mathbb{N}_0 \tag{24}$$

due to $\sum_{0\leqslant\iota\leqslant q}\delta_\iota^{\frac{1}{2}} \leqslant \frac{3}{2}$. The fact that $\sum_{1\leqslant\iota\leqslant q}\delta_\iota^{\frac{1}{2}} \leqslant \frac{3}{2}$ due to (23) will also justify the second inequality in (26a). Hereafter, we denote

$$C_{t,x,q} \triangleq C\left([t_q,t]\times\mathbb{T}^2\right), \quad C_{t,q}L^p_x \triangleq C\left([t_q,t];L^p\left(\mathbb{T}^2\right)\right).$$

For $q \in \mathbb{N}_0$ we consider on $[t_q,T]$

$$\partial_{tt}v_q + \partial_t v_q + (-\Delta)^m v_q + \text{div}(v_q \otimes v_q) + \nabla\pi_q = \text{div}\mathring{R}_q, \tag{25a}$$

$$\nabla\cdot v_q = 0, \tag{25b}$$

where $\mathring{R}_q$ is a trace-free symmetric matrix. We explain our inductive estimates.

Hypothesis 4.1 Inductive Hypothesis at level *q*. We impose on $[t_q,T]$,

$$\|v_q\|_{C_{t,q}L^2_x} \leqslant L\left(1 + \sum_{1\leqslant r\leqslant q}\delta_r^{\frac{1}{2}}\right)\bar{e}^{\frac{1}{2}} \leqslant 3L\bar{e}^{\frac{1}{2}}, \tag{26a}$$

$$\|v_q\|_{C^1_{t,x,q}} \leqslant \lambda_q^3\bar{e}^{\frac{1}{2}}, \tag{26b}$$

$$\|\mathring{R}_q\|_{C_{t,q}L^1_x} \leqslant \frac{\epsilon_\gamma}{36}\delta_{q+2}e(t), \tag{26c}$$

$$\frac{3}{4}\delta_{q+1}e(t) \leqslant e(t) - \|v_q(t)\|_{L^2_x}^2 \leqslant \frac{5}{4}\delta_{q+1}e(t), \tag{26d}$$

for a universal constant L sufficiently large to be determined subsequently (see (77)).

Proposition 4.1 (Initial step $q=0$). *Together with $\pi_0 \equiv 0$, the pair $(v_0, \mathring{R}_0) = (0,0)$ solves (25) and satisfies hypothesis 4.1 at level $q=0$.*

Proof of proposition 4.1. Equations (25) and (26*a*)–(26*c*) are all readily verified. Verification of (26*d*) follows making use of the fact that $\delta_1 = 1$ due to (21). □

Proposition 4.2 (Step $q+1$ assuming the step q). *Under the hypothesis of theorem 2.1, there exists a choice of parameters a_0, β, and b (see (34)) such that for all $(v_q, \mathring{R}_q)$ that solves (25) and satisfies hypothesis 4.1, there exists $(v_{q+1}, \mathring{R}_{q+1})$ that solves (25) and satisfies hypothesis 4.1 at level $q+1$ such that for all $t \in [t_{q+1}, T]$*

$$\|v_{q+1} - v_q\|_{C_{t,q+1}L_x^2} \leqslant L\delta_{q+1}^{\frac{1}{2}} \bar{e}^{\frac{1}{2}}. \tag{27}$$

Next, we prove theorem 2.1 assuming proposition 4.2.

Proof of theorem 2.1 assuming proposition 4.2. We can start from $(v_0, \mathring{R}_0) = (0,0)$ in proposition 4.1 and then rely on proposition 4.2 to obtain $(v_q, \mathring{R}_q)$ for all $q \in \mathbb{N}_0$ that solves (25) and satisfies hypothesis 4.1. By interpolation and that $b^{q+1} \geqslant b(q+1)$ for all $b \in \mathbb{N}$ such that $b \geqslant 2$, for all $\beta' \in (0, \frac{\beta}{3+\beta})$ and all $t \in [-\frac{1}{2}, T]$, we can compute

$$\begin{aligned} \sum_{q \geqslant 0} \|v_{q+1} - v_q\|_{C\left([-\frac{1}{2},t];H_x^{\beta'}\right)} &\overset{(27)}{\lesssim} \sum_{q \geqslant 0} L^{1-\beta'} \delta_{q+1}^{\frac{1-\beta'}{2}} \left(\|v_{q+1}\|_{C([-\frac{1}{2},T];C_x^1)} + \|v_q\|_{C([-\frac{1}{2},T];C_x^1)} \right)^{\beta'} \\ &\overset{(26b)}{\lesssim} L^{1-\beta'} \lambda_1^{\beta(1-\beta')} \sum_{q \geqslant 0} \delta_{q+1}^{\frac{1-\beta'}{2}} \left(\lambda_{q+1}^3\right)^{\beta'} \overset{(21)}{\lesssim} L^{1-\beta'} \lambda_1^{\beta(1-\beta')}. \end{aligned} \tag{28}$$

Identical computations show

$$\sum_{q \geqslant 0} \|v_{q+1} - v_q\|_{C^{\beta'}([-\frac{1}{2},t];L_x^2)} \lesssim L^{1-\beta'} \lambda_1^{\beta(1-\beta')}.$$

Therefore, we obtain the limit of a Cauchy sequence $v \triangleq \lim_{q \to \infty} v_q \in C([-\frac{1}{2}, T]; H^{\beta'}(\mathbb{T}^2)) \cap C^{\beta'}([-\frac{1}{2}, T]; L^2(\mathbb{T}^2))$ which implies the regularity in (4). Due to $\|\mathring{R}_q\|_{C_{t,q}L_x^1} \overset{(26c)}{\leqslant} \frac{\epsilon_\gamma}{36} \delta_{q+2} e(t) \to 0$ as $q \to \infty$, it follows that v is a weak solution of (1). On the other hand, taking $q \to \infty$ in (26*d*) leads to (5).

Finally, the argument concerning non-uniqueness is as follows. If we start with two different energies e_1 and e_2 that satisfy (3) and $e_1 \equiv e_2$ on $[0, t \wedge T]$ for some $t > 0$, then the corresponding perturbations $\{w_{q+1}^1\}_{q \in \mathbb{N}_0}$ and $\{w_{q+1}^2\}_{q \in \mathbb{N}_0}$ corresponding respectively to e_1 and e_2 are identical on $[0, t \wedge T]$ (see (62)). Thus, we can start with identical initial choices $(v_0^1, \mathring{R}_0^1) = (v_0^2, \mathring{R}_0^2) = (0,0)$ according to proposition 4.1 and see that $\{v_q^1\}_{q \in \mathbb{N}_0}$ and $\{v_q^2\}_{q \in \mathbb{N}_0}$ corresponding to e_1 and e_2 are also identical on $[0, t \wedge T]$. As a result, the constructed limiting solutions $v_1, v_2 \in C([-\frac{1}{2}, T]; H^{\beta'}(\mathbb{T}^2)) \cap C^{\beta'}([-\frac{1}{2}, T]; L^2(\mathbb{T}^2))$ corresponding respectively to e_1 and e_2 are identical on $[0, t \wedge T]$. This completes the proof of theorem 2.1. □

4.2. *Proof of proposition 4.2*

We now prove proposition 4.2 which is the heart of the matter.

Remark 4.1. The very first idea of our proof, which ended up not working immediately, is to consider $\partial_{tt} v_q$ as a linear force on the Navier–Stokes equations. In a typical convex integration scheme, the key ingredient consists of the construction of building blocks and that is based on the nonlinear term, especially the most technical oscillation term therein (see (98)). Because the Navier–Stokes equations and the hyperbolic Navier–Stokes equations (1) share the same main nonlinear term $(v \cdot \nabla) v$ (pressure, the other nonlinear term, is readily handled), this implies that their building blocks would be the same. Once the building blocks are determined, a linear force such as the diffusion term $(-\Delta)^m v_q$ would appear only in the last step of estimating the Reynolds stress. Hence, our initial idea was to treat $\partial_{tt} v_q$ similarly to $(-\Delta)^m v_q$.

The reason why this ended up not working in the 3D case is because the term $\partial_{tt} v_q$ is too singular. We can easily get a glimpse of why this is the case by considering a typical convex integration scheme for the Euler equations, for which its perturbation can be

$$w_{(\xi)}(t,x) = a_{(\xi)}(t,x) e^{i\lambda_{q+1}\xi \cdot (\Phi_j(t,x)-x)} B_\xi e^{i\lambda_{q+1}\xi \cdot x},$$

where $a_{(\xi)}$ is a certain amplitude function, B_ξ is a certain $\mathbb{C}$-valued vector, and Φ_j is a solution to a certain transport equation (see [9, section 5.5.4 on p 208] for details). We can see that if ∂_{tt} falls on such a perturbation, then we get λ_{q+1}^2 from chain rule which is too large and we will not be able to close its estimate.

The next idea would then be to turn to intermittency approach using Mikado flows. Application of intermittency via Mikado flows has been done for the 3D Navier–Stokes equations in [5]; however, its choice of parameters were

$$r_\parallel \triangleq \lambda_{q+1}^{\frac{13-20m}{12}}, \quad r_\perp \triangleq \lambda_{q+1}^{\frac{1-20m}{24}}, \quad \mu \triangleq \frac{\lambda_{q+1}^{2m-1} r_\parallel}{r_\perp} = \lambda_{q+1}^{2m-1} \lambda_{q+1}^{\frac{25-20m}{24}} \tag{29}$$

under the constraints of

$$r_\perp \ll r_\parallel \ll 1 \text{and } r_\perp^{-1} \ll \lambda_{q+1} \tag{30}$$

(see [5, equation (2.23) on p 3344]). This choice of $r_\parallel \triangleq \lambda_{q+1}^{\frac{13-20m}{12}}$ and the constraint of $r_\parallel \ll 1$ immediately requires $m > \frac{13}{20}$. As explained on [5, pp 3343–3344], these parameters are optimized for their specific case; e.g. in [5], considering λ_{q+1}^{2m} from the diffusive term $(-\Delta)^m$ and $\partial_t w_{q+1}^{(p)}$ that gives $\frac{r_\perp \lambda_{q+1} \mu}{r_\parallel}$, they optimized by matching

$$\lambda_{q+1}^{2m} = \frac{r_\perp \lambda_{q+1} \mu}{r_\parallel} \text{and equivalently } \mu = \frac{\lambda_{q+1}^{2m-1} r_\parallel}{r_\perp} \text{as in (29).}$$

To fit to our case, we would need to choose a different choice of parameters. Upon this attempt, we listed all the necessary conditions on all parameters but unfortunately ended up with an empty range of the parameters.

Then, we realized that simply considering $\partial_{tt} v_q$ as a force is not a good idea. The reason is that the anti-divergence operator $\mathcal{R}$ is applied on $\partial_{tt} w_{q+1} = \partial_{tt}(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)})$ in the $\|\mathring{R}_{q+1}\|_{C_t L_x^1}$-estimate (see (62)). Typically, $w_{q+1}^{(p)} + w_{q+1}^{(c)}$ is of some form of a curl (because $w_{q+1}^{(c)}$ is the divergence corrector; see (66)) so that $\mathcal{R}$ can reduce a derivative from $w_{q+1}^{(p)} + w_{q+1}^{(c)}$; however, $w_{q+1}^{(t)}$ is not of such a form and this *loss* of one derivative was the main reason why our previous attempts failed. Our novel approach is to consider an integral of the *usual* $w_{q+1}^{(t)}$

in the convex integration scheme for the Navier–Stokes equations; this way, our $\partial_{tt} w_{q+1}^{(t)}$ for the hyperbolic Navier–Stokes equations can play the role of $\partial_t w_{q+1}^{(t)}$ for the Navier–Stokes equations (see (63*c*)), and fortunately this modification did not destroy any key identity (a time integral on $w_{q+1}^{(p)}$ or $w_{q+1}^{(c)}$ would destroy a necessary identity such as (66)). Finally, even with this new approach, after completing all the estimates, we ended up with an empty range of parameters in the 3D case. Yet, upon exploring different spatial dimensions, we finally saw a non-trivial range of parameters in the 2D case; upon optimizing to gain the largest interval for m, we were able to conclude theorem 2.1 with $m \in (0, \frac{2}{3})$. We describe the difficulty in the 3D case furthermore in remark 4.5.

Lastly, let us comment that it is tempting to integrate in time all of w_{q+1}, the perturbation for the Navier–Stokes equations, rather than just $w_{q+1}^{(t)}$. The problem then would be that $w_{q+1}^{(p)}$ would be of an integral form and $w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}$ would not be able to cancel out $\mathring{R}_l$ as needed (see (181)).

We start the proof of proposition 4.2 with a remark.

Remark 4.2. As we mentioned already, in convex integration scheme, the diffusive term does not play any role until the very end. To be specific, verifying that $\mathring{R}_{q+1}$ satisfies (26*c*) at level $q+1$ requires $\|\mathring{R}_{q+1}\|_{C_{t,q+1}L_x^1} \leqslant \frac{\epsilon_\gamma}{36}\delta_{q+3}e(t)$ and in particular, due to (97*a*), we will need to estimate $\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3}e(t)$ for some $p^* \in (1,2)$. Therefore, the proof becomes more difficult as m becomes larger; in fact, in case $m \in (0, \frac{1}{2}]$, we can bound $\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_{t,q+1}L_x^{p^*}} \lesssim \|w_{q+1}\|_{C_{t,q+1}L_x^{p^*}}$ so that bounding by any small constant multiple of $\delta_{q+3}e(t)$ is straightforward (see e.g. [50, equation (118)]) . Therefore, we present the proof of proposition 4.2 that applies for $m \in (\frac{1}{2}, \frac{2}{3})$, considering that the case $m \in (0, \frac{1}{2}]$ can be obtained via a straightforward modification of the case $m \in (\frac{1}{2}, \frac{2}{3})$.

Choice of parameters There are many parameters, namely

$$r, \mu, \sigma^{-1}, \text{and } l \ll 1, b \in \mathbb{N} \text{such that } b \geqslant 2, \text{and } p^* \in (1,2), \tag{31}$$

where l is a mollifier parameter, to appear in (36). We need to optimize over r, μ, and σ^{-1}, where the upper bound of $m < \frac{2}{3}$ appears, and then find the corresponding appropriate range for the rest of the parameters. The selection of these parameters is crucial and detail will be explained in the appendices A and B for completeness. The heuristic outline of how we determined these parameters is as follows.

(1) Considering l to be arbitrarily small, b to be large, and $p^* \in (1,2)$ to be arbitrarily close to 1, we can complete the proof entirely leaving free the specific choices of r, μ, and σ^{-1}. As we will see, the diffusive term will give us a condition of $\lambda_{q+1}^{2m-1} \ll r$ (see (104)) while the term involving the second-order derivative in time will require $\mu \ll \lambda_{q+1}^{-\frac{1}{2}}\sigma^{-1}r^{-\frac{1}{2}}$ (see (124)). Optimizing together with (12) leads us to our choices of

$$r \triangleq \lambda_{q+1}^{\frac{11m-5}{7}}, \quad \mu \triangleq \lambda_{q+1}^{\frac{8m-3}{7}}, \quad \sigma^{-1} \triangleq \lambda_{q+1}^{\frac{3m}{2}} \tag{32}$$

(see appendix A for details.) It can be readily verified that such r, μ, and σ^{-1} satisfy

$$1 \ll r \ll \mu \ll \sigma^{-1} \ll \lambda$$

from (12) with 'λ' $= \lambda_{q+1}$ as needed. We postpone the verification of the other conditions from (12) to remark 4.3.

(2) Once we fix such r, μ, and σ^{-1}, we can plug them in to our estimates and determine the necessary choices of l. The following choices turned out to be sufficient:

$$l \triangleq \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}} \tag{33}$$

(see appendix B for details).

(3) Once such r, μ, σ^{-1}, and l have been fixed, we can take the maximum among all the lower bounds on b from (71), (88), (93), (103), (106), (115), (120), (127), (133), (139), and (147), and choose any $b \in \mathbb{N}$ that satisfies

$$b > \frac{(42)(56)}{2-3m}. \tag{34}$$

(4) At last, with r, μ, σ^{-1}, l, and b fixed, we choose

$$p^* \in \left(1, \frac{8(112)(11m-5)}{10795m-5106}\right) \tag{35}$$

to accommodate the necessary estimates (107), (116), (121), (128), (134), (140), and (148); the fact that $1 < \frac{8(112)(11m-5)}{10795m-5106}$ can be verified using the hypothesis that $m \in (\frac{1}{2}, \frac{2}{3})$.

Remark 4.3. The other conditions of $\lambda_{q+1}, \lambda_{q+1}\sigma \in 5\mathbb{N}$ in (12) can be verified by straightforward modifications of (32) as follows. Let us point out that our choices of r, μ, and σ^{-1} in (32) satisfy $1 \ll r \ll \mu \ll \sigma^{-1} \ll \lambda$ simply by their exponents on a^b; i.e. $0 < \frac{11m-5}{7} < \frac{8m-3}{7} < \frac{3m}{2} < 1$ (and additionally (177)). By denseness of the rationals in the reals, we can easily choose a rational $\frac{d_1}{d_2}$ for $d_1, d_2 \in \mathbb{N}$ that is arbitrarily close to $\frac{11m-5}{7}$ and another rational $\frac{d_3}{d_4}$ for $d_3, d_4 \in \mathbb{N}$ that is arbitrarily close to $\frac{3m}{2}$ so that the required relationship such as $\lambda_{q+1}, \lambda_{q+1}\sigma \in 5\mathbb{N}$ in (12) or (177) continue to hold even when r is replaced by $\lambda_{q+1}^{\frac{d_1}{d_2}}$ and σ^{-1} is replaced by $\lambda_{q+1}^{\frac{d_3}{d_4}}$. Then we can choose b with the lower bound of (34) to be a natural number that is a multiple of $d_2 d_4$ so that $r = a^{b^{q+1}(\frac{d_1}{d_2})} \in \mathbb{N}$ and $\lambda_{q+1}\sigma = a^{b^{q+1}(1-\frac{d_3}{d_4})} \in \mathbb{N}$ too; we refer to the same explanation after [49, equation (68)]. The process of finding other parameters l, b and p^* can be executed much more clearly when the dependence on m is explicit as in (32). Thus, we will keep the r, μ, and σ^{-1} in (32), choose all other parameters, complete the proof, and afterwards, informally replace $\frac{11m-5}{7}$ and $\frac{3m}{2}$ with an arbitrarily close rationals and choose b accordingly to satisfy the conditions of $r, \lambda_{q+1}\sigma \in 5\mathbb{N}$ in (12) to conclude this proof of proposition 4.2.

Throughout the rest of the proof, if not described otherwise, we will always assume that $t \in [t_{q+1}, T]$. We let $\{\phi_\epsilon\}_{\epsilon>0}$ and $\{\varphi_\epsilon\}_{\epsilon>0}$ respectively be families of standard mollifiers on $\mathbb{R}^2$ and $\mathbb{R}$ with mass one where the latter has compact support in $(0, \delta_{q+1})$ and mollify $(v_q, \mathring{R}_q)$ in space-time to obtain over $[t_{q+1}, T]$

$$v_l \triangleq (v_q *_x \phi_l) *_t \varphi_l, \quad \mathring{R}_l \triangleq \left(\mathring{R}_q *_x \phi_l\right) *_t \varphi_l, \quad \text{where } \phi_l(\cdot) \triangleq \frac{1}{l^2}\phi\left(\frac{\cdot}{l}\right), \ \varphi_l(\cdot) \triangleq \frac{1}{l}\varphi\left(\frac{\cdot}{l}\right). \tag{36}$$

It follows that $(v_l, \mathring{R}_l)$ satisfies over $[t_{q+1}, T]$

$$\partial_{tt} v_l + \partial_t v_l + (-\Delta)^m v_l + \operatorname{div}(v_l \otimes v_l) + \nabla \pi_l = \operatorname{div}\left(\mathring{R}_l + R_{\text{com}}\right) \tag{37}$$

where

$$R_{\text{com}} \triangleq v_l \mathring{\otimes} v_l - \left(v_q \mathring{\otimes} v_q\right) *_x \phi_l *_t \varphi_l, \tag{38a}$$

$$\pi_l \triangleq \pi_q *_x \phi_l *_t \varphi_l - \frac{1}{2}\left(|v_l|^2 - |v_q|^2 *_x \phi_l *_t \varphi_l\right). \tag{38b}$$

We obtain basic estimates for the mollified velocity as follows: for any $N \geqslant 1$,

$$\|v_l - v_q\|_{C_{t,q+1}L_x^2} \lesssim \|v_q - v_l\|_{C_{t,x,q+1}} \lesssim l\|v_q\|_{C^1_{t,x,q+1}} \overset{(26b)}{\lesssim} l\lambda_q^3 \bar{e}^{\frac{1}{2}}, \tag{39a}$$

$$\|v_l\|_{C_{t,q+1}L_x^2} \leqslant \|v_q\|_{C_{t,q+1}L_x^2} \overset{(26a)}{\leqslant} L\left(1 + \sum_{1\leqslant r\leqslant q} \delta_r^{\frac{1}{2}}\right)\bar{e}^{\frac{1}{2}}, \tag{39b}$$

where we used Young's inequality for convolution and the fact that the ϕ and ψ have mass one. Now we define for $\epsilon_\gamma > 0$ from the lemma 3.1,

$$\rho(t,x) \triangleq \epsilon_\gamma^{-1}\sqrt{l^2 + |\mathring{R}_l(t,x)|^2} + \gamma_l(t), \tag{40a}$$

$$\gamma_l(t) \triangleq (\gamma_q *_t \varphi_l)(t), \text{ where } \gamma_q(t) \triangleq \frac{1}{2(2\pi)^2}\left[e(t)(1-\delta_{q+2}) - \|v_q(t)\|_{L_x^2}^2\right]. \tag{40b}$$

By (26d), we see that

$$\|v_q(t)\|_{L_x^2}^2 \leqslant e(t)\left(1 - \frac{3}{4}\delta_{q+1}\right) \quad \text{for all } t \in [t_q, T]. \tag{41}$$

On the other hand, considering (23) and that $b > 2$ due to (34) immediately allows us to verify that

$$\delta_{q+2} \leqslant \frac{3}{4}\delta_{q+1}. \tag{42}$$

Considering (41) and (42) together shows that

$$\gamma_q \geqslant 0. \tag{43}$$

As $\varphi_l \geqslant 0$, this implies by definition of γ_l from (40b) that

$$\gamma_l \geqslant 0. \tag{44}$$

We note two immediate consequences:

$$\left|\text{Id} - \left(\text{Id} - \frac{\mathring{R}_l}{\rho}\right)\right| = \left|\frac{\mathring{R}_l}{\rho}\right| \overset{(40a)}{=} \left|\frac{\mathring{R}_l}{\epsilon_\gamma^{-1}\sqrt{l^2 + |\mathring{R}_l|^2} + \gamma_l}\right| \overset{(44)}{\leqslant} \left|\frac{\mathring{R}_l}{\epsilon_\gamma^{-1}\sqrt{l^2 + |\mathring{R}_l|^2}}\right| \leqslant \epsilon_\gamma, \tag{45a}$$

$$\rho(t,x) \geqslant \max\left\{\epsilon_\gamma^{-1} l, \epsilon_\gamma^{-1}|\mathring{R}_l(t,x)|, \gamma_l(t)\right\}. \tag{45b}$$

Proposition 4.3. *The function ρ defined in (40a) satisfies*

$$\|\rho\|_{C_{t,q+1}L_x^p} \leqslant \epsilon_\gamma^{-1}\left(l(2\pi)^{\frac{2}{p}} + \|\mathring{R}_l\|_{C_{t,q+1}L_x^p}\right) + (2\pi)^{\frac{2}{p}-2}\frac{5}{8}\delta_{q+1}\bar{e}, \tag{46a}$$

$$\|\rho\|_{C_{t,x,q+1}} \lesssim l^{-3}\delta_{q+2}\bar{e}, \tag{46b}$$

$$\|\rho\|_{C^N_{t,x,q+1}} \lesssim l^{-5N+1}\delta_{q+2}\bar{e} \quad \forall N \in \mathbb{N}. \tag{46c}$$

Proof of proposition 4.3. First, let us observe that

$$0 \overset{(42)}{\leqslant} \left(\frac{3}{4}\delta_{q+1} - \delta_{q+2}\right) e(t) \overset{(26d)}{\leqslant} e(t)(1-\delta_{q+2}) - \|v_q(t)\|^2_{L_x^2} \overset{(26d)}{\leqslant} \frac{5}{4}\delta_{q+1}\bar{e}. \tag{47}$$

This allows us to estimate for all $p \in [1,\infty)$,

$$\begin{aligned}\|\rho\|_{C_{t,q+1}L_x^p} &\overset{(40a)}{\leqslant} \epsilon_\gamma^{-1}\left\|l + |\mathring{R}_l|\right\|_{C_{t,q+1}L_x^p} + \|\gamma_q\|_{C_{t,q+1}L_x^p} \\ &\overset{(40b)(47)}{\leqslant} \epsilon_\gamma^{-1}\left(l(2\pi)^{\frac{2}{p}} + \|\mathring{R}_l\|_{C_{t,q+1}L_x^p}\right) + (2\pi)^{\frac{2}{p}-2}\frac{5}{8}\delta_{q+1}\bar{e},\end{aligned} \tag{48}$$

which is (46a). Next, for any $N \geqslant 0$ and $t \in [t_{q+1}, T]$, we have due to $W^{3,1}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$,

$$\|\mathring{R}_l\|_{C^N_{t,x,q+1}} \lesssim \sum_{0 \leqslant n+|\alpha| \leqslant N} \|\partial_t^n D^\alpha \mathring{R}_l\|_{C_{t,q+1}W_x^{3,1}} \overset{(26c)}{\lesssim} l^{-3-N}\delta_{q+2}\bar{e}. \tag{49}$$

We apply (49) and straightforward estimates of $\max\{l, \delta_{q+1}\} \lesssim l^{-3}\delta_{q+2}$ for $\beta > 0$ sufficiently small and a_0 sufficiently large to deduce (46b). Finally, to prove (46c), we first compute by [6, equation (130)],

$$\left\|\sqrt{l^2 + |\mathring{R}_l|^2}\right\|_{C^N_{t,x,q+1}} \overset{(49)}{\lesssim} l^{-3-N}\delta_{q+2}\bar{e} + l^{-(N-1)}\left(l^{-4}\delta_{q+2}\bar{e}\right)^N \lesssim l^{-5N+1}\delta_{q+2}\bar{e}. \tag{50}$$

Using (50) and a straightforward estimate of $l^{-N}\delta_{q+1} \lesssim l^{-5N+1}\delta_{q+2}$, we compute

$$\begin{aligned}\|\rho\|_{C^N_{t,x,q+1}} &\overset{(40a)(50)}{\lesssim} l^{-5N+1}\delta_{q+2}\bar{e} + l^{-N}\|\gamma_q\|_{C_{t,q+1}} \\ &\overset{(40b)(47)}{\lesssim} l^{-5N+1}\delta_{q+2}\bar{e} + l^{-N}\delta_{q+1}\bar{e} \lesssim l^{-5N+1}\delta_{q+2}\bar{e}.\end{aligned} \tag{51}$$

This completes the proof of proposition 4.3. □

Next, we define the amplitude function

$$a_\zeta(t,x) \triangleq a_{\zeta,q+1}(t,x) \triangleq \frac{1}{2}\rho(t,x)^{\frac{1}{2}}\gamma_\zeta\left(\mathrm{Id} - \frac{\mathring{R}_l(t,x)}{\rho(t,x)}\right), \tag{52}$$

where $\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l(t,x)}{\rho(t,x)}\right)$ is well-defined due to (45a). For convenience, let us compute the following identities making use of $b_\zeta\mathring{\otimes}b_{-\zeta}=\zeta^\perp\otimes\zeta^\perp-\frac{1}{2}\mathrm{Id}$ whereas $b_\zeta\otimes b_{-\zeta}=\zeta^\perp\otimes\zeta^\perp$, (8), as well as $\fint_{\mathbb{T}^2}\eta_\zeta^2\mathrm{d}x=1$ from (14b):

$$\sum_{\zeta,\vartheta\in\Lambda}a_\zeta(t)a_\vartheta(t)\fint_{\mathbb{T}^2}W_\zeta\otimes W_\vartheta(t)\,\mathrm{d}x=-\mathring{R}_l(t)+\rho(t)\,\mathrm{Id},\tag{53a}$$

$$\sum_{\zeta,\vartheta\in\Lambda}a_\zeta(t)a_\vartheta(t)\fint_{\mathbb{T}^2}W_\zeta\mathring{\otimes}W_\vartheta(t)\,\mathrm{d}x=-\mathring{R}_l(t)+\rho(t)\left(1-\frac{1}{8}\sum_{\zeta\in\Lambda}\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l(t)}{\rho(t)}\right)^2\right)\mathrm{Id},\tag{53b}$$

(see [8, equation (3.15)], [49, equation (83)], and [38, equation (5.4)]).

Proposition 4.4. *The function a_ζ defined in (52) satisfies*

$$\|a_\zeta\|_{C_{t,q+1}L_x^2}\leqslant\frac{1}{4}\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\frac{M}{|\Lambda|},\tag{54a}$$

$$\|a_\zeta\|_{C_{t,x,q+1}}\lesssim l^{-\frac{3}{2}}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}},\tag{54b}$$

$$\|a_\zeta\|_{C_{t,x,q+1}^N}\lesssim l^{-6N-8}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\quad\forall N\in\mathbb{N}\tag{54c}$$

where M is the constant from (9).

Proof of proposition 4.4. Along with a straightforward estimate of $l\leqslant\frac{\delta_{q+1}\bar{e}}{4}$ by taking $\beta>0$ sufficiently small, we verify (54a) as follows:

$$\begin{aligned}\|a_\zeta\|_{C_{t,q+1}L_x^2}&\overset{(52)(45a)}{\leqslant}\frac{1}{2}\|\rho\|_{C_{t,q+1}L_x^1}^{\frac{1}{2}}\|\gamma_\zeta\|_{C\left(B_{\epsilon_\gamma}(\mathrm{Id})\right)}\\&\leqslant\frac{1}{2}\left[\epsilon_\gamma^{-1}\left(\delta_{q+1}\bar{e}\pi^2+\frac{\epsilon_\gamma}{48}\delta_{q+1}\bar{e}\right)+\frac{5}{8}\delta_{q+1}\bar{e}\right]^{\frac{1}{2}}\left(\frac{M}{C_\Lambda}\right)\overset{(9)}{=}\frac{1}{4}\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\frac{M}{|\Lambda|}.\end{aligned}\tag{55}$$

Next, we estimate

$$\|a_\zeta\|_{C_{t,x,q+1}}\overset{(52)(45a)}{\lesssim}\|\rho\|_{C_{t,x,q+1}}^{\frac{1}{2}}\|\gamma_\zeta\|_{C\left(B_{\epsilon_\gamma}(\mathrm{Id})\right)}\overset{(46b)}{\lesssim}l^{-\frac{3}{2}}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}},$$

which verifies (54b).

Finally, to verify (54c) for $N\in\mathbb{N}$, we compute relying on [6, equation (130)], (45b), and (46c), for any $k\in\{0,1,\ldots,N-r\}$ and $r\in\{0,1,\ldots,N\}$,

$$\left\|\frac{1}{\rho}\right\|_{C_{t,x,q+1}^{N-r-k}}\lesssim l^{-2}\left[l^{-5(N-r-k)+1}\delta_{q+2}\bar{e}\right]+l^{-(N-r-k+1)}\left[l^{-4}\delta_{q+2}\bar{e}\right]^{N-r-k}\lesssim l^{-5(N-r-k)-1}\delta_{q+2}\bar{e}.\tag{56}$$

This leads to, for any $r\in\{0,1,\ldots,N\}$,

$$\left\|\frac{\mathring{R}_l}{\rho}\right\|_{C_{t,x,q+1}^{N-r}}\lesssim\sum_{k=0}^{N-r}\|\mathring{R}_l\|_{C_{t,x,q+1}^k}\left\|\frac{1}{\rho}\right\|_{C_{t,x,q+1}^{N-r-k}}\overset{(49)(56)}{\lesssim}l^{-6(N-r)-5}.\tag{57}$$

We can furthermore compute for all $r \in \{0,1,\ldots,N\}$,

$$\|D_{t,x}\mathring{R}_l\|_{C_{t,x,q+1}}^{N-r} \lesssim \left(l^{-4}\|\mathring{R}_q\|_{C_{t,q+1}L_x^1}\right)^{N-r} \overset{(26c)}{\lesssim} l^{-5(N-r)}, \tag{58a}$$

$$\left\|\frac{\mathring{R}_l}{\rho^2}\right\|_{C_{t,x,q+1}}^{N-r} \overset{(45a)}{\lesssim} \left\|\frac{1}{\rho}\right\|_{C_{t,x,q+1}}^{N-r} \overset{(45b)}{\lesssim} l^{-(N-r)}, \tag{58b}$$

$$\|D_{t,x}\rho\|_{C_{t,x,q+1}}^{N-r} \overset{(46c)}{\lesssim} \left(l^{-4}\delta_{q+2}\bar{e}\right)^{N-r} \lesssim l^{-5(N-r)}. \tag{58c}$$

Combining (57) and (58), we can deduce by another application of [6, equation (130)], for any $r \in \{0,1,\ldots,N\}$,

$$\begin{aligned}\left\|\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l}{\rho}\right)\right\|_{C_{t,x,q+1}^{N-r}} \lesssim \left\|\frac{\mathring{R}_l}{\rho}\right\|_{C_{t,x,q+1}^{N-r}} + l^{-(N-r)}\|D_{t,x}\mathring{R}_l\|_{C_{t,x,q+1}}^{N-r} \\ + \left\|\frac{\mathring{R}_l}{\rho^2}\right\|_{C_{t,x,q+1}}^{N-r}\|D_{t,x}\rho\|_{C_{t,x,q+1}}^{N-r} \overset{(45b)(57)(58)}{\lesssim} l^{-6(N-r)-6}.\end{aligned} \tag{59}$$

Finally, we can compute by another application of [6, equation (130)], for all $r \in \{0,1,\ldots,N\}$,

$$\|\rho^{\frac{1}{2}}\|_{C_{t,x,q+1}^r} \overset{(45b)}{\lesssim} l^{-\frac{1}{2}}\|\rho\|_{C_{t,x,q+1}^r} + l^{\frac{1}{2}-r}\|\rho\|_{C_{t,x,q+1}^1}^r \overset{(56c)}{\lesssim} l^{-6r}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}. \tag{60}$$

At last, we are ready to conclude that for all $N \in \mathbb{N}$,

$$\|a_\zeta\|_{C_{t,x,q+1}^N} \overset{(52)}{\lesssim} \sum_{r=0}^N \|\rho^{\frac{1}{2}}\|_{C_{t,x,q+1}^r}\left\|\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l}{\rho}\right)\right\|_{C_{t,x,q+1}^{N-r}} \overset{(56b)(59)(60)(45a)}{\lesssim} l^{-6N-8}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}. \tag{61}$$

□

Next, we recall $\psi_\zeta, \eta_\zeta, W_\zeta$, and μ respectively from (10), (12), (13) and (15), and define the perturbation

$$w_{q+1} \triangleq w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \text{and } v_{q+1} \triangleq v_l + w_{q+1} \tag{62}$$

where

$$w_{q+1}^{(p)} \triangleq \sum_{\zeta\in\Lambda} a_\zeta W_\zeta, \tag{63a}$$

$$w_{q+1}^{(c)} \triangleq \sum_{\zeta\in\Lambda} \nabla^\perp(a_\zeta\eta_\zeta)\psi_\zeta, \tag{63b}$$

$$w_{q+1}^{(t)} \triangleq \mu^{-1}\left(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-}\right)\mathbb{P}\mathbb{P}_{\neq 0}\int_{t_{q+1}}^t \left(a_\zeta^2\mathbb{P}_{\neq 0}\eta_\zeta^2\zeta\right)\mathrm{d}s. \tag{63c}$$

Remark 4.4. We come back to continue from remark 2.2 in explaining why we cannot follow the proof of [38, corollary 1.2] and deduce the existence of a non-trivial weak solution to the hyperbolic Navier–Stokes equations with compact support in time. Luo and Qu defines

$$N_\epsilon(S) \triangleq \{t \in [0,T] : \text{there exists } s \in S \text{such that } |t-s| \leqslant \epsilon\}$$

in [38, equation (2.21)] and included an inductive hypothesis of

$$\mathrm{supp}_t v_{q+1} \cup \mathrm{supp}_t \mathring{R}_{q+1} \subset N_{\delta_{q+1}} \left(\mathrm{supp}_t v_q \cup \mathrm{supp}_t \mathring{R}_q \right) \tag{64}$$

in [38, equation (2.17)]. Given any smooth divergence-free vector field $u(t,x)$ that is mean-zero and $\epsilon > 0$, the authors completed the inductive step $q = 0$ with $v_0 = u$. As $v_q \to v$ in $C([0,T];L^2(\mathbb{T}^2))$ and $\mathring{R}_q \to 0$ in $C([0,T];L^1(\mathbb{T}^2))$ and

$$\sum_{q=1}^{\infty} \delta_q = \sum_{q=1}^{\infty} \lambda_q^{-2\beta} \leqslant \sum_{q=1}^{\infty} a^{-b2\beta q} = \frac{a^{-b2\beta}}{1 - a^{-b2\beta}} < \epsilon$$

for $\beta > 0$ sufficiently small so that

$$\mathrm{supp}_t v \subset N_\epsilon \left(\mathrm{supp}_t v_0 \right) = N_\epsilon \left(\mathrm{supp}_t u \right),$$

taking $u \equiv 0$ allowed them to deduce a solution with the compact temporal support. Now, in order to verify (64) at level $q+1$, Luo and Qu defines a temporal cut-off function Φ_q as a smooth function such that $\mathrm{supp}\Phi_q(t) \subset N_l(\mathrm{supp}_t \mathring{R}_l^*)$ where $\mathring{R}_l^* \triangleq \mathring{R}_l + R_{\mathrm{com}}$ in [38, equation (5.1)] and then $a_\zeta \triangleq A^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \gamma_\zeta (A^{-1} \delta_{q+1}^{-1} \mathring{R}_l^*(t,x)) \Phi_q(t)$ so that $\mathrm{supp}_t a_\zeta \subset N_l(\mathrm{supp}_t \mathring{R}_l^*)$ for all $\zeta \in \Lambda$ (see [38, equation (5.2)] for details) which in turn leads to

$$\mathrm{supp}_t w_{q+1} \subset \cup_{\zeta \in \Lambda} \mathrm{supp}_t a_\zeta \subset N_l \left(\mathrm{supp}_t \mathring{R}_l^* \right); \tag{65}$$

here, the first inclusion crucially relies on their choice of

$$w_{q+1}^{(t)} = \mu^{-1} \left(\sum_{\zeta \in \Lambda^+} - \sum_{\zeta \in \Lambda^-} \right) \mathbb{P}\mathbb{P}_{\neq 0} \left(a_\zeta^2 \mathbb{P}_{\neq 0} \eta_\zeta^2 \zeta \right).$$

In contrast, our choice of $w_{q+1}^{(t)}$ in (63c) does not lead to (65) because $\mathrm{supp}_t \int_{t_{q+1}}^t (a_\zeta^2 \mathbb{P}_{\neq 0} \eta_\zeta^2 \zeta) \mathrm{d}s \not\subset \cup_{\zeta \in \Lambda} \mathrm{supp}_t a_\zeta$.

We have the identity of

$$\left(w_{q+1}^{(p)} + w_{q+1}^{(c)} \right)(t,x) \overset{(11a)(15)}{=} \nabla^\perp \left(\sum_{\zeta \in \Lambda} a_\zeta(t,x) \eta_\zeta(t,x) \psi_\zeta(x) \right). \tag{66}$$

It follows that w_{q+1} is divergence-free and mean-zero. By (13) η_ζ is $(\mathbb{T}/\lambda_{q+1}\sigma)^2$-periodic, while by (10) and (11) b_ζ is $(\mathbb{T}/\lambda_{q+1})^2$-periodic. It follows that W_ζ in (15) is $(\mathbb{T}/\lambda_{q+1}\sigma)^2$-periodic. Thus, we can apply lemma 3.4 to deduce

$$\begin{aligned} \|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^2} &\overset{(63)(19)}{\leqslant} \sum_{\zeta \in \Lambda} \|a_\zeta\|_{C_{t,q+1}L_x^2} \|W_\zeta\|_{C_{t,q+1}L_x^2} + C(\lambda_{q+1}\sigma)^{-\frac{1}{2}} \|a_\zeta\|_{C_{t,q+1}C_x^1} \|W_\zeta\|_{C_{t,q+1}L_x^2} \\ &\overset{(17a)(54a)(54c)}{\leqslant} |\Lambda| \frac{M}{4C_\Lambda} \delta_{q+1}^{\frac{1}{2}} \bar{e}^{\frac{1}{2}} + C \lambda_{q+1}^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} l^{-14} \delta_{q+2}^{\frac{1}{2}} \bar{e}^{\frac{1}{2}}. \end{aligned} \tag{67}$$

Now the process of determining the optimal choices of parameters based on the minimum constraints from (12) and (31) starts here. First, for the subsequent estimates (73) and (77), we need to bound this $\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^2}$ by a constant multiple of $\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}$. We notice that $1 \ll \lambda_{q+1}\sigma$ from (12) making $\lambda_{q+1}^{-\frac{1}{2}}\sigma^{-\frac{1}{2}} \ll 1$, and therefore, as long as l satisfies

$$\lambda_{q+1}^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}l^{-14} \lesssim 1, \tag{68}$$

we can conclude from (67) that

$$\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^2} \lesssim \delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}} \tag{69}$$

without imposing any condition on the precise choice of σ^{-1} because $\delta_{q+2} \leqslant \delta_{q+1}$. Our choices of r, μ, and σ^{-1} in (32) will be determined in appendix A after collecting all the conditions similarly. Second, in order to determine l after choosing r, μ, and σ^{-1} in (32), let us observe that due to (32) and (68) is implied by

$$l^{-1} \ll \lambda_{q+1}^{\frac{2-3m}{56}}. \tag{70}$$

Again, our choice of l in (33) will be determined in appendix B after collection all such conditions similarly. Third, in order to determine b after choosing l in (33), let us observe that applying our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}}\lambda_q^{-\frac{3}{2}}$ from (33), the estimate (70) holds if

$$\frac{168}{2-3m} < b; \tag{71}$$

thus, we incorporate this condition to our choice of b in (34) to claim (69).

Next, we can show that the functions $w_{q+1}^{(p)}, w_{q+1}^{(c)}$, and $w_{q+1}^{(t)}$ defined in (63) satisfies for all $p \in (1,\infty)$ and $t \in [t_{q+1}, T]$,

$$\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^p} \overset{(63)}{\leqslant} \sup_{s\in[t_{q+1},t]}\sum_{\zeta\in\Lambda}\|a_\zeta(s)\|_{L_x^\infty}\|W_\zeta(s)\|_{L_x^p} \overset{(17a)(54c)}{\lesssim} \delta_{q+2}^{\frac{1}{2}}l^{-\frac{3}{2}}r^{1-\frac{2}{p}}\bar{e}^{\frac{1}{2}}, \tag{72a}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^p} &\overset{(63)}{\lesssim} \sup_{s\in[t_{q+1},t]}\sum_{\zeta\in\Lambda}\|\nabla^\perp(a_\zeta\eta_\zeta)(s)\|_{L_x^p}\|\psi_\zeta\|_{L_x^\infty} \\ &\overset{(11b)(17b)(54)}{\lesssim} \lambda_{q+1}^{-1}\delta_{q+2}^{\frac{1}{2}}l^{-\frac{3}{2}}\bar{e}^{\frac{1}{2}}r^{1-\frac{2}{p}}\left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r\right],\end{aligned} \tag{72b}$$

$$\|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^p} \overset{(63)}{\lesssim} \left\|\mu^{-1}\left(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-}\right)\mathbb{P}\mathbb{P}_{\neq 0}\left(a_\zeta^2\mathbb{P}_{\neq 0}\eta_\zeta^2\zeta\right)\right\|_{C_{t,q+1}L_x^p} \tag{72c}$$

$$\lesssim \mu^{-1}\sum_{\zeta\in\Lambda}\|a_\zeta\|_{C_{t,q+1}L_x^\infty}^2\|\eta_\zeta\|_{C_{t,q+1}L_x^{2p}}^2 \overset{(17b)(54c)}{\lesssim} \mu^{-1}\delta_{q+2}l^{-3}r^{2-\frac{2}{p}}\bar{e}. \tag{72d}$$

It follows from (62), (69) and (72b)–(72d) that

$$\|w_{q+1}\|_{C_{t,q+1}L_x^2} \lesssim \delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}} + \lambda_{q+1}^{-1}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}l^{-\frac{3}{2}}\left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r\right] + \mu^{-1}\delta_{q+2}l^{-3}r\bar{e}. \tag{73}$$

For subsequent estimates in (78) and (79), we need to bound this by a constant multiple of $\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}$. In fact, for our subsequent verification of the inductive hypothesis (26*d*) (precisely (173)), it would be convenient that we dominate all these terms by δ_{q+2} so that

$$\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^2} + \|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^2} \ll \delta_{q+2} \leqslant \delta_{q+2}^{\frac{1}{2}} \tag{74}$$

for a_0 sufficiently large and $\beta > 0$ sufficiently small, and therefore we shall impose

$$l^{-14} \ll \lambda_{q+1}, \quad l^{-\frac{3}{2}}\sigma r \ll 1, \quad \mu^{-1} l^{-3} r \ll 1, \tag{75}$$

to deduce for $\beta > 0$ sufficiently small

$$\lambda_{q+1}^{-1}\delta_{q+2}^{\frac{1}{2}} l^{-14}\bar{e}^{\frac{1}{2}} \ll \delta_{q+2}, \quad \delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}} l^{-\frac{3}{2}}\sigma r \ll \delta_{q+2}, \quad \mu^{-1}\delta_{q+2} l^{-3} r\bar{e} \ll \delta_{q+2}, \tag{76}$$

respectively. In the last inequality of (76), we used the fact that $\delta_q \leqslant 1$ for all $q \geqslant 1$. We note that none of the conditions in (75) imposes any constraint on our choice of σ, r, or μ^{-1}, because $\sigma r \ll 1$ and $\mu^{-1} r \ll 1$ from (12). In order to determine l after r, μ, and σ^{-1} from (32) are chosen in appendix A after collecting all their constraints, we see that the first condition $l^{-14} \ll \lambda_{q+1}$ in (75) is implied by $l^{-1} \ll \lambda_{q+1}^{\frac{2-3m}{56}}$ from (70) while we plug in (32) to the second and third conditions of (75) to see that they are implied by

$$l^{-\frac{3}{2}}\lambda_{q+1}^{\frac{m-10}{14}} \ll 1, \quad l^{-3}\lambda_{q+1}^{\frac{3m-2}{7}} \ll 1$$

which are both implied by (70), respectively. Therefore, applying (76) to (73) we obtain by choosing the universal constant $L \gg 1$,

$$\|w_{q+1}\|_{C_{t,q+1}L_x^2} \leqslant \frac{3}{4} L \delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}. \tag{77}$$

It now follows that

$$\|v_{q+1}\|_{C_{t,q+1}L_x^2} \overset{(62)}{\leqslant} \|v_l\|_{C_{t,q+1}L_x^2} + \|w_{q+1}\|_{C_{t,q+1}L_x^2} \overset{(39b)(77)}{\leqslant} L\left(1 + \sum_{1 \leqslant r \leqslant q+1} \delta_r^{\frac{1}{2}}\right)\bar{e}^{\frac{1}{2}} \tag{78}$$

which verifies (26*a*) at level $q+1$. We can also verify (27) as for all $t \in [t_{q+1}, T]$, we can compute

$$\begin{aligned}\|v_{q+1}(t) - v_q(t)\|_{L_x^2} &\leqslant \|w_{q+1}(t)\|_{L_x^2} + \|v_l(t) - v_q(t)\|_{L_x^2} \\ &\overset{(77)(39a)}{\leqslant} \frac{3}{4} L\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}} + Cl\lambda_q^3\bar{e}^{\frac{1}{2}} \leqslant L\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\end{aligned} \tag{79}$$

assuming that

$$l\lambda_q^3 \ll 1 \tag{80}$$

and taking $\beta > 0$ sufficiently small after the b is already fixed so that $Cl\lambda_q^3 \leqslant \frac{1}{4} L\delta_{q+1}^{\frac{1}{2}}$. The estimate (80) can be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}}\lambda_q^{-\frac{3}{2}}$ from (33) if

$$\frac{168}{2-3m} < b,$$

which is same as (71), to claim (80) and therefore (79). Next, we estimate $\|w_{q+1}^{(p)}\|_{C^1_{t,x,q+1}}$ starting from (63*a*); for simplicity we will not keep track of bound by $\bar{e}$ because such $C_{t,x,q+1}$-norm estimates are for the purpose of verifying (26*b*) which is independent of $\bar{e}$ anyway. We compute

$$\|w_{q+1}^{(p)}\|_{C^1_{t,x,q+1}} \lesssim \sum_{\zeta\in\Lambda} \|a_\zeta\|_{C^1_{t,x,q+1}} \|W_\zeta\|_{L^\infty_{t,x,q+1}} + \|a_\zeta\|_{L^\infty_{t,x,q+1}} \|W_\zeta\|_{C^1_{t,x,q+1}}$$
$$\overset{(54c)(54b)(17a)}{\lesssim} \left(\delta_{q+2}^{\frac{1}{2}} l^{-14}\right) r + \left(\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\right) [\lambda_{q+1} + \lambda_{q+1}\sigma r\mu] r \lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1} r [1+\sigma r\mu] \tag{81}$$

where we assumed that $l^{-\frac{25}{2}} \ll \lambda_{q+1}$ which follows from $l^{-14} \ll \lambda_{q+1}$ from (75).

Next, we compute from (63)

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x,q+1}} \lesssim \sum_{\zeta\in\Lambda} \|a_\zeta\|_{C_{t,q+1}C^2_x} \|\eta_\zeta\|_{C_{t,q+1}C_x} \|\psi_\zeta\|_{C_x} + \|a_\zeta\|_{C_{t,q+1}C^1_x} \|\eta_\zeta\|_{C_{t,q+1}C^1_x} \|\psi_\zeta\|_{C_x}$$
$$+ \|a_\zeta\|_{C_{t,q+1}C^1_x} \|\eta_\zeta\|_{C_{t,q+1}C_x} \|\psi_\zeta\|_{C^1_x} + \|a_\zeta\|_{C_{t,q+1}C_x} \|\eta_\zeta\|_{C_{t,q+1}C^2_x} \|\psi_\zeta\|_{C_x}$$
$$+ \|a_\zeta\|_{C_{t,q+1}C_x} \|\eta_\zeta\|_{C_{t,q+1}C^1_x} \|\psi_\zeta\|_{C^1_x} + \|a_\zeta\|_{C^1_{t,q+1}C^1_x} \|\eta_\zeta\|_{C_{t,q+1}C_x} \|\psi_\zeta\|_{C_x}$$
$$+ \|a_\zeta\|_{C_{t,q+1}C^1_x} \|\eta_\zeta\|_{C^1_{t,q+1}C_x} \|\psi_\zeta\|_{C_x} + \|a_\zeta\|_{C^1_{t,q+1}C_x} \|\eta_\zeta\|_{C_{t,q+1}C^1_x} \|\psi_\zeta\|_{C_x}$$
$$+ \|a_\zeta\|_{C_{t,q+1}C_x} \|\eta_\zeta\|_{C^1_{t,q+1}C^1_x} \|\psi_\zeta\|_{C_x}$$
$$\overset{(54)(17b)(11b)}{\lesssim} \delta_{q+2}^{\frac{1}{2}} \Big[l^{-20} r\lambda_{q+1}^{-1} + l^{-14}\sigma r^2 + l^{-14} r + l^{-\frac{3}{2}} \lambda_{q+1}\sigma^2 r^3 + l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2$$
$$+ l^{-14}\sigma r^2 \mu + l^{-\frac{3}{2}} \lambda_{q+1}\sigma^2 r^3 \mu \Big]. \tag{82}$$

For a subsequent estimate in (90), we would like to bound this by a constant multiple of $\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2 [1+\sigma r\mu]$. In order to do so, we observe that $r \ll \sigma^{-1}$ from (12) so that $r\sigma \ll 1$ and consequently

$$l^{-14}\sigma r^2 \ll l^{-14} r, \quad l^{-\frac{3}{2}} \lambda_{q+1}\sigma^2 r^3 \ll l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2.$$

Concerning rest of the terms, because $\mu \ll \lambda_{q+1}$ from (12), we only have to impose $l^{-\frac{25}{2}} \ll \lambda_{q+1}$, or more strongly

$$l^{-\frac{25}{2}}\mu \ll \lambda_{q+1} \tag{83}$$

to assure

$$l^{-14}\sigma r^2 \mu \ll l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2. \tag{84}$$

Finally, it follows from (70) that we assumed already that

$$l^{-20} r\lambda_{q+1}^{-1} \ll l^{-14} r \lambda_{q+1}^{6\left(\frac{2-3m}{56}\right)} \lambda_{q+1}^{-1} \leqslant l^{-14} r.$$

Applying such estimates already gives us

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x,q+1}} \lesssim \delta_{q+2}^{\frac{1}{2}} \left[l^{-14} r + l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2 + l^{-\frac{3}{2}} \lambda_{q+1}\sigma^2 r^3 \mu \right].$$

Moreover, because $1 \ll r$ and $\lambda_{q+1}\sigma \gg 1$ from (12), it does not cost any additional constraint on σ, r, or μ^{-1} to ask for

$$l^{-\frac{25}{2}} \ll \lambda_{q+1}\sigma r, \tag{85}$$

which would imply $l^{-14} r \ll l^{-\frac{3}{2}}\lambda_{q+1}\sigma r^2$ and therefore

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x,q+1}} \lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1}\sigma r^2 \left[1+\sigma r\mu\right]. \tag{86}$$

Now, in order to determine l after r, μ, and σ^{-1} in (32) are already selected, we observe that plugging (32) into (83) and (85) leads to

$$l^{-\frac{25}{2}} \ll \lambda_{q+1}^{\frac{m+4}{14}}. \tag{87}$$

In order to determine the condition on b after l in (33) is chosen, we plug in $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) to see that (87) is satisfied if

$$\frac{(75)(56)}{91m+14} < b; \tag{88}$$

we incorporate this condition upon choosing b in (34) to claim (87) and hence (86).

Finally, we give up λ_{q+1}^{ϵ} for $\epsilon > 0$ arbitrarily small to bound $\mathbb{P}$ in C_x and compute

$$\begin{aligned} &\|w_{q+1}^{(t)}\|_{C^1_{t,x,q+1}} \\ &\lesssim \mu^{-1}\lambda_{q+1}^{\epsilon} \sum_{\zeta\in\Lambda} \left[\|a_\zeta\|_{C_{t,x,q+1}} \|a_\zeta\|_{C_{t,q+1}C_x^1} \|\eta_\zeta\|^2_{C_{t,x,q+1}} + \|a_\zeta\|^2_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_{t,q+1}C_x^1}\right] \\ &\overset{(54c)(17b)}{\lesssim} \mu^{-1}\lambda_{q+1}^{\epsilon}\delta_{q+2} l^{-3} r^2 \left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r\right] \overset{(85)}{\lesssim} \mu^{-1}\lambda_{q+1}^{1+\epsilon}\delta_{q+2} l^{-3} r^3\sigma \end{aligned} \tag{89}$$

where the last inequality used an assumption that

$$l^{-\frac{25}{2}} \ll \lambda_{q+1}\sigma r$$

and we notice that this is same as (85). Thus, we conclude

$$\begin{aligned} \|v_{q+1}\|_{C^1_{t,x,q+1}} &\overset{(62)}{\leqslant} \|v_q\|_{C^1_{t,x,q+1}} + \|w_{q+1}^{(p)}\|_{C^1_{t,x,q+1}} + \|w_{q+1}^{(c)}\|_{C^1_{t,x,q+1}} + \|w_{q+1}^{(t)}\|_{C^1_{t,x,q+1}} \\ &\overset{(26b)(86)(81)(89)}{\lesssim} \lambda_q^3 + \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\lambda_{q+1} r\left[1+\sigma r\mu\right] \\ &\quad + \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\lambda_{q+1}\sigma r^2\left[1+\sigma r\mu\right] + \mu^{-1}\lambda_{q+1}^{1+\epsilon}\delta_{q+2} l^{-3} r^3\sigma \\ &\lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\lambda_{q+1} r\left[1+\sigma r\mu\right] + \mu^{-1}\lambda_{q+1}^{1+\epsilon}\delta_{q+2} l^{-3} r^3\sigma \end{aligned} \tag{90}$$

where the last inequality used the fact that $\sigma r \ll 1$ due to (12).

We continue from (90) to verify (26b) at level $q+1$ using our choices of r, μ, and σ^{-1} in (32); some experience with convex integration suggests that this should not depend on the choice of r, μ, and σ^{-1}, informally because the upper bound of λ_{q+1}^3 in (26b) at level $q+1$ is so large that this verification is expected to not create any significant difficulties. We compute for any $\varepsilon > 0$,

$$\|v_{q+1}\|_{C^1_{t,x,q+1}} \overset{(90)(32)}{\lesssim} \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{11m+2}{7}} \left[1 + \lambda_{q+1}^{\frac{17m-16}{14}}\right] + \lambda_{q+1}^{\epsilon + \frac{29m-10}{14}} \delta_{q+2} l^{-3}$$
$$\lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{11m+2}{7}} + \lambda_{q+1}^{\epsilon + \frac{29m-10}{14}} \delta_{q+2} l^{-3}, \tag{91}$$

which can be further bounded by $\lambda_{q+1}^{3} \bar{e}^{\frac{1}{2}}$ if l satisfies

$$l^{-\frac{3}{2}} \ll \lambda_{q+1}^{\frac{19-11m}{7}} \text{ and } l^{-3} \ll \lambda_{q+1}^{\frac{52-29m}{14} - \epsilon}. \tag{92}$$

Applying our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33), and choosing e.g. $\epsilon = \frac{m}{14}$ shows that (92) holds if

$$\frac{3(1176)}{2870 - 1617m} < b, \tag{93}$$

and we incorporate this condition to our choice of b in (34) to claim that (26*b*) at level $q+1$ was verified.

Lastly, for $p \in (1, \infty)$ we compute the $W_x^{1,p}$-norms of the perturbation for the purpose of subsequent estimates (100). First, we estimate using (66)

$$\begin{aligned}
&\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_{t,q+1} W_x^{1,p}} \\
&\lesssim \sum_{\zeta \in \Lambda} \|a_\zeta\|_{C_{t,q+1} C_x^2} \|\eta_\zeta\|_{C_{t,q+1} L_x^p} \|\psi_\zeta\|_{C_x} + \|a_\zeta\|_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_{t,q+1} W_x^{2,p}} \|\psi_\zeta\|_{C_x} \\
&\quad + \|a_\zeta\|_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_t L_x^p} \|\psi_\zeta\|_{C_x^2} \\
&\overset{(54)(17b)(11b)}{\lesssim} \left(\delta_{q+2}^{\frac{1}{2}} l^{-20}\right) r^{1-\frac{2}{p}} \lambda_{q+1}^{-1} + \left(\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\right) (\lambda_{q+1} \sigma r)^2 r^{1-\frac{2}{p}} \lambda_{q+1}^{-1} + \left(\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}}\right) r^{1-\frac{2}{p}} \lambda_{q+1} \\
&\lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \left[\lambda_{q+1} \sigma^2 r^{3-\frac{2}{p}} + r^{1-\frac{2}{p}} \lambda_{q+1}\right] \lesssim \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p}} \lambda_{q+1}
\end{aligned} \tag{94}$$

where in the second to last inequality we assumed that $l^{-\frac{37}{2}} \lambda_{q+1}^{-1} \lesssim \lambda_{q+1}$ which follows from $l^{-\frac{25}{2}} \ll \lambda_{q+1}$ that already appeared before (83), while the last inequality used the fact that $r \ll \sigma^{-1}$ due to (12) so that $\sigma^2 r^{3-\frac{2}{p}} = (\sigma r)^2 r^{1-\frac{2}{p}} \ll r^{1-\frac{2}{p}}$. Second, we estimate

$$\begin{aligned}
\|w_{q+1}^{(t)}\|_{C_{t,q+1} W_x^{1,p}} &\overset{(63)}{\lesssim} \mu^{-1} \sum_{\zeta \in \Lambda} \|a_\zeta\|_{C_{t,q+1} C_x} \|a_\zeta\|_{C_{t,q+1} C_x^1} \|\eta_\zeta\|^2_{C_{t,q+1} L_x^{2p}} \\
&\quad + \|a_\zeta\|^2_{C_{t,q+1} C_x} \|\eta_\zeta\|_{C_{t,q+1} L_x^{2p}} \|\eta_\zeta\|_{C_{t,q+1} W_x^{1,2p}} \\
&\overset{(54c)(17b)}{\lesssim} \mu^{-1} \delta_{q+2} l^{-3} r^{2-\frac{2}{p}} \left[l^{-\frac{25}{2}} + \lambda_{q+1} \sigma r\right] \lesssim \mu^{-1} \delta_{q+2} l^{-3} \lambda_{q+1} \sigma r^{3-\frac{2}{p}}
\end{aligned} \tag{95}$$

where the last inequality relied on the previous assumption (85).

We now verify (26c) at level $q+1$. We write using (25), (62), and (37),

$$\begin{aligned}
&\operatorname{div}\mathring{R}_{q+1} - \nabla\pi_{q+1}\\
&= \underbrace{(-\Delta)^m w_{q+1} + \partial_t\left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) + \operatorname{div}\left(v_l \otimes w_{q+1} + w_{q+1} \otimes v_l\right)}_{\operatorname{div}R_{\text{lin }1} + \nabla\pi_{\text{lin }1}}\\
&\quad + \underbrace{\partial_{tt}\left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right)}_{\operatorname{div}R_{\text{lin }2}} + \underbrace{\operatorname{div}\left(\left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\right)}_{\operatorname{div}R_{\text{cor}} + \nabla\pi_{\text{cor}}}\\
&\quad + \underbrace{\operatorname{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_l\right) + \partial_{tt} w_{q+1}^{(t)}}_{\operatorname{div}R_{\text{osc}} + \nabla\pi_{\text{osc}}} + \operatorname{div}R_{\text{com}} - \nabla\pi_l; \qquad (96)
\end{aligned}$$

i.e.

$$\operatorname{div}\mathring{R}_{q+1} - \nabla\pi_{q+1} = \operatorname{div}\left(R_{\text{lin }1} + R_{\text{lin }2} + R_{\text{corr}} + R_{\text{osc}} + R_{\text{com}}\right) - \nabla\left(\pi_{\text{lin }1} + \pi_{\text{corr}} + \pi_{\text{osc}} + \pi_l\right)$$

where

$$R_{\text{lin }1} \triangleq \mathcal{R}(-\Delta)^m w_{q+1} + \mathcal{R}\partial_t\left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) + v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l, \qquad (97a)$$

$$\pi_{\text{lin }1} \triangleq v_l \cdot w_{q+1}, \qquad (97b)$$

$$R_{\text{lin }2} \triangleq \mathcal{R}\partial_{tt}\left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right), \qquad (97c)$$

$$R_{\text{cor}} \triangleq \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right), \qquad (97d)$$

$$\pi_{\text{cor}} \triangleq \frac{1}{2}\left[\left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) \cdot w_{q+1} + w_{q+1}^{(p)} \cdot \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\right]. \qquad (97e)$$

Concerning R_{osc} that is arguably the most technical, we can write

$$R_{\text{osc}} \triangleq \mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,1} + \frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\sum_{k=1,3,4}\mathcal{E}_{\zeta,\vartheta,2,k} + A_2 + A_3\right), \qquad (98a)$$

$$\begin{aligned}
\pi_{\text{osc}} \triangleq &\frac{1}{2}|w_{q+1}^{(p)}|^2 + \rho\left(1 - \frac{1}{8}\sum_{\zeta\in\Lambda}\gamma_\zeta\left(\operatorname{Id} - \frac{\mathring{R}_l}{\rho}\right)^2\right) \qquad (98b)\\
&+ \frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathbb{P}_{\neq 0}\left(a_\zeta a_\vartheta \mathbb{P}_{\geqslant \frac{\lambda_{q+1}}{2}}\left(\eta_\zeta\eta_\vartheta\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)\right)1_{\zeta+\vartheta\neq 0}\\
&+ \frac{1}{2}\sum_{\zeta\in\Lambda}a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2 - \Delta^{-1}\nabla\cdot\mu^{-1}\left(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-}\right)\mathbb{P}_{\neq 0}\partial_t\left(a_\zeta^2\mathbb{P}_{\neq 0}\eta_\zeta^2\zeta\right)
\end{aligned}$$

where $\mathcal{E}_{\zeta,\vartheta,1}$ can be found in (183a), $\mathcal{E}_{\zeta,\vartheta,2,1}$ in (188a), and A_2 and A_3 are defined in (192b)-(192c). We leave the detailed derivation of (98) in appendix C for completeness. Considering (96) we define

$$\pi_{q+1} \triangleq \pi_l - \pi_{\text{lin }1} - \pi_{\text{cor}} - \pi_{\text{osc}} \text{and } \mathring{R}_{q+1} \triangleq R_{\text{lin }1} + R_{\text{lin }2} + R_{\text{cor}} + R_{\text{osc}} + R_{\text{com}}. \qquad (99)$$

Proposition 4.5. *$R_{lin\,1}$ defined in (97a) satisfies for a_0 sufficiently large and $\beta > 0$ sufficiently small*

$$\|R_{lin\,1}\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3} e(t).$$

Proof of proposition 4.5. First, we estimate the diffusive term, recalling the $m \in (\frac{1}{2}, \frac{2}{3})$ due to remark 4.2:

$$\begin{aligned}
&\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_{t,q+1}L_x^{p^*}} \\
&\lesssim \|w_{q+1}\|_{C_{t,q+1}L_x^{p^*}}^{2-2m} \|\nabla w_{q+1}\|_{C_{t,q+1}L_x^{p^*}}^{2m-1} \\
&\overset{(72)(94)(95)}{\lesssim} \left[\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \left(1 + \lambda_{q+1}^{-1} l^{-\frac{25}{2}} + \sigma r\right) + \mu^{-1}\delta_{q+2} l^{-3} r^{2-\frac{2}{p^*}}\right]^{2-2m} \\
&\quad \times \left[\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1} + \mu^{-1}\delta_{q+2} l^{-3}\lambda_{q+1}\sigma r^{3-\frac{2}{p^*}}\right]^{2m-1} \\
&\lesssim \left[\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + \mu^{-1}\delta_{q+2} l^{-3} r^{2-\frac{2}{p^*}}\right]^{2-2m} \\
&\quad \times \left[\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1} + \mu^{-1}\delta_{q+2} l^{-3}\lambda_{q+1}\sigma r^{3-\frac{2}{p^*}}\right]^{2m-1} \overset{(101)}{\lesssim} \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1}^{2m-1}
\end{aligned} \tag{100}$$

where we used that $\sigma r \ll 1$ due to (12), and $\lambda_{q+1}^{-1} l^{-\frac{25}{2}} \lesssim 1$ due to (83) in the second to last inequality and also assumed

$$\mu^{-1}\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r \lesssim 1, \quad \mu^{-1}\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} r^2\sigma \lesssim 1 \tag{101}$$

in the last inequality, none of which implies condition on μ, σ, or r because $\mu^{-1} r \ll 1$ and $r\sigma \ll 1$ due to (12) so that $\mu^{-1} r^2 \sigma \ll 1$. Now, after r, μ, and σ^{-1} have been selected in (32), in order to determine l, let us observe that due to (32) and (101) is equivalent to $\lambda_{q+1}^{\frac{3m-2}{7}} \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lesssim 1$ which is implied by (70), and

$$\lambda_{q+1}^{\frac{m-2}{7}} l^{-\frac{3}{2}} \lesssim 1. \tag{102}$$

After r, μ, and σ^{-1} in (32) and l in (33) have been selected, we see that the estimate in (102) can be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) if

$$\frac{9(56)}{58-3m} < b \tag{103}$$

holds, as well as by taking $\beta \in (0,1)$ sufficiently small after b in (34) is fixed; we incorporate this condition to our choice of b in (34) to claim (102) and hence (100).

Now that we have claimed (100), considering $r^{1-\frac{2}{p^*}} \lambda_{q+1}^{2m-1}$ in (100) for $p^* > 1$ arbitrarily close, we now impose

$$r^{-1}\lambda_{q+1}^{2m-1} \ll 1. \tag{104}$$

In fact, let us make a stronger assumption that

$$l^{-\frac{3}{2}} r^{-1} \lambda_{q+1}^{2m-1} \ll 1$$

or equivalently due to (32)

$$l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{3m-2}{7}} \ll 1. \tag{105}$$

The estimate (105) can be seen to be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) if

$$\frac{9(112)}{58(2-3m)} < b, \tag{106}$$

and we incorporate this condition to our choice of b in (34) to claim (105).

At last, with all the parameters chosen thus far, let us make our argument continuing from (100) formal by selecting the appropriate $p^* \in (1,2)$ in (35) and making use of the fact that $e(t) \geqslant \underline{e} \geqslant 4$ due to (3): for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\begin{aligned} &\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_{t,q+1}L_x^{p^*}} \\ &\overset{(100)(33)(32)}{\lesssim} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}}\right)^{\frac{3}{2}} \left(\lambda_{q+1}^{\frac{11m-5}{7}}\right)^{1-\frac{2}{p^*}} \lambda_{q+1}^{2m-1} \\ &\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{448m-224}{224} + \frac{3(2-3m)}{224} + \frac{9}{4}\left(\frac{2-3m}{(42)(56)}\right) + \frac{11m-5}{7} - \frac{(11m-5)}{7}\frac{2}{p^*}} \approx \lambda_{q+1}^{\frac{11065m-5286}{14(224)} - \frac{(11m-5)2}{7p^*}} \ll \delta_{q+3} e(t) \end{aligned} \tag{107}$$

where the last inequality is because $\frac{11065m-5286}{14(224)} - \frac{(11m-5)2}{7p^*} < 0$ if and only if $p^* < \frac{(11m-5)(28)(32)}{11065m-5286}$ and this holds due to our choice from p^* in (35).

Second, we estimate

$$\|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\|_{C_{t,q+1}L_x^{p^*}} \leqslant \|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right)\|_{C_{t,q+1}L_x^{p^*}} + \|\mathcal{R}\partial_t w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^{p^*}}. \tag{108}$$

We can compute separately

$$\begin{aligned} \|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right)\|_{C_{t,q+1}L_x^{p^*}} &\overset{(66)}{\lesssim} \sum_{\zeta \in \Lambda} \|\partial_t a_\zeta\|_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_{t,q+1}L_x^{p^*}} \|\psi_\zeta\|_{C_x} \\ &\quad + \|a_\zeta\|_{C_{t,x,q+1}} \|\partial_t \eta_\zeta\|_{C_{t,q+1}L_x^{p^*}} \|\psi_\zeta\|_{C_x} \\ &\overset{(54)(11b)(17b)}{\lesssim} \delta_{q+2}^{\frac{1}{2}} r^{1-\frac{2}{p^*}} \left[l^{-14}\lambda_{q+1}^{-1} + l^{-\frac{3}{2}} \sigma \mu r\right], \end{aligned} \tag{109a}$$

$$\begin{aligned} \|\mathcal{R}\partial_t w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^{p^*}} &\overset{(63)}{\lesssim} \|\mu^{-1} \left(\sum_{\zeta \in \Lambda^+} - \sum_{\zeta \in \Lambda^-}\right) \mathbb{P}\mathbb{P}_{\neq 0}\left(a_\zeta^2 \mathbb{P}_{\neq 0} \eta_\zeta^2 \zeta\right)\|_{C_{t,q+1}L_x^{p^*}} \\ &\overset{(72d)}{\lesssim} \mu^{-1} \delta_{q+2} l^{-3} r^{2-\frac{2}{p^*}}. \end{aligned} \tag{109b}$$

Applying (109) to (108) gives us

$$\|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\|_{C_{t,q+1}L_x^{p^*}} \lesssim \delta_{q+2}^{\frac{1}{2}} r^{1-\frac{2}{p^*}} \left[l^{-14}\lambda_{q+1}^{-1} + l^{-\frac{3}{2}} \sigma \mu r\right] + \mu^{-1} \delta_{q+2} l^{-3} r^{2-\frac{2}{p^*}}. \tag{110}$$

Concerning the first term $\delta_{q+2}^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-14} \lambda_{q+1}^{-1}$, paying attention to only the dominant term $r^{1-\frac{2}{p^*}} \lambda_{q+1}^{-1}$ for $p^* > 1$ arbitrarily close, we see that the first term does not impose any condition on r, μ, or σ^{-1} because $1 \ll r\lambda_{q+1}$ due to (12). Similarly, for the second term $\delta_{q+2}^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \sigma\mu r$, $r^{1-\frac{2}{p^*}}(\sigma\mu r) \ll 1$ is expected to hold for $p^* > 1$ arbitrarily close to 1 due to $\mu \ll \sigma^{-1}$ from (12). Finally, for the third term $\mu^{-1}\delta_{q+2} l^{-3} r^{2-\frac{2}{p^*}}$, we can rely on $\mu^{-1} \ll 1$ due to (12). Therefore, bounding (110) by a small constant multiple of $\delta_{q+3} e(t)$ does not impose any additional conditions on our choice of r, μ, and σ^{-1}.

That being said, in order to see the conditions on l, let us continue to estimate from (110) using (32) as follows:

$$\begin{aligned}
&\|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)\|_{C_{t,q+1} L_x^{p^*}} \\
&\overset{(110)(32)}{\lesssim} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} \left[l^{-14} \lambda_{q+1}^{-1} + l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-16}{14}} \right] + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(2-\frac{2}{p^*}\right)} \\
&\lesssim \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-16}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(2-\frac{2}{p^*}\right)}
\end{aligned} \tag{111}$$

where in the last inequality we assumed that

$$l^{-\frac{25}{2}} \lesssim \lambda_{q+1}^{\frac{17m-2}{14}} \tag{112}$$

so that $l^{-14}\lambda_{q+1}^{-1} \lesssim l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-16}{14}}$. Thus, assuming $p^* \in (1,2)$ to be taken is arbitrarily close to 1, we see the need to impose

$$\lambda_{q+1}^{\left(\frac{11m-5}{7}\right)(-1)} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-16}{14}} \ll 1 \text{and } \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \ll 1 \tag{113}$$

or equivalently

$$l^{-\frac{3}{2}} \ll \lambda_{q+1}^{\frac{6+5m}{14}} \text{ and } l^{-3} \ll \lambda_{q+1}^{\frac{8m-3}{7}} \tag{114}$$

respectively.

After our choices of r, μ, and σ^{-1} in (32) and thereafter l in (33) are determined determined, in order to find the conditions on b, we observe that the estimates (112) and (114) can be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) if

$$\frac{21\,(200)}{347m - 82} < b \tag{115}$$

and we incorporate this condition to our choice of b in (34) to claim (112) and (114).

We now use all our parameters chosen and make these arguments precise, continuing from (111) by formally selecting the appropriate $p^* \in (1,2)$ from (35), making use of the fact that $e(t) \geqslant \underline{e} \geqslant 4$: for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$
\begin{aligned}
&\|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(111)(33)(32)}{\lesssim} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)} \lambda_{q+1}^{-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}}\right)^{\frac{3}{2}} \lambda_{q+1}^{\frac{17m-16}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}}\right)^{3} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)2} \lambda_{q+1}^{-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{11m-5}{7}} \lambda_{q+1}^{-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \lambda_{q+1}^{\frac{3(2-3m)}{2(112)}} \lambda_{q+1}^{\frac{9}{4}\left[\frac{2-3m}{(42)(56)}\right]} \lambda_{q+1}^{\frac{17m-16}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} \lambda_{q+1}^{\frac{3(2-3m)}{112}} \lambda_{q+1}^{\frac{9}{2}\left[\frac{2-3m}{(42)(56)}\right]} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)2} \lambda_{q+1}^{-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\approx \lambda_{q+1}^{\frac{2867(3m-2)}{14(224)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{3001m-1478}{14(112)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \ll \delta_{q+3} e(t)
\end{aligned}
\tag{116}
$$

where the last inequality used the fact that $\frac{2867(3m-2)}{14(224)} - (\frac{11m-5}{7})\frac{2}{p^*} < 0$ as $m < \frac{2}{3}$ and $\frac{3001m-1478}{14(112)} - (\frac{11m-5}{7})\frac{2}{p^*} < 0$ due to our choice from (35).

Third, we estimate

$$
\begin{aligned}
&\|v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l\|_{C_{t,q+1}L_x^{p^*}} \\
&\lesssim \|v_q\|_{C^1_{t,x,q+1}} \left(\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^{p^*}} + \|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^{p^*}} + \|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^{p^*}}\right) \\
&\overset{(26b)(72)}{\lesssim} \lambda_q^3 \left(l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + \lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r\right] + \mu^{-1} l^{-3} r^{2-\frac{2}{p^*}}\right).
\end{aligned}
\tag{117}
$$

Concerning the first two terms $\lambda_q^3 l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}}$ and $\lambda_q^3 \lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{25}{2}}$, paying attention to only $r^{1-\frac{2}{p^*}}$, when $p^* > 1$ is arbitrarily close to 1, we see that no condition is imposed on r because $r^{-1} \ll 1$ due to (12). Similarly, the third term $\lambda_q^3 \lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1}\sigma r = \lambda_q^3 l^{-\frac{3}{2}} \sigma r^{2-\frac{2}{p^*}}$ does not cause an additional condition because $\sigma \ll 1$ due to (12). Finally, the fourth term $\lambda_q^3 \mu^{-1} l^{-3} r^{2-\frac{2}{p^*}}$ does not require any additional condition on μ because $\mu^{-1} \ll 1$ due to (12).

In order to determine conditions on l after r, μ, and σ^{-1} are determined in (32), we continue from (117) with our choice from (32) as follows:

$$
\begin{aligned}
&\|v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(117)(85)}{\lesssim} \lambda_q^3 \left(l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + \lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1}\sigma r + \mu^{-1} l^{-3} r^{2-\frac{2}{p^*}}\right) \\
&\overset{(32)}{\approx} \lambda_q^3 \left(l^{-\frac{3}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} + l^{-\frac{3}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} \lambda_{q+1}^{\frac{m-10}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(2-\frac{2}{p^*}\right)}\right) \\
&\lesssim \lambda_q^3 \left(l^{-\frac{3}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} r^{\left(\frac{11m-5}{7}\right)\left(2-\frac{2}{p^*}\right)}\right)
\end{aligned}
\tag{118}
$$

where in the last inequality, we used $\lambda_{q+1}^{\frac{m-10}{14}} \lesssim 1$. Considering $p^* > 1$ arbitrarily close to 1, we impose additionally

$$
\lambda_q^3 l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{5-11m}{7}} \ll 1, \quad \lambda_q^3 \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \ll 1. \tag{119}
$$

The estimate (119) can be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) if

$$
\frac{21(56)}{361m - 166} < b, \tag{120}
$$

and we incorporate this condition to our choice of b in (34) to claim (119).

Now let us use our choices of parameters and make this argument precise, continuing from (118) as follows by formally selecting the appropriate $p^* \in (1,2)$ from (35), using the fact that $e(t) \geqslant \underline{e} \geqslant 4$ due to (3): for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\begin{aligned}
&\|v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(118)(33)}{\lesssim} \lambda_q^3 \left[\left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}} \right)^{\frac{3}{2}} \lambda_{q+1}^{\frac{11m-5}{7} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{3-8m}{7}} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}} \right)^3 \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)2 - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \right] \\
&\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{3(2-3m)}{2(112)} + \frac{21}{4}\left(\frac{2-3m}{(42)(56)}\right) + \frac{11m-5}{7} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{3-8m}{7} + \frac{3(2-3m)}{112} + \frac{15}{2}\left(\frac{2-3m}{(42)(56)}\right) + \frac{2(11m-5)}{7} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\approx \lambda_{q+1}^{\frac{-306+683m}{4(112)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{-1474+2995m}{14(112)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \ll \delta_{q+3} e(t)
\end{aligned} \tag{121}$$

where the last inequality is de to $\frac{-306+683m}{4(112)} - (\frac{11m-5}{7})\frac{2}{p^*} < 0$ and $\frac{-1474+2995m}{14(112)} - (\frac{11m-5}{7})\frac{2}{p^*} < 0$ due to our choice of p^* from (35).

We are now able to conclude that for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\begin{aligned}
\|R_{\text{lin }1}\|_{C_{t,q+1}L_x^1} &\overset{(97a)}{\leqslant} \|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_{t,q+1}L_x^1} + \|\mathcal{R}\partial_t \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)\|_{C_{t,q+1}L_x^1} \\
&\quad + \|v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l\|_{C_{t,q+1}L_x^1} \overset{(107)(116)(121)}{\ll} \delta_{q+3} e(t).
\end{aligned} \tag{122}$$

□

Proposition 4.6. *$R_{lin\,2}$ defined in (97c) satisfies for a_0 sufficiently large and $\beta > 0$ sufficiently small*

$$\|R_{lin\,2}\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3} e(t).$$

Proof of proposition 4.6. We come to the unique term that we must estimate for the hyperbolic Navier–Stokes equations. This term is singular due to the second derivatives with respect to time variable t and creates constraints in the choice of parameters, and ultimately the upper bound of $m < \frac{2}{3}$, as it will be explained in detail appendix A. We estimate

$$\begin{aligned}
&\|\mathcal{R}\partial_{tt} \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} \right)\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(66)}{\lesssim} \sum_{\zeta \in \Lambda} \|\partial_{tt} [a_\zeta(t,x) \eta_\zeta(t,x)] \psi_\zeta(x)\|_{C_{t,q+1}L_x^{p^*}} \\
&\lesssim \sum_{\zeta \in \Lambda} \left(\|\partial_{tt} a_\zeta\|_{C_{t,x,q+1}} \|\eta_\zeta\|_{C_{t,q+1}L_x^{p^*}} + \|a_{(\zeta)}\|_{C_{t,x,q+1}} \|\partial_{tt}\eta_\zeta\|_{C_{t,q+1}L_x^{p^*}} \right) \|\psi_\zeta\|_{C_x} \\
&\overset{(54)(11b)(17b)}{\lesssim} \left(\left[\delta_{q+2}^{\frac{1}{2}} l^{-20} \right] r^{1-\frac{2}{p^*}} + \left[\delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \right] (\lambda_{q+1}\sigma r \mu)^2 r^{1-\frac{2}{p^*}} \right) \lambda_{q+1}^{-1}.
\end{aligned} \tag{123}$$

The first term in (123), in which we only pay attention to the dominant term $r^{1-\frac{2}{p^*}} \lambda_{q+1}^{-1}$, does not require any additional condition on r because $1 \ll \lambda_{q+1} r$ due to (12). However, for the second term in (123), paying attention to only $(\lambda_{q+1}\sigma r \mu)^2 r^{1-\frac{2}{p^*}} \lambda_{q+1}^{-1}$ when $p^* > 1$ is arbitrarily close to 1, we obtain a new condition of

$$\mu \ll \lambda_{q+1}^{-\frac{1}{2}} \sigma^{-1} r^{-\frac{1}{2}}. \tag{124}$$

After having selected r, μ, and σ^{-1} in (32), in order to find conditions on l, continuing from (123), we estimate by using our choices from (32) as follows:

$$\|\mathcal{R}\partial_{tt}\left(w_{q+1}^{(p)}+w_{q+1}^{(c)}\right)\|_{C_{t,q+1}L_x^{p^*}} \lesssim \left(\delta_{q+2}^{\frac{1}{2}} l^{-20} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)} + \delta_{q+2}^{\frac{1}{2}} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-2}{7}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(1-\frac{2}{p^*}\right)}\right) \lambda_{q+1}^{-1}. \tag{125}$$

Considering $p^* > 1$ arbitrarily close to 1, this requires that we impose additionally

$$l^{-20} \lambda_{q+1}^{\frac{5-11m}{7}} \lambda_{q+1}^{-1} \ll 1, \tag{126}$$

as well as $l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{17m-2}{7}} \lambda_{q+1}^{\frac{5-11m}{7}} \lambda_{q+1}^{-1} \ll 1$, but this condition is implied by (70).

After l has been determined in (33), in order to find conditions on b, we use our choice from (33) and see that (126) can be satisfied if

$$\frac{30(28)}{59m-2} < b. \tag{127}$$

and we incorporate this condition to our choice of b in (34) to claim (126).

Now with all the parameters chosen, let us make these arguments precise continuing from (125) by formally selecting the appropriate $p^* \in (1,2)$ from (35) and utilizing the fact that $e(t) \geqslant \underline{e} \geqslant 4$ due to (3): for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$\begin{aligned}
&\|\mathcal{R}\partial_{tt}\left(w_{q+1}^{(p)}+w_{q+1}^{(c)}\right)\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(125)(33)}{\lesssim} \left(\left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}}\right)^{20} \lambda_{q+1}^{\frac{11m-5}{7}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}}\right)^{\frac{3}{2}} \lambda_{q+1}^{\frac{17m-2}{7}} \lambda_{q+1}^{\frac{11m-5}{7}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}}\right) \lambda_{q+1}^{-1} \\
&\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{5(2-3m)}{28}+30\left(\frac{2-3m}{(42)(56)}\right)+\frac{11m-12}{7}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{3(2-3m)}{224}+\frac{9}{4}\left(\frac{2-3m}{(42)(56)}\right)+4m-2-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\approx \lambda_{q+1}^{\frac{391m-522}{7(56)}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} + \lambda_{q+1}^{\frac{12319m-6122}{(14)(224)}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \ll \delta_{q+3} e(t)
\end{aligned} \tag{128}$$

where the last inequality is due to $\frac{391m-522}{7(56)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*} < 0$ which is immediate because $m \in \left(\frac{1}{2}, \frac{2}{3}\right)$, and $\frac{12319m-6122}{(14)(224)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*} < 0$ due to our choice of p^*.

Hence, we conclude that for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$\|R_{\text{lin }2}\|_{C_{t,q+1}L_x^1} \overset{(97c)}{=} \|\mathcal{R}\partial_{tt}\left(w_{q+1}^{(p)}+w_{q+1}^{(c)}\right)\|_{C_{t,q+1}L_x^1} \ll \delta_{q+3} e(t). \tag{129}$$

□

Proposition 4.7. *R_{cor} defined in (97d) satisfies for a_0 sufficiently large and $\beta > 0$ sufficiently small,*

$$\|R_{cor}\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3} e(t).$$

Proof of proposition 4.7. We estimate

$$\begin{aligned}
&\| \left(w_{q+}^{(c)} + w_{q+1}^{(t)}\right) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) \|_{C_{t,q+1}L_x^{p^*}} \\
&\lesssim \left(\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^{2p^*}} \right) \\
&\quad \times \left(\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^{2p^*}} + \|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^{2p^*}} \right) \\
&\overset{(72)}{\lesssim} \left(\lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{1}{p^*}} \left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r \right] + \mu^{-1} l^{-3} r^{2-\frac{1}{p^*}} \right) \\
&\quad \times \left(l^{-\frac{3}{2}} r^{1-\frac{1}{p^*}} + \lambda_{q+1}^{-1} l^{-\frac{3}{2}} r^{1-\frac{1}{p^*}} \left[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r \right] + \mu^{-1} l^{-3} r^{2-\frac{1}{p^*}} \right).
\end{aligned} \tag{130}$$

Considering the dominant terms for $p^* > 1$ arbitrarily close to 1, we see that it suffices that $\lambda_{q+1}^{-1}[l^{-\frac{25}{2}} + \lambda_{q+1}\sigma r] + \mu^{-1} r \ll 1$; thus, this does not impose any condition on our choices of r, μ, or σ^{-1} because $\sigma r \ll 1$ and $\mu^{-1} r \ll 1$ due to (12).

After μ, r, and σ^{-1} are determined in (32), in order to determine the conditions on l, we continue from (130) with our choices from (32) as follows:

$$\begin{aligned}
&\| (w_{q+}^{(c)} + w_{q+1}^{(t)}) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(130)(32)(85)}{\lesssim} \left(\lambda_{q+1}^{-1} l^{-\frac{3}{2}} \lambda_{q+1}^{(\frac{11m-5}{7})(1-\frac{1}{p^*})} \lambda_{q+1}^{\frac{4+m}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{(\frac{11m-5}{7})(2-\frac{1}{p^*})} \right) \\
&\quad \times \left(l^{-\frac{3}{2}} \lambda_{q+1}^{(\frac{11m-5}{7})(1-\frac{1}{p^*})} + \lambda_{q+1}^{-1} l^{-\frac{3}{2}} \lambda_{q+1}^{(\frac{11m-5}{7})(1-\frac{1}{p^*})} \lambda_{q+1}^{\frac{4+m}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{(\frac{11m-5}{7})(2-\frac{1}{p^*})} \right) \\
&\lesssim \left(\lambda_{q+1}^{-1} l^{-\frac{3}{2}} \lambda_{q+1}^{(\frac{11m-5}{7})(1-\frac{1}{p^*})} \lambda_{q+1}^{\frac{4+m}{14}} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{(\frac{11m-5}{7})(2-\frac{1}{p^*})} \right) \\
&\quad \times \left(l^{-\frac{3}{2}} \lambda_{q+1}^{(\frac{11m-5}{7})(1-\frac{1}{p^*})} + \lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{(\frac{11m-5}{7})(2-\frac{1}{p^*})} \right).
\end{aligned} \tag{131}$$

Considering $p^* > 1$ arbitrarily close to one, we see that we need to impose

$$l^{-3} \lambda_{q+1}^{\frac{m-10}{14}} \ll 1 \text{and } l^{-\frac{9}{2}} \lambda_{q+1}^{\frac{3m-2}{7}} \ll 1 \tag{132}$$

so that $(\lambda_{q+1}^{-1} l^{-\frac{3}{2}} \lambda_{q+1}^{\frac{4+m}{14}}) l^{-\frac{3}{2}} \ll 1$ and $(\lambda_{q+1}^{\frac{3-8m}{7}} l^{-3} \lambda_{q+1}^{\frac{11m-5}{7}}) l^{-\frac{3}{2}} \ll 1$, respectively.

After l is chosen in (33), in order to determine the conditions on b, we see that the estimate (132) can be satisfied by our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) if

$$\frac{(27)(56)}{23(2-3m)} < b, \tag{133}$$

and we incorporate this condition to our choice of b in (34).

Now with all the selected parameters, let us make these estimates more precise continuing from (131) and formally selecting the appropriate $p^* \in (1,2)$ from (35): for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$
\begin{aligned}
&\left\| \left(w_{q+}^{(c)} + w_{q+1}^{(t)} \right) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right) \right\|_{C_{t,q+1} L_x^{p^*}} \\
&\overset{(131)(33)}{\lesssim} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}} \right)^3 \lambda_{q+1}^{\frac{2(11m-5)}{7} - \frac{2(11m-5)}{7p^*} + \frac{m-10}{14}} + \left(\lambda_{q+1}^{\frac{2-7m}{112}} \lambda_q^{\frac{3}{2}} \right)^{\frac{9}{2}} \lambda_{q+1}^{\frac{3(11m-5)}{7} - \frac{2(11m-5)}{7p^*} + \frac{3-8m}{7}} \\
&\quad + \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}} \right)^6 \lambda_{q+1}^{\frac{4(11m-5)}{7} - \frac{2(11m-5)}{7p^*} + \frac{2(3-8m)}{7}} \\
&\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{3(2-3m)}{112} + \frac{9}{2}\left(\frac{2-3m}{(42)(56)}\right) + \frac{2(11m-5)}{7} + \frac{m-10}{14} - \frac{2(11m-5)}{7p^*}} + \lambda_{q+1}^{\frac{9(2-3m)}{224} + \frac{27}{4}\left(\frac{2-3m}{(42)(56)}\right) + \frac{25m-12}{7} - \frac{2(11m-5)}{7p^*}} \\
&\quad + \lambda_{q+1}^{\frac{3(2-3m)}{56} + 9\left(\frac{2-3m}{(42)(56)}\right) + \frac{28m-14}{7} - \frac{2(11m-5)}{7p^*}} \\
&\approx \lambda_{q+1}^{\frac{1635(3m-2)}{(14)(112)} - \frac{2(11m-5)}{7p^*}} + \lambda_{q+1}^{\frac{10795m-5106}{(28)(112)} - \frac{2(11m-5)}{7p^*}} + \lambda_{q+1}^{\frac{3001m-1478}{(14)(56)} - \frac{2(11m-5)}{7p^*}} \ll \delta_{q+3} e(t)
\end{aligned}
\tag{134}
$$

where the last inequality is due to $\frac{1635(3m-2)}{(14)(112)} - \frac{2(11m-5)}{7p^*} < 0$ due to $m < \frac{2}{3}$ while $\frac{10795m-5106}{(28)(112)} - \frac{2(11m-5)}{7p^*} < 0$ and $\frac{3001m-1478}{(14)(56)} - \frac{2(11m-5)}{7p^*} < 0$ due to our choice of p^* from (35).

Therefore, we conclude that for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$
\|R_{\text{cor}}\|_{C_{t,q+1} L_x^1} \overset{(97d)}{=} \left\| \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right) \right\|_{C_{t,q+1} L_x^1} \ll \delta_{q+3} e(t). \tag{135}
$$

□

Proposition 4.8. *R_{osc} defined in (98a) satisfies for a_0 sufficiently large and $\beta > 0$ sufficiently small*

$$
\|R_{osc}\|_{C_{t,q+1} L_x^{p^*}} \ll \delta_{q+3} e(t).
$$

Proof of proposition 4.8. First, we estimate using lemma 3.5

$$
\begin{aligned}
&\left\| \mathcal{R} \left(\frac{1}{2} \sum_{\zeta, \vartheta \in \Lambda} \mathcal{E}_{\zeta, \vartheta, 1} \right) \right\|_{C_{t,q+1} L_x^{p^*}} \\
&\overset{(183)(20)}{\lesssim} \lambda_{q+1}^{-1} \sigma^{-1} \sum_{\zeta, \vartheta \in \Lambda} \left(\|a_\zeta\|_{C_{t,q+1} C_x^3} \|a_\vartheta\|_{C_{t,x,q+1}} + \|a_\zeta\|_{C_{t,x,q+1}} \|a_\vartheta\|_{C_{t,q+1} C_x^3} \right) \|W_\zeta\|_{C_{t,q+1} L_x^{2p^*}} \|W_\vartheta\|_{C_{t,q+1} L_x^{2p^*}} \\
&\overset{(54)(17a)}{\lesssim} \lambda_{q+1}^{-1} \sigma^{-1} \delta_{q+2} l^{-\frac{55}{2}} r^{2 - \frac{2}{p^*}}.
\end{aligned}
\tag{136}
$$

For $p^* > 1$ arbitrarily close to 1, this does not impose any condition on our choice of r, μ, or σ^{-1} because $\lambda_{q+1}^{-1} \sigma^{-1} \ll 1$ due to (12).

After selecting r, μ, and σ^{-1} in (32), in order to find conditions on l we continue from (136) with our choices from (32) as follows:

$$
\left\| \mathcal{R} \left(\frac{1}{2} \sum_{\zeta, \vartheta \in \Lambda} \mathcal{E}_{\zeta, \vartheta, 1} \right) \right\|_{C_{t,q+1} L_x^{p^*}} \overset{(136)(32)}{\lesssim} \lambda_{q+1}^{-1} \lambda_{q+1}^{\frac{3m}{2}} \delta_{q+2} l^{-\frac{55}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)\left(2 - \frac{2}{p^*}\right)}. \tag{137}
$$

Considering $p^* > 1$ arbitrarily close to 1, this implies that we need additional condition of

$$
\lambda_{q+1}^{-1 + \frac{3m}{2}} l^{-\frac{55}{2}} \ll 1. \tag{138}
$$

To see the necessary conditions on b, we plug in our choices of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) and see that (138) can be satisfied if

$$\frac{56(165)}{57(2-3m)} < b, \tag{139}$$

and we incorporate this condition to our choice of b in (34) to claim (138).

We now use our choice of parameters to make these arguments precise, continuing from (137) by formally selecting the appropriate $p^* \in (1,2)$ from (35): for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\left\| \mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,1}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \overset{(137)(33)}{\lesssim} \lambda_{q+1}^{\frac{3m}{2}-1}\left(\lambda_{q+1}^{\frac{2-3m}{112}}\lambda_q^{\frac{3}{2}}\right)^{\frac{55}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)2-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}}$$
$$\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{3m-2}{2}+\frac{55(2-3m)}{2(112)}+\frac{165}{4}\left[\frac{2-3m}{(42)(56)}\right]+\frac{2(11m-5)}{7}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \approx \lambda_{q+1}^{\frac{12085m-5966}{28(112)}-\left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \ll \delta_{q+3}e(t) \tag{140}$$

where the last inequality is due to $\frac{12085m-5966}{28(112)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}$ due to (35).

Second, we estimate

$$\left\| \mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,3}\right)\right\|_{C_{t,q+1}L_x^{p^*}}$$
$$\overset{(188c)(20)}{\lesssim} \sum_{\zeta,\vartheta\in\Lambda} \lambda_{q+1}^{-1}\left(\|a_\zeta\|_{C_{t,q+1}C_x^3}\|a_\vartheta\|_{C_{t,x,q+1}} + \|a_\zeta\|_{C_{t,x,q+1}}\|a_\vartheta\|_{C_{t,q+1}C_x^3}\right)$$
$$\times \|\eta_\zeta\|_{C_{t,q+1}L_x^{2p^*}}\|\eta_\vartheta\|_{C_{t,q+1}L_x^{2p^*}}\lambda_{q+1}^2\|\psi_\zeta\|_{C_x}\|\psi_\vartheta\|_{C_x} \overset{(54c)(11b)(17b)}{\lesssim} \lambda_{q+1}^{-1}\delta_{q+2}l^{-\frac{55}{2}}r^{2-\frac{2}{p^*}}. \tag{141}$$

We mention that in the application of (20), it is required that $\frac{\lambda_{q+1}}{10} \in \mathbb{N}$ which is the reason why we chose to impose $a \in 10\mathbb{N}$. For $p^* > 1$ arbitrarily close to 1, this term does not impose any condition on our choice of r, μ, or σ^{-1} because $\lambda_{q+1}^{-1} \ll 1$.

Hence, in order to see conditions on other parameters, we continue from (141) using or choice of r, μ, and σ^{-1} from (32) as follows:

$$\left\| \mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,3}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \overset{(141)(32)}{\lesssim} \lambda_{q+1}^{-1}\delta_{q+2}l^{-\frac{55}{2}}\left(\lambda_{q+1}^{\frac{11m-5}{7}}\right)^{2-\frac{2}{p^*}}. \tag{142}$$

The right hand side of (142) is bounded by the right hand side of (137) and thus we immediately conclude that for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\left\| \mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,3}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3}e(t). \tag{143}$$

Third, we compute

$$\begin{aligned}
&\left\| \mathcal{R}\left(\frac{1}{2} \sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,1} \right) \right\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(188a)(20)}{\lesssim} \sum_{\zeta,\vartheta\in\Lambda} \lambda_{q+1}^{-1} \left(\|a_\zeta\|_{C_{t,q+1}C_x^2} \|a_\vartheta\|_{C_{t,x,q+1}} + \|a_\zeta\|_{C_{t,x,q+1}} \|a_\vartheta\|_{C_{t,q+1}C_x^2} \right) \|b_\zeta\|_{C_x} \|b_\vartheta\|_{C_x} \\
&\quad \times \left(\|\nabla \eta_\zeta\|_{C_{t,q+1}L_x^{2p^*}} \|\eta_\vartheta\|_{C_{t,q+1}L_x^{2p^*}} + \|\eta_\zeta\|_{C_{t,q+1}L_x^{2p^*}} \|\nabla \eta_\vartheta\|_{C_{t,q+1}L_x^{2p^*}} \right) \\
&\overset{(54c)(11b)(17b)}{\lesssim} \delta_{q+2} l^{-\frac{43}{2}} \sigma r^{3-\frac{2}{p^*}}. \qquad (144)
\end{aligned}$$

Considering $p^* > 1$ arbitrarily close to 1, we see that this term does not impose any additional condition because $\sigma r \ll 1$ due to (12).

After r, μ, and σ^{-1} in (32) are selected, in order to determine l, we continue from (144) with our choices from (32) as follows:

$$\left\| \mathcal{R}\left(\frac{1}{2} \sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,1} \right) \right\|_{C_{t,q+1}L_x^{p^*}} \overset{(144)(32)}{\lesssim} \delta_{q+2} l^{-\frac{43}{2}} \lambda_{q+1}^{-\frac{3m}{2}} \left(\lambda_{q+1}^{\frac{11m-5}{7}} \right)^{3-\frac{2}{p^*}}. \qquad (145)$$

Considering $p^* > 1$ arbitrarily close to 1, this leads to additional condition of

$$l^{-1} \ll \lambda_{q+1}^{\frac{1}{43}\left(\frac{10-m}{7}\right)} = \lambda_{q+1}^{\frac{2}{43}\left(\frac{10-m}{14}\right)} \qquad (146)$$

so that $l^{-\frac{43}{2}} \lambda_{q+1}^{-\frac{3m}{2}} \lambda_{q+1}^{\frac{11m-5}{7}} \ll 1$.

After l in (33) is selected, in order to determine conditions on b, we apply our choice of $l = \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}}$ from (33) and see that (146) is satisfied if

$$\frac{3(42)}{2m+1} < b, \qquad (147)$$

and we incorporate this condition to our choice of b in (34) to claim (146).

We now utilize our choice of parameters and make these arguments precise, continuing from (145) as follows: for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$\begin{aligned}
\left\| \mathcal{R}\left(\frac{1}{2} \sum_{\zeta,\vartheta\in\Lambda} \mathcal{E}_{\zeta,\vartheta,2,1} \right) \right\|_{C_{t,q+1}L_x^{p^*}} &\overset{(145)(33)}{\lesssim} \left(\lambda_{q+1}^{\frac{2-3m}{112}} \lambda_q^{\frac{3}{2}} \right)^{\frac{43}{2}} \lambda_{q+1}^{-\frac{3m}{2}} \lambda_{q+1}^{\left(\frac{11m-5}{7}\right)3 - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\overset{(34)}{\lesssim} \lambda_{q+1}^{\frac{43(2-3m)}{2(112)} + \frac{129}{4}\left[\frac{2-3m}{(42)(56)}\right]} \lambda_{q+1}^{\frac{-21m+6(11m-5)}{14} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \\
&\approx \lambda_{q+1}^{\frac{8145(3m-2)}{2(42)(112)} - \left(\frac{11m-5}{7}\right)\frac{2}{p^*}} \ll \delta_{q+3} e(t) \qquad (148)
\end{aligned}$$

where the last inequality is due to $m < \frac{2}{3}$.

Fourth, we compute

$$\begin{aligned}
&\left\|\mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,2,4}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(188d)(20)}{\lesssim} \lambda_{q+1}\sum_{\zeta,\vartheta\in\Lambda}\left(\|a_\zeta\|_{C_{t,q+1}C_x^2}\|a_\vartheta\|_{C_{t,x,q+1}}+\|a_\zeta\|_{C_{t,x,q+1}}\|a_\vartheta\|_{C_{t,q+1}C_x^2}\right) \\
&\quad\times\left(\|\nabla\eta_\zeta\|_{C_{t,q+1}L_x^{2p^*}}\|\eta_\vartheta\|_{C_{t,q+1}L_x^{2p^*}}+\|\eta_\zeta\|_{C_{t,q+1}L_x^{2p^*}}\|\nabla\eta_\vartheta\|_{C_{t,q+1}L_x^{2p^*}}\right)\|\psi_\zeta\|_{C_x}\|\psi_\vartheta\|_{C_x} \\
&\overset{(54c)(11b)(17b)}{\lesssim} \delta_{q+2}l^{-\frac{43}{2}}\sigma r^{3-\frac{2}{p^*}}. \qquad (149)
\end{aligned}$$

The upper bound of (149) is identical to that of (144) and hence we are immediately able to conclude that for a_0 sufficiently large and $\beta>0$ sufficiently small

$$\left\|\mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,2,4}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3}e(t). \qquad (150)$$

Finally,

$$\begin{aligned}
&\|\mathcal{R}(A_2+A_3)\|_{C_{t,q+1}L_x^{p^*}} \\
&\overset{(192)(20)}{\lesssim} \sum_{\zeta\in\Lambda}(\lambda_{q+1}\sigma)^{-1}\left[\|a_\zeta\|_{C_{t,q+1}C_x^3}\|a_\zeta\|_{C_{t,x,q+1}}+\mu^{-1}\|a_\zeta\|_{C^1_{t,q+1}C_x^2}\|a_\zeta\|_{C_{t,x,q+1}}\right]\|\eta_\zeta\|^2_{C_{t,q+1}L_x^{2p^*}} \\
&\overset{(54c)(17b)}{\lesssim} \lambda_{q+1}^{-1}\sigma^{-1}\left[\delta_{q+2}l^{-\frac{55}{2}}+\mu^{-1}\delta_{q+2}l^{-\frac{55}{2}}\right]r^{2-\frac{2}{p^*}} \lesssim \lambda_{q+1}^{-1}\sigma^{-1}\delta_{q+2}l^{-\frac{55}{2}}r^{2-\frac{2}{p^*}} \qquad (151)
\end{aligned}$$

where the last inequality used the fact that $\mu^{-1}\ll 1$ due to (12). We realize that the bound in (151) is identical to that of (136) and hence we conclude that for a_0 sufficiently large and $\beta>0$ sufficiently small

$$\|\mathcal{R}(A_2+A_3)\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3}e(t). \qquad (152)$$

At last, we deduce that for a_0 sufficiently large and $\beta>0$ sufficiently small

$$\begin{aligned}
\|R_{\mathrm{osc}}\|_{C_{t,q+1}L_x^1} &\overset{(98a)}{\leqslant} \left\|\mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,1}\right)\right\|_{C_{t,q+1}L_x^1}+\left\|\mathcal{R}\left(\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\sum_{k=1,3,4}\mathcal{E}_{\zeta,\vartheta,2,k}\right)\right\|_{C_{t,q+1}L_x^1} \\
&\quad+\|\mathcal{R}(A_2+A_3)\|_{C_{t,q+1}L_x^1} \overset{(140)(143)(148)(150)(152)}{\ll} \delta_{q+3}e(t). \qquad (153)
\end{aligned}$$

□

Proposition 4.9. *R_{com} defined in (38) satisfies for a_0 sufficiently large and $\beta>0$ sufficiently small*

$$\|R_{com}\|_{C_{t,q+1}L_x^{p^*}} \ll \delta_{q+3}e(t).$$

Proof of proposition 4.9. We estimate

$$\|R_{\text{com}}\|_{C_{t,q+1}L_x^1} \lesssim l\|v_q\|_{C^1_{t,x,q+1}}\|v_q\|_{C_{t,q+1}L_x^2} \overset{(26a)(26b)}{\lesssim} lL\bar{e}\lambda_q^3. \tag{154}$$

The condition on l that we obtain from (154) is $l\lambda_q^3 \ll 1$ which is same as (80); thus, we conclude that for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\|R_{\text{com}}\|_{C_{t,q+1}L_x^1} \ll \delta_{q+3}e(t). \tag{155}$$

□

Finally, combining Propositions 4.5–4.9, we have proven that for a_0 sufficiently large and $\beta > 0$ sufficiently small

$$\begin{aligned}\|\mathring{R}_{q+1}\|_{C_{t,q+1}L_x^1} &\overset{(99)}{\leqslant} \|R_{\text{lin }1}\|_{C_{t,q+1}L_x^1} + \|R_{\text{lin }2}\|_{C_{t,q+1}L_x^1} + \|R_{\text{cor}}\|_{C_{t,q+1}L_x^1} \\ &+ \|R_{\text{osc}}\|_{C_{t,q+1}L_x^1} + \|R_{\text{com}}\|_{C_{t,q+1}L_x^1} \overset{(122)(129)(135)(153)(155)}{\ll} \delta_{q+3}e(t). \end{aligned} \tag{156}$$

This concludes the proof of (26c) at level $q+1$.

Remark 4.5. In order to find the appropriate choice of parameters in (32), we just described lower and upper bounds on μ. It is here that we faced difficulty upon attempting similarly using the 3D intermittent jets from [9, section 7.4]; we explain briefly with notations from [9, section 7.4]. Here, we emphasize that $r_{\|}$ and $r_{\perp}$ below are different from r in (117). For example, analogous computations to (123) on $R_{\text{lin }2}$ gave us

$$\left\|\mathcal{R}\partial_{tt}\left(w_{q+1}^{(p)} + w_{q+1}^{(c)}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \lesssim l^{-2}\lambda_{q+1}^{-1}r_{\perp}^{\frac{2}{p^*}-1}r_{\|}^{\frac{1}{p^*}-\frac{1}{2}}\left(\frac{r_{\perp}\lambda_{q+1}\mu}{r_{\|}}\right)^2$$

so that for $p^* \approx 1$, considering only the dominant terms leads us to

$$\lambda_{q+1}r_{\perp}^3 r_{\|}^{-\frac{3}{2}}\mu^2 \ll 1; \text{i.e. } \mu \ll \lambda_{q+1}^{-\frac{1}{2}}r_{\perp}^{-\frac{3}{2}}r_{\|}^{\frac{3}{4}}. \tag{157}$$

On the other hand, analogous computations to (130) on R_{cor} gave us

$$\begin{aligned}&\left\|\left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\mathring{\otimes} w_{q+1} + w_{q+1}^{(p)}\mathring{\otimes}\left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right)\right\|_{C_{t,q+1}L_x^{p^*}} \\ &\lesssim l^{-2}r_{\perp}^{\frac{2}{2p^*}-1}r_{\|}^{\frac{1}{2p^*}-\frac{1}{2}} + l^{-12}r_{\perp}^{\frac{2}{2p^*}}r_{\|}^{\frac{1}{2p^*}-\frac{3}{2}} + \mu^{-1}l^{-4}r_{\perp}^{\frac{2}{2p^*}-2}r_{\|}^{\frac{1}{2p^*}-1}\end{aligned}$$

so that for $p^* \approx 1$, considering only the dominant terms leads us to requiring

$$\mu^{-1}r_{\perp}^{-1}r_{\|}^{-\frac{1}{2}} \ll 1 \text{i.e., } r_{\perp}^{-1}r_{\|}^{-\frac{1}{2}} \ll \mu. \tag{158}$$

Considering (157) and (158) leads to a requirement of

$$r_{\perp}^{-1}r_{\|}^{-\frac{1}{2}} \ll \lambda_{q+1}^{-\frac{1}{2}}r_{\perp}^{-\frac{3}{2}}r_{\|}^{\frac{3}{4}}$$

which is equivalent to

$$r_{\perp}\lambda_{q+1} \ll r_{\|}^{\frac{5}{2}}. \tag{159}$$

Unfortunately, $r_\perp^{-1} \ll \lambda_{q+1}$ from (30) so that $1 \ll r_\perp \lambda_{q+1}$ and $r_\| \ll 1$ from (30) imply that (159) is impossible.

At last, faced with such a difficulty, we actually attempted analogous approach in higher dimension $d \geqslant 3$ using the generalized intermittent jets in higher dimension from [37, section 3]; however, it led to a requirement of

$$r_\perp^{-\frac{d}{2}+\frac{1}{2}} r_\|^{-\frac{1}{2}} \ll \mu \ll \lambda_{q+1}^{-\frac{1}{2}} r_\perp^{-\frac{d}{4}-\frac{3}{4}} r_\|^{\frac{3}{4}}$$

which is an analogue of (157) and (158); this reduces to

$$r_\perp^{\frac{5-d}{2}} \lambda_{q+1} \ll r_\|^{\frac{5}{2}}, \tag{160}$$

which is an analogue of (159). Unfortunately, the constraints that we need in such generalized intermittent jets in higher dimension is the same as (30). Thus, for any $d \geqslant 3$ we realize that (160) is impossible again because $1 \ll r_\perp \lambda_{q+}$ while $r_\| \ll 1$. However, this is where we come to the crucial observation. Because $r_\perp \ll 1$ from (30), the condition (160) becomes increasingly more difficult as d rises; this is the reason why we realized that the only pathway possible for us with our current approach is the case $d = 2$, which led to theorem 2.1 after optimizing all the parameters thereafter.

Proposition 4.10. *Define*

$$\delta E(t) \triangleq \left| e(t)(1-\delta_{q+2}) - \|v_{q+1}(t)\|_{L_x^2}^2 \right|. \tag{161}$$

Then, for all $t \in [t_{q+1}, \mathfrak{t}]$

$$\delta E(t) \leqslant \frac{1}{4} \delta_{q+2} e(t) \tag{162}$$

so that (26d) holds at level $q+1$.

Proof of proposition 4.10. The following computations follow those of [29], and have similarities to previous estimates. Not surprisingly, the constraints on the parameters l and b we have already determined in previous sections turn out to suffice. Therefore, we will not mention new constraints and simply use the parameters l and b respectively from (33) and (34) and complete this proof. We compute using (40b) and (62)

$$\begin{aligned} \delta E(t) = \Bigg| & \gamma_q(t) 2(2\pi)^2 + \int_{\mathbb{T}^2} |v_q(t)|^2 - |v_l(t)|^2 - 2v_l(t) \cdot w_{q+1}^{(p)}(t) - 2v_l(t) \cdot \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)(t) \\ & - |w_{q+1}^{(p)}(t)|^2 - 2w_{q+1}^{(p)}(t) \cdot \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)(t) - \left| \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)(t) \right|^2 \mathrm{d}x \Bigg| \leqslant \sum_{k=1}^{5} I_k \end{aligned} \tag{163}$$

where

$$I_1 \triangleq \left| \gamma_q(t) 2 (2\pi)^2 - \| w_{q+1}^{(p)}(t) \|_{L_x^2}^2 \right|, \tag{164a}$$

$$I_2 \triangleq \left\| \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)(t) \right\|_{L_x^2}^2, \tag{164b}$$

$$I_3 \triangleq 2 \left| \int_{\mathbb{T}^2} \left(v_l + w_{q+1}^{(p)} \right) \cdot \left(w_{q+1}^{(c)} + w_{q+1}^{(t)} \right)(t) \, \mathrm{d}x \right|, \tag{164c}$$

$$I_4 \triangleq 2 \left| \int_{\mathbb{T}^2} v_l \cdot w_{q+1}^{(p)}(t) \, \mathrm{d}x \right|, \tag{164d}$$

$$I_5 \triangleq \left| \int_{\mathbb{T}^2} |v_q(t)|^2 - |v_l(t)|^2 \mathrm{d}x \right|. \tag{164e}$$

For I_1, we use the fact that $\mathring{R}_q$ and therefore $\mathring{R}_l$ is trace-free to write using (63a) and (53a),

$$|w_{q+1}^{(p)}(t)|^2 - \gamma_q(t) 2 = \sum_{\zeta, \vartheta \in \Lambda} (a_\zeta a_\vartheta)(t) \mathrm{Tr} \mathbb{P}_{\neq 0} (W_\zeta \otimes W_\vartheta)(t) + 2 (\rho(t) - \gamma_q(t)). \tag{165}$$

This leads us to

$$\begin{aligned} I_1 \leqslant\ & 2\epsilon_\gamma^{-1} (2\pi)^2 l + 2\epsilon_\gamma^{-1} \| \mathring{R}_l(t) \|_{L_x^1} + 2 (2\pi)^2 |\gamma_l(t) - \gamma_q(t)| \\ & + \sum_{\zeta, \vartheta \in \Lambda} \left| \int_{\mathbb{T}^2} (a_\zeta a_\vartheta)(t) \mathrm{Tr} \mathbb{P}_{\neq 0} (W_\zeta \otimes W_\vartheta)(t) \, \mathrm{d}x \right|. \end{aligned} \tag{166}$$

We estimate the first term of (166) using the fact that $e(t) \geqslant \underline{e} \geqslant 4$, for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$2\epsilon_\gamma^{-1} (2\pi)^2 l \overset{(33)}{=} \epsilon_\gamma^{-1} 8\pi^2 \lambda_{q+1}^{-\frac{2-3m}{112}} \lambda_q^{-\frac{3}{2}} \ll \lambda_1^{2\beta} \lambda_{q+2}^{-2\beta} e(t) = \delta_{q+2} e(t). \tag{167}$$

We estimate the second term of (166) by

$$2\epsilon_\gamma^{-1} \| \mathring{R}_l(t) \|_{L_x^1} \leqslant 2\epsilon_\gamma^{-1} \| \mathring{R}_q(t) \|_{L_x^1} \overset{(26c)}{\leqslant} \frac{1}{18} \delta_{q+2} e(t). \tag{168}$$

We estimate the third term of (166) using the fact that $e(t) \geqslant \underline{e} \geqslant 4$, for a_0 sufficiently large and $\beta > 0$ sufficiently small,

$$\begin{aligned} 2 (2\pi)^2 |\gamma_l(t) - \gamma_q(t)| & \overset{(3)(40b)}{\lesssim} l\tilde{e} + l \| v_q \|_{C_{t,q+1} L_x^2} \| v_q \|_{C_{t,q+1}^1 L_x^2} \\ & \overset{(26)}{\lesssim} l \lambda_q^3 \overset{(33)(34)}{\lesssim} \lambda_{q+1}^{-\frac{13}{14} \left(\frac{2-3m}{112} \right)} \ll \delta_{q+2} e(t). \end{aligned} \tag{169}$$

Next, we take

$$M > \frac{4073m - 1670}{4(2-3m)(347)} \tag{170}$$

and estimate the fourth term of (166) by

$$\begin{aligned}
&\sum_{\zeta,\vartheta\in\Lambda}\left|\int_{\mathbb{T}^2}(a_\zeta a_\vartheta)(t)\operatorname{Tr}\mathbb{P}_{\neq 0}(W_\zeta\otimes W_\vartheta)(t)\,dx\right| \\
&\overset{(16c)(54)}{\lesssim}\sum_{\zeta,\vartheta\in\Lambda}\left(l^{-\frac{3}{2}}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\right)\left(l^{-6M-8}\delta_{q+2}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\right)(\lambda_{q+1}\sigma)^{-M}\|W_\zeta\|_{C_{t,q+1}L_x^4}\|W_\vartheta\|_{C_{t,q+1}L_x^4} \\
&\overset{(17a)(32)(33)}{\lesssim}\delta_{q+2}\left(\lambda_{q+1}^{\frac{2-3m}{112}}\lambda_q^{\frac{3}{2}}\right)^{6M+\frac{19}{2}}\lambda_{q+1}^{\left(\frac{2-3m}{2}\right)(-M)}\lambda_{q+1}^{\frac{11m-5}{7}} \\
&\overset{(34)}{\lesssim}\delta_{q+2}\left(\lambda_{q+1}^{\frac{2-3m}{112}+\frac{3}{2}\left[\frac{2-3m}{(42)(56)}\right]}\right)^{6M+\frac{19}{2}}\lambda_{q+1}^{\left(\frac{2-3m}{2}\right)(-M)+\frac{11m-5}{7}} \\
&\approx\delta_{q+2}\lambda_{q+1}^{-\frac{(2-3m)(347)M}{7(112)}+\frac{4073m-1670}{28(112)}}\overset{(170)}{\ll}\delta_{q+2}e(t).
\end{aligned}\tag{171}$$

Applying (167)–(169), and (171) to (166) gives us now for $\beta>0$ sufficiently small,

$$I_1\leqslant\frac{1}{9}\delta_{q+2}e(t).\tag{172}$$

Next, we are able to take advantage of previous estimate (74) make quick work of

$$I_2\overset{(164b)}{=}\left\|\left(w_{q+1}^{(c)}+w_{q+1}^{(t)}\right)(t)\right\|_{L_x^2}^2\lesssim\left(\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^2}+\|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^2}\right)^2\overset{(74)}{\ll}\delta_{q+2}e(t),\tag{173a}$$

$$\begin{aligned}
I_3&\overset{(164c)}{\leqslant}2\left(\|v_l\|_{C_{t,q+1}L_x^2}+\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^2}\right)\left[\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^2}+\|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^2}\right] \\
&\overset{(39b)(69)}{\lesssim}\left(L\bar{e}^{\frac{1}{2}}+\delta_{q+1}^{\frac{1}{2}}\bar{e}^{\frac{1}{2}}\right)\left[\|w_{q+1}^{(c)}\|_{C_{t,q+1}L_x^2}+\|w_{q+1}^{(t)}\|_{C_{t,q+1}L_x^2}\right]\overset{(74)}{\ll}\delta_{q+2}e(t),
\end{aligned}\tag{173b}$$

for a_0 sufficiently large and $\beta>0$ sufficiently small.

Next, to estimate I_4 from (164d), we take $\varepsilon>0$ such that

$$\epsilon<\frac{91m-42}{8(11m-5)}<2\tag{174}$$

and estimate from (164d) for a_0 sufficiently large and $\beta>0$ sufficiently small,

$$\begin{aligned}
I_4&\lesssim\|v_l\|_{C_{t,q+1}C_x^1}\|w_{q+1}^{(p)}\|_{C_{t,q+1}L_x^{\frac{2}{2-\epsilon}}}\overset{(26b)(72a)(33)}{\lesssim}\lambda_q^{\frac{21}{2}}\lambda_{q+1}^{\frac{3(2-3m)}{224}+\left(\frac{11m-5}{7}\right)(-1+\epsilon)} \\
&\overset{(34)}{\lesssim}\lambda_{q+1}^{\frac{21}{2}\left[\frac{2-3m}{(42)(56)}\right]+\frac{3(2-3m)+32(11m-5)(-1+\epsilon)}{2(112)}}\approx\lambda_{q+1}^{\frac{42-91m+8(11m-5)\epsilon}{56}}\overset{(174)}{\ll}\delta_{q+2}e(t).
\end{aligned}\tag{175}$$

Finally, we estimate from (164e), for a_0 sufficiently large and $\beta>0$ sufficiently small

$$\begin{aligned}
I_5\lesssim\|v_q\|_{C_{t,q+1}L_x^2}l\|v_q\|_{C^1_{t,x,q+1}}&\overset{(26a)(26b)(33)}{\lesssim}\lambda_{q+1}^{-\left(\frac{2-3m}{112}\right)}\lambda_q^{-\frac{3}{2}}\lambda_q^3 \\
&\overset{(34)}{\lesssim}\lambda_{q+1}^{-\left(\frac{2-3m}{112}\right)+\frac{3}{2}\left[\frac{2-3m}{(42)(56)}\right]}\approx\lambda_{q+1}^{-\left(\frac{13}{14}\right)\left[\frac{2-3m}{112}\right]}\ll\delta_{q+2}e(t).
\end{aligned}\tag{176}$$

Applying (172), (173), (175), and (176) to (163) we are finally able to conclude that

$$\delta E(t) \leqslant \sum_{k=1}^{5} I_k \leqslant \frac{1}{4}\delta_{q+2} e(t)$$

for a_0 sufficiently large and $\beta > 0$ sufficiently small; this completes the proof of proposition 4.10.

□

Proof of proposition 4.2. We proved (27) in (79), (26*a*) in (78), (26*b*) in (91) together with (92), (93), (26*c*) in (156), and (26*d*) in proposition 4.10, completing the proof of proposition 4.2. □

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University where the authors had fruitful discussions. The first author is partially supported by the National Science Foundation (DMS 2104682 and DMS 2309748). The second author is supported by the Simons Foundation (962572, K Y). We thank Prof. Yao Yao and Prof. Hongjie Dong for fruitful discussions. Finally, we express our deep gratitude to the editors and anonymous referees for the careful reading and valuable suggestions and comments that have improved this manuscript significantly.

Appendix A. Derivation of r, μ, and σ^{-1} from (32)

Let us explain our choice of r, μ, and σ^{-1} from (32). Recall from remark 4.2 that we consider only $m \in (\frac{1}{2}, \frac{2}{3})$. We need $\lambda_{q+1}^{2m-1} \ll r$ from (104) while $\mu \ll \lambda_{q+1}^{-\frac{1}{2}} \sigma^{-1} r^{-\frac{1}{2}}$ from (124). We have $r \gg 1$ and $\lambda_{q+1} \gg 1$ from (12) so that $\lambda_{q+1}^{-\frac{1}{2}} r^{-\frac{1}{2}} \ll 1$. These imply that we need

$$\lambda_{q+1}^{2m-1} \overset{(104)}{\ll} r \overset{(12)}{\ll} \mu \overset{(124)}{\ll} \lambda_{q+1}^{-\frac{1}{2}} \sigma^{-1} r^{-\frac{1}{2}} \ll \sigma^{-1} \overset{(12)}{\ll} \lambda_{q+1}. \tag{177}$$

We optimize and choose

$$\mu = \sqrt{r\left(\lambda_{q+1}^{-\frac{1}{2}} \sigma^{-1} r^{-\frac{1}{2}}\right)} = r^{\frac{1}{4}} \lambda_{q+1}^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} \tag{178}$$

so that (177) reduces to

$$\lambda_{q+1}^{2m-1} \ll r \ll r^{\frac{1}{4}} \lambda_{q+1}^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} \ll \lambda_{q+1}^{-\frac{1}{2}} \sigma^{-1} r^{-\frac{1}{2}} \ll \sigma^{-1} \ll \lambda_{q+1}.$$

We furthermore optimize from this to choose

$$r = \sqrt{\lambda_{q+1}^{2m-1}\left(r^{\frac{1}{4}}\lambda_{q+1}^{-\frac{1}{4}}\sigma^{-\frac{1}{2}}\right)} = \lambda_{q+1}^{m-\frac{1}{2}} r^{\frac{1}{8}}\lambda_{q+1}^{-\frac{1}{8}}\sigma^{-\frac{1}{4}},$$

which implies $r^{\frac{7}{8}}\lambda_{q+1}^{\frac{5}{8}-m} = \sigma^{-\frac{1}{4}}$ and hence

$$r^{\frac{7}{2}}\lambda_{q+1}^{\frac{5}{2}-4m} = \sigma^{-1}. \tag{179}$$

From (177) we know we need $\sigma^{-1} \ll \lambda_{q+1}$ which implies that we require $r^{\frac{7}{2}}\lambda_{q+1}^{-4m+\frac{5}{2}} \ll \lambda_{q+1}$ or equivalently $r \ll \lambda_{q+1}^{\frac{2}{7}(4m-\frac{3}{2})}$. From (177) we know we need $\lambda_{q+1}^{2m-1} \ll r$ and thus we optimize over $\frac{(2m-1)+\frac{2}{7}(4m-\frac{3}{2})}{2} = \frac{11m-5}{7}$ and hence define $r = \lambda_{q+1}^{\frac{11m-5}{7}}$. Applying this choice of r to (179) leads us to $\sigma^{-1} = \lambda_{q+1}^{\frac{3m}{2}}$. At last, we apply this definition of $r = \lambda_{q+1}^{\frac{11m-5}{7}}$ and $\sigma^{-1} = \lambda_{q+1}^{\frac{3m}{2}}$ to (178) to conclude $\mu = \lambda_{q+1}^{\frac{8m-3}{7}}$.

Appendix B. Derivation of *l* in (33)

We have conditions on l from (70), (80), (87), (92), (102), (112), (114), (119), (126), (132), (138), and (146). In short, all of these conditions boil down to $\lambda_q^3 \ll l^{-1}$ from (80) and $l^{-1} \ll \lambda_{q+1}^{\frac{2-3m}{56}}$ from (70) assuming that $b \in \mathbb{N}$ is sufficiently large and yet to be determined; thus, we optimize and select

$$l^{-1} = \sqrt{\lambda_{q+1}^{\frac{2-3m}{56}}\lambda_q^3} = \lambda_{q+1}^{\frac{2-3m}{112}}\lambda_q^{\frac{3}{2}}$$

as we did in (33).

Appendix C. Derivation of (98)

We sketch the derivation of (98). In contrast to previous works such as [38, 49], our $\mathrm{div} R_{\mathrm{osc}} + \nabla\pi_{\mathrm{osc}}$ consists of $\partial_{tt} w_{q+1}^{(t)}$ instead of $\partial_t w_{q+1}^{(t)}$; we designed our $w_{q+1}^{(t)}$ in (63*c*) so that this difference does not create major difficulties in the following computations (see [49, equations (101)–(115)] for details). First, we write

$$\mathrm{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}\right) = \mathrm{div}\left(w_{q+1}^{(p)} \mathring{\otimes} w_{q+1}^{(p)}\right) + \nabla\frac{1}{2}|w_{q+1}^{(p)}|^2, \tag{180}$$

while

$$\begin{aligned} w_{q+1}^{(p)} \mathring{\otimes} w_{q+1}^{(p)} + \mathring{R}_l \overset{(53b)(63)}{=} & \sum_{\zeta,\vartheta\in\Lambda} a_\zeta a_\vartheta \mathbb{P}_{\geqslant \frac{\lambda_{q+1}\sigma}{2}}\left(W_\zeta \mathring{\otimes} W_\vartheta\right) \\ & + \rho\left(1 - \frac{1}{8}\sum_{\zeta\in\Lambda}\gamma_\zeta\left(\mathrm{Id} - \frac{\mathring{R}_l}{\rho}\right)^2\right)\mathrm{Id} \end{aligned} \tag{181}$$

due to (16). The identity (181) leads to

$$\operatorname{div}\left(w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(p)}+\mathring{R}_l\right)=\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,1}+\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,2}$$
$$+\nabla\left(\rho\left(1-\frac{1}{8}\sum_{\zeta\in\Lambda}\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l}{\rho}\right)^2\right)\right) \tag{182}$$

where

$$\mathcal{E}_{\zeta,\vartheta,1}\triangleq\mathbb{P}_{\neq 0}\left(\nabla\left(a_\zeta a_\vartheta\right)\cdot\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\left(W_\zeta\mathring{\otimes}W_\vartheta+W_\vartheta\mathring{\otimes}W_\zeta\right)\right), \tag{183a}$$
$$\mathcal{E}_{\zeta,\vartheta,2}\triangleq\mathbb{P}_{\neq 0}\left(a_\zeta a_\vartheta\nabla\cdot\left(W_\zeta\mathring{\otimes}W_\vartheta+W_\vartheta\mathring{\otimes}W_\zeta\right)\right), \tag{183b}$$

in which we used symmetry. Now for any $\zeta,\vartheta\in\Lambda\subset\mathbb{S}^1$, we can compute

$$\left(\zeta^\perp\otimes\vartheta^\perp+\vartheta^\perp\otimes\zeta^\perp\right)(\zeta+\vartheta)=\left(\zeta^\perp\cdot\vartheta^\perp-1\right)\mathrm{Id}\,(\zeta+\vartheta). \tag{184}$$

It follows from (10) and (184) that

$$\nabla\cdot\left(b_\zeta\mathring{\otimes}b_\vartheta+b_\vartheta\mathring{\otimes}b_\zeta\right)(x)=\nabla\left(\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)(x). \tag{185}$$

Consequently, via (15) and (185),

$$\nabla\cdot\left(W_\zeta\mathring{\otimes}W_\vartheta+W_\vartheta\mathring{\otimes}W_\zeta\right)=\left(b_\zeta\mathring{\otimes}b_\vartheta+b_\vartheta\mathring{\otimes}b_\zeta\right)\cdot\nabla\left(\eta_\zeta\eta_\vartheta\right)+\left(\eta_\zeta\eta_\vartheta\right)\nabla\left(\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right). \tag{186}$$

After splitting $\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,2}=\frac{1}{2}(\sum_{\zeta,\vartheta\in\Lambda:\,\zeta+\vartheta\neq0}+\sum_{\zeta,\vartheta\in\Lambda:\,\zeta+\vartheta=0})\mathcal{E}_{\zeta,\vartheta,2}$, this allows us to write

$$\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda:\,\zeta+\vartheta\neq0}\mathcal{E}_{\zeta,\vartheta,2}\overset{(16b)(183b)(186)}{=}\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\sum_{k=1}^{4}\mathcal{E}_{\zeta,\vartheta,2,k} \tag{187}$$

where

$$\mathcal{E}_{\zeta,\vartheta,2,1}\triangleq\mathbb{P}_{\neq0}\left(a_\zeta a_\vartheta\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{10}}\left[\left(b_\zeta\mathring{\otimes}b_\vartheta+b_\vartheta\mathring{\otimes}b_\zeta\right)\cdot\nabla\left(\eta_\zeta\eta_\vartheta\right)\right]\right)1_{\zeta+\vartheta\neq0}, \tag{188a}$$
$$\mathcal{E}_{\zeta,\vartheta,2,2}\triangleq\nabla\mathbb{P}_{\neq0}\left(a_\zeta a_\vartheta\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{10}}\left(\eta_\zeta\eta_\vartheta\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)\right)1_{\zeta+\vartheta\neq0}, \tag{188b}$$
$$\mathcal{E}_{\zeta,\vartheta,2,3}\triangleq-\mathbb{P}_{\neq0}\left(\nabla\left(a_\zeta a_\vartheta\right)\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{10}}\left(\eta_\zeta\eta_\vartheta\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)\right)1_{\zeta+\vartheta\neq0}, \tag{188c}$$
$$\mathcal{E}_{\zeta,\vartheta,2,4}\triangleq-\mathbb{P}_{\neq0}\left(a_\zeta a_\vartheta\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{10}}\left(\nabla\left(\eta_\zeta\eta_\vartheta\right)\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)\right)1_{\zeta+\vartheta\neq0}. \tag{188d}$$

On the other hand, in case $\zeta+\vartheta=0$ we have $\nabla(\lambda_{q+1}^2\psi_\zeta\psi_{-\zeta})\overset{(10)}{=}\lambda_{q+1}^2\nabla(\frac{1}{\lambda_{q+1}^2}e^{i\lambda_{q+1}(\zeta-\zeta)\cdot x})$ $=0$, while we can multiply (14a) by $2\eta_\zeta$ to deduce $\mu^{-1}\partial_t|\eta_\zeta|^2=\pm(\zeta\cdot\nabla)|\eta_\zeta|^2$ for all $\zeta\in\Lambda^\pm$ and hence

$$\nabla\cdot\left(W_\zeta\mathring{\otimes}W_{-\zeta}+W_{-\zeta}\mathring{\otimes}W_\zeta\right)\overset{(15)(185)(10)}{=}2\zeta^\perp\mathring{\otimes}\zeta^\perp\nabla\eta_\zeta^2=\nabla\eta_\zeta^2\mp2\mu^{-1}\left(\partial_t\eta_\zeta^2\right)\zeta. \tag{189}$$

This allows us to write

$$\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda:\,\zeta+\vartheta=0}\mathcal{E}_{\zeta,\vartheta,2}\overset{(183b)(189)}{=}\frac{1}{2}\sum_{\zeta\in\Lambda}\nabla\left(a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2\right)-\mathbb{P}_{\neq0}\left(\nabla a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2\right)$$
$$-\mu^{-1}\left(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-}\right)\partial_t\mathbb{P}_{\neq0}\left(a_\zeta^2\mathbb{P}_{\neq0}\left(\eta_\zeta^2\zeta\right)\right)$$
$$-\mathbb{P}_{\neq0}\left(\partial_t a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\left(\eta_\zeta^2\zeta\right)\right)\tag{190}$$

where we also used that $\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2=\mathbb{P}_{\neq0}\eta_\zeta^2$. At last, we obtain by using the definition of $\mathbb{P}=\mathrm{Id}-\nabla\Delta^{-1}\nabla\cdot$

$$\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda:\,\zeta+\vartheta=0}\mathcal{E}_{\zeta,\vartheta,2}+\partial_{tt}w_{q+1}^{(t)}\overset{(63)(190)}{=}\sum_{k=1}^{4}A_k\tag{191}$$

where

$$A_1\triangleq\frac{1}{2}\sum_{\zeta\in\Lambda}\nabla\left(a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2\right),\tag{192a}$$

$$A_2\triangleq-\frac{1}{2}\sum_{\zeta\in\Lambda}\mathbb{P}_{\neq0}\left(\nabla a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2\right),\tag{192b}$$

$$A_3\triangleq\mu^{-1}\left(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-}\right)\mathbb{P}_{\neq0}\left(\partial_t a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\left(\eta_\zeta^2\zeta\right)\right),\tag{192c}$$

$$A_4\triangleq-\nabla\Delta^{-1}\nabla\cdot\mu^{-1}\left(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-}\right)\mathbb{P}_{\neq0}\partial_t\left(a_\zeta^2\mathbb{P}_{\neq0}\eta_\zeta^2\zeta\right).\tag{192d}$$

Therefore, combining (180), (182), (187), and (191) gives us

$$\begin{aligned}&\operatorname{div}\left(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}+\mathring{R}_l\right)+\partial_{tt}w_{q+1}^{(t)}\\&=\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathcal{E}_{\zeta,\vartheta,1}+\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\sum_{k=1,3,4}\mathcal{E}_{\zeta,\vartheta,2,k}+A_2+A_3\\&\quad+\nabla\Bigg[\frac{1}{2}|w_{q+1}^{(p)}|^2+\left(\rho\left(1-\frac{1}{8}\sum_{\zeta\in\Lambda}\gamma_\zeta\left(\mathrm{Id}-\frac{\mathring{R}_l}{\rho}\right)^2\right)\right)\\&\quad+\frac{1}{2}\sum_{\zeta,\vartheta\in\Lambda}\mathbb{P}_{\neq0}\left(a_\zeta a_\vartheta\mathbb{P}_{\geqslant\frac{\lambda_{q+1}}{10}}\left(\eta_\zeta\eta_\vartheta\lambda_{q+1}^2\psi_\zeta\psi_\vartheta\right)\right)1_{\zeta+\vartheta\neq0}\\&\quad+\frac{1}{2}\sum_{\zeta\in\Lambda}a_\zeta^2\mathbb{P}_{\geqslant\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2-\Delta^{-1}\nabla\cdot\mu^{-1}\left(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-}\right)\mathbb{P}_{\neq0}\partial_t\left(a_\zeta^2\mathbb{P}_{\neq0}\eta_\zeta^2\zeta\right)\Bigg],\end{aligned}$$

which finally leads us to (98).

Appendix D. Construction of a solution to (1) that doubles its energy

Here, we briefly sketch the proof of the construction of a solution that doubles its energy from initial time by time $t = 1$ in belief of its independent mathematical interest.

Theorem D.1. *Fix $m \in (0, \frac{2}{3})$. Then there exists a constant $\beta = \beta(m) \in (0,1)$ sufficiently small such that the following holds. There exists a mean-zero weak solution $v \in C([0,1]; H^{\beta}(\mathbb{T}^2)) \cap C^{\beta}([0,1]; L^2(\mathbb{T}^2))$ to the hyperbolic Navier–Stokes equations (1) such that*

$$\|v(1)\|_{L_x^2} > 2\|v(0)\|_{L_x^2}. \tag{193}$$

The existence time interval of the solution is taken to be $[0,1]$ in theorem D.1 for simplicity and can be replaced by $[0,T]$ for any $T > 0$ fixed *a priori*.

We sketch the proof of theorem D.1. In contrast to (21), we can set simply define $\lambda_q \triangleq a^{b^q}, \delta_q \triangleq \lambda_q^{-2\beta}$; i.e. it is no longer necessary that $\delta_1 = 1$. Requiring $a^{-2b\beta} \leqslant \frac{1}{49}$ assure that $t_q \leqslant -\frac{5}{6}$. We can consider the same iteration (25) with $T = 1$. We can simplify the induction hypothesis (26) as follows: on $[t_q, 1]$

$$\|v_q\|_{C_{t,q}L_x^2} \leqslant L^{\frac{1}{2}} \left(1 + \sum_{1 \leqslant r \leqslant q} \delta_r^{\frac{1}{2}}\right), \tag{194a}$$

$$\|v_q\|_{C^1_{t,x,q}} \leqslant L^{\frac{1}{2}} \lambda_q^3, \tag{194b}$$

$$\|\mathring{R}_q\|_{C_{t,q}L_x^1} \leqslant c_R L \delta_{q+1} \tag{194c}$$

for a universal constant $c_R > 0$ and L sufficiently large so that

$$\frac{(4\pi + 8)^2 (49)^2}{c_R^2} < L. \tag{195}$$

The step $q = 0$ will become more complicated than proposition 4.1 as follows:

Proposition D.2 (Initial step $q = 0$). *Define*

$$v_0(t,x) \triangleq \frac{tL^{\frac{1}{2}}}{2\pi} \begin{pmatrix} \sin(x^2) \\ 0 \end{pmatrix}$$

and then

$$\mathring{R}_0(t,x) \triangleq \frac{L^{\frac{1}{2}}}{2\pi} \begin{pmatrix} 0 & -\cos(x^2) \\ -\cos(x^2) & 0 \end{pmatrix} + \mathcal{R}(-\Delta)^m v_0(t,x).$$

Then $(v_0, \mathring{R}_0)$ solves (25) with $T = 1$, satisfies (194) provided

$$\left(4\pi\sqrt{2} + 8\right) 49 \leqslant \left(4\pi\sqrt{2} + 8\right) a^{2b\beta} \leqslant c_R L^{\frac{1}{2}};$$

moreover, v_0 satisfies

$$\|v_0(t)\|_{L_x^2} = \frac{|t| L^{\frac{1}{2}}}{\sqrt{2}} \leqslant L^{\frac{1}{2}}. \tag{196}$$

Next, proposition 4.2 is replaced by the following:

Proposition D.3 (Step $q+1$ assuming the step q). *Let $L>0$ be sufficiently large so that (195) holds. Under the hypothesis of theorem D.1, there exists a choice of parameters a, b, and β such that for all $(v_q, \mathring{R}_q)$ that solves (25) and satisfies (194), there exists $(v_{q+1}, \mathring{R}_{q+1})$ that solves (25) and satisfies (194) at level $q+1$ such that for all $t \in [t_{q+1}, 1]$*

$$\|v_{q+1} - v_q\|_{C_{t,q+1}L_x^2} \leqslant L^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}. \tag{197}$$

The main difference in the proof of propositions D.3 and 4.2 is that we would let χ be a smooth function such that

$$\chi(z) \triangleq \begin{cases} 1 & \text{if } z \in [0,1], \\ z & \text{if } z \in [2,\infty), \end{cases} \tag{198}$$

and $z \leqslant 2\chi(z) \leqslant 4z$ for $z \in (1,2)$ and thereby define

$$\rho(t,x) \triangleq 4c_R \delta_{q+1} L \chi \left((c_R \delta_{q+1} L)^{-1} |\mathring{R}_l(t,x)| \right).$$

Proof of theorem D.1. We only highlight the difference from the proof of theorem 2.1, namely (193). We can compute

$$\|v(t) - v_0(t)\|_{L_x^2} \leqslant \sum_{q\geqslant 0} \|v_{q+1}(t) - v_q(t)\|_{L_x^2} \overset{(197)}{\leqslant} \sum_{q\geqslant 0} L^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \leqslant \frac{L^{\frac{1}{2}}}{6}. \tag{199}$$

Recalling $\|v_0(t)\|_{L_x^2} = \frac{tL^{\frac{1}{2}}}{\sqrt{2}}$ for all $t \in [0,1]$ from (196), we are ready to conclude

$$\begin{aligned} 2\|v(0)\|_{L_x^2} &\leqslant 2\|v_0(0)\|_{L_x^2} + 2\|v(0) - v_0(0)\|_{L_x^2} \\ &\overset{(196)(199)}{\leqslant} \frac{L^{\frac{1}{2}}}{3} < \frac{L^{\frac{1}{2}}}{\sqrt{2}} - \frac{L^{\frac{1}{2}}}{6} \overset{(196)(199)}{=} \|v_0(1)\|_{L_x^2} - \|v(1) - v_0(1)\|_{L_x^2} \leqslant \|v(1)\|_{L_x^2}. \end{aligned}$$

□

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