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Algebraic calming for the 2D Kuramoto-Sivashinsky equations

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Abstract

We propose an approximate model for the 2D Kuramoto–Sivashinsky equations (KSE) of flame fronts and crystal growth. We prove that this new 'calmed' version of the KSE is globally well-posed, and moreover, its solutions converge to solutions of the KSE on the time interval of existence and uniqueness of the KSE at an algebraic rate. In addition, we provide simulations of the calmed KSE, illuminating its dynamics. These simulations also indicate that our analytical predictions of the convergence rates are sharp. We also discuss analogies with the 3D Navier–Stokes equations of fluid dynamics.

Keywords: Kuramoto–Sivashinsky equation, calming, approximate models, Navier–Stokes equations, global well-posedness, multi-dimensional

Mathematics Subject Classification numbers: 35K25, 35K58, 35B65, 35A35, 65M70

1. Introduction

The Kuramoto–Sivashinsky equation (KSE) is a captivating model for flame fronts, crystal growth, and many other phenomena. It is both satisfying and frustrating. In one space

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dimension, the model acts as a fantastic toy model: it has highly non-trivial chaotic dynamics while still being amenable to a wide range of analytical tools. However, in higher dimensions, it has so far resisted nearly every analytical attack due to its lack of any known conserved quantity, and the basic question of global well-posedness of solutions remains open, even in two dimensions. Moreover, the nonlinearity of the system has many similarities with the nonlinearity of the Navier–Stokes equations (NSE), making investigation of the KSE even more intriguing.

How does one proceed in the face of such difficulty? In the case of the NSE, at least one approach has been fruitful since at least the work of Smagorinsky in 1963 [58], where a modification of the Navier–Stokes system was proposed, resulting in a system which is both globally well-posed [36], and less computationally demanding to simulate. Since then, hundreds of so-called 'turbulence models' have arisen (see, e.g. [15, 52] for a survey), which typically modify the equations in some way. It is therefore natural to ask whether such an approach might work for the 2D KSE[3]. However, one quickly realises that approaches which work for the NSE are unlikely to work for the KSE. Indeed, for the NSE, the problem is the growth of large gradients; more specifically, the problem is the development of large vorticity, $\boldsymbol{\omega} := \nabla \times \mathbf{u}$ (see, e.g. [3]). One possible reason may be the cubic nature of the vorticity equation: $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_{L^2}^2 \sim (\boldsymbol{\omega}\cdot\nabla\mathbf{u},\boldsymbol{\omega})$. Hence, in order to handle the NSE (say, in numerical simulations or to obtain analytical approximations), one typically attempts to control the gradient of the solution, for example, by strengthening the viscosity or weakening the nonlinear term, since the nonlinear term cascades energy from large scales to small scales, intensifying the gradient. That is, the vorticity growth seems to be associated with growth of *small scales*. On the other hand, for the KSE, one problem seems to be the growth of large scales (another is still the lack of any known maximum principle). One possible reason may be the cubic nature of the energy equation: $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^2}^2 \sim (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{u})$. In the 1D case (and in the NSE case), this latter term vanishes, but not in the 2D KSE case. Moreover, controlling the small scales is not a major problem, as the KSE has a fourth-order diffusion term, strongly curbing the growth of gradients. Therefore, the problem for the KSE appears to be the exact opposite of the problem for the NSE. That is, the problems inherent in the KSE seem to be associated with the growth of *large scales*. Hence, the standard approaches that work for the NSE might not work for the KSE (see [32] for investigations of this notion in the 1D case), and searching for new approaches to handling the KSE seems justified. The purpose of the present work is to propose and investigate one such approach.

In [39] a modification of the 2D KSE, called the 'reduced KSE' (r-KSE), was proposed and studied, which featured an adjustment made to the linear term in one component. This system admits a maximum principle, allowing for a proof of globally well-posedness. Moreover, simulations in [39] indicate that the dynamics of the r-KSE are arguably qualitatively similar to KSE. However, r-KSE suffers from the drawback that there is no clear way to see solutions of the r-KSE converge to solutions of the KSE, as any introduction of a 'turning' parameter interpolating between the r-KSE and the KSE would immediately violate the maximum principle. In contrast, the model introduced in this present paper allows for such a parameter $\epsilon > 0$, which we call the 'calming parameter.' In particular, by adjusting the nonlinear term in the (1.2), we create a globally well-posed PDE that approximates solutions to the 2D KSE to arbitrary precision, at least on the time interval of existence and uniqueness of solutions to the KSE. Perhaps

[3] Since the KSE governs the evolution of a surface, its natural space dimension is two. Moreover, it is not clear that the 3D case for the KSE is fundamentally more difficult than the 2D case, due to the already strong dissipation. Hence, we focus on the 2D case.

surprisingly, our construction does not require the use of a maximum principle, nor does it add artificial viscosity to the system.

The N-dimensional Kuramoto–Sivashinsky equation (KSE) is given in scalar form by

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + \triangle\phi + \triangle^2\phi = 0. \tag{1.1}$$

with periodic boundary conditions on a domain $[0,L]^N$. By setting $\mathbf{u} = \nabla\phi$ in (1.1), one formally[4] obtains the vector formulation of KSE:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \triangle\mathbf{u} + \triangle^2\mathbf{u} = \mathbf{0}, \tag{1.2}$$

These equations were originally proposed in the 1970's by Kuramoto and Tsuzuki in the studies of crystal growth [34, 35] as well as by Sivashinsky in the study of flame-front instabilities [55] (see also [56]). It has since found many other applications in the sciences, such as describing the flow of fluid down inclined planes [57], and has shown to be a generic feature of many physical phenomena involving bifurcations [43].

Many results on the 1D equation have been obtained since its origination, and the equation has been shown to be rich with interesting dynamics. It is globally well-posed [46, 59], solutions continue to exhibit chaotic dynamics at large times (see, e.g. [9, 26, 42, 47, 49]), and a large body of work has been published on quantitative results pertaining to the global attractor (see, e.g. [9–12, 18–20, 22–24, 26, 27, 32, 48, 51, 59, 61]).

There are far fewer results on the KSE in the 2D case. Global well-posedness for sufficiently small initial data was first shown in [54] on a domain $[0,2\pi]\times[0,2\pi\epsilon]$ with $\epsilon > 0$ sufficiently small. This result was improved upon in [45] by showing global existence on a domain $[0,L_1]\times[0,L_2]$ with $L_2 \leqslant CL_1^q$ for some particular q. Later works continued to improve on the sharpness of this bound (see, e.g. [4, 33, 41, 44] and references therein). Other works employ control of the domain size as a means to control the instability in Fourier modes. It was shown in [1] that for small enough domains (on which no growing Fourier modes are present in the linear terms), global existence holds when the initial data is sufficiently small in a certain Wiener algebra. This result was then extended in [2] to domains in which there is one linearly growing mode in each direction. Further studies have investigated modified equations [13, 17, 25, 39, 44, 63] or have looked at the equations with different boundary conditions [21, 38, 50]. For other results on the case $N > 1$, see also [5, 6, 37].

The intent of the present work is to propose a modification of the 2D KSE in vector form which is globally well-posed for any size of domain or initial data. To do this, we make use of what we call an *algebraic calming function* or simply a *calming function*[5] which constrains the advective velocity of the solution.

We propose the following modification of system (1.2),

$$\partial_t \mathbf{u} + (\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\,\mathbf{u} + \triangle\mathbf{u} + \triangle^2\mathbf{u} = \mathbf{0}, \tag{1.3a}$$

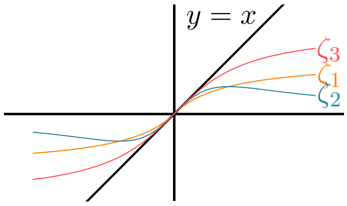
$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \tag{1.3b}$$

[4] We do not claim that $\nabla\phi$ is a unique solution to (1.2) when ϕ is a solution to (1.1). We only observe that one can formally obtain the set of equation (1.2) by taking the gradient of equation (1.1). In particular, it may be the case that there exist solutions to (1.2) that are not gradients of solutions to (1.1), or of any other function.

[5] Such a function is simply a bounded smooth truncation function, but we call it a 'calming' function due to the way it is used in the nonlinearity to suppress the algebraic growth of the nonlinear term. We do not call it a 'regularization,' since we reserve this term for techniques which smooth the equations by modifying derivative operators.

with L-periodic[6] boundary conditions on the 2-dimensional periodic torus $\mathbb{T}^2 := \mathbb{R}^2/(L\mathbb{Z})^2 = [0,L]^2$ for some $L > 0$. We call $\epsilon > 0$ the *calming parameter*, and $\boldsymbol{\zeta}^\epsilon$ the *calming function*. We require that $\boldsymbol{\zeta}^\epsilon$ satisfies the conditions described in definition 1.3. For the sake of concreteness, we consider several example choices for $\boldsymbol{\zeta}^\epsilon$; namely

$$\boldsymbol{\zeta}^\epsilon(\mathbf{x}) = \begin{cases} \zeta_1^\epsilon(\mathbf{x}) & := \frac{\mathbf{x}}{1+\epsilon|\mathbf{x}|} \\ \zeta_2^\epsilon(\mathbf{x}) & := \frac{\mathbf{x}}{1+\epsilon^2|\mathbf{x}|^2} \\ \zeta_3^\epsilon(\mathbf{x}) & := \frac{1}{\epsilon}\arctan(\epsilon\mathbf{x}) \end{cases} \tag{1.4}$$



where the arctangent acts componentwise; that is, for a given vector $\mathbf{z} = \binom{z_1}{z_2}$, we denote $\arctan(\mathbf{z}) = \binom{\arctan(z_1)}{\arctan(z_2)}$.

Remark 1.1. For a given choice of ζ_i^ϵ, $i = 1, 2$, or 3, ϵ can be thought of as a parameter which limits the velocity scale advecting the flow. However, it is not immediately clear that velocity scales are limited in the same way between choices of ζ_i^ϵ for a fixed ϵ. One could imagine trying to find a more meaningful comparison by rescaling the ζ_i^ϵ to have the same supremum. However, in doing so, $\zeta_i^\epsilon(\mathbf{x})$ is no longer a good pointwise approximation for $\mathbf{x}$. Therefore, it may not be meaningful to compare different convergence rates (or errors) between different types for fixed ϵ. Yet, for the sake of convenience, we plot all error curves, etc on the same plot. Moreover, the goal of the present work is not to compare different types of ζ_i^ϵ but rather to exhibit the robustness of this approach to different choices of calming function ζ_i^ϵ.

Remark 1.2. We see no major difficulty in extending our work to the case of physical boundary conditions, i.e. $\mathbf{u}\big|_{\partial\Omega} = \triangle\mathbf{u}\big|_{\partial\Omega} = \mathbf{0}$. However, for the sake of simplicity, we only consider periodic boundary conditions in the present work.

Section 1.1 lists our main definitions and results, and section 2 lists some preliminaries. Section 3 contains a proof of global well-posedness, which is mostly standard Galerkin methods, but with some subtle differences due to the non-polynomial form of the nonlinearity. Section 4 contains a proof of higher-order (but not arbitrary order) regularity of solutions. Section 5 contains a proof of convergence of solutions of the calmed equation to solutions to the original KSE as the calming parameter $\epsilon \to 0$. The proof here is not so straight-forward due to issues with commutator terms involving the calming function. As we will see, these issues are circumvented by taking advantage of structural properties of the calming function, and then using a boot-strapping argument in time. In addition, our techniques yield an explicit convergence rate. In section 6 we extend our ideas to the scalar form of the KSE. In particular, we consider a modification to system (1.1),

$$\partial_t\phi + \tfrac{1}{2}\boldsymbol{\zeta}^\epsilon(\nabla\phi)\cdot\nabla\phi + \triangle\phi + \triangle^2\phi = 0, \tag{1.5a}$$

$$\phi(\mathbf{x},0) = \phi_0(\mathbf{x}). \tag{1.5b}$$

[6] Note: One could easily consider rectangular non-square periodic domains, say $\mathbb{R}^2/((L_1\mathbb{Z})\times(L_2\mathbb{Z}))$ as well with slight modification of the techniques we use here. For the sake of keeping the discussion focused, we do not pursue such matters here.

Other formulations are of course possible. For example, one could consider a nonlinearity of the form $\frac{1}{2}\boldsymbol{\zeta}^\epsilon(|\nabla\phi|^2)$, or $\frac{1}{2}\boldsymbol{\zeta}^\epsilon(|\nabla\phi|^\delta)|\nabla\phi|^{2-\delta}$ $(0<\delta<2)$, or $\frac{\frac{1}{2}|\nabla\phi|^2}{1+\epsilon^2|\phi|^2}$, or many other possibilities. However, in the present work, we choose to focus only on the form in (1.5), as the advective nature of the nonlinearity seems perhaps closest in spirit to the nature of the original equation.

Section 7 exhibits results from simulations and provides computational evidence that the convergence rates we obtained in section 5 are sharp (at least, in terms of convergence order). Concluding remarks are in section 8.

1.1. Main results

Definition 1.3. We say $\boldsymbol{\zeta}^\epsilon:\mathbb{R}^2\to\mathbb{R}^2$ is a *calming function* if the following conditions hold:

(i) $\boldsymbol{\zeta}^\epsilon$ is Lipschitz continuous with Lipschitz constant 1,
(ii) For $\epsilon>0$ fixed, $\boldsymbol{\zeta}^\epsilon$ is bounded.

These two conditions are sufficient to show that (1.3) is globally well-posed. In section 5, we impose this third condition to obtain convergence:

(iii) There exists $C>0$, $\alpha>0$ and $\beta\geqslant 1$ such that for any $\mathbf{x}\in\mathbb{R}^2$,

$$|\boldsymbol{\zeta}^\epsilon(\mathbf{x})-\mathbf{x}|\leqslant C\epsilon^\alpha|\mathbf{x}|^\beta. \tag{1.6}$$

Proposition 1.4. *Consider $\boldsymbol{\zeta}^\epsilon_i$ as described in (1.4). Then $\boldsymbol{\zeta}^\epsilon_i$ satisfies conditions (i), (ii), and (iii) of definition 1.3 for each $i=1,2,3$. In particular, the following explicit bounds hold for $\epsilon>0$.*

(i) For $\boldsymbol{\zeta}^\epsilon_1$,

$$\|\boldsymbol{\zeta}^\epsilon_1\|_{L^\infty}=\frac{1}{\epsilon}\textit{and } |\boldsymbol{\zeta}^\epsilon_1(\mathbf{x})-\mathbf{x}|\leqslant\epsilon|\mathbf{x}|^2.$$

(ii) For $\boldsymbol{\zeta}^\epsilon_2$,

$$\|\boldsymbol{\zeta}^\epsilon_2\|_{L^\infty}=\frac{1}{2\epsilon}\textit{and } |\boldsymbol{\zeta}^\epsilon_2(\mathbf{x})-\mathbf{x}|\leqslant\epsilon^2|\mathbf{x}|^3.$$

(iii) For $\boldsymbol{\zeta}^\epsilon_3$,

$$\|\boldsymbol{\zeta}^\epsilon_3\|_{L^\infty}=\frac{\sqrt{2}\pi}{2\epsilon}\textit{and } |\boldsymbol{\zeta}^\epsilon_3(\mathbf{x})-\mathbf{x}|\leqslant\epsilon^2|\mathbf{x}|^3.$$

Proof. Straightforward computations (using a Taylor series expansions in the case $i=3$) yield the result. □

Definition 1.5. Let $\mathbf{u}_0\in L^2(\mathbb{T}^2)$ and let $T>0$. We say that $\mathbf{u}$ is a *weak solution* to calmed KSE (1.3) on the interval $[0,T]$ if $\mathbf{u}\in L^2([0,T];H^2(\mathbb{T}^2))\cap C([0,T];L^2(\mathbb{T}^2))$, $\partial_t\mathbf{u}\in L^2(0,T;H^{-2}(\mathbb{T}^2))$, and $\mathbf{u}$ satisfies (1.3*a*) in the sense of $L^2(0,T;H^{-2}(\mathbb{T}^2))$ and satisfies (1.3*b*) in the sense of $C([0,T];L^2(\mathbb{T}^2))$.

Theorem 1.6 (Global well-posedness). *Let $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, let $T > 0$ and fix $\epsilon > 0$. Suppose ζ^ϵ is a calming function which satisfies Conditions (i) and (ii) of definition 1.3. Then weak solutions to (1.3) on $[0,T]$ exist, are unique, and depend continuously on the initial data in $L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$.*

Theorem 1.7 (Regularity). *Suppose that ζ^ϵ is calming function which satisfies Conditions (i), and (ii) of definition 1.3. Let $m \in \{1,2\}$, and suppose that $\boldsymbol{u}$ is a weak solution to (1.3) on $[0,T]$ for some $T > 0$. If $\mathbf{u}_0 \in H^m(\mathbb{T}^2)$, then $\mathbf{u} \in L^\infty(0,T;H^m(\mathbb{T}^2)) \cap L^2(0,T;H^{m+2}(\mathbb{T}^2))$.*

Theorem 1.8 (Convergence). *Given $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, let*

$$\mathbf{u} \in C\left([0,T];L^2\left(\mathbb{T}^2\right)\right) \cap L^2\left(0,T;H^2\left(\mathbb{T}^2\right)\right). \tag{1.7}$$

be the corresponding weak solution of (1.2) with maximal time of existence and uniqueness $T^* > 0$ and with $T \in (0,T^*)$. Suppose ζ^ϵ satisfies conditions (i) and (ii) of definition 1.3. Furthermore, suppose ζ^ϵ satisfies condition (iii), so that (1.6) holds for some fixed $C, \alpha > 0$ and any $\beta \in [1,3]$. Let $\boldsymbol{u}^\epsilon$ be the corresponding weak solution of (1.3) with calming function ζ^ϵ and initial data $\boldsymbol{u}_0$. Then for any $\epsilon > 0$, it holds that

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \leqslant K_2 \epsilon^\alpha,$$
$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^2(0,T;H^2)} \leqslant K_4 \epsilon^\alpha,$$

where K_2 and K_4 are positive constants which depend on T, β, and $\boldsymbol{u}$, but not on ϵ or α.

Remark 1.9.

- The exact dependence of K_2 and K_4 on T, β, and $\mathbf{u}$ are explicitly shown in the proof of theorem 1.8.
- This convergence result may not hold on the maximal interval $[0,T^*]$. Indeed. as $T \to T^*$ it may be the case that $K_2, K_4 \to \infty$, but this remains an open question for KSE.
- The upper bound $\beta \leqslant 3$ is a technical limitation that appears in the proof of theorem 1.8, in particular to ensure the integrability of terms derived in estimate (5.7). This limitation can be removed by choosing smoother initial data $\mathbf{u}_0 \in H^2(\mathbb{T}^2)$. Additionally, we remark that any bound on β only affects the choice of calming function that one uses. Since each example of a calming function (1.4) satisfies $\beta \in [1,3]$ (see proposition 1.4), we do not consider this bound to be restrictive.

Definition 1.10. Let $\phi_0 \in L^2(\mathbb{T}^2)$ and let $T > 0$. We say that ϕ is a *weak solution* to (1.5) on the interval $[0,T]$ if $\phi \in L^2([0,T];H^2(\mathbb{T}^2)) \cap C([0,T];L^2(\mathbb{T}^2))$, $\partial_t \phi \in L^2(0,T;H^{-2}(\mathbb{T}^2))$, and ϕ satisfies (1.5*a*) in the sense of $L^2(0,T;H^{-2}(\mathbb{T}^2))$ and satisfies (1.5*b*) in the sense of $C([0,T];L^2(\mathbb{T}^2))$.

Theorem 1.11 (Global well-posedness in scalar form). *Let initial data $\phi_0 \in L^2(\mathbb{T}^2)$ be given, and let $T > 0$, $\epsilon > 0$ be fixed. Suppose ζ^ϵ is a calming function which satisfies Conditions (i) and (ii) of definition 1.3. Then weak solutions to (1.5) on $[0,T]$ exist, are unique, and depend continuously on the initial data in $L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$.*

Theorem 1.12 (Convergence in scalar form). *Choose $\phi_0 \in L^2(\mathbb{T}^2)$ and let ϕ be the corresponding weak solution of the scalar KSE (1.1) with maximal time of existence T^*. We assume that ϕ is in the natural energy space: for $T < T^*$,*

$$\phi \in C\left([0,T];L^2\left(\mathbb{T}^2\right)\right) \cap L^2\left(0,T;H^2\left(\mathbb{T}^2\right)\right). \tag{1.8}$$

Suppose ζ^ϵ *satisfies (i), (ii), and (iii) of definition* 1.3*, so that there exists* C*,* $\alpha > 0$*, and* $\beta \in \left[1, \frac{3}{2}\right]$ *for which* (1.6) *holds. and let* ϕ^ϵ *be the corresponding weak solution of the scalar calmed KSE* (1.5) *with calming function* ζ^ϵ *and with initial data* ϕ_0*. Consider the convergence of* ϕ^ϵ *to* ϕ *on the interval* $[0,T]$*. The difference* $\phi^\epsilon - \phi$ *satisfies*

$$\begin{aligned} \|\phi^\epsilon - \phi\|_{L^\infty(0,T;L^2)} &\leqslant K\epsilon^\alpha, \\ \|\phi^\epsilon - \phi\|_{L^2(0,T;H^2)} &\leqslant K'\epsilon^\alpha, \end{aligned}$$

where $K, K' > 0$ *depend on* T*,* β*, and various norms of* ϕ*, but not on* ϵ *or* α*.*

Remark 1.13. Similar to the comments made in remark 1.9, The upper bound $\beta \leqslant \frac{3}{2}$ is needed to establish the integrability of terms present in (6.16), and this bound can be removed by selecting initial data in, say, $H^2(\mathbb{T})$.

2. Preliminaries

In this section, we lay out some notation and preliminary notions that will be used later in the text. We denote the 2π-periodic box $\mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2 = [0,2\pi)^2$. We denote the set of real vector-valued L^2 functions on $\mathbb{T}^2$ by

$$L^2\left(\mathbb{T}^2\right) := \left\{ \mathbf{u} \middle| \mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} \hat{\mathbf{u}}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}, \overline{\hat{\mathbf{u}}_\mathbf{k}} = \hat{\mathbf{u}}_{-\mathbf{k}}, \text{ and } \sum_{\mathbf{k}\in\mathbb{Z}^2} |\hat{\mathbf{u}}_\mathbf{k}|^2 < \infty \right\}$$

(with the usual convention of equivalence up to sets of measure zero). We also denote the (real) L^2 inner-product and H^s Sobolev norm, $s \in \mathbb{R}$, by

$$(\mathbf{u},\mathbf{v}) := \sum_{i=1}^{2} \int_{\mathbb{T}^2} u_i(\mathbf{x})\, v_i(\mathbf{x})\, \mathrm{d}\mathbf{x}, \qquad \|\mathbf{u}\|_{H^s} := \left(\sum_{\mathbf{k}\in\mathbb{Z}^2} (1+|\mathbf{k}|)^{2s} |\hat{\mathbf{u}}_\mathbf{k}|^2 \right)^{1/2},$$

and the corresponding space $H^s(\mathbb{T}^2) = \left\{\mathbf{u} \in L^2(\mathbb{T}^2) \middle| \quad \|\mathbf{u}\|_{H^s} < \infty\right\}$. The space $L^2(\mathbb{T}^2)$ has an orthogonal basis of eigenfunctions of the Laplacian operator $-\triangle$ given by

$$\left\{ \left(e^{i\mathbf{k}\cdot\mathbf{x}}, 0\right), \left(0, e^{i\mathbf{k}\cdot\mathbf{x}}\right) : \mathbf{k} \in \mathbb{Z}^2 \right\},$$

with corresponding eigenvalues $\left\{ |\mathbf{k}|^2 : \mathbf{k} \in \mathbb{Z}^2 \right\}$.

For any $m \in \mathbb{N}$, we denote by $P_m : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ the projection onto finitely many eigenfunctions of the operator $-\triangle$,

$$P_m\mathbf{u} = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^2 \\ |\mathbf{k}|\leqslant m}} \hat{\mathbf{u}}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Denote $Q_m := I - P_m$. We recall the following projection estimates for any $\mathbf{u} \in H^s(\mathbb{T}^2)$, $s > 0$,

$$\left\| (-\triangle)^s P_m\mathbf{u} \right\|_{L^2} \leqslant m^s \|P_m\mathbf{u}\|_{L^2} \tag{2.1}$$

$$\|Q_m\mathbf{u}\|_{L^2} \leqslant \frac{1}{m^s} \|\mathbf{u}\|_{H^s}. \tag{2.2}$$

Using integration by parts, the Cauchy–Schwarz, and Young's inequalities, we obtain, for any $\delta > 0$, the estimate

$$\|\nabla \mathbf{u}\|_{L^2}^2 \leqslant \frac{1}{2\delta}\|\mathbf{u}\|_{L^2}^2 + \frac{\delta}{2}\|\triangle \mathbf{u}\|_{L^2}^2. \tag{2.3}$$

We also recall Agmon's inequality on $\mathbb{T}^2$, for $s_1 < 1 < s_2$,

$$\|\mathbf{u}\|_{L^\infty} \leqslant C\|\mathbf{u}\|_{H^{s_1}}^{\theta}\|\mathbf{u}\|_{H^{s_2}}^{1-\theta}, \tag{2.4}$$

where $\theta s_1 + (1-\theta)s_2 = 1$. We also frequently use a special case of the Gagliardo–Nirenberg–Sobolev inequality,

$$\|\mathbf{u}\|_{L^4}^2 \leqslant C\|\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^1}. \tag{2.5}$$

Notice that this is similar to but not the same as Ladyzhenskaya's inequality, since it involves the full H^1 norm. This is necessary because neither KSE nor calmed KSE preserve mean-free vector fields. However, using (2.5) and elliptic regularity, and due to our periodic boundary conditions, we have the following Ladyzhenskaya-type inequality on higher-order derivatives, since they are mean-free (denoting the average $\overline{\mathbf{u}} := \frac{1}{|\Omega|}\int_\Omega \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}$):

$$\begin{aligned}\|\nabla \mathbf{u}\|_{L^4}^2 &\leqslant C\|\nabla \mathbf{u}\|_{L^2}\|\nabla \mathbf{u}\|_{H^1} = C\|\nabla \mathbf{u}\|_{L^2}\|\nabla(\mathbf{u}-\overline{\mathbf{u}})\|_{H^1}\\ &\leqslant C\|\nabla \mathbf{u}\|_{L^2}\|\triangle(\mathbf{u}-\overline{\mathbf{u}})\|_{L^2} = C\|\nabla \mathbf{u}\|_{L^2}\|\triangle \mathbf{u}\|_{L^2}.\end{aligned} \tag{2.6}$$

Applying integration by parts and the Cauchy–Schwarz inequality in conjunction with (2.5) also yields the following useful estimate:

$$\|\mathbf{u}\|_{L^4}^2 \leqslant C\|\mathbf{u}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \tag{2.7}$$

We also denote by C a positive constant which may change from line to line.

3. Global well-posedness for calmed KSE

In this section, we prove the results stated section 1, along with some auxilliary results. We begin with some lemmata.

Lemma 3.1. *Suppose that $\boldsymbol{\zeta}^\epsilon$ satisfies conditions (i) and (ii) of definition 1.3. Then the following statements hold.*

(i) Given $1 \leqslant p \leqslant \infty$, if $\mathbf{u} \in L^p(\mathbb{T}^2)$ then $\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \in L^p(\mathbb{T}^2)$ and $\boldsymbol{\zeta}^\epsilon$ is Lipschitz in $L^p(\mathbb{T}^2)$ with Lipschitz constant 1.

(ii) Fix $\mathbf{u}, \mathbf{w} \in L^2(0,T;L^2(\mathbb{T}^2))$ and $T > 0$, let $I_{\mathbf{u},\mathbf{w}} : L^2(0,T;H^1(\mathbb{T}^2)) \to \mathbb{R}$ be the map

$$I_{\mathbf{u},\mathbf{w}}(\phi) = \int_0^T ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\phi, \mathbf{w})\,\mathrm{d}t. \tag{3.1}$$

Then $I_{\mathbf{u},\mathbf{w}}$ is continuous.

Proof. *(i)*. The result follows immediately from the definition of the L^p norm and from condition (i) of definition 1.3.

(ii). Let $\zeta_j^\epsilon(\mathbf{u})$ denote the jth component of $\boldsymbol{\zeta}^\epsilon(\mathbf{u})$.

For $\phi \in L^2(0,T;H^1(\mathbb{T}^2))$, we estimate

$$
\begin{aligned}
|I_{\mathbf{u},\mathbf{w}}(\phi)| &\leqslant \sum_{j=1}^{2} \int_0^T \left|\left(\zeta_j^\epsilon(\mathbf{u})\,\partial_j\phi, \mathbf{w}\right)\right| \mathrm{d}t \\
&\leqslant \sum_{j=1}^{2} \int_0^T \left\|\zeta_j^\epsilon(\mathbf{u})\right\|_{L^\infty} \|\partial_j\phi\|_{L^2} \|\mathbf{w}\|_{L^2}\,\mathrm{d}t \\
&\leqslant \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \int_0^T \|\phi\|_{H^1} \|\mathbf{w}\|_{L^2}\,\mathrm{d}t \\
&\leqslant \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty} \|\mathbf{w}\|_{L^2(0,T;L^2)} \|\phi\|_{L^2(0,T;H^1)}
\end{aligned}
$$

by the Cauchy–Schwarz inequality. □

Using the projection operator P_m, define the finite-dimensional space $H_m := P_m(L^2(\mathbb{T}^2))$. Consider the following initial value problem obtained via Galerkin approximation: Given $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, find $\mathbf{u} \in H_m$ which satisfies

$$
\partial_t\mathbf{u} + P_m\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\right)\mathbf{u}\right) + \triangle\mathbf{u} + \triangle^2\mathbf{u} = \mathbf{0}, \tag{3.2a}
$$

$$
\mathbf{u}(\mathbf{x},0) = P_m\mathbf{u}_0(\mathbf{x}). \tag{3.2b}
$$

Lemma 3.2. *If $\boldsymbol{\zeta}^\epsilon$ satisfies (i) of definition 1.3, then the map $F: H_m \to H_m$ defined by*

$$
F(\mathbf{u}) = -P_m\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\right)\mathbf{u}\right) - \triangle\mathbf{u} - \triangle^2\mathbf{u}
$$

is locally Lipschitz on H_m. As a consequence, solutions to (3.2) exist and are unique in $C^1([0,T],H_m)$ for some $T>0$.

Proof. Fix $\mathbf{u} \in H_m$ and let $\mathbf{v} \in H_m$ be arbitrary. Rewrite the difference $F(\mathbf{u}) - F(\mathbf{v})$ as

$$
\begin{aligned}
F(\mathbf{u}) - F(\mathbf{v}) = &-\triangle(\mathbf{u}-\mathbf{v}) - \triangle^2(\mathbf{u}-\mathbf{v}) - P_m\left(\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\right)\cdot\nabla\right)\mathbf{u}\right) \\
&- P_m\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla\right)(\mathbf{u}-\mathbf{v})\right).
\end{aligned}
$$

From Condition (i) of definition 1.3, Estimate (2.1), and Agmon's inequality, it follows that

$$
\begin{aligned}
&\|F(\mathbf{u}) - F(\mathbf{v})\|_{L^2} \\
&\quad\leqslant \|\triangle(\mathbf{u}-\mathbf{v})\|_{L^2} + \left\|\triangle^2(\mathbf{u}-\mathbf{v})\right\|_{L^2} \\
&\qquad + \left\|\left(\left(\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\right)\cdot\nabla\right)\mathbf{u}\right\|_{L^2} + \left\|\left(\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla\right)(\mathbf{u}-\mathbf{v})\right\|_{L^2} \\
&\quad\leqslant \left(m+m^2\right)\|\mathbf{u}-\mathbf{v}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty}\|\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \boldsymbol{\zeta}^\epsilon(\mathbf{v})\|_{L^2} \\
&\qquad + \|\boldsymbol{\zeta}^\epsilon(\mathbf{v})\|_{L^\infty}\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2} \\
&\quad\leqslant \left(m+m^2\right)\|\mathbf{u}-\mathbf{v}\|_{L^2} + \|\mathbf{u}\|_{H^3}\|\mathbf{u}-\mathbf{v}\|_{L^2} + m^{\frac{1}{2}}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\mathbf{u}-\mathbf{v}\|_{L^2}.
\end{aligned}
$$

Since $\mathbf{u}$ is a finite linear combination of eigenfunctions of $-\triangle$, $\|\mathbf{u}\|_{H^3} < \infty$. Thus F is locally Lipschitz at $\mathbf{u} \in H_m$. Existence and uniqueness of solutions to (3.2) in $C^1([0,T],H_m)$ now follows as a consequence of the Picard-Lindelöf theorem □

Due to the presence of the calming function ζ^ϵ, the Galerkin system here is not necessarily quadratic such as in the case of the 2D NSEs or the 2D KSEs. Thus we give a fully rigorous proof of well-posedness here.

Proof of theorem 1.6. We will show that a solution exists using Galerkin approximation. Given $\mathbf{u}_0 \in L^2(\mathbb{T}^2)$, suppose $\mathbf{u}^m \in C([0,T_m];H_m)$ is a solution to (3.2) on the interval $[0,T_m]$ for some $T_m > 0$ with initial data $\mathbf{u}_0^m = P_m\mathbf{u}_0$. We take the inner product of (3.2) with $\mathbf{u}^m$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}^m\|_{L^2}^2 + \|\triangle\mathbf{u}^m\|_{L^2}^2 = -\left(\triangle\mathbf{u}^m,\mathbf{u}^m\right) - \left(\left(\zeta^\epsilon\left(\mathbf{u}^m\right)\cdot\nabla\right)\mathbf{u}^m,\mathbf{u}^m\right)$$

We estimate the first term by $-\left(\triangle\mathbf{u}^m,\mathbf{u}^m\right) \leqslant \frac{1}{4}\|\triangle\mathbf{u}^m\|_{L^2}^2 + \|\mathbf{u}^m\|_{L^2}^2$. For the nonlinear term, we estimate

$$\begin{aligned}
\left|\left(\zeta^\epsilon\left(\mathbf{u}^m\right)\cdot\nabla\right)\mathbf{u}^m,\mathbf{u}^m\right)\right| &\leqslant \|\zeta^\epsilon(\mathbf{u}^m)\|_{L^\infty}\|\nabla\mathbf{u}^m\|_{L^2}\|\mathbf{u}^m\|_{L^2} \\
&\leqslant \|\zeta^\epsilon\|_{L^\infty}\|\mathbf{u}^m\|_{L^2}^{\frac{1}{2}}\|\triangle\mathbf{u}^m\|_{L^2}^{\frac{1}{2}}\|\mathbf{u}^m\|_{L^2} \\
&\leqslant \frac{3}{4}\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}}\|\mathbf{u}^m\|_{L^2}^2 + \frac{1}{4}\|\triangle\mathbf{u}^m\|_{L^2}^2
\end{aligned}$$

Combining the above estimates and denoting $K_\epsilon := \frac{3}{2}\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}} + 2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}^m\|_{L^2}^2 + \|\triangle\mathbf{u}^m\|_{L^2}^2 \leqslant K_\epsilon\|\mathbf{u}^m\|_{L^2}^2. \tag{3.3}$$

After dropping the second term of (3.3), Grönwall's inequality yields for all $t \in [0,T_m]$,

$$\|\mathbf{u}^m(t)\|_{L^2}^2 \leqslant e^{K_\epsilon t}\|\mathbf{u}^m(0)\|_{L^2}^2 \leqslant e^{K_\epsilon T_m}\|\mathbf{u}_0\|_{L^2}^2. \tag{3.4}$$

Since $\mathbf{u}^m \in C([0,T_m],\mathbb{T}^2)$, via a bootstrapping argument, it holds that for any $T > 0$ and any $t \in [0,T]$,

$$\|\mathbf{u}^m(t)\|_{L^2}^2 \leqslant e^{K_\epsilon t}\|\mathbf{u}_0\|_{L^2}^2 \leqslant e^{K_\epsilon T}\|\mathbf{u}_0\|_{L^2}^2. \tag{3.5}$$

Next, we integrate (3.3) on $[0,T]$ and apply estimate (3.5):

$$\begin{aligned}
&\|\mathbf{u}^m(T)\|_{L^2}^2 + \frac{1}{2}\int_0^T\|\triangle\mathbf{u}^m(s)\|_{L^2}^2\,\mathrm{d}s \\
&\quad\leqslant \int_0^T K_\epsilon\|\mathbf{u}^m(s)\|_{L^2}^2\,\mathrm{d}s + \|\mathbf{u}^m(0)\|_{L^2}^2 \\
&\quad\leqslant \int_0^T K_\epsilon e^{K_\epsilon s}\|\mathbf{u}_0\|_{L^2}^2\,\mathrm{d}s + \|\mathbf{u}_0\|_{L^2}^2 \\
&\quad= e^{K_\epsilon T}\|\mathbf{u}_0\|_{L^2}^2.
\end{aligned} \tag{3.6}$$

Hence, for all $T > 0$,

$$\{\mathbf{u}^m\}_{m=1}^{\infty} \text{ is bounded in } L^\infty\left([0,T];L^2\right) \cap L^2\left([0,T];H^2\right). \tag{3.7}$$

To bound the time derivative, we estimate

$$\begin{aligned}
\|\partial_t \mathbf{u}^m\|_{H^{-2}} &\leqslant \left\|\triangle^2 \mathbf{u}^m\right\|_{H^{-2}} + \|\triangle \mathbf{u}^m\|_{H^{-2}} + \sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}} |\langle P_m\left((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m\right),\phi\rangle| \\
&\leqslant C_1\|\mathbf{u}^m\|_{H^2} + C_2\|\mathbf{u}^m\|_{L^2} + \sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}} |\langle \boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m, P_m(\phi)\rangle| \\
&\leqslant C\|\mathbf{u}^m\|_{H^2} + C\|\mathbf{u}^m\|_{L^2} + \sup_{\substack{\phi\in H^2\\ \|\phi\|_{H^2}=1}} \|\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\|_{L^\infty}\|\mathbf{u}^m\|_{H^1}\|\phi\|_{L^2} \\
&\leqslant C\|\mathbf{u}^m\|_{H^2} + C\|\mathbf{u}^m\|_{L^2} + \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\mathbf{u}^m\|_{H^1}.
\end{aligned}$$

Hence, $\{\partial_t \mathbf{u}^m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-2}(\mathbb{T}^2))$. By the Banach–Alaoglu theorem, there exists $\mathbf{u} \in L^2(0,T;H^2(\mathbb{T}^2)) \cap L^\infty(0,T;L^2(\mathbb{T}^2))$ and a subsequence (which we will still label as $\mathbf{u}^m$) such that

$$\mathbf{u}^m \rightharpoonup^* \mathbf{u} \text{ weak-}^* \text{ in } L^\infty\left(0,T;L^2\left(\mathbb{T}^2\right)\right), \tag{3.8}$$

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ weakly in } L^2\left(0,T;H^2\left(\mathbb{T}^2\right)\right), \tag{3.9}$$

$$\partial_t\mathbf{u}^m \rightharpoonup \partial_t\mathbf{u} \text{ weakly in } L^2\left(0,T;H^{-2}\left(\mathbb{T}^2\right)\right). \tag{3.10}$$

Moreover, by the Aubin–Lions lemma we may pass to another subsequence, relabelled to be $\mathbf{u}^m$, such that

$$\mathbf{u}^m \to \mathbf{u} \text{ strongly in } C\left(0,T;L^2\left(\mathbb{T}^2\right)\right). \tag{3.11}$$

Now we can show that $\mathbf{u}$ is a weak solution to (1.3). Given $\mathbf{w} \in L^2(0,T;H^2(\mathbb{T}^2))$, we compute

$$\begin{aligned}
&(\langle\partial_t\mathbf{u},\mathbf{w}\rangle + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) + (\triangle\mathbf{u},\mathbf{w}) + (\triangle\mathbf{u},\triangle\mathbf{w})) \\
&\quad - (\langle\partial_t\mathbf{u}^m,\mathbf{w}\rangle + (P_m((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m),\mathbf{w}) + (\triangle\mathbf{u}^m,\mathbf{w}) + (\triangle\mathbf{u}^m,\triangle\mathbf{w})) \\
&= \langle\partial_t(\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle + (\triangle(\mathbf{u}-\mathbf{u}^m),\mathbf{w}) + (\triangle(\mathbf{u}-\mathbf{u}^m),\triangle\mathbf{w}) \\
&\quad + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) - (P_m((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m),\mathbf{w}) \\
&= \langle\partial_t(\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle + (\triangle(\mathbf{u}-\mathbf{u}^m),\mathbf{w}) + (\triangle(\mathbf{u}-\mathbf{u}^m),\triangle\mathbf{w}) \\
&\quad + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m,\mathbf{w}) + (Q_m((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m),\mathbf{w}) \\
&= \langle\partial_t(\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle + (\triangle(\mathbf{u}-\mathbf{u}^m),\mathbf{w}) + (\triangle(\mathbf{u}-\mathbf{u}^m),\triangle\mathbf{w}) \\
&\quad + (((\boldsymbol{\zeta}^\epsilon(\mathbf{u})-\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m))\cdot\nabla)\mathbf{u}^m,\mathbf{w}) + ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)(\mathbf{u}-\mathbf{u}^m),\mathbf{w}) \\
&\quad + (((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla)\mathbf{u}^m),Q_m\mathbf{w}). \\
&:= \sum_{k=1}^{6} I_k.
\end{aligned}$$

Integrate $\sum_{k=1}^{6} I_k$ in time for $t \in [0,T]$. We observe that I_1, I_2, and I_3 all vanish as $m \to \infty$ by (3.8)–(3.10). Using Condition (i) of definition 1.3, Agmon's inequality, and Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^T I_4 \mathrm{d}t &\leqslant \int_0^T \|\zeta^\epsilon(\mathbf{u}) - \zeta^\epsilon(\mathbf{u}^m)\|_{L^2} \|\nabla \mathbf{u}^m\|_{L^2} \|\mathbf{w}\|_{L^\infty} \mathrm{d}t \\
&\leqslant C \int_0^T \|\mathbf{u} - \mathbf{u}^m\|_{L^2} \|\mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{H^2}^{\frac{1}{2}} \mathrm{d}t \\
&\leqslant C \|\mathbf{u} - \mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \\
&\quad \times \int_0^T \|\mathbf{u} - \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}^m\|_{L^2}^{\frac{1}{2}} \|\mathbf{w}\|_{H^2} \mathrm{d}t \\
&\leqslant C \|\mathbf{u} - \mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{u}^m\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \\
&\quad \times \|\triangle \mathbf{u}^m\|_{L^2(0,T;L^2)}^{\frac{1}{2}} \|\mathbf{w}\|_{L^2(0,T;H^2)} \|\mathbf{u} - \mathbf{u}^m\|_{L^2(0,T;L^2)}^{\frac{1}{2}},
\end{aligned} \tag{3.12}$$

which is bounded due to (3.5), (3.9), and (3.11).

For I_5,

$$\int_0^T I_5 \mathrm{d}t = I_{\mathbf{u},\mathbf{w}}(\mathbf{u} - \mathbf{u}^m) \tag{3.13}$$

for $I_{\mathbf{u},\mathbf{w}}$ as defined in (3.1), which convergences due to lemma 3.1. Finally, using Hölder's inequality, condition (ii) of definition 1.3 and (2.2),

$$\begin{aligned}
\int_0^T I_6 \mathrm{d}t &\leqslant \int_0^T \|\zeta^\epsilon(\mathbf{u}^m)\|_{L^\infty} \|\nabla \mathbf{u}^m\|_{L^2} \|Q_m \mathbf{w}\|_{L^2} \mathrm{d}t \\
&\leqslant \|\zeta^\epsilon\|_{L^\infty} \left(\int_0^T \|\nabla \mathbf{u}^m\|_{L^2}^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \|Q_m \mathbf{w}\|_{L^2}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leqslant \|\zeta^\epsilon\|_{L^\infty} \left(\int_0^T \|\nabla \mathbf{u}^m\|_{L^2}^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \frac{1}{m^4} \|\mathbf{w}\|_{H^2}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leqslant \|\zeta^\epsilon\|_{L^\infty} \|\mathbf{u}^m\|_{L^2(0,T;H^2)} \|\mathbf{w}\|_{L^2(0,T;H^2)} \frac{1}{m^2},
\end{aligned} \tag{3.14}$$

which converges to zero by (3.7).

Invoking (3.9)–(3.14),

$$\lim_{m \to \infty} \int_0^T \left(\sum_{k=1}^{6} I_k\right) \mathrm{d}t = 0.$$

Therefore solutions to the ODE system (3.2) converge to a solution of the PDE system (1.3). Thus $\mathbf{u}$ is indeed a solution to (1.3).

Now we show that the solution $\mathbf{u}$ satisfies $\mathbf{u}(0)=\mathbf{u}_0$ in the sense of $C([0,T],L^2)$. Applying lemma 1.1 from chapter 3 of [62, p 250], for all $\mathbf{v}\in H^2(\mathbb{T}^2)$, it follows that

$$\langle \partial_t \mathbf{u},\mathbf{v}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v}) = -(\triangle\mathbf{u},\mathbf{v}) - (\triangle\mathbf{u},\triangle\mathbf{v}) - (\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\mathbf{u},\mathbf{v}) \tag{3.15}$$

in the scalar distribution sense on $[0,T]$. Now, suppose that $\psi\in C^1([0,T])$ and satisfies $\psi(0)=1$, $\psi(T)=0$. We then integrate (3.15) in time with ψ and apply integration by parts to obtain

$$\begin{aligned}\int_0^T(\mathbf{u},\mathbf{v})\psi'(t)\,\mathrm{d}t = &-\int_0^T(\triangle\mathbf{u},\mathbf{v})\psi(t)\,\mathrm{d}t - \int_0^T(\triangle\mathbf{u},\triangle\mathbf{v})\psi(t)\,\mathrm{d}t\\ &-\int_0^T(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\mathbf{u},\mathbf{v})\psi(t)\,\mathrm{d}t + (\mathbf{u}(0),\mathbf{v}).\end{aligned} \tag{3.16}$$

On the other hand, if we take the inner product of (3.2) with $\mathbf{v}$ then integrate in time with ψ we obtain

$$\begin{aligned}\int_0^T(\mathbf{u}^m,\mathbf{v})\psi'(t)\,\mathrm{d}t = &-\int_0^T(\triangle\mathbf{u}^m,\mathbf{v})\psi(t)\,\mathrm{d}t - \int_0^T(\triangle\mathbf{u}^m,\triangle\mathbf{v})\psi(t)\,\mathrm{d}t\\ &-\int_0^T(P_m(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^m)\cdot\nabla\mathbf{u}^m),\mathbf{v})\psi(t)\,\mathrm{d}t + (\mathbf{u}_0^m,\mathbf{v}).\end{aligned}$$

Passing to the limit as $m\to\infty$ then yields

$$\begin{aligned}\int_0^T(\mathbf{u},\mathbf{v})\psi'(t)\,\mathrm{d}t = &-\int_0^T(\triangle\mathbf{u},\mathbf{v})\psi(t)\,\mathrm{d}t - \int_0^T(\triangle\mathbf{u},\triangle\mathbf{v})\psi(t)\,\mathrm{d}t\\ &-\int_0^T(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla\mathbf{u},\mathbf{v})\psi(t)\,\mathrm{d}t + (\mathbf{u}_0,\mathbf{v}).\end{aligned} \tag{3.17}$$

By then comparing (3.16) and (3.17), we obtain $(\mathbf{u}(0)-\mathbf{u}_0,\mathbf{v})=0$ for all $\mathbf{v}\in H^2(\mathbb{T}^2)$. Since $H^2(\mathbb{T}^2)$ is dense in $L^2(\mathbb{T}^2)$, it follows that $(\mathbf{u}(0)-\mathbf{u}_0,\mathbf{v})=0$ for all $\mathbf{v}\in L^2(\mathbb{T}^2)$. Thus $\mathbf{u}$ satisfies $\mathbf{u}(0)=\mathbf{u}_0$. Next, we show that weak solutions are unique. Set $\mathbf{w}=\mathbf{u}-\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are both weak solutions of calmed KSE (1.3) on the interval $[0,T]$ with $\mathbf{u}_0=\mathbf{v}_0$. After taking the difference of the two equations, we then take the action of the difference equation with w, which yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{L^2}^2 + \|\triangle\mathbf{w}\|_{L^2}^2\\ &\quad= -(\triangle\mathbf{w},\mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{u},\mathbf{w}) + ((\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla)\mathbf{v},\mathbf{w})\\ &\quad= -(\triangle\mathbf{w},\mathbf{w}) + (((\boldsymbol{\zeta}^\epsilon(\mathbf{v})-\boldsymbol{\zeta}^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{u},\mathbf{w}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{v})\cdot\nabla)\mathbf{w},\mathbf{w})\\ &\quad= J_1+J_2+J_3,\end{aligned} \tag{3.18}$$

where we have used the Lions–Magenes lemma to write $\langle\partial_t\mathbf{w},\mathbf{w}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_{L^2}^2$. Then,

$$J_1 \leqslant C\|\mathbf{w}\|_{L^2}^2 + \frac{1}{6}\|\triangle\mathbf{w}\|_{L^2}^2.$$

Also, using the Lipschitz condition (i) of ζ^ϵ, (2.7), and Young's inequality, we have

$$\begin{aligned} J_2 &:= (((\zeta^\epsilon(\mathbf{v}) - \zeta^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{u},\mathbf{w}) \\ &\leqslant \|\zeta^\epsilon(\mathbf{v}) - \zeta^\epsilon(\mathbf{u})\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^4} \\ &\leqslant \|\nabla\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^4}^2 \\ &\leqslant C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^2}^{\frac{3}{2}}\|\triangle\mathbf{w}\|_{L^2}^{\frac{1}{2}} \\ &\leqslant C\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}}\right)\|\mathbf{w}\|_{L^2}^2 + \frac{1}{6}\|\triangle\mathbf{w}\|_{L^2}^2 \end{aligned}$$

and, finally,

$$\begin{aligned} J_3 &:= -((\zeta^\epsilon(\mathbf{v})\cdot\nabla)\mathbf{w},\mathbf{w}) \\ &\leqslant \|\zeta^\epsilon(\mathbf{v})\|_{L^\infty}\|\nabla\mathbf{w}\|_{L^2}\|\mathbf{w}\|_{L^2} \\ &\leqslant \left(\|\zeta^\epsilon\|_{L^\infty}\|\mathbf{w}\|_{L^2}^{\frac{3}{2}}\right)\left(\|\triangle\mathbf{w}\|_{L^2}^{\frac{1}{2}}\right) \\ &\leqslant C\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}}\|\mathbf{w}\|_{L^2}^2 + \frac{1}{6}\|\triangle\mathbf{w}\|_{L^2}^2. \end{aligned}$$

From the above estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|_{L^2}^2 + \|\triangle\mathbf{w}(t)\|_{L^2}^2 \leqslant C\left(1 + \|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}} + \|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}}\right)\|\mathbf{w}(t)\|_{L^2}^2.$$

We observe that $K_1(t) = C(1 + \|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}} + \|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}})$ is integrable on $[0,T]$. Thus we conclude, recalling that $\mathbf{w} = \mathbf{u} - \mathbf{v}$,

$$\|\mathbf{w}(t)\|_{L^2}^2 \leqslant \|\mathbf{w}_0\|_{L^2}^2 \exp\left(\int_0^T K_1(t)\,\mathrm{d}t\right). \tag{3.19}$$

Therefore solutions to (1.3) are unique. If we now integrate (3.18) on the interval $[0,T]$ and apply estimate (3.19), we obtain

$$\int_0^T \|\triangle\mathbf{w}(t)\|_{L^2}^2\,\mathrm{d}t \leqslant \left(\int_0^T K_1(t)\exp\left(\int_0^t K_1(s)\,\mathrm{d}s\right)\mathrm{d}t\right)\|\mathbf{w}_0\|_{L^2}^2 \tag{3.20}$$

From estimates (3.19) and (3.20) we conclude that solutions depend continuously on the initial data in $L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$.

□

4. Higher-order regularity of solutions

In this section, we only work formally, but the results can be made rigorous by using, e.g. the Galerkin method. We will show that the regularity of a weak solution $\mathbf{u}$ to (1.3) is dependent on the regularity of the calming function ζ^ϵ and the initial data $\mathbf{u}_0$.

Remark 4.1. It seems likely that higher-order regularity ($m > 2$) also holds, but we do not pursue such matters here.

Proof of theorem 1.7. We first show the case $m = 1$. We take the (formal) inner product of (1.3) with $-\triangle \mathbf{u}$ and integrate by parts to obtain

$$(\partial_t \mathbf{u}, -\triangle \mathbf{u}) - ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla) \mathbf{u}, \triangle \mathbf{u}) - (\triangle \mathbf{u}, \triangle \mathbf{u}) - \left(\triangle^2 \mathbf{u}, \triangle \mathbf{u}\right) = 0$$

which we will rewrite as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \triangle \mathbf{u}\|_{L^2}^2 &= (((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla) \mathbf{u}), \triangle \mathbf{u}) - (\nabla \mathbf{u}, \nabla \triangle \mathbf{u}) \\ &= ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla) \mathbf{u}, \triangle \mathbf{u}) - (\nabla \mathbf{u}, \nabla \triangle \mathbf{u}).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \triangle \mathbf{u}\|_{L^2}^2 \\ &\quad\leqslant \|\boldsymbol{\zeta}^\epsilon(\mathbf{u})\|_{L^\infty}\|\nabla \mathbf{u}\|_{L^2}\|\triangle \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}\|\nabla \triangle \mathbf{u}\|_{L^2} \\ &\quad\leqslant \|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}\|\nabla \mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\nabla \triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\nabla \mathbf{u}\|_{L^2}\|\nabla \triangle \mathbf{u}\|_{L^2} \\ &\quad\leqslant \left(\frac{3}{4}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} + \frac{1}{2}\right)\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{3}{4}\|\nabla \triangle \mathbf{u}\|_{L^2}^2.\end{aligned}$$

This estimate can then be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla \triangle \mathbf{u}\|_{L^2}^2 \leqslant \left(\frac{3}{2}\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} + 1\right)\|\nabla \mathbf{u}\|_{L^2}^2. \tag{4.1}$$

Then, by Grönwall's inequality,

$$\begin{aligned}\|\nabla \mathbf{u}(t)\|_{L^2}^2 &\leqslant \|\nabla \mathbf{u}_0\|_{L^2}^2 \exp\left(\frac{3}{2}t\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} + t\right) \\ &\leqslant \|\nabla \mathbf{u}_0\|_{L^2}^2 \exp\left(\frac{3}{2}T\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} + T\right)\end{aligned} \tag{4.2}$$

Now, after integrating (4.1) on the interval $[0, T]$ and applying estimate (4.2), it follows that

$$\int_0^T \|\nabla \triangle \mathbf{u}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leqslant 2\|\nabla \mathbf{u}_0\|_{L^2}^2 \exp\left(\frac{3}{2}T\|\boldsymbol{\zeta}^\epsilon\|_{L^\infty}^{\frac{4}{3}} + T\right). \tag{4.3}$$

Thus, $\mathbf{u} \in L^2(0, T; H^3(\mathbb{T}^2)) \cap L^\infty(0, T; H^1(\mathbb{T}^2))$ whenever $\mathbf{u}_0 \in H^1(\mathbb{T}^2)$.

The case $m = 2$ proceeds in a similar way. We take the inner product with $\triangle^2\mathbf{u}$, then use (2.5) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle\mathbf{u}\|_{L^2}^2 + \left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 \\
&\quad\leqslant \left|\left(\triangle\mathbf{u},\triangle^2\mathbf{u}\right)\right| + \left|\left(\left(\boldsymbol{\zeta}^{\epsilon}(\mathbf{u})\cdot\nabla\right)\mathbf{u},\triangle^2\mathbf{u}\right)\right| \\
&\quad\leqslant \frac{1}{2}\|\triangle\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 + \|\boldsymbol{\zeta}^{\epsilon}(\mathbf{u})\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\left\|\triangle^2\mathbf{u}\right\|_{L^2} \\
&\quad\leqslant \frac{1}{2}\|\triangle\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 + C\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + \frac{1}{4}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 \\
&\quad\leqslant \frac{1}{2}\|\triangle\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 + C\left(\|\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{H^1}\right)\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1} + \frac{1}{4}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 \\
&\quad\leqslant \frac{1}{2}\|\triangle\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2 + C\left(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2\right)\|\triangle\mathbf{u}\|_{L^2}^2 + \frac{1}{4}\left\|\triangle^2\mathbf{u}\right\|_{L^2}^2,
\end{aligned}\tag{4.4}$$

in which we have used elliptic regularity and the mean-free condition of $\nabla\mathbf{u}$ to write $\|\nabla\mathbf{u}\|_{L^4}^2 \leqslant C\|\triangle\mathbf{u}\|_{L^2}^2$. Similar to the case $m = 1$, this estimate reveals that $\mathbf{u}\in L^\infty(0,T;H^2(\mathbb{T}^2))\cap L^2(0,T;H^4(\mathbb{T}^2))$ whenever $\mathbf{u}_0\in H^2(\mathbb{T}^2)$. □

5. Convergence to Kuramoto–Sivashinsky Solutions

It is known that, for any initial data $\mathbf{u}_0\in L^2(\mathbb{T}^2)$, solutions to 2D KSE exist and are unique in $C([0,T];L^2(\mathbb{T}^2))\cap L^2(0,T;H^2(\mathbb{T}^2))$ for some (possibly only small) $T>0$ (see, e.g. [5, 17]). In this section we show that as $\epsilon\to 0$, solutions $\mathbf{u}^\epsilon$ of the calmed KSE (1.3) converge to solutions $\mathbf{u}$ of KSE (1.2) prior to its potential blowup time. For this result, it seems necessary that our calming function $\boldsymbol{\zeta}^\epsilon$ satisfies Condition (iii) of definition 1.3. Indeed, if one wants to show that $(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\mathbf{u}^\epsilon\to(\mathbf{u}\cdot\nabla)\mathbf{u}$ in some sense as $\epsilon\to 0$, then one expects that at least $\boldsymbol{\zeta}^\epsilon(\mathbf{x})\to\mathbf{x}$ as $\epsilon\to 0$. We do not find this imposition to be restrictive, as our example choices for $\boldsymbol{\zeta}^\epsilon$ satisfy this condition, as seen in proposition 1.4.

In order to prove theorem 1.8, we need the following abstract bootstrapping/continuity argument (see, e.g. [60, p 20]).

Remark 5.1. It was pointed out by one of the anonymous referees that to prove theorem 1.8, one can avoid the abstractness of lemma 5.2 by employing more elementary means based on nonlinear Grönwall-type arguments. However, for the sake of brevity, we use lemma 5.2.

Lemma 5.2. *Let $T>0$. Assume that two statements C(t) and H(t) with $t\in[0,T]$ satisfy the following conditions:*

(a) If H(t) holds for some $t\in[0,T]$, then C(t) holds for the same t;
(b) If C(t) holds for some $t_0\in[0,T]$, then H(t) holds for t in a neighbourhood of t_0;
(c) If C(t) holds for $t_m\in[0,T]$ and $t_m\to t$, then C(t) holds;
(d) H(t) holds for at least one $t_1\in[0,T]$.

Then C(t) holds for all $t\in[0,T]$.

Proof of theorem 1.8. Set

$$\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}$$

and take the difference between (1.3) and (1.2) to obtain

$$\partial_t \mathbf{w}^\epsilon + \Delta \mathbf{w}^\epsilon + \Delta^2 \mathbf{w}^\epsilon = -(\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\mathbf{u}^\epsilon + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Testing each side by $\mathbf{w}^\epsilon$ we obtain, after integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 = \|\nabla\mathbf{w}^\epsilon\|_{L^2}^2 + N, \tag{5.1}$$

where N is given by

$$N := -\int_{\mathbb{T}^2} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon)\cdot\nabla)\mathbf{u}^\epsilon - (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{w}^\epsilon\,\mathrm{d}\mathbf{x}.$$

By inequality (2.3),

$$\|\nabla\mathbf{w}^\epsilon\|_{L^2}^2 \leqslant \frac{1}{2}\|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\mathbf{w}^\epsilon\|_{L^2}^2. \tag{5.2}$$

Inserting (5.2) in (5.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\Delta\mathbf{w}^\epsilon\|_{L^2}^2 \leqslant \|\mathbf{w}^\epsilon\|_{L^2}^2 + 2N. \tag{5.3}$$

N can be written as

$$\begin{aligned} N = &-\int_{\mathbb{T}^2} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{w}^\epsilon\cdot\mathbf{w}^\epsilon\,\mathrm{d}\mathbf{x} - \int_{\mathbb{T}^2}(\boldsymbol{\zeta}^\epsilon(\mathbf{u})\cdot\nabla)\mathbf{w}^\epsilon\cdot\mathbf{w}^\epsilon\,\mathrm{d}\mathbf{x} \\ &-\int_{\mathbb{T}^2} ((\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u}))\cdot\nabla)\mathbf{u}\cdot\mathbf{w}^\epsilon\,\mathrm{d}\mathbf{x} - \int_{\mathbb{T}^2}((\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u})\cdot\nabla)\mathbf{u}\cdot\mathbf{w}^\epsilon\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Using the Lipschitz property of $\boldsymbol{\zeta}^\epsilon$ and (1.6), we see that N is bounded by

$$\begin{aligned} |N| \leqslant &\int_{\mathbb{T}^2} |\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x} + \int_{\mathbb{T}^2}|\boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x} \\ &+ \int_{\mathbb{T}^2} |\boldsymbol{\zeta}^\epsilon(\mathbf{u}^\epsilon) - \boldsymbol{\zeta}^\epsilon(\mathbf{u})|\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x} + \int_{\mathbb{T}^2}|\boldsymbol{\zeta}^\epsilon(\mathbf{u}) - \mathbf{u}|\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x}. \\ \leqslant &\int_{\mathbb{T}^2} |\mathbf{w}^\epsilon|^2\,|\nabla\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x} + \int_{\mathbb{T}^2}|\mathbf{u}|\,|\nabla\mathbf{w}^\epsilon|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x} \\ &+ \int_{\mathbb{T}^2} |\mathbf{w}^\epsilon|^2\,|\nabla\mathbf{u}|\,\mathrm{d}\mathbf{x} + C\epsilon^\alpha\int_{\mathbb{T}^2}|\mathbf{u}|^\beta\,|\nabla\mathbf{u}|\,|\mathbf{w}^\epsilon|\,\mathrm{d}\mathbf{x}. \\ = &\, N_1 + N_2 + N_3 + N_4. \end{aligned}$$

These terms can be bounded as follows. By Hölder's inequality and (2.7), we obtain

$$\begin{aligned} N_1 &\leqslant \|\mathbf{w}^\epsilon\|_{L^4}^2 \|\nabla \mathbf{w}^\epsilon\|_{L^2} \leqslant C\left(\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}}\right) \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \\ &= C\|\mathbf{w}^\epsilon\|_{L^2}^{\frac{5}{2}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} + C\|\mathbf{w}^\epsilon\|_{L^2}^2 \|\triangle \mathbf{w}^\epsilon\|_{L^2} \\ &\leqslant C\|\mathbf{w}^\epsilon\|_{L^2}^{\frac{10}{3}} + C\|\mathbf{w}^\epsilon\|_{L^2}^4 + \frac{1}{16}\|\triangle \mathbf{w}^\epsilon\|_{L^2} \end{aligned} \tag{5.4}$$

and, using similar estimates along with (2.6),

$$\begin{aligned} N_2 &\leqslant \|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^4} \|\nabla \mathbf{w}^\epsilon\|_{L^4} \\ &\leqslant C\|\mathbf{u}\|_{L^2}\left(\|\mathbf{w}^\epsilon\|_{L^2} + \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{4}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{4}}\right) \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{4}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{4}} \\ &= C\|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{5}{4}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{4}} + C\|\mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2} \|\triangle \mathbf{w}^\epsilon\|_{L^2} \\ &\leqslant C\left(\|\mathbf{u}\|_{L^2}^{\frac{8}{5}} + \|\mathbf{u}\|_{L^2}^2\right) \|\mathbf{w}^\epsilon\|_{L^2}^2 + \frac{1}{16}\|\triangle \mathbf{w}^\epsilon\|_{L^2}^2 \end{aligned} \tag{5.5}$$

and also, by (2.7),

$$\begin{aligned} N_3 &\leqslant \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^4}^2 \leqslant C\|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2} \|\mathbf{w}^\epsilon\|_{L^2}^{\frac{3}{2}} \|\triangle \mathbf{w}^\epsilon\|_{L^2}^{\frac{1}{2}} \\ &\leqslant C\left(\|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}^{\frac{4}{3}}\right) \|\mathbf{w}^\epsilon\|_{L^2}^2 + \frac{1}{16}\|\triangle \mathbf{w}^\epsilon\|_{L^2}^2 . \end{aligned} \tag{5.6}$$

Finally, by Agmon's inequality,

$$\begin{aligned} N_4 &\leqslant C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^\infty}^\beta \|\nabla \mathbf{u}\|_{L^2} \\ &\leqslant C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} \|\mathbf{u}\|_{H^2}^{\frac{\beta}{2}} \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \\ &\leqslant C\epsilon^\alpha \|\mathbf{w}^\epsilon\|_{L^2} \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} \left(\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}} + \|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}}\right) \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} \\ &= \left(C\epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} + C\epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right) \|\mathbf{w}^\epsilon\|_{L^2} . \end{aligned} \tag{5.7}$$

We now insert the bounds for N in (5.1) to obtain the following,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2}^2 + \|\triangle \mathbf{w}^\epsilon\|_{L^2}^2 \\ &\quad\leqslant \frac{6}{16}\|\triangle \mathbf{w}^\epsilon\|_{L^2}^2 + C\|\mathbf{w}^\epsilon\|_{L^2}^4 + C\|\mathbf{w}^\epsilon\|_{L^2}^{\frac{10}{3}} \\ &\qquad + C\left(1 + \|\mathbf{u}\|_{L^2}^{\frac{8}{5}} + \|\mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}^{\frac{4}{3}}\right) \|\mathbf{w}^\epsilon\|_{L^2}^2 \\ &\qquad + C\left(\epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{1}{2}} + \epsilon^\alpha \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}} \|\triangle \mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right) \|\mathbf{w}^\epsilon\|_{L^2} . \end{aligned} \tag{5.8}$$

Due to the presence of the terms $\|\mathbf{w}^\epsilon\|_{L^2}^4$ and $\|\mathbf{w}^\epsilon\|_{L^2}^{\frac{10}{3}}$, we cannot apply Grönwall's inequality directly. However, since $\|\mathbf{w}^\epsilon\|_{L^2}$ is supposed to be small, these terms are not 'bad' and are even smaller than $\|\mathbf{w}^\epsilon\|_{L^2}^2$. We just need to apply a bootstrapping argument, as stated in lemma 5.2. Denote by $H(t)$ with $t \in [0,T]$ the statement that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant 1$$

and by $C(t)$ the statement that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant e^{A(T)}B(T)\,\epsilon^\alpha \leqslant \tfrac{1}{2},$$

where $A(t)$ and $B(t)$ are defined as in (5.11) and (5.12) below and ϵ is taken to be sufficiently small such that

$$e^{A(T)}B(T)\,\epsilon^\alpha \leqslant \tfrac{1}{2}.$$

Clearly, $C(t)$ is a stronger statement than $H(t)$, and thus (b) of lemma 5.2 holds. When the solutions are regular enough, then $\|\mathbf{w}^\epsilon(t)\|_{L^2}$ is continuous in time. Indeed, this regularity is given by condition 1.7 and definition 1.5 and thus (c) of lemma 5.2 holds. For $t=0$, $\|\mathbf{w}^\epsilon(t)\|_{L^2}$ is zero and thus (d) of lemma 5.2 holds. In order to apply lemma 5.2, it remains to verify (a). That is, if $H(t)$ holds for some $t \in [0,T]$, namely

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant 1,$$

then $C(t)$ holds at the same t, namely

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant e^{A(T)}B(T)\,\epsilon^\alpha < \tfrac{1}{2}.$$

We assume that, for some $t \in [0,T]$,

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant 1 \tag{5.9}$$

and then show that (5.9) leads to a desired smaller bound at this same t. Now we replace $\|\mathbf{w}^\epsilon\|_{L^2}^4$ and $\|\mathbf{w}^\epsilon\|_{L^2}^{\frac{10}{3}}$ by $\|\mathbf{w}^\epsilon\|_{L^2}^2$ in (5.8), remove the higher-order diffusive terms, and eliminate $\|\mathbf{w}^\epsilon\|_{L^2}$ from each term to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2} \leqslant{}& C\left(1+\|\mathbf{u}\|_{L^2}^{\frac{8}{5}}+\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}}\right)\|\mathbf{w}^\epsilon\|_{L^2}\\ &+C\epsilon^\alpha\left(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}}+\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right),\end{aligned}$$

in which we also use the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2}^2 = 2\|\mathbf{w}^\epsilon\|_{L^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}^\epsilon\|_{L^2}$.

Due to the regularity assumption on $\mathbf{u}$ in 1.7, the terms $C(1+\|\mathbf{u}\|_{L^2}^{\frac{8}{5}}+\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}})$ and $\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}}$ are integrable for $\beta \geqslant 0$. Furthermore, for $\beta \leqslant 3$, $\frac{\beta}{2}+\frac{1}{2} \leqslant 2$ and thus $\|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}$ is integrable. It then follows from Grönwall's inequality that

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant e^{A(t)}\|\mathbf{w}^\epsilon(0)\|_{L^2} + e^{A(t)}B(t)\,\epsilon^\alpha \leqslant e^{A(T)}B(T)\,\epsilon^\alpha, \tag{5.10}$$

where we have used the fact that the initial difference $\mathbf{w}^\epsilon(0) = 0$ and have written

$$A(t) := C\int_0^t \left(1 + \|\mathbf{u}\|_{L^2}^{\frac{8}{5}} + \|\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}}\right)\mathrm{d}s, \tag{5.11}$$

$$B(t) := C\int_0^t \left(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right)\mathrm{d}s. \tag{5.12}$$

By taking ϵ sufficiently small, from (5.10) we deduce that any $t \in [0, T]$ which satisfies

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} < 1.$$

must also satisfy

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant e^{A(T)}B(T)\,\epsilon^\alpha < \tfrac{1}{2}.$$

Thus the bootstrapping argument holds, and we conclude as claimed that for all $t \leqslant T$,

$$\|\mathbf{w}^\epsilon(t)\|_{L^2} \leqslant K_2\epsilon^\alpha \tag{5.13}$$

where $K_2 = e^{A(T)}B(T)$ depends on T, $\mathbf{u}$, and β. In particular, $\mathbf{u}^\epsilon \to \mathbf{u}$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$ as $\epsilon \to 0^+$. Next, integrate (5.8) for all $t \in [0,T]$ (again replacing $\|\mathbf{w}^\epsilon\|_{L^2}^4$ by $\|\mathbf{w}^\epsilon\|_{L^2}^2$) to obtain

$$\begin{aligned}
&\frac{10}{16}\int_0^T \|\triangle\mathbf{w}^\epsilon\|_{L^2}^2\,\mathrm{d}t \\
&\quad\leqslant C\int_0^T \left(1 + \|\mathbf{u}\|_{L^2}^{\frac{8}{5}} + \|\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{4}{3}}\right)\|\mathbf{w}^\epsilon\|_{L^2}^2\,\mathrm{d}t \\
&\qquad + C\epsilon^\alpha\int_0^T \left(\|\mathbf{u}\|_{L^2}^{\beta+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\|\triangle\mathbf{u}\|_{L^2}^{\frac{\beta}{2}+\frac{1}{2}}\right)\|\mathbf{w}^\epsilon\|_{L^2}\,\mathrm{d}t
\end{aligned}$$

In which we are again using the fact that $\mathbf{w}^\epsilon(0) = 0$. Applying (5.11)–(5.13) then yields

$$\int_0^T \|\triangle\mathbf{w}^\epsilon\|_{L^2}^2\,\mathrm{d}t \leqslant \frac{16}{10}A(T)K_2^2\epsilon^{2\alpha} + \frac{16}{10}B(T)K_2\epsilon^{2\alpha}. \tag{5.14}$$

For $K_3 = \frac{16}{10}A(T)K_2^2 + \frac{16}{10}B(T)K_2$ (again only depending on T, $\|\mathbf{u}\|_{L^\infty(0,T;L^2)}$, $\|\mathbf{u}\|_{L^2(0,T;H^2)}$, and β), we obtain

$$\|\triangle\mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \leqslant K_3^{\frac{1}{2}}\epsilon^\alpha. \tag{5.15}$$

Finally, we use an interpolation inequality:

$$\begin{aligned}
\|\mathbf{w}^\epsilon\|_{L^2(0,T;H^2)} &\leqslant C\|\mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} + C\|\triangle\mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \\
&\leqslant CT^{\frac{1}{2}}\|\mathbf{w}^\epsilon\|_{L^\infty(0,T;L^2)} + C\|\triangle\mathbf{w}^\epsilon\|_{L^2(0,T;L^2)} \\
&\leqslant C\left(T^{\frac{1}{2}}K_2 + K_3^{\frac{1}{2}}\right)\epsilon^\alpha \\
&\leqslant K_4\epsilon^\alpha,
\end{aligned} \tag{5.16}$$

where $K_4 = C(T^{\frac{1}{2}}K_2 + K_3^{\frac{1}{2}})$. Thus we see that $\mathbf{u}^\epsilon \to \mathbf{u}$ in $L^2(0,T;H^2(\mathbb{T}^2))$ as $\epsilon \to 0^+$. □

Corollary 5.3. *Consider the calming functions ζ^ϵ as described in (1.4). Let $\mathbf{u}, \mathbf{u}^\epsilon$ be as in the statement of theorem 1.8 with the same initial data, where $\boldsymbol{u}^\epsilon$ is determined by ζ_i^ϵ, $i = 1, 2$, or 3. Then for $T < T^*$, there exists $K_i' > 0$ independent of ϵ such that*

(1) For $\zeta^\epsilon = \zeta_1^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \leqslant K_1'\epsilon, \tag{5.17}$$

(2) For $\zeta^\epsilon = \zeta_2^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \leqslant K_2'\epsilon^2, \tag{5.18}$$

(3) For $\zeta^\epsilon = \zeta_3^\epsilon$,

$$\|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^\infty(0,T;L^2)} \leqslant K_3'\epsilon^2. \tag{5.19}$$

Proof. The proof follows immediately from theorem 1.8 and proposition 1.4. □

6. The scalar form

Here we investigate the scalar formulation (1.5). The analysis is similar to the analysis of (1.3), so we only briefly present formal energy estimates.

For the sake of brevity, we work formally rather than rigorously. However, the proof below can be made rigorous, e.g. via the use of Galerkin methods as in the proof of theorem 1.6.

Proof of theorem 1.11. Take a (formal) inner product of (1.5*a*) with ϕ and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2 + \|\triangle\phi\|_{L^2}^2 = -(\triangle\phi,\phi) - \left(\tfrac{1}{2}\zeta^\epsilon(\nabla\phi)\cdot\nabla\phi,\phi\right). \tag{6.1}$$

Using (2.3), Hölder's, and Young's inequality,

$$\begin{aligned}\left|\left(\tfrac{1}{2}\zeta^\epsilon(\nabla\phi)\cdot\nabla\phi,\phi\right)\right| &\leqslant \frac{1}{2}\|\zeta^\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\phi\|_{L^2}\\ &\leqslant C\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}}\|\phi\|_{L^2}^2 + \frac{1}{4}\|\triangle\phi\|_{L^2}^2.\end{aligned}$$

thus we obtain from (6.1) the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2 + \|\triangle\phi\|_{L^2}^2 \leqslant \left(2 + C\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}}\right)\|\phi\|_{L^2}^2. \tag{6.2}$$

Hence from Grönwall's inequality, dropping the second term in (6.2), we obtain

$$\|\phi(t)\|_{L^2}^2 \leqslant e^{K_\epsilon T}\|\phi_0\|_{L^2}^2, \tag{6.3}$$

where $K_\epsilon = (2 + C\|\zeta^\epsilon\|_{L^\infty}^{\frac{4}{3}})$. Hence $\phi \in L^\infty(0,T;L^2(\mathbb{T}^2))$. Next, we integrate (6.2) in time on the interval $[0,T]$ and drop any unnecessary terms:

$$\begin{aligned}\int_0^T \frac{1}{2}\|\triangle\phi\|_{L^2}^2 &\leqslant \int_0^T K_\epsilon \|\phi(t)\|_{L^2}^2 \,\mathrm{d}t + \|\phi_0\|_{L^2}^2 \\ &\leqslant \int_0^T K_\epsilon e^{K_\epsilon T}\|\phi_0\|_{L^2}^2 \,\mathrm{d}t + \|\phi_0\|_{L^2}^2 \\ &= \left(K_\epsilon T e^{K_\epsilon T} + 1\right)\|\phi_0\|_{L^2}^2 . \end{aligned} \tag{6.4}$$

Therefore $\phi \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$. Now we obtain estimates on $\partial_t\phi$: For any $\psi \in L^2(0,T;H^2(\mathbb{T}^2))$,

$$\begin{aligned}|\langle\partial_t\phi,\psi\rangle| &= \left|\int_0^T \partial_t\phi\,\psi\,\mathrm{d}t\right| \\ &= \left|\int_0^T\left(\frac{1}{2}\zeta^\epsilon(\nabla\phi)\cdot\nabla\phi\right)\psi\,\mathrm{d}t + \int_0^T(\triangle\phi)\,\psi\,\mathrm{d}t + \int_0^T(\triangle\phi)\triangle\psi\,\mathrm{d}t\right| \\ &\leqslant \frac{1}{2}\int_0^T|\zeta^\epsilon(\nabla\phi)||\nabla\phi||\psi|\,\mathrm{d}t + \int_0^T|\triangle\phi||\psi|\,\mathrm{d}t + \int_0^T|\triangle\phi||\triangle\psi|\,\mathrm{d}t \\ &\leqslant \frac{1}{2}\|\zeta^\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2(0,T;L^2)}\|\psi\|_{L^2(0,T;L^2)} \\ &\quad + \|\triangle\phi\|_{L^2(0,T;L^2)}\|\psi\|_{L^2(0,T;L^2)} + \|\triangle\phi\|_{L^2(0,T;L^2)}\|\triangle\psi\|_{L^2(0,T;L^2)} \\ &\leqslant \left(\frac{1}{2}\|\zeta^\epsilon\|_{L^\infty}\|\phi\|_{L^2(0,T;H^2)} + 2\|\phi\|_{L^2(0,T;H^2)}\right)\|\psi\|_{L^2(0,T;H^2)}. \end{aligned} \tag{6.5}$$

It follows from Estimate (6.4) that $\|\partial_t\phi\|_{L^2(0,T;H^{-2})} < \infty$, hence $\partial_t\phi \in L^2(0,T;H^{-2}(\mathbb{T}^2))$. From this we deduce that a solution ϕ to (1.5) exists, with

$$\phi \in C\left(0,T;L^2\left(\mathbb{T}^2\right)\right) \cap L^2\left(0,T;H^2\left(\mathbb{T}^2\right)\right).$$

Now, let ϕ and ψ be two solutions to (1.5) with $\phi(0) = \psi(0) = \phi_0$. Let $\delta = \phi - \psi$. Then δ satisfies the equation

$$\partial_t\delta + \triangle^2\delta = -\triangle\delta + \zeta^\epsilon(\nabla\psi)\cdot\nabla\psi - \zeta^\epsilon(\nabla\phi)\cdot\nabla\phi \tag{6.6}$$

with $\delta(0) = 0$. We can then rewrite the nonlinear term as

$$\zeta^\epsilon(\nabla\psi)\cdot\nabla\psi - \zeta^\epsilon(\nabla\phi)\cdot\nabla\phi = (\zeta^\epsilon(\nabla\psi) - \zeta^\epsilon(\nabla\phi))\cdot\nabla\psi - \zeta^\epsilon(\nabla\phi)\cdot\nabla\delta. \tag{6.7}$$

We now insert (6.7) into (6.6) and apply integration by parts to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\|_{L^2}^2 + \|\triangle\delta\|_{L^2}^2 \\ &\quad\leqslant |(\triangle\delta,\delta)| + |((\zeta^\epsilon(\nabla\psi) - \zeta^\epsilon(\nabla\phi))\cdot\nabla\psi,\delta)| + |(\zeta^\epsilon(\nabla\phi)\cdot\nabla\delta,\delta)|. \end{aligned} \tag{6.8}$$

Handling the first term is straightforward. For the second term, we use Condition (i) of definition 1.3, Hölder's, (2.5), and Young's inequality to obtain

$$
\begin{aligned}
&|((\zeta^{\epsilon}(\nabla\psi)-\zeta^{\epsilon}(\nabla\phi))\cdot\nabla\psi,\delta)| \\
&\quad\leqslant \|\zeta^{\epsilon}(\nabla\psi)-\zeta^{\epsilon}(\nabla\phi)\|_{L^4}\|\nabla\psi\|_{L^2}\|\delta\|_{L^4} \\
&\quad\leqslant \|\nabla\psi\|_{L^2}\|\delta\|_{L^4}\|\nabla\delta\|_{L^4} \\
&\quad\leqslant C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}^{\frac{1}{2}}\|\delta\|_{H^1}^{\frac{1}{2}}\|\nabla\delta\|_{L^2}^{\frac{1}{2}}\|\nabla\delta\|_{H^1}^{\frac{1}{2}} \\
&\quad\leqslant C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}^{\frac{1}{2}}\left(\|\delta\|_{L^2}^{\frac{1}{2}}+\|\nabla\delta\|_{L^2}^{\frac{1}{2}}\right)\|\nabla\delta\|_{L^2}^{\frac{1}{2}}\|\triangle\delta\|_{L^2}^{\frac{1}{2}} \\
&\quad\leqslant C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}\|\nabla\delta\|_{L^2}^{\frac{1}{2}}\|\triangle\delta\|_{L^2}^{\frac{1}{2}} \\
&\qquad+C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}^{\frac{1}{2}}\|\nabla\delta\|_{L^2}\|\triangle\delta\|_{L^2}^{\frac{1}{2}} \\
&\quad\leqslant C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}^{\frac{5}{4}}\|\triangle\delta\|_{L^2}^{\frac{3}{4}} \\
&\qquad+C\|\nabla\psi\|_{L^2}\|\delta\|_{L^2}\|\triangle\delta\|_{L^2} \\
&\quad\leqslant C\left(\|\nabla\psi\|_{L^2}^{\frac{8}{5}}+\|\nabla\psi\|_{L^2}^{2}\right)\|\delta\|_{L^2}^{2}+\frac{1}{6}\|\triangle\delta\|_{L^2}^{2}.
\end{aligned}
\tag{6.9}
$$

In the third term, we apply condition (ii) of definition 1.3, use Young's inequality, and use interpolation inequalities to obtain

$$
\begin{aligned}
|(\zeta^{\epsilon}(\nabla\phi)\cdot\nabla\delta,\delta)| &\leqslant \|\zeta^{\epsilon}\|_{L^\infty}\|\nabla\delta\|_{L^2}\|\delta\|_{L^2} \\
&\leqslant \|\zeta^{\epsilon}\|_{L^\infty}\|\delta\|_{L^2}^{\frac{3}{2}}\|\triangle\delta\|_{L^2}^{\frac{1}{2}} \\
&\leqslant C\|\zeta^{\epsilon}\|_{L^\infty}^{\frac{4}{3}}\|\delta\|_{L^2}^{2}+\frac{1}{6}\|\triangle\delta\|_{L^2}^{2}.
\end{aligned}
\tag{6.10}
$$

After inserting (6.9) and (6.10) into (6.8) and rearranging the terms, the inequality becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\|_{L^2}^{2}+\|\triangle\delta\|_{L^2}^{2}\leqslant C\left(1+\|\nabla\psi\|_{L^2}^{\frac{8}{5}}+\|\nabla\psi\|_{L^2}^{2}+\|\zeta^{\epsilon}\|_{L^\infty}^{\frac{4}{3}}\right)\|\delta\|_{L^2}^{2}. \tag{6.11}
$$

Then applying Grönwall's inequality, we obtain

$$
\|\phi(t)-\psi(t)\|_{L^2}^{2}\leqslant e^{\widetilde{K}_1(T)}\|\phi_0-\psi_0\|_{L^2}^{2}, \tag{6.12}
$$

where $\widetilde{K}_1(T)=\int_0^T 1+\|\nabla\psi(t)\|_{L^2}^2+\|\zeta^{\epsilon}\|_{L^\infty}^{\frac{4}{3}}\,\mathrm{d}t$. Since $\psi\in L^2(0,T;H^2(\mathbb{T}^2))$, and ζ^{ϵ} is bounded, $\widetilde{K}_1(T)<\infty$. So $\phi(t)=\psi(t)$ for all $t\in[0,T]$, hence solutions to (1.5) are unique. Now, we integrate (6.11) on the interval $[0,T]$ and apply (6.12), which yields

$$
\int_0^T\|\triangle\phi(t)-\triangle\psi(t)\|_{L^2}^{2}\,\mathrm{d}t\leqslant\widetilde{K}_2\|\phi_0-\psi_0\|_{L^2}^{2} \tag{6.13}
$$

for some $\widetilde{K}_2$ which depends on T, $\|\nabla\psi(t)\|_{L^2}$, and $\|\zeta^{\epsilon}\|_{L^\infty}$. From estimates (6.12) and (6.13) we conclude that solutions depend continuously on the initial data in $L^\infty(0,T;L^2(\mathbb{T}^2))\cap L^2(0,T;H^2(\mathbb{T}^2))$.

□

Here, we will show the convergences of solutions to (1.5) to that of (1.1) as $\epsilon\to0^+$. This proof has only minor variations from the proof of theorem 1.8.

Proof of theorem 1.12. We set $\delta^\epsilon = \phi - \phi^\epsilon$ take the difference between (1.1) and (1.5*a*), and take the inner product with δ^ϵ. to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta^\epsilon\|_{L^2}^2 + \|\triangle\delta^\epsilon\|_{L^2}^2 \leqslant \|\delta^\epsilon\|_{L^2}^2 + N_1 + N_2 + N_3 + N_4, \tag{6.14}$$

where

$$N_1 = |((\boldsymbol{\zeta}^\epsilon(\nabla\phi^\epsilon) - \boldsymbol{\zeta}^\epsilon(\nabla\phi)) \cdot \nabla\delta^\epsilon, \delta^\epsilon)| \leqslant C\|\delta^\epsilon\|_{L^2}^6 + \frac{3}{4}\|\triangle\delta^\epsilon\|_{L^2}^2,$$
$$N_2 = |((\boldsymbol{\zeta}^\epsilon(\nabla\phi^\epsilon) - \boldsymbol{\zeta}^\epsilon(\nabla\phi)) \cdot \nabla\phi, \delta^\epsilon)| \leqslant C\|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}}\|\delta^\epsilon\|_{L^2}^2 + \frac{1}{8}\|\triangle\delta^\epsilon\|_{L^2}^2,$$
$$N_3 = |(\boldsymbol{\zeta}^\epsilon(\nabla\phi) \cdot \nabla\delta^\epsilon, \delta^\epsilon)| \leqslant C\|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}\|\delta^\epsilon\|_{L^2}^2 + \frac{1}{16}\|\triangle\delta^\epsilon\|_{L^2}^2,$$

and

$$\begin{aligned} N_4 &= |((\boldsymbol{\zeta}^\epsilon(\nabla\phi) - \nabla\phi) \cdot \nabla\phi, \delta^\epsilon)| \\ &\leqslant C\epsilon^\alpha \int_{\mathbb{T}^2} |\nabla\phi|^{\beta+1}|\delta^\epsilon|\,\mathrm{d}\mathbf{x} \\ &\leqslant C\epsilon^\alpha \|\nabla\phi\|_{L^{2\beta+2}}^{\beta+1}\|\delta^\epsilon\|_{L^2}. \end{aligned}$$

Applying the Sobolev inequality, we deduce that

$$\|\nabla\phi\|_{L^{2\beta+2}}^{\beta+1} \leqslant C\|\nabla\phi\|_{L^2}\|\nabla\phi\|_{H^1}^{\beta} \leqslant C\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}.$$

Inserting our bounds for each N_i into (6.14) and rearranging then yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\delta^\epsilon\|_{L^2}^2 + \frac{1}{16}\|\triangle\delta^\epsilon\|_{L^2}^2 &\leqslant C\|\delta^\epsilon\|_{L^2}^6 + \|\delta^\epsilon\|_{L^2}^2 + C\epsilon^\alpha\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}\|\delta^\epsilon\|_{L^2} \\ &\quad + C\left(\|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}} + \|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}\right)\|\delta^\epsilon\|_{L^2}^2. \end{aligned} \tag{6.15}$$

Now we apply the ansatz

$$\|\delta^\epsilon\|_{L^2} < 1$$

to obtain the bound

$$\|\delta^\epsilon\|_{L^2}^6 \leqslant \|\delta^\epsilon\|_{L^2}^2.$$

We apply this estimate to (6.15) and eliminate $\|\delta^\epsilon\|_{L^2}$ from each term to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\delta^\epsilon\|_{L^2} &\leqslant C\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}\epsilon^\alpha \\ &\quad + C\left(1 + \|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}} + \|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}\right)\|\delta^\epsilon\|_{L^2}. \end{aligned} \tag{6.16}$$

The term

$$1 + \|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}} + \|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}$$

is always integrable and the term $\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}$ is integrable for $\beta\in[1,\frac{3}{2}]$. It now follows from Grönwall's inequality that

$$\|\delta^\epsilon(t)\|_{L^2}\leqslant e^{A(t)}\|\delta^\epsilon(0)\|_{L^2}+e^{A(t)}B(t)\epsilon^\alpha\leqslant e^{A(T)}B(T)\epsilon^\alpha, \tag{6.17}$$

using the fact that $\delta^\epsilon(0)=0$, and with

$$A(t)=C\int_0^t 1+\|\phi\|_{L^2}^{\frac{2}{5}}\|\triangle\phi\|_{L^2}^{\frac{6}{5}}+\|\phi\|_{L^2}^{\frac{1}{4}}\|\triangle\phi\|_{L^2}^{\frac{3}{4}}\,\mathrm{d}s,$$
$$B(t)=C\int_0^t\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi\|_{H^2}^{\beta+\frac{1}{2}}\,\mathrm{d}s.$$

By taking ϵ sufficiently small, we have for all $0\leqslant t\leqslant T$

$$\|\delta^\epsilon(t)\|_{L^2}<1.$$

It follows from a bootstrapping argument that

$$\|\delta^\epsilon(t)\|_{L^\infty(0,T;L^2)}\leqslant e^{A(T)}B(T)\epsilon^\alpha. \tag{6.18}$$

Now we integrate (6.15) on $[0,T]$, again using that $\|\delta^\epsilon\|_{L^2}^6\leqslant\|\delta^\epsilon\|_{L^2}^2$, and apply to obtain

$$\begin{aligned}\int_0^T\|\triangle\delta^\epsilon\|_{L^2}^2\,\mathrm{d}t&\leqslant C\epsilon^\alpha B(T)e^{A(T)}B(T)\epsilon^\alpha+A(T)\left(e^{A(T)}B(T)\epsilon^\alpha\right)^2\\&\leqslant K(T)^2\epsilon^{2\alpha},\end{aligned} \tag{6.19}$$

where

$$K(T)^2=CB(T)^2e^{A(T)}+A(T)B(T)e^{A(T)}.$$

Therefore we obtain

$$\|\delta^\epsilon\|_{L^2(0,T;H^2)}\leqslant\left(Te^{A(T)}B(T)+K(T)\right)\epsilon^\alpha. \tag{6.20}$$

□

7. Computational results

In this section, we examine the calmed KSEs computationally via several simulations, where the calming function $\boldsymbol{\zeta}^\epsilon=\boldsymbol{\zeta}_i^\epsilon$ is described in (1.4). We include snapshots of the evolution of solutions for the different choices of $\boldsymbol{\zeta}^\epsilon$ in figure 1, and for different choices of ϵ in figure 2(we show results for $\boldsymbol{\zeta}_3^\epsilon$ only for the sake of brevity; $\boldsymbol{\zeta}_1^\epsilon$ and $\boldsymbol{\zeta}_2^\epsilon$ yielded qualitatively similar results). The former illustrates the different effects of the choice of $\boldsymbol{\zeta}^\epsilon$ on the dynamics, while the latter indicates the uniform convergence of $\mathbf{u}^\epsilon$ to $\mathbf{u}$.

In addition, we examine convergence rates in $L^\infty(0,T;L^2)$, $L^\infty(0,T;L^\infty)$, and $L^2(0,T;H^2)$ for $\boldsymbol{\zeta}_1^\epsilon$ (figure 3), $\boldsymbol{\zeta}_2^\epsilon$ (figure 4), and $\boldsymbol{\zeta}_3^\epsilon$ (figure 5) with initial data (7.2) as $\epsilon\to0^+$ (for simplicity, we set $T=1$, since with all our initial data, solutions to KSE appear to be quite stable on $[0,1]$). We find that the powers on the $L^\infty(0,T;L^2)$ and $L^2(0,T;H^2)$ convergence rates in corollary 5.3 appear to be sharp.

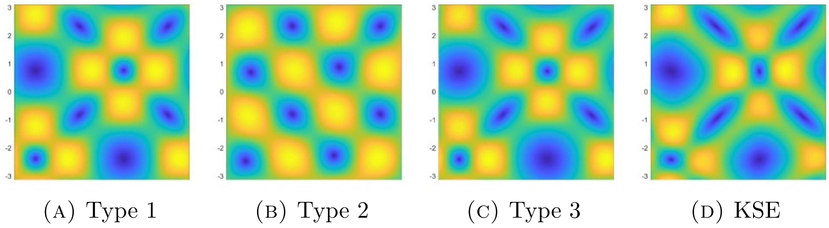


Figure 1. Solutions to calmed KSE of each type compared with a solution to KSE at time $t = 2$, with $\epsilon = 0.1$, $\lambda = 4.1$, and $\mathbf{u}_0$ given by (7.2).

Finally, in figures 6–8 we check the robustness of the convergence with respect to larger initial data (7.3) for ζ_1^ϵ, ζ_2^ϵ, and ζ_3^ϵ. In comparing initial data (7.2) with (7.3), we find very little qualitative variation in the error rates, indicating that changes in initial data will only marginally change the error between solutions to KSE and solutions to calmed KSE for $\epsilon > 0$ sufficiently small.

7.1. Numerical methods

All computations were done in Matlab (R2021a) using pseudo-spectral methods with the standard $2/3's$ dealiasing for the nonlinear term. To evolve the system, we used a well-known modification of the Runge-Kutta-4 time-stepping scheme adapted to handle the linear terms implicitly via an integrating factor to handle the nonlinear terms implicitly (see, e.g. [29]) with time step $\Delta t \approx 4.2943 \times 10^{-4}$ chosen to respect the maximum advective CFL condition in figures 1–5, with later figures having a rescaled time step $\Delta t = 1.0736 \times 10^{-4}$. Our simulations for KSE and cKSE were resolved[7] with a spatial mesh of 128^2. All computations were done using the nondimensionalised calmed KSEs,

$$\partial_t \mathbf{u} + (\boldsymbol{\zeta}^\epsilon(\mathbf{u}) \cdot \nabla)\mathbf{u} + \lambda \triangle \mathbf{u} + \triangle^2 \mathbf{u} = \mathbf{0}, \tag{7.1a}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{7.1b}$$

over the periodic domain $\Omega = [-\pi, \pi)^2$ for $\lambda > 0$.

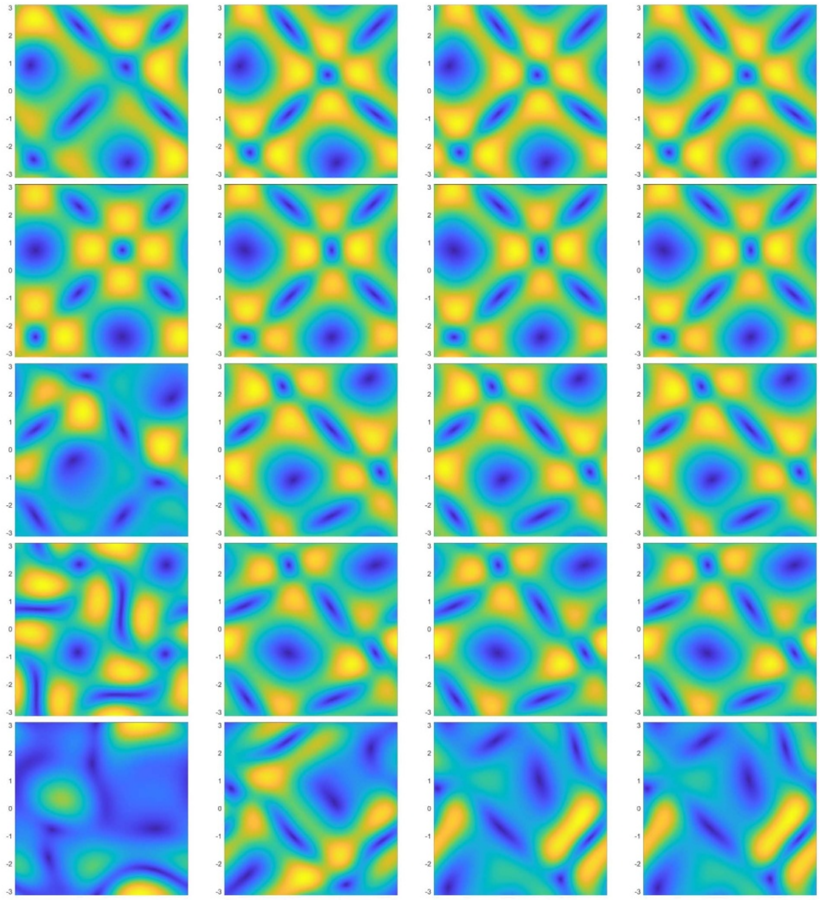
Throughout this section, a *type* 1, *type* 2, or *type* 3 solution is a solution to calmed KSE with calming function ζ_1^ϵ, ζ_2^ϵ, or ζ_3^ϵ respectively.

7.2. Simulations

Here, we take initial conditions to be

$$\mathbf{u}_0(x, y) = \begin{pmatrix} \cos(x + y) + \cos(x) \\ \cos(x + y) + \cos(y) \end{pmatrix} \tag{7.2}$$

[7] Note: For the KSEs (calmed or otherwise), even in fairly chaotic regimes, one often does not need especially high resolution, due to the strong hyperdiffusion term. Moreover, so long as the solution is well-resolved, which we take to mean that the energy spectrum at the modes higher than the 2/3's dealiasing cut-off is at or below machine precision (roughly 2.22×10^{-16}), increasing the resolution only increases round-off error, due to the additional computations being performed. Hence, to minimise roundoff error, we purposely chose the fairly low resolution of 128^2, although our higher-resolution tests, not reported here, produced qualitatively similar results.



(A) $\epsilon_1 = 0.1$ (B) $\epsilon_2 = 0.01$ (C) $\epsilon_3 = 0.001$ (D) KSE

Figure 2. Column (d) is a solution to KSE (1.2) for $t = 1, \ldots, 5$, whereas columns (a)–(c) are type 3 solutions to calmed KSE (1.3) on the same time interval with $\epsilon \in \{0.1, 0.01, 0.001\}$. In this figure, $\lambda = 4.1$ is fixed and initial data $\mathbf{u}_0$ is given in (7.2). Viewing the pictures from left to right, we can see that $\mathbf{u}^\epsilon \to \mathbf{u}$ as $\epsilon \to 0$.

and all colour plots seen below are plots of the magnitude $|\mathbf{u}| = |(u, v)| = \sqrt{u^2 + v^2}$. In all plots of solutions, the horizontal axis corresponds to the y-axis and the vertical axis corresponds to the x-axis.

Our choice for initial data $\mathbf{u}_0$ was motivated by the choice of scalar initial data found in [28, 39], and [37]; namely,

$$\phi_0(x, y) = \sin(x + y) + \sin(x) + \sin(y).$$

Hence, we set $\mathbf{u}_0 = \nabla \phi_0$.

Though some differences can be seen among the images above, one can see that each type of calmed KSE solution approximates the overall behaviour of a KSE solution. One can also observe that the accuracy of the approximation varies by type.

In figure 2 we focus only on type 3 approximations to better illustrate how well calmed KSE solutions can approximate KSE solutions over time for various choices of ϵ. Indeed,

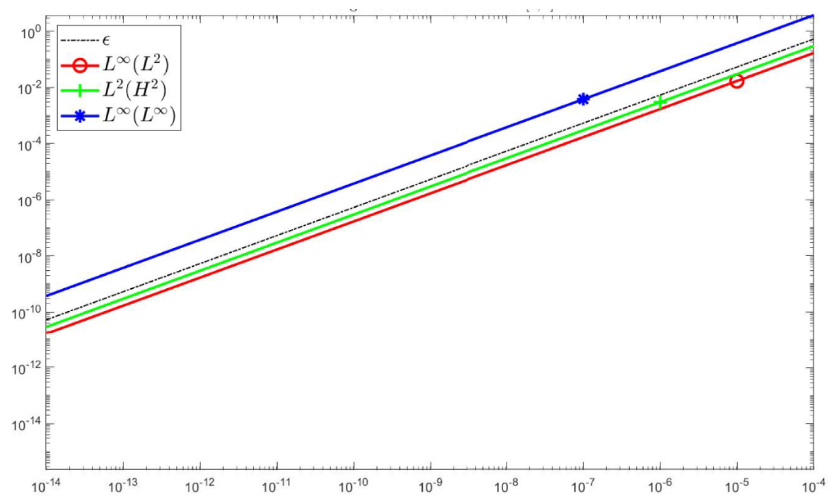


Figure 3. Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 1 solution and with initial data given by (7.2). These estimates show a linear convergence rate.

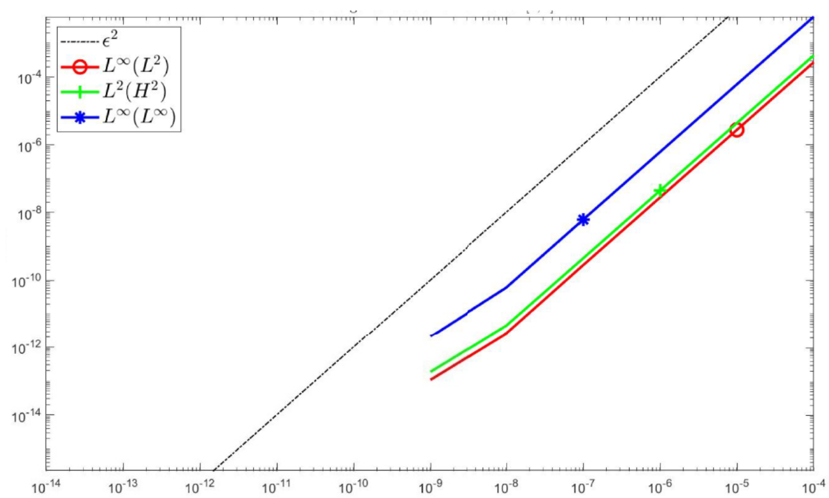


Figure 4. Estimates of $\mathbf{u} - \mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T = 1$ with $\mathbf{u}^\epsilon$ a type 2 solution and with initial data given by (7.2). These estimates show a quadratic convergence rate. Note: for $\epsilon \lesssim 10^{-9}$, the error in our simulations was exactly 0, hence it does not appear in this log-log plot.

when viewed from left to right we can observe the convergence of our calmed KSE solutions to the original KSE solution.

In accordance with corollary 5.3 we see that solutions to calmed KSE corresponding to calming function ζ_1^ϵ yield a linear convergence rate whereas solutions to calmed KSE corresponding to calming functions ζ_2^ϵ or ζ_3^ϵ yield quadratic convergence rates.

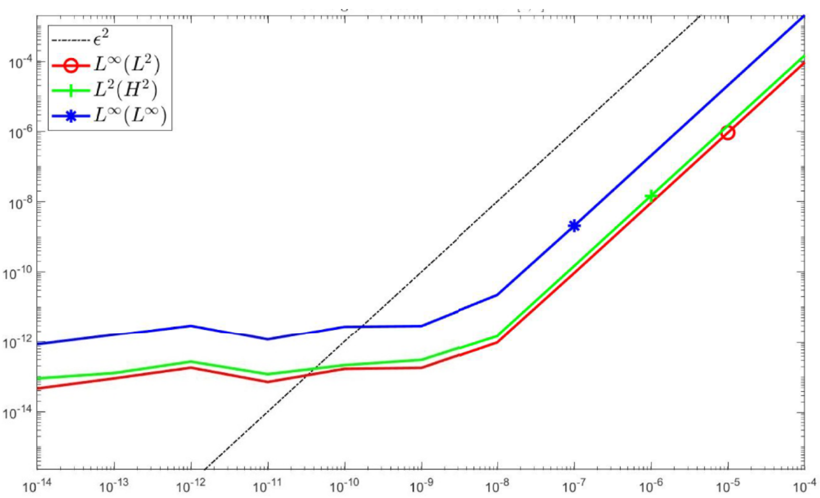


Figure 5. Estimates of $\mathbf{u}-\mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T=1$ with $\mathbf{u}^\epsilon$ a type 3 solution and with initial data given by (7.2). These estimates show a quadratic convergence rate.

For additional testing, we choose initial data with higher oscillation and higher magnitude,

$$\mathbf{u}_0(x,y) = \begin{pmatrix} 4(\cos(x+y)+\sin(3x)) \\ 4(\cos(x+y)+\cos(4y)) \end{pmatrix}, \tag{7.3}$$

and examine the convergence rates for each solution type. For each convergence test, we have the fixed parameters $N=128$, $T=1$, and $\lambda=4.1$.

We observe that even with larger choice of initial data, figures 6–8 remain qualitatively similar to figures 3–5. This computational result is again in accordance with corollary 5.3.

8. Conclusions

We introduced new modifications of the 2D KSE, in both scalar and vector forms, with a 'calming-parameter' $\epsilon>0$ that we call the 'calmed KSE,' and proved that associated PDEs are globally well-posed in the sense of Hadamard. Moreover, we proved that, under suitable conditions on the calming function $\boldsymbol{\zeta}^\epsilon$, that (on the time interval of existence and uniqueness of solutions to the KSE) the solutions of the calmed equation converge to solutions of the KSE as $\epsilon\to 0^+$ at a certain algebraic rate. Moreover, our computational simulations indicate that this rate is sharp. To the best of our knowledge, this is the first globally well-posed PDE model whose solutions approximate solutions to the 2D KSE with arbitrary precision, at least before the potential blow-up time of the latter.

In addition, we note that this 'calming' technique can be applied to a wide variety of other equations, which we will investigate in several forthcoming works. In particular, in [16], we consider applications of calming to the 3D NSEs.

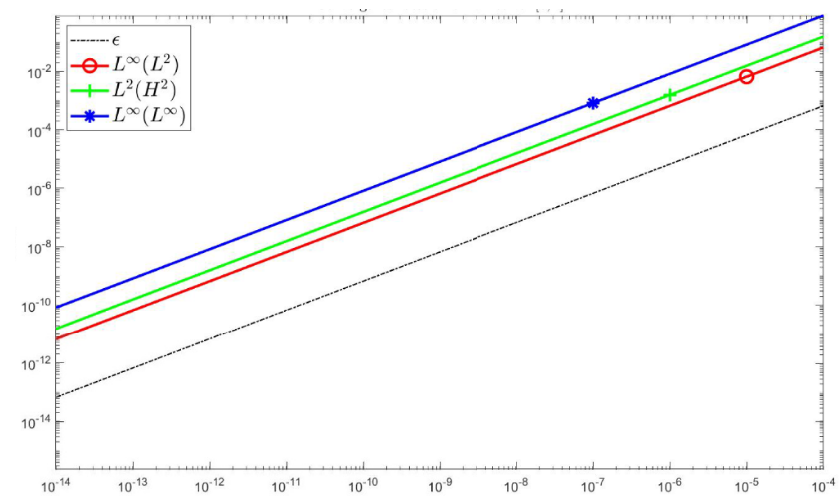


Figure 6. Estimates of $\mathbf{u}-\mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T=1$ with $\mathbf{u}^\epsilon$ a type 1 solution and with initial data given by (7.3). These estimates show a linear convergence rate.

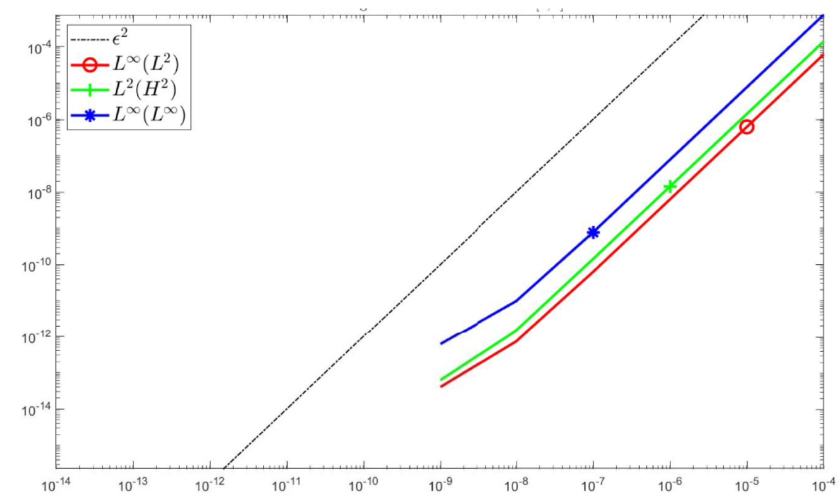


Figure 7. Estimates of $\mathbf{u}-\mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T=1$ with $\mathbf{u}^\epsilon$ a type 2 solution and with initial data given by (7.3). These estimates show a quadratic convergence rate. Note: for $\epsilon \lesssim 10^{-9}$, the error in our simulations was exactly 0, hence it does not appear in this log-log plot.

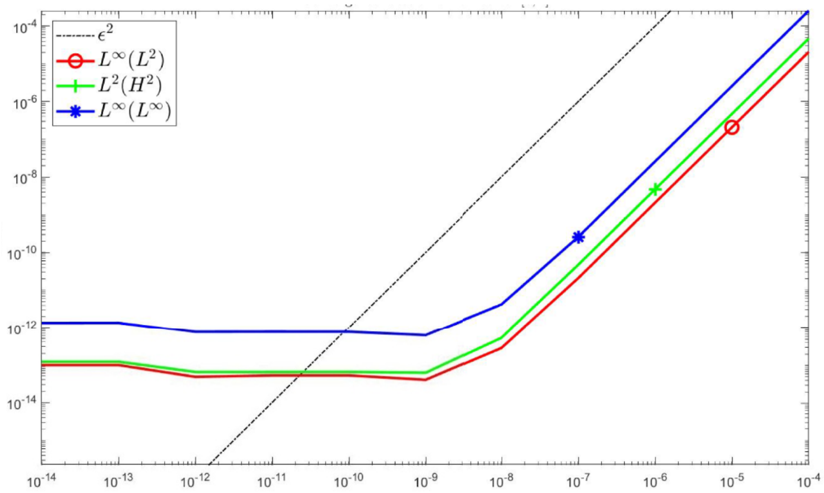


Figure 8. Estimates of $\mathbf{u}-\mathbf{u}^\epsilon$ vs. ϵ in norms $\|\cdot\|_{L^\infty(0,T;L^2)}$, $\|\cdot\|_{L^\infty(0,T;L^\infty)}$, and $\|\cdot\|_{L^2(0,T;H^2)}$, at time $T=1$ with $\mathbf{u}^\epsilon$ a type 3 solution and with initial data given by (7.3). These estimates show a quadratic convergence.

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: https://github.com/menlow2/KSE_code/tree/main [64].

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