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Global regularity results of the 2D fractional Boussinesq equations

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Abstract

This paper presents some of the sharpest global existence and regularity results on the two-dimensional incompressible Boussinesq equations with fractional dissipation, $\Lambda^\alpha u$ and $\Lambda^\beta \theta$, where $\Lambda = \sqrt{-\Delta}$ is the Zygmund operator. For the subcritical regime $\alpha + \beta > 1$ with $\alpha > \frac{2}{3}$, any initial data in the Sobolev space $H^s(\mathbb{R}^2)$ with $s > 2$ leads to a unique global solution. For any (α, β) in the critical regime $\alpha + \beta = 1$ with $\alpha > \frac{2}{3}$, an extra smallness condition on the L^∞-norm of the initial temperature would also guarantee the global regularity. This paper introduces an iterative procedure to minimize the dissipation requirement.

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1 Introduction

This paper focuses on the following two-dimensional (2D) Boussinesq equations with fractional dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \Lambda^\alpha u + \nabla p = \theta e_2, & x \in \mathbb{R}^2, \ t > 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ denotes the fluid velocity, $\theta = \theta(x,t)$ the temperature in the content of thermal convection and p the scalar pressure. $e_2 = (0,1)$ is the unit vector in the vertical direction. The numbers α and β are nonnegative real parameters. The fractional Laplacian operator $\Lambda^\delta \triangleq (-\Delta)^{\frac{\delta}{2}}$ is defined via the Fourier transform, namely

$$\widehat{\Lambda^\delta f}(\xi) = |\xi|^\delta \hat{f}(\xi).$$

The 2D Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Rayleigh-Bénard convection (see [29, 31]). Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations (see [29]). We adopt the convention that $\alpha = 0$ means no dissipation term in the velocity equation, and $\beta = 0$ means no diffusion in the temperature equation.

It should be mentioned that although (1.1) with fractional dissipation appears to be a merely mathematical generalization, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. The typical example is that flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [4, 18] for details).

Due to their prominent roles in modeling many phenomena in astrophysics and geophysics, the classical Boussinesq equations and their fractional dissipation counterparts have been studied extensively. The global regularity of the 2D Boussinesq equations with both Δu and $\Delta\theta$ can be established following a similar process as that for the 2D Navier–Stokes (see, e.g., [3]). In contrast, the fundamental issue of whether classical solutions to the inviscid Boussinesq equations can develop finite-time singularities is extremely challenging. Some significant progress on the inviscid Boussinseq equations ($\alpha = \beta = 0$) finite-time blowup problem has been made recently (see, e.g., [8, 16, 17]).

The issue that arises naturally is how much dissipation is really needed to ensure the global regularity. There are substantial recent developments on the global regularity problem on the Boussinesq equations with partial or fractional dissipation. Chae [7] and Hou-Li [24] successfully established the global regularity to (1.1) with $\alpha = 2$,

$\beta = 0$ or $\alpha = 0,\ \beta = 2$ (see [37] for the general case $\alpha + \beta = 2$). For the case $\alpha = 0,\ \beta = 2$ with the rough initial data, we refer to [15, 20]. For $\alpha = 0, \beta \in (1, 2]$, Hmidi and Zerguine [23] established the global well-posedness of (1.1) with rough initial data. By making use of the combined quantities, Hmidi, Keraani and Rousset [21, 22] were able to establish the global well-posedness for the cases when either $\alpha = 1,\ \beta = 0$ or $\alpha = 0,\ \beta = 1$. An alternative proof for the case $\alpha = 0,\ \beta = 1$ is given in [41], where the above mentioned combined quantity is no longer required.

[25] studied the global regularity of the 2D Boussinesq with general fractional dissipation $\Lambda^\alpha u$ and $\Lambda^\beta \theta$, namely (1.1). [25] was able to convert the Boussinesq regularity problem to a corresponding regularity problem on the generalized surface quasi-geostrophic (SQG) equation. This key observation in [25] leads to the fact that the size of $\alpha + \beta$ plays a crucial role in the global regularity problem on the fractional Boussinesq system. As a consequence, $\alpha + \beta = 1$ is classified as the critical case while $\alpha + \beta > 1$ as the subcritical case and $\alpha + \beta < 1$ as the supercritical case.

We explain why this classification is important and summarize some of the main results for each case. Applying curl to $(1.1)_1$ yields the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda^\alpha \omega = \partial_{x_1} \theta. \tag{1.2}$$

To deal with the "vortex stretching" term $\partial_{x_1}\theta$, we consider the combined quantity

$$G = \omega - \mathcal{R}_\alpha \theta, \qquad \mathcal{R}_\alpha \triangleq \partial_{x_1} \Lambda^{-\alpha}.$$

It is easy to check that G obeys

$$\partial_t G + (u \cdot \nabla)G + \Lambda^\alpha G = [\mathcal{R}_\alpha,\ u \cdot \nabla]\theta + \Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{1.3}$$

As explained in [25], G actually enjoys better regularity than ω. This motivates [25] to decompose the velocity field u into two pieces, one associated with G and the other with θ. In fact, by the Biot-Savart law,

$$u = \nabla^\perp \Delta^{-1}\omega = \nabla^\perp \Delta^{-1}(G + \mathcal{R}_\alpha \theta) = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1}\mathcal{R}_\alpha \theta \triangleq u_G + u_\theta.$$

For α and β in suitable range, G can be shown to have enough regularity such that

$$u_G = \nabla^\perp \Delta^{-1} G$$

becomes Lipschitz. Then the equation of θ becomes

$$\partial_t \theta + (u_G \cdot \nabla)\theta + (u_\theta \cdot \nabla)\theta + \Lambda^\beta \theta = 0, \quad u_\theta = \nabla^\perp \Delta^{-1}\mathcal{R}_\alpha \theta.$$

Since u_G is Lipschitz, the regularity problem on the Boussinesq system is then reduced to the problem on the corresponding generalized SQG equation

$$\begin{cases} \partial_t \theta + (u_\theta \cdot \nabla)\theta + \Lambda^\beta \theta = 0, \\ u_\theta = \nabla^\perp \Delta^{-1}\mathcal{R}_\alpha \theta. \end{cases} \tag{1.4}$$

The generalized SQG equation has been studied extensively and significant results have been obtained (see, e.g., [2, 9–14, 26]). For α and β in the subcritical regime $\alpha+\beta>1$ or in the critical regime $\alpha+\beta=1$, (1.4) always possesses a unique global classical solution. The global regularity problem for the supercritical case $\alpha+\beta<1$ appears to be out of reach at this moment.

Correspondingly, no large data global regularity result for the supercritical Boussinesq system is currently available. Existing global regularity results are for the subcritical or critical cases $\alpha+\beta\geq 1$. A few results have been established for the general critical case. [25] obtained the global regulatory for any α and β satisfying

$$\alpha+\beta=1, \quad \alpha>\frac{23-\sqrt{145}}{12}\approx 0.9132.$$

Subsequent efforts are devoted to enlarge the range of α. The work of Stefanov and Wu [34] extended the global regularity to α and β satisfying

$$\alpha+\beta=1, \quad \alpha>\frac{\sqrt{1777}-23}{12}\approx 0.7981.$$

Wu, Xu, Xue and Ye [36] further improved the global regularity to the range

$$\alpha+\beta=1, \quad \alpha>\frac{10}{13}\approx 0.7692.$$

Subsequent investigations appear to indicate that the largest possible range can be reached by the approach of [25] is

$$\alpha+\beta=1, \quad \alpha>\frac{2}{3}. \tag{1.5}$$

One goal of this paper is to prove the global regularity for the critical regime in (1.5), although we need to impose a minor condition on θ_0.

There are quite a number of global regularity results for the subcritical case $\alpha+\beta>1$. It is worth emphasizing that the global regularity of (1.1) in the subcritical ranges is not a trivial problem. In fact, many subcritical cases haven't been resolved. In particular, we do not know the global regularity for the case when α and β are close to one half and $\alpha+\beta>1$. Actually, direct energy estimates are not sufficient to obtain the desired global a priori bounds due to $\alpha,\beta<1$. To give an accurate account of current results, we further divide the subcritical ranges into two cases, $\alpha\geq\beta$ (the velocity dissipation dominated regime) and $\alpha<\beta$ (the thermal diffusion dominated regime). In the thermal diffusion dominated case, Constantin and Vicol [13] verified the global regularity of (1.1) with

$$0<\alpha<1, \;\; 0<\beta<1, \;\; \beta>\frac{2}{2+\alpha}.$$

Yang, Jiu and Wu [39] proved the global regularity of (1.1) for

$$0 < \alpha < 1, \ \ 0 < \beta < 1, \ \ \beta > 1 - \frac{\alpha}{2}, \ \ \beta \geq \frac{2+\alpha}{3}, \ \ \beta > \frac{10-5\alpha}{10-4\alpha}.$$

The ranges in [13, 39] were further enlarged by [42] to

$$\beta > \begin{cases} \max\left\{\frac{2}{3}, \ \frac{4-\alpha^2}{4+3\alpha}\right\}, & 0 < \alpha \leq \frac{2}{3}, \\ \frac{2-\alpha}{2}, & \frac{2}{3} \leq \alpha < 1. \end{cases}$$

For the velocity dominated case, Miao and Xue [30] obtained the global regularity for system (1.1) with

$$0.8876 \approx \frac{6-\sqrt{6}}{4} < \alpha < 1, \ \ 1-\alpha < \beta < \min\left\{\frac{7+2\sqrt{6}}{5}\alpha - 2, \ \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, \ \ 2-2\alpha\right\}.$$

This result was further refined by [40] to the range

$$0.7351 \approx \frac{10-2\sqrt{10}}{5} < \alpha < 1, \quad 1-\alpha < \beta < \min\left\{3-3\alpha, \ \frac{\alpha}{2}, \ \frac{3\alpha^2+4\alpha-4}{8(1-\alpha)}\right\}.$$

Zhou, Li, Shang, Wu, Yuan and Zhao [43], making use of [40] and the nonlinear maximum principle for fractional Laplacian operators developed by Constantin and Vicol [13], were able to establish the global regularity for

$$\frac{2}{3} < \alpha < 1, \ \ 0 < \beta < 1, \ \ \beta > \frac{1-\alpha}{\alpha}.$$

The main goal of this paper is to prove the global regularity for the subcritical regime

$$\frac{2}{3} < \alpha < 1, \quad \alpha + \beta > 1.$$

More precisely, we obtain the following result.

Theorem 1.1 *Let* $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$. *If* $\alpha + \beta > 1$ *and* $\frac{2}{3} < \alpha < 1$, *then there exists a unique global solution to (1.1) such that for any given* $T > 0$

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\alpha}{2}}(\mathbb{R}^2)),$$

$$\theta \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\beta}{2}}(\mathbb{R}^2)).$$

Due to the obvious fact $\frac{1-\alpha}{\alpha} > 1-\alpha$ for $\alpha \in (0,1)$, Theorem 1.1 improves the result in [43]. Since [40] has previously obtained the global regularity for $\alpha + \beta > 1$ with $\alpha > \frac{10-2\sqrt{10}}{5} \approx 0.7351$, it suffices to deal with the case $\alpha + \beta > 1$ with $\frac{2}{3} < \alpha \leq \frac{10-2\sqrt{10}}{5}$ in the proof.

The proof for Theorem 1.1 can actually be adapted to prove the global regularity for the critical case $\alpha + \beta = 1$ with $\frac{2}{3} < \alpha \leq \frac{10}{13}$ provided that the L^∞-norm of initial data θ_0 is small enough. More precisely, we have the following result. Its proof is given in Appendix A.

Theorem 1.2 *Consider the 2D Boussinesq equations*

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \Lambda^\alpha u + \nabla p = \nu \theta e_2, \\ \partial_t \theta + (u \cdot \nabla)\theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.6}$$

Assume $\alpha + \beta = 1$ with $\frac{2}{3} < \alpha \leq \frac{10}{13}$. Let $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 2$. If

$$|\nu| \, \|\theta_0\|_{L^\infty} \leq C_0 \tag{1.7}$$

for an absolute constant C_0 (independent of the initial data), then there exists a unique global solution to (1.6) such that, for any given $T > 0$,

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\alpha}{2}}(\mathbb{R}^2)),$$

$$\theta \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\beta}{2}}(\mathbb{R}^2)).$$

The global regularity for the case $\alpha + \beta = 1$ with $\alpha > \frac{10}{13}$ has already been established by [36] with general initial data. Therefore, the proof focuses on $\frac{2}{3} < \alpha \leq \frac{10}{13}$.

Finally, we investigate the case when the thermal diffusion dominates. We prove the following global regularity result.

Theorem 1.3 *Let $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 2$. If $\alpha, \beta \in (0,1)$ satisfy*

$$\beta > \max\left\{\alpha, \ \frac{4-\alpha^2}{4+3\alpha}\right\}, \quad 0 < \alpha \leq \frac{2}{3}, \tag{1.8}$$

then there exists a unique global solution to (1.1) such that, for any $T > 0$

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\alpha}{2}}(\mathbb{R}^2)),$$

$$\theta \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\frac{\beta}{2}}(\mathbb{R}^2)).$$

It is easy to see that this result improves the previous work [42]. Furthermore, Theorem 1.3 is still valid for the case $\beta = \alpha \geq \frac{2}{3}$. The details are provided in the end of Sect. 4.

The rest of this paper is divided into three sections and an appendix. Section 2 recalls the Littlewood-Paley decomposition, the definition of Besov spaces and some other useful facts. Section 3 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.3 is given in Sect. 4. In Appendix A, we sketch the proof of Theorem 1.2.

2 Preliminaries

This section provides the definition of the Littlewood-Paley decomposition and the definition of Besov spaces. Some useful facts are also included.

We first recall the definition of the Littlewood–Paley decomposition. More details can be found in [1, 35]. Let $\chi \in C_0^\infty(\mathbb{R}^2)(0 \le \chi \le 1)$ be a radial non-increasing function supported in the ball $\mathcal{B} \triangleq \{\xi \in \mathbb{R}^2, |\xi| \le \frac{4}{3}\}$ and with values 1 on $\{\xi \in \mathbb{R}^2, |\xi| \le \frac{3}{4}\}$. Let $\varphi(\xi) \triangleq \chi\left(\frac{\xi}{2}\right) - \chi(\xi)$. It is clear that $\varphi \in C_0^\infty(\mathbb{R}^2)$ is supported in the annulus $\mathcal{C} \triangleq \{\xi \in \mathbb{R}^2, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and satisfies

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2; \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \neq 0.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$. The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{2j} \int_{\mathbb{R}^2} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{Z},$$

while the low-frequency cut-off by

$$\dot{S}_j u = \chi(2^{-j}D)u = \sum_{k\le j-1} \dot{\Delta}_k u = 2^{2j} \int_{\mathbb{R}^2} \widetilde{h}(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{Z}.$$

The inhomogeneous dyadic blocks Δ_j are set by

$$\Delta_j u = 0, \quad j \le -2; \qquad \Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^2} \widetilde{h}(y)u(x-y)\,dy;$$
$$\Delta_j u = \varphi(2^{-j}D)u = 2^{2j} \int_{\mathbb{R}^2} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}\cup\{0\}.$$

We denote the function spaces of rapidly decreasing functions by $S(\mathbb{R}^2)$, tempered distributions by $S'(\mathbb{R}^2)$, and polynomials by $\mathcal{P}(\mathbb{R}^2)$. The homogeneous Besov spaces are defined via the Littlewood-Paley decomposition as follows.

Definition 2.1 Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)$ such that

$$\dot{B}^s_{p,r} = \{f \in S'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2); \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} \triangleq \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jrs} \|\dot{\Delta}_j f\|^r_{L^p}\right)^{\frac{1}{r}}, & 1 \le r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}, & r = \infty. \end{cases}$$

The inhomogeneous Besov spaces are defined as follows.

Definition 2.2 Let $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^2)$ such that

$$B^s_{p,r} = \{f \in S'(\mathbb{R}^2);\ \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} \triangleq \begin{cases} \Big(\sum_{j\ge -1} 2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\ge -1} 2^{js}\|\Delta_j f\|_{L^p}, & r=\infty. \end{cases}$$

We will use the following Bernstein inequalities (see [1, Lemma 2.1]), which are very useful in dealing with Fourier localized functions.

Lemma 2.1 *Let $\sigma > 0$, $1 \le a \le b \le \infty$, $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball of $\mathbb{R}^2$. Then the following estimates are true*

$$\mathrm{Supp}\widehat{f} \subset \lambda\mathcal{B} \ \Rightarrow\ \|\Lambda^\sigma f\|_{L^b(\mathbb{R}^2)} \le C_1\,\lambda^{\sigma+2(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a(\mathbb{R}^2)},$$

$$\mathrm{Supp}\widehat{f} \subset \lambda\mathcal{C} \ \Rightarrow\ C_2\,\lambda^\sigma\|f\|_{L^b(\mathbb{R}^2)} \le \|\Lambda^\sigma f\|_{L^b(\mathbb{R}^2)} \le C_3\,\lambda^{\sigma+2(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a(\mathbb{R}^2)},$$

where C_1, C_2 and C_3 are constants depending on σ, a and b only.

We next recall the following commutator estimates (see [34]).

Lemma 2.2 *Let $\frac{1}{2} < \alpha < 1$ and $1 < p_2 < \infty$, $1 < p_1, p_3 \le \infty$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$. Then for $0 \le s_1 < 1-\alpha$ and $s_1+s_2 > 1-\alpha$, it holds true*

$$\Big|\int_{\mathbb{R}^2} F[\mathcal{R}_\alpha,\ u_G\cdot\nabla]\theta\,dx\Big| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|G\|_{L^{p_3}}. \tag{2.1}$$

Similarly, for $0 \le s_1 < 1-\alpha$ and $s_1+s_2 > 2-2\alpha$, it holds true

$$\Big|\int_{\mathbb{R}^2} F[\mathcal{R}_\alpha,\ u_\theta\cdot\nabla]H\,dx\Big| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|H\|_{L^{p_3}}. \tag{2.2}$$

The following commutator estimate will also be used later.

Lemma 2.3 *For any $0 < \sigma < 1$, we have*

$$\|[\Lambda^\sigma,\ f\cdot\nabla]g\|_{L^p} \le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}}, \tag{2.3}$$

where $p, r_1, r_2 \in (1,\infty)$ such that $\frac{1}{p} = \frac{1}{r_1}+\frac{1}{r_2}$.

Proof Using the summation convention on repeated indices, we have

$$\begin{aligned}[\Lambda^\sigma, f\cdot\nabla]g &= \Lambda^\sigma(f_k\partial_k g) - f_k\Lambda^\sigma\partial_k g\\ &= \Lambda^\sigma\partial_k(f_k g) - f_k\Lambda^\sigma\partial_k g - \Lambda^\sigma(\partial_k f_k g)\\ &= \Lambda^\sigma\partial_k(f_k g) - f_k\Lambda^\sigma\partial_k g - g\Lambda^\sigma\partial_k f_k + g\Lambda^\sigma\partial_k f_k - \Lambda^\sigma(g\partial_k f_k).\end{aligned} \tag{2.4}$$

Letting $A^{\sigma+1} = \Lambda^\sigma\partial_k$, it follows from (2.4) that

$$\begin{aligned}[\Lambda^\sigma, f\cdot\nabla]g &= A^{\sigma+1}(f_k g) - f_k A^{\sigma+1}g - gA^{\sigma+1}f_k - [\Lambda^\sigma, g]\partial_k f_k\\ &= \Big(A^{\sigma+1}(f_k g) - f_k A^{\sigma+1}g - \nabla f_k A^{\sigma+1,\nabla}g - gA^{\sigma+1}f_k\Big)\\ &\quad + \nabla f_k A^{\sigma+1,\nabla}g - [\Lambda^\sigma, g]\partial_k f_k\\ &= \Big(A^{\sigma+1}(f_k g) - f_k A^{\sigma+1}g - \nabla f_k A^{\sigma+1,\nabla}g - gA^{\sigma+1}f_k\Big)\\ &\quad + \nabla f_k A^{\sigma+1,\nabla}g - [\Lambda^\sigma, g]\partial_k f_k,\end{aligned} \tag{2.5}$$

where the symbol $A^{\sigma+1,\nabla}$ is defined via Fourier transform as

$$\widehat{A^{\sigma+1,\nabla}}(\xi) = -i\nabla_\xi(\widehat{A^{\sigma+1}}(\xi)).$$

Taking $s_1 = 1$, $s_2 = \sigma$ in (1.9) of [28], we have

$$\|A^{\sigma+1}(f_k g) - f_k A^{\sigma+1}g - \nabla f_k A^{\sigma+1,\nabla}g - gA^{\sigma+1}f_k\|_{L^p} \le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}}. \tag{2.6}$$

Moreover, one gets

$$\|[\Lambda^\sigma, g]\partial_k f_k\|_{L^p} \le C\|\Lambda^\sigma g\|_{L^{r_2}}\|\partial_k f_k\|_{L^{r_1}} \le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}}, \tag{2.7}$$

where we have used the inequality (see [28, Corollary 5.2] and [27, Theorem 6.1])

$$\|[\Lambda^\sigma, u]v\|_{L^p} \le C\|v\|_{L^{r_1}}\|\Lambda^\sigma u\|_{L^{r_2}}.$$

Direct computations yield

$$\begin{aligned}\widehat{A^{\sigma+1,\nabla}f}(\xi) &= -i\nabla_\xi(\widehat{A^{\sigma+1}}(\xi))\widehat{f}(\xi)\\ &= i\nabla_\xi(|\xi|^\sigma i\xi_k)\widehat{f}(\xi)\\ &= -\nabla_\xi(|\xi|^\sigma\xi_k)|\xi|^{-\sigma}\widehat{\Lambda^\sigma f}(\xi)\\ &\triangleq m(\xi)\widehat{\Lambda^\sigma f}(\xi),\end{aligned}$$

where $m(\xi) = (m_1(\xi), m_2(\xi))$ is given by

$$
\begin{aligned}
m_j(\xi) &= -\nabla_{\xi_j}(|\xi|^\sigma \xi_k)|\xi|^{-\sigma} \\
&= -(\sigma|\xi|^{\sigma-2}\xi_j\xi_k + |\xi|^\sigma \delta_{jk})|\xi|^{-\sigma} \\
&= -(\sigma|\xi|^{-2}\xi_j\xi_k + \delta_{jk}).
\end{aligned}
$$

Hence, $m(\xi)$ of course obeys the Hörmander-Mihlin condition. Now invoking the well-known Hörmander-Mihlin theorem, we are able to show

$$
\|A^{\sigma+1,\nabla} g\|_{L^{r_2}} \le C\|\Lambda^\sigma g\|_{L^{r_2}}, \quad 1 < r_2 < \infty. \tag{2.8}
$$

Putting the above estimates (2.6), (2.7) and (2.8) into (2.5), we show that

$$
\begin{aligned}
\|[\Lambda^\sigma, f\cdot\nabla]g\|_{L^p} &\le \|A^{\sigma+1}(f_k g) - f_k A^{\sigma+1} g - \nabla f_k A^{\sigma+1,\nabla} g - g A^{\sigma+1} f_k\|_{L^p} \\
&\quad + \|\nabla f_k A^{\sigma+1,\nabla} g\|_{L^p} + \|[\Lambda^\sigma, g]\partial_k f_k\|_{L^p} \\
&\le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}} + \|\nabla f_k A^{\sigma+1,\nabla} g\|_{L^p} \\
&\le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}} + C\|\nabla f_k\|_{L^{r_1}}\|A^{\sigma+1,\nabla} g\|_{L^{r_2}} \\
&\le C\|\nabla f\|_{L^{r_1}}\|\Lambda^\sigma g\|_{L^{r_2}}.
\end{aligned}
$$

This finishes the proof of the lemma. □

The next lemma concerns the bilinear estimate (see [36]).

Lemma 2.4 *Let $2 < m < \infty$ and $0 < s < 1$, then it holds*

$$
\begin{aligned}
\|\Lambda^s(|f|^{m-2}f)\|_{L^p} &\le C\|f\|_{\dot{B}^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}}, \\
\left\||f|^{m-2}f\right\|_{W^{s,p}} &\le C\|f\|_{B^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}},
\end{aligned} \tag{2.9}
$$

where $p, q, r \in (1,\infty)^3$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

The following lemma is concerned with the Hölder continuity of the advection fractional-diffusion equation (see [32]).

Lemma 2.5 *Consider the following advection fractional-diffusion equation with $0 < \beta < 1$ in $\mathbb{R}^2$*

$$
\begin{cases}
\partial_t\theta + (u\cdot\nabla)\theta + \Lambda^\beta\theta = 0, \\
\nabla\cdot u = 0, \\
\theta(x,0) = \theta_0(x).
\end{cases}
$$

Let $\theta_0 \in L^\infty(\mathbb{R}^2)$ and u be a vector field in $L^\infty((0,\ T],\ C^{1-\beta}(\mathbb{R}^2))$ for given $T > 0$. Then the solution θ is Hölder continuous for any positive time $0 < t \le T$. Moreover, it holds

$$
\|\theta\|_{L^\infty((0,T];\ C^\ell(\mathbb{R}^2))} \le C\|\theta_0\|_{L^\infty},
$$

where the constant C and $\ell > 0$ depend on β and $\|u\|_{C^{1-\beta}}$ only.

Finally, we recall the differentiability of the advection fractional-diffusion equation (see [33, 38]).

Lemma 2.6 *Consider the following advection fractional-diffusion equation with* $0 < \beta < 1$ *in* $\mathbb{R}^2$

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta + \Lambda^{\beta}\theta = 0, \\ \nabla \cdot u = 0, \\ \theta(x, 0) = \theta_0(x). \end{cases}$$

Assume $T > 0$ *be given. Let* $\theta_0 \in L^{\infty}(\mathbb{R}^2)$ *and* u *be a vector field in* $L^{\infty}((0,\ T],\ C^{1-\beta+\zeta}(\mathbb{R}^2))$ *for any* $\zeta \in (0,\ \beta)$*. Then the solution* θ *actually belongs to space* $C^{1,\,\zeta}$*. Moreover, it holds*

$$\|\theta\|_{L^{\infty}((0,\ T],\ C^{1,\,\zeta}(\mathbb{R}^2))} \le C\|\theta_0\|_{L^{\infty}},$$

where the constant C *depends on* β *and* $\|u\|_{C^{1-\beta+\zeta}}$ *only.*

3 The proof of Theorem 1.1

This section proves Theorem 1.1. Since the local well-posedness of (1.1) for smooth initial data is well-known (see for instance [5, 29]), the main efforts are devoted to obtaining global *a priori* bounds for (u, θ) on $[0,\ T]$ for any given $T > 0$. Throughout this paper, we denote by C an universal positive constant whose value may change from line to line. The symbol $C(a, b, ...)$ means that C depends on variables a, b and so on.

Let us begin with the natural energy estimates.

Proposition 3.1 *Assume that* $u_0 \in L^2$ *and* $\theta_0 \in L^2 \cap L^{\infty}$*. Then*

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2\, d\tau \le \|\theta_0\|_{L^2}^2, \quad \|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [2, \infty],$$
$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}u(\tau)\|_{L^2}^2\, d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Based on (1.3), we are able to show the following estimate.

Proposition 3.2 *If* $\alpha + \beta > 1$ *and* $\frac{2}{3} < \alpha < 1$*, then*

$$\|G(t)\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{\frac{\alpha}{2}}G(\tau)\|_{L^2}^2 + \|\Lambda^{\beta}\theta(\tau)\|_{L^2}^2\right) d\tau \le C(t,\ u_0,\ \theta_0). \tag{3.1}$$

Proof Testing (1.3) by G and integrating in the space variable, one finds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 &= \int_{\mathbb{R}^2}[\mathcal{R}_\alpha,\,u\cdot\nabla]\theta\;G\,dx+\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_{x_1}\theta\;G\,dx\\ &=\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,\,u_G\cdot\nabla]\theta\;G\,dx+\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,\,u_\theta\cdot\nabla]\theta\;G\,dx\\ &\quad+\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_{x_1}\theta\;G\,dx\\ &\triangleq N_1+N_2+N_3. \end{aligned}\tag{3.2}$$

Applying $\Lambda^{\frac{\beta}{2}}$ to $(1.1)_2$, multiplying by $\Lambda^{\frac{\beta}{2}}\theta$ and integrating on $\mathbb{R}^2$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2+\|\Lambda^{\beta}\theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2}\Lambda^{\frac{\beta}{2}}\big(u\cdot\nabla\theta\big)\Lambda^{\frac{\beta}{2}}\theta\,dx\\ &=-\int_{\mathbb{R}^2}[\Lambda^{\frac{\beta}{2}},u\cdot\nabla]\theta\;\Lambda^{\frac{\beta}{2}}\theta\,dx\\ &\triangleq N_4. \end{aligned}\tag{3.3}$$

Summing up (3.2) and (3.3) yields

$$\frac{1}{2}\frac{d}{dt}(\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2)+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\|\Lambda^{\beta}\theta\|_{L^2}^2=N_1+N_2+N_3+N_4.\tag{3.4}$$

Thanks to (2.1) with $s_1=0$, $s_2=\frac{\alpha}{2}$, $p_1=\infty$ and $p_2=p_3=2$, one has

$$\begin{aligned}N_1&\le C\|\theta\|_{L^\infty}\|G\|_{W^{\frac{\alpha}{2},2}}\|G\|_{L^2}\\ &\le C\|\theta_0\|_{L^\infty}(\|G\|_{L^2}+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2})\|G\|_{L^2}\\ &\le \frac{1}{8}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+C\|G\|_{L^2}^2.\end{aligned}$$

In view of (2.2) with $s_2=\frac{\alpha}{2}$, $2-\frac{5\alpha}{2}<s_1<1-\alpha$, $p_1=2$ and $p_2=2$, $p_3=\infty$, we have

$$\begin{aligned}N_2&\le C\|\Lambda^{s_1}\theta\|_{L^2}\|G\|_{W^{\frac{\alpha}{2},2}}\|\theta\|_{L^\infty}\\ &\le C\|\Lambda^{s_1}\theta\|_{L^2}(\|G\|_{L^2}+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2})\|\theta_0\|_{L^\infty}\\ &\le C\|\theta\|_{L^2}^{\frac{\beta-s_1}{\beta}}\|\Lambda^{\beta}\theta\|_{L^2}^{\frac{s_1}{\beta}}(\|G\|_{L^2}+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2})\\ &\le \frac{1}{8}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C(1+\|G\|_{L^2}^2).\end{aligned}$$

Moreover, by interpolation,

$$
\begin{aligned}
N_3 &\leq C\|\Lambda^{\beta}\theta\|_{L^2}\|\Lambda^{1-\alpha}G\|_{L^2} \\
&\leq C\|\Lambda^{\beta}\theta\|_{L^2}\|G\|_{L^2}^{\frac{3\alpha-2}{\alpha}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{\frac{2(1-\alpha)}{\alpha}} \\
&\leq \frac{1}{8}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C\|G\|_{L^2}^2.
\end{aligned}
$$

To control the last term N_4, we should restrict α and β to the subcritical case $\alpha+\beta>1$. Indeed, an application of (2.3) gives

$$
\begin{aligned}
N_4 &\leq C\|[\Lambda^{\frac{\beta}{2}}, u\cdot\nabla]\theta\|_{L^{\frac{4}{3}}}\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^4} \\
&\leq C\|\nabla u\|_{L^2}\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^4}^2 \\
&\leq C\|\nabla u\|_{L^2}\|\Lambda^{\frac{\beta}{2}}\theta\|_{\dot{B}_{\infty,\infty}^{-\frac{\beta}{2}}}\|\Lambda^{\frac{\beta}{2}}\theta\|_{\dot{B}_{2,2}^{\frac{\beta}{2}}} \\
&\leq C\|\omega\|_{L^2}\|\theta\|_{\dot{B}_{\infty,\infty}^{0}}\|\Lambda^{\beta}\theta\|_{L^2} \\
&\leq C(\|G\|_{L^2}+\|\Lambda^{1-\alpha}\theta\|_{L^2})\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2} \\
&\leq C(\|G\|_{L^2}+\|\theta\|_{L^2}^{\frac{\alpha+\beta-1}{\beta}}\|\Lambda^{\beta}\theta\|_{L^2}^{\frac{1-\alpha}{\beta}})\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2} \\
&\leq \frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|G\|_{L^2}^2+C+\|\theta\|_{L^2}^2\|\theta\|_{L^\infty}^{\frac{2\beta}{\alpha+\beta-1}} \\
&\leq \frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C(1+\|G\|_{L^2}^2),
\end{aligned}
\tag{3.5}
$$

where the sharp interpolation inequality has been used (see [1, Theorem 2.42])

$$
\|f\|_{L^p}\leq C\|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{\frac{p-2}{p}}\|f\|_{\dot{B}_{2,2}^{\gamma}}^{\frac{2}{p}},\quad \gamma=\frac{\alpha(p-2)}{2},\quad p\in(2,\infty). \tag{3.6}
$$

We note that the condition $\alpha+\beta>1$ is first used (3.5). Inserting the above estimates into (3.4), we get

$$
\frac{d}{dt}(\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2)+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\|\Lambda^{\beta}\theta\|_{L^2}^2\leq C(1+\|G\|_{L^2}^2). \tag{3.7}
$$

The desired (3.1) follows from (3.7) and the Gronwall inequality. □

Naturally, the next step is to show the global *a priori* bound for $\|G(t)\|_{L^m}$ with $m>2$. To this end, we appeal to the following iterative approach. Comparing with the previous works, this iterative approach is a new idea which is an effective approach to deal with the subcritical case. Now we are in the position to prove the following proposition, which plays a crucial role in proving our main result.

Proposition 3.3 *Let* $\alpha+\beta>1$ *and* $\frac{2}{3}<\alpha<1$. *If it holds*

$$
\|G(t)\|_{L^{m_k}}^{m_k}+\int_0^t\|G(\tau)\|_{L^{\frac{2m_k}{2-\alpha}}}^{m_k}\,d\tau\leq M, \tag{3.8}
$$

then

$$\|G(t)\|_{L^{m_{k+1}}}^{m_{k+1}} + \int_0^t \|G(\tau)\|_{L^{\frac{2m_{k+1}}{2-\alpha}}}^{m_{k+1}} \, d\tau \le C(t, \ M, \ u_0, \ \theta_0), \tag{3.9}$$

where

$$m_{k+1} < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k}.$$

Furthermore, we may restrict m_k *and* m_{k+1} *to the range*

$$2 \le m_k, \ m_{k+1} < \min\left\{\frac{8}{2-\alpha}, \ \frac{1}{1-\alpha}, \ \frac{2(2+\beta)}{(2+\alpha)\beta}\right\}. \tag{3.10}$$

Proof We first claim that under the assumption of (3.8), it holds

$$\|\Lambda^{\delta_k}\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\delta_k+\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2 \, d\tau \le C(t, \ u_0, \ \theta_0), \tag{3.11}$$

where δ_k is given by

$$\delta_k = \frac{\beta[(2+\alpha)m_k - 2]}{4} < 1.$$

To prove (3.11), we apply Λ^{δ_k} to $(1.1)_2$ and multiply it by $\Lambda^{\delta_k}\theta$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^{\delta_k}\theta(t)\|_{L^2}^2 + \|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Lambda^{\delta_k}\big(u\cdot\nabla\theta\big)\Lambda^{\delta_k}\theta \, dx \\
&= -\int_{\mathbb{R}^2} [\Lambda^{\delta_k}, u\cdot\nabla]\theta \ \ \Lambda^{\delta_k}\theta \, dx \\
&= -\int_{\mathbb{R}^2} [\Lambda^{\delta_k}, u_G\cdot\nabla]\theta \ \ \Lambda^{\delta_k}\theta \, dx \\
&\quad - \int_{\mathbb{R}^2} [\Lambda^{\delta_k}, u_\theta\cdot\nabla]\theta \ \ \Lambda^{\delta_k}\theta \, dx \\
&\triangleq N_5 + N_6.
\end{aligned} \tag{3.12}$$

Thanks to (2.3) and (3.6),

$$\begin{aligned}
N_5 &\le C\|[\Lambda^{\delta_k}, u_G\cdot\nabla]\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k+\beta}}} \|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}} \\
&\le C\|\nabla u_G\|_{L^{\frac{2\delta_k+\beta}{\beta}}} \|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}} \|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}} \\
&\le C\|G\|_{L^{\frac{2\delta_k+\beta}{\beta}}} \|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}^2 \\
&\le C\left(\|G\|_{L^{m_k}}^{\frac{2\beta m_k-(2-\alpha)(2\delta_k+\beta)}{\alpha(2\delta_k+\beta)}} \|G\|_{L^{\frac{2m_k}{2-\alpha}}}^{\frac{2(2\delta_k+\beta-\beta m_k)}{\alpha(2\delta_k+\beta)}}\right)\left(\|\Lambda^{\delta_k}\theta\|_{\dot{B}^{-\delta_k}_{\infty,\infty}}^{\frac{2\beta}{2\delta_k+\beta}} \|\Lambda^{\delta_k}\theta\|_{\dot{B}^{\frac{\beta}{2}}_{2,2}}^{\frac{4\delta_k}{2\delta_k+\beta}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\|G\|_{L^{m_k}}^{\frac{\alpha}{2+\alpha}}\|G\|_{L^{\frac{2m_k}{2-\alpha}}}^{\frac{2}{2+\alpha}}\right)\left(\|\theta\|_{L^\infty}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\right)\\
&\leq \frac{1}{4}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|G\|_{L^{m_k}}^{\frac{\alpha(2\delta_k+\beta)}{(2+\alpha)\beta}}\|G\|_{L^{\frac{2m_k}{2-\alpha}}}^{\frac{2(2\delta_k+\beta)}{(2+\alpha)\beta}}\\
&=\frac{1}{4}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|G\|_{L^{m_k}}^{\frac{\alpha m_k}{2}}\|G\|_{L^{\frac{2m_k}{2-\alpha}}}^{m_k}.
\end{aligned}
\tag{3.13}
$$

Using again (2.3) and (3.6), due to $\alpha+\beta>1$, one has

$$
\begin{aligned}
N_6 &\leq C\|[\Lambda^{\delta_k}, u_\theta\cdot\nabla]\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k+\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}\\
&\leq C\|\nabla u_\theta\|_{L^{\frac{2\delta_k+\beta}{\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}\\
&\leq C\|\Lambda^{1-\alpha}\theta\|_{L^{\frac{2\delta_k+\beta}{\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}^2\\
&\leq C\left(\|\Lambda^{1-\alpha}\theta\|_{\dot B^{-(1-\alpha)}_{\infty,\infty}}^{\frac{2\delta_k-\beta}{2\delta_k+\beta}}\|\Lambda^{1-\alpha}\theta\|_{\dot B_{2,2}^{\frac{(1-\alpha)(2\delta_k-\beta)}{2\beta}}}^{\frac{2\beta}{2\delta_k+\beta}}\right)\left(\|\Lambda^{\delta_k}\theta\|_{\dot B^{-\delta_k}_{\infty,\infty}}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k}\theta\|_{\dot B_{2,2}^{\frac{\beta}{2}}}^{\frac{4\delta_k}{2\delta_k+\beta}}\right)\\
&\leq C\left(\|\theta\|_{L^\infty}^{\frac{2\delta_k-\beta}{2\delta_k+\beta}}\|\Lambda^{\frac{(1-\alpha)(2\delta_k+\beta)}{2\beta}}\theta\|_{L^2}^{\frac{2\beta}{2\delta_k+\beta}}\right)\left(\|\theta\|_{L^\infty}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\right)\\
&\leq C\|\theta\|_{L^\infty}\|\Lambda^{\frac{(1-\alpha)(2\delta_k+\beta)}{2\beta}}\theta\|_{L^2}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\\
&\leq C\|\theta\|_{L^\infty}\left(\|\theta\|_{L^2}^{\frac{2(\alpha+\beta-1)}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{2(1-\alpha)}{2\delta_k+\beta}}\right)\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\\
&\leq \frac{1}{4}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2+C\|\theta\|_{L^2}^{\frac{2\delta_k+\beta}{\alpha+\beta-1}}\|\theta\|_{L^\infty}^2.
\end{aligned}
\tag{3.14}
$$

We point out that this is the last place where α and β are required to be in the subcritical case $\alpha+\beta>1$. Inserting (3.13) and (3.14) into (3.12) implies

$$
\frac{1}{2}\frac{d}{dt}\|\Lambda^{\delta_k}\theta(t)\|_{L^2}^2+\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2\leq C\|\theta\|_{L^\infty}^2\|G\|_{L^{m_k}}^{\frac{\alpha m_k}{2}}\|G\|_{L^{\frac{2m_k}{2-\alpha}}}^{m_k}+C\|\theta\|_{L^2}^{\frac{2\delta_k+\beta}{\alpha+\beta-1}}\|\theta\|_{L^\infty}^2.
\tag{3.15}
$$

Keeping in mind (3.8) and integrating (3.15) in time, we are able to show that (3.11) is valid. Now with the help of (3.11), we are in the position to show (3.9). By (3.11), we get from (3.6) that

$$
\|\Lambda^{\gamma\beta}\theta\|_{L_t^{\frac{(\alpha+2)m_k}{2\gamma}}L_x^{\frac{(\alpha+2)m_k}{2\gamma}}}\leq C\|\Lambda^{\frac{(\alpha+2)\beta m_k}{4}}\theta\|_{L_t^2L_x^2}^{\frac{4\gamma}{(\alpha+2)m_k}}\|\theta\|_{L_t^\infty L_x^\infty}^{1-\frac{4\gamma}{(\alpha+2)m_k}},
$$

where

$$0 < \gamma < \frac{(\alpha+2)m_k}{4}.$$

Taking the inner product of (1.3) with $|G|^{m-2}G$, we obtain, after integrating by parts

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\|G(t)\|_{L^m}^m + \int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx &= \int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_{x_1}\theta\ |G|^{m-2}G\,dx \\ &\quad + \int_{\mathbb{R}^2}[\mathcal{R}_\alpha,\, u_\theta\cdot\nabla]\theta\ |G|^{m-2}G\,dx \\ &\quad + \int_{\mathbb{R}^2}[\mathcal{R}_\alpha,\, u_G\cdot\nabla]\theta\ |G|^{m-2}G\,dx \\ &\triangleq K_1 + K_2 + K_3. \end{aligned} \tag{3.16}$$

By the maximum principle (see [6] for example) and the Sobolev embedding, we observe

$$\int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx \ge \widetilde{C}\|\Lambda^{\frac{\alpha}{2}}G^{\frac{m}{2}}\|_{L^2}^2 \ge \widetilde{C}\|G\|_{L^{\frac{2m}{2-\alpha}}}^m, \tag{3.17}$$

where $\widetilde{C} > 0$ is an absolute constant. It thus follows from the Hölder inequality and (2.9) that

$$\begin{aligned}|K_1| &\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|\Lambda^{1-\alpha+(1-\gamma)\beta}(|G|^{m-2}G)\|_{L^{\frac{(\alpha+2)m_k}{(\alpha+2)m_k-2\gamma}}} \\ &\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{B^{1-\alpha+(1-\gamma)\beta}_{2,\frac{(\alpha+2)m_k}{(\alpha+2)m_k-2\gamma}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}} \\ &\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}},\end{aligned}$$

where we have used the fact

$$1-\alpha+(1-\gamma)\beta < \frac{\alpha}{2} \quad \text{or} \quad \gamma > \frac{2\beta+2-3\alpha}{2\beta}. \tag{3.18}$$

Consequently, we have

$$|K_1| \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}}. \tag{3.19}$$

To handle the term K_2, we choose $s_1 = \gamma\beta \in [0, 1-\alpha)$ and s_2 satisfying

$$2-2\alpha-\gamma\beta < s_2 < \frac{\alpha}{2},$$

which requires the following restriction to ensure the existence of such s_2 above

$$2-2\alpha-\gamma\beta < \frac{\alpha}{2}. \tag{3.20}$$

Thanks to Lemmas 2.2 and 2.4, we conclude

$$\begin{aligned}|K_2| &\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|\theta\|_{L^\infty}\||G|^{m-2}G\|_{W^{s_2,\frac{(\alpha+2)m_k}{(\alpha+2)m_k-2\gamma}}}\\&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|\theta_0\|_{L^\infty}\|G\|_{B^{s_2}_{2,\frac{(\alpha+2)m_k}{(\alpha+2)m_k-2\gamma}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}}\\&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}}.\end{aligned} \tag{3.21}$$

To bound the term K_3, we take

$$s_2 = 1-\alpha+\delta,$$

where $\delta > 0$ is sufficiently small. For $m-1 < q \le 2(m-1)$ and $\frac{1}{p}+\frac{1}{q} = 1$, we apply Lemma 2.2 and Lemma 2.4 to obtain

$$\begin{aligned}|K_3| &\le C\|G\|_{L^q}\|\theta\|_{L^\infty}\||G|^{m-2}G\|_{W^{s_2-\frac{\delta}{2},p}}\\&\le C\|\theta_0\|_{L^\infty}\|G\|_{L^q}\|G\|^{m-2}_{L^{(m-2)\times\frac{q}{m-2}}}\|G\|_{B^{s_2-\frac{\delta}{2}}_{\frac{q}{q-(m-1)},p}}\\&\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{B^{s_2-\frac{\delta}{2}}_{\frac{q}{q-(m-1)},p}}\\&\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{H^{s_2-1+\frac{2(m-1)}{q}}}.\end{aligned}$$

We further require $q > \frac{4(m-1)}{3\alpha-2\delta}$ to obtain the following interpolation inequality

$$\|G\|_{H^{-\alpha+\delta+\frac{2(m-1)}{q}}} \le C\|G\|^{1-\mu}_{L^2}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}},$$

where

$$\mu = \frac{2(\delta-\alpha)q+4(m-1)}{\alpha q} \in (0,\ 1).$$

Therefore, for $\frac{4(m-1)}{3\alpha-2\delta} < q < 2(m-1)$, one finds that

$$\begin{aligned}|K_3| &\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|^{1-\mu}_{L^2}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}\\&\le C\|G\|^{m-1}_{L^q}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}.\end{aligned} \tag{3.22}$$

Substituting (3.17), (3.19), (3.21) and (3.22) into (3.16), one may readily check that

$$\begin{aligned}\frac{d}{dt}\|G(t)\|^m_{L^m} + \|G\|^m_{L^{\frac{2m}{2-\alpha}}} \le & C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}}\\&+C\|G\|^{m-1}_{L^q}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}.\end{aligned} \tag{3.23}$$

It follows from the Gagliardo-Nirenberg inequalities that

$$\|G\|_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}} \le C\|G\|_{L^m}^{1-\lambda_1}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{\lambda_1}, \tag{3.24}$$

$$\|G\|_{L^q} \le C\|G\|_{L^m}^{1-\lambda_2}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{\lambda_2}, \tag{3.25}$$

where $\lambda_1,\ \lambda_2 \in (0,\ 1)$ are given by

$$\lambda_1 = \frac{[(\alpha+2)m_k+4\gamma]m - 4(\alpha+2)m_k}{\alpha(\alpha+2)m_k(m-2)}, \qquad \lambda_2 = \frac{2(q-m)}{\alpha q}.$$

In order for $\lambda_1,\ \lambda_2 \in (0,\ 1)$, we impose the following restrictions

$$\begin{cases} \dfrac{(\alpha+2)m_k(4-m)}{4m} \le \gamma \le \dfrac{\alpha(\alpha+2)m_k(m-2)+(\alpha+2)m_k(4-m)}{4m}, \\ m \le q \le \dfrac{2m}{2-\alpha}. \end{cases} \tag{3.26}$$

In view of (3.24), we obtain

$$\begin{aligned} & C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(\alpha+2)m_k(m-2)}{(\alpha+2)m_k-4\gamma}}} \\ & \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|_{L^m}^{(m-2)(1-\lambda_1)}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-2)\lambda_1} \\ & \le \frac{\widetilde{C}}{10}\|G\|^m_{L^{\frac{2m}{2-\alpha}}} + C\left(\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\right)^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}}. \end{aligned} \tag{3.27}$$

Coming back to (3.25), one observes

$$\begin{aligned} C\|G\|_{L^q}^{m-1}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}} &\le C\|G\|_{L^m}^{(m-1)(1-\lambda_2)}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-1)\lambda_2}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}} \\ &\le \frac{\widetilde{C}}{10}\|G\|^m_{L^{\frac{2m}{2-\alpha}}} + C\|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned} \tag{3.28}$$

Inserting (3.27) and (3.28) into (3.23), it holds that

$$\begin{aligned} \frac{d}{dt}\|G(t)\|^m_{L^m} + \|G\|^m_{L^{\frac{2m}{2-\alpha}}} \le & C\left(\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\right)^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}} \\ & + C\|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned} \tag{3.29}$$

Obviously, we are able to check that

$$\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1} \le m, \quad \frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2} \le m,$$

and the following is valid

$$\frac{m\mu}{m-(m-1)\lambda_2} \le 2$$

due to $m < \frac{1}{1-\alpha}$. Thanks to $m < \frac{8}{2-\alpha}$, we further choose γ such that

$$\gamma \le \frac{[8-(2-\alpha)m]m_k}{4m}. \tag{3.30}$$

Later we explain why such γ can be selected. Then, for γ satisfying (3.30), one gets

$$\frac{m}{m-(m-2)\lambda_1} \le \frac{2(\alpha+2)m_k}{(\alpha+2)m_k+4\gamma}.$$

Therefore, we deduce from (3.29) that

$$\frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|^m_{L^{\frac{2m}{2-\alpha}}} \le C\Big(1+\|\Lambda^{\gamma\beta}\theta\|^{\frac{(\alpha+2)m_k}{2\gamma}}_{L^{\frac{(\alpha+2)m_k}{2\gamma}}} + \|G\|^2_{H^{\frac{\alpha}{2}}}\Big)(1+\|G\|^m_{L^m}). \tag{3.31}$$

Let us now explain that q and γ can be selected to satisfy all the restrictions stated above. The number q should satisfy

$$\max\left\{\frac{4(m-1)}{3\alpha-2\delta},\ m\right\} < q < \min\left\{2(m-1),\ \frac{2m}{2-\alpha}\right\}.$$

Direct computations yield that the number q can be fixed if we select $\delta < \frac{3\alpha-2}{2}$. Putting all the restrictions (3.18), (3.20) and (3.26), (3.30) on γ, we have

$$\underline{\mathcal{B}}(\alpha) < \gamma < \overline{\mathcal{B}}(\alpha), \tag{3.32}$$

where

$$\underline{\mathcal{B}}(\alpha) = \max\left\{0,\ \frac{2\beta+2-3\alpha}{2\beta},\ \frac{4--5\alpha}{2\beta},\ \frac{(\alpha+2)m_k(4-m)}{4\,m}\right\},$$

$$\overline{\mathcal{B}}(\alpha) = \min\left\{\frac{(\alpha+2)m_k}{4},\ \frac{1-\alpha}{\beta},\ \frac{\alpha(\alpha+2)m_k(m-2)+(\alpha+2)m_k(4-m)}{4m},\ \frac{[8-(2-\alpha)m]m_k}{4m}\right\}.$$

Moreover, the number m should obey

$$2 < m < \min\left\{\frac{8}{2-\alpha},\ \frac{1}{1-\alpha}\right\}.$$

Invoking direct computation yields that for $m > 2$

$$\frac{(\alpha+2)m_k}{4} \geq \frac{\alpha(\alpha+2)m_k(m-2)+(\alpha+2)m_k(4-m)}{4m} \geq \frac{[8-(2-\alpha)m]m_k}{4m}.$$

As a consequence of the above fact, the condition (3.32) reduces to

$$\max\left\{0,\ \frac{2\beta+2-3\alpha}{2\beta},\ \frac{(\alpha+2)m_k(4-m)}{4m}\right\} < \gamma < \min\left\{\frac{1-\alpha}{\beta},\ \frac{[8-(2-\alpha)m]m_k}{4m}\right\}.$$

Now we take m as

$$m < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k}, \tag{3.33}$$

then it is not difficult to check that the γ would work (see *Remark* 3.1 below for details). Therefore, we deduce from (3.31) that

$$\frac{d}{dt}\|G(t)\|_{L^{m_{k+1}}}^{m_{k+1}} + \|G\|_{L^{\frac{2m_{k+1}}{2-\alpha}}}^{m_{k+1}} \leq C\Big(1+\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{(\alpha+2)m_k}{2\gamma}}}^{\frac{(\alpha+2)m_k}{2\gamma}} + \|G\|_{H^{\frac{\alpha}{2}}}^2\Big) \times (1+\|G\|_{L^{m_{k+1}}}^{m_{k+1}}), \tag{3.34}$$

where in addition to (3.10), m_{k+1} should further satisfy

$$m_{k+1} < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k}.$$

Applying the Gronwall inequality to (3.34) implies the desired result (3.9). This finishes the proof of Proposition 3.3. □

Remark 3.1 In order to ensure the existence of γ, we need a restriction on the upper bound of β, namely $\beta < \frac{\alpha}{2}$. Here are the details.

$$\frac{2\beta+2-3\alpha}{2\beta} < \frac{1-\alpha}{\beta} \Leftarrow \beta < \frac{\alpha}{2};$$
$$\frac{2\beta+2-3\alpha}{2\beta} < \frac{[8-(2-\alpha)m]m_k}{4m} \Leftarrow m < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k};$$
$$\frac{(\alpha+2)m_k(4-m)}{4m} < \frac{1-\alpha}{\beta} \Leftarrow m > \frac{4(\alpha+2)\beta m_k}{(\alpha+2)\beta m_k+4(1-\alpha)};$$
$$\frac{(\alpha+2)m_k(4-m)}{4m} < \frac{[8-(2-\alpha)m]m_k}{4m} \Leftarrow m > 2.$$

By the direct computations, we achieve

$$\begin{aligned}
&\frac{4(\alpha+2)\beta m_k}{(\alpha+2)\beta m_k+4(1-\alpha)} < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k} \\
&\Leftarrow (\alpha^2+2\alpha)\beta m_k + 8(1-\alpha) + 2(\alpha+2)(3\alpha-2-2\beta) > 0 \\
&\Leftarrow 2(\alpha^2+2\alpha)\beta + 8(1-\alpha) + 2(\alpha+2)(3\alpha-2-2\beta) > 0 \\
&\Leftarrow \beta < \frac{3\alpha^2}{4-\alpha^2},
\end{aligned}$$

where we have used the fact $m_k \geq 2$. Concerning the above estimates, we take m as (3.33). Moreover, β should satisfy

$$1-\alpha < \beta < \min\left\{\frac{\alpha}{2},\ \frac{3\alpha^2}{4-\alpha^2}\right\} = \frac{\alpha}{2}.$$

Proposition 3.3 allows us to show the following key estimate.

Proposition 3.4 *Let $\alpha+\beta > 1$ and $\frac{2}{3} < \alpha < 1$, then it holds*

$$\|G(t)\|_{L^m}^m + \int_0^t \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m \, d\tau \leq C(t,\ u_0,\ \theta_0), \tag{3.35}$$

where m satisfies

$$2 \leq m < \min\left\{\frac{8}{2-\alpha},\ \frac{1}{1-\alpha},\ \frac{2(2+\beta)}{(2+\alpha)\beta},\ \frac{2(2\beta+3\alpha-2)}{(2-\alpha)\beta}\right\}.$$

Proof Before proving this proposition we point out that it suffices to consider the case $\alpha \in (\frac{2}{3},\ \frac{4}{5})$ as the global regularity of the remainder case $\alpha \in [\frac{4}{5},\ 1)$ has been proven by [36, 40]. Now recalling (3.9) and (3.1), we have for $k = 0, 1, 2, \cdots$

$$\|G(t)\|_{L^{m_{k+1}}}^{m_{k+1}} + \int_0^t \|G(\tau)\|_{L^{\frac{2m_{k+1}}{2-\alpha}}}^{m_{k+1}} \, d\tau \leq C(t,\ u_0,\ \theta_0),$$

where $m_1 = 2$ and

$$m_{k+1} < \frac{8\beta m_k}{2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k}.$$

Due to $\alpha \in (\frac{2}{3},\ \frac{4}{5})$ and $\beta \geq 1-\alpha$,

$$2(2\beta+2-3\alpha)+(2-\alpha)\beta m_k > 0, \quad \forall\, m_k \geq 2.$$

We choose small $\epsilon > 0$ and take m_{k+1} as

$$m_{k+1} = \frac{8\beta m_k}{2(2\beta+2-3\alpha+\epsilon)+(2-\alpha)\beta m_k},$$

where $\epsilon > 0$ will be specified later. By means of the direct computations, m_k can be solved as

$$m_k = \frac{2(2\beta + 3\alpha - 2 - \epsilon)}{(2-\alpha)\beta + (\alpha\beta + 3\alpha - 2 - \epsilon)\left(\frac{2\beta+2-3\alpha+\epsilon}{4\beta}\right)^{k-1}}, \quad k \geq 1.$$

If we fix $\epsilon > 0$ as

$$3\alpha - 2 - 2\beta < \epsilon < 3\alpha - 2 + \alpha\beta,$$

then the sequence $\{m_k\}_{k\in\mathbb{N}}$ is increasing. Notice that $\alpha \in (\frac{2}{3}, \frac{4}{5})$ and $\beta \geq 1 - \alpha$, it yields

$$3\alpha - 2 - 2\beta \leq 0,$$

which leads to

$$0 < \epsilon < \alpha\beta + 3\alpha - 2.$$

Moreover, we are able to show

$$\lim_{k\to\infty} m_k = \frac{2(2\beta + 3\alpha - 2 - \epsilon)}{(2-\alpha)\beta}.$$

Due to the arbitrariness of $\epsilon > 0$, it allows us to derive (3.35) for any m satisfying

$$2 \leq m < \frac{2(2\beta + 3\alpha - 2)}{(2-\alpha)\beta}.$$

Furthermore, due to $\alpha > \frac{2}{3}$, it is obvious to see that

$$\min\left\{\frac{8}{2-\alpha}, \frac{1}{1-\alpha}, \frac{2(2+\beta)}{(2+\alpha)\beta}, \frac{2(2\beta+3\alpha-2)}{(2-\alpha)\beta}\right\} > \frac{2}{\alpha}.$$

Consequently, (3.35) is valid and the proof of Proposition 3.4 is completed. □

The following proposition allows us to obtain more higher regularity estimate of the combined quantity G.

Proposition 3.5 *Consider (1.3), namely*

$$\partial_t G + (u\cdot\nabla)G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u\cdot\nabla]\theta + \Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{3.36}$$

Let $\beta \geq 1-\alpha$ and $\alpha > \frac{1}{2}$. Suppose G admits the following bound

$$\sup_{0\leq t\leq T} \|G(t)\|_{L^q} < \infty, \qquad q > \frac{2}{\alpha} \quad \left(\textit{we may assume } q < \frac{2}{1-\alpha}\right),$$

for any given $T > 0$, then for any $0 < s \leq 2\alpha - 1 - \beta$, it holds

$$\sup_{0\leq t\leq T} \|G(t)\|_{B^s_{r,\infty}} < \infty,$$

where r is given by

$$\frac{2}{2\alpha-1}<r\le\frac{2q}{2-(1-\alpha)q}.$$

Proof The proof is inspired by [36, Lemma 2.5]. For the sake of completeness we present here the full argument. Applying Δ_k to (3.36), one obtains

$$\partial_t\Delta_k G+\Lambda^{\alpha}\Delta_k G=\Delta_k([\mathcal{R}_\alpha,\, u\cdot\nabla]\theta)-\Delta_k(u\cdot\nabla G)+\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{3.37}$$

Multiplying (3.37) by $|\Delta_k G|^{r-2}\Delta_k G$, integrating the result over space $\mathbb{R}^2$ and using the divergence-free condition, we get

$$\frac{1}{r}\frac{d}{dt}\|\Delta_k G(t)\|_{L^r}^r+\int_{\mathbb{R}^2}(\Lambda^{\alpha}\Delta_k G)|\Delta_k G|^{r-2}\Delta_k G\,dx=I_1^k+I_2^k+I_3^k, \tag{3.38}$$

where

$$\begin{aligned}I_1^k&=\int_{\mathbb{R}^2}\Delta_k([\mathcal{R}_\alpha,\, u\cdot\nabla]\theta)\;|\Delta_k G|^{r-2}\Delta_k G\,dx,\\ I_2^k&=-\int_{\mathbb{R}^2}\Delta_k(u\cdot\nabla G)\;|\Delta_k G|^{r-2}\Delta_k G\,dx,\\ I_3^k&=\int_{\mathbb{R}^2}\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta\;|\Delta_k G|^{r-2}\Delta_k G\,dx.\end{aligned}$$

We recall the following lower bound (see [9])

$$\int_{\mathbb{R}^2}(\Lambda^{\alpha}\Delta_k G)|\Delta_k G|^{r-2}\Delta_k G\,dx\ge c2^{\alpha k}\|\Delta_k G\|_{L^r}^r,\quad k\ge 0$$

with an absolute constant $c>0$ independent of k. According to $r\le\frac{2q}{2-(1-\alpha)q}$, one has

$$\begin{aligned}\|u\|_{\dot B^{\alpha}_{r,\infty}}&\le C\|\Lambda^{\alpha}u\|_{L^r}\\ &\le C\|\Lambda^{\alpha}u_G\|_{L^r}+C\|\Lambda^{\alpha}u_\theta\|_{L^r}\\ &\le C\|\Lambda^{\alpha-1}G\|_{L^r}+C\|\Lambda^{\alpha-1}\mathcal{R}_\alpha\theta\|_{L^r}\\ &\le C\|G\|_{L^q}+C\|G\|_{L^2}+C\|\theta\|_{L^r}\\ &\le C(T,\, u_0,\, \theta_0).\end{aligned} \tag{3.39}$$

Let us recall the following estimate (see (A.8) of [36])

$$\|\Delta_k([\mathcal{R}_\alpha,\, u\cdot\nabla]\theta)\|_{L^r}\le C\Big(2^{(2-2\alpha)k}\|u\|_{\dot B^{\alpha}_{r,\infty}}+\|u\|_{L^2}+\|\theta\|_{L^2}\Big)\|\theta\|_{L^\infty},$$

which together with (3.39) directly gives

$$\begin{aligned}\|\Delta_k([\mathcal{R}_\alpha,\, u\cdot\nabla]\theta)\|_{L^r} &\le C2^{(2-2\alpha)k}\|u\|_{\dot B^{\alpha}_{r,\infty}}\|\theta\|_{L^\infty} + C(\|u\|_{L^2}+\|\theta\|_{L^2})\|\theta\|_{L^\infty}\\ &\le C2^{(2-2\alpha)k}.\end{aligned}$$

As a result, we obtain

$$|I_1^k| \le C2^{2(1-\alpha)k}\|\Delta_k G\|_{L^r}^{r-1}. \tag{3.40}$$

Noticing (3.39), it follows from the proof of the estimate (7.17) in [25] that

$$\begin{aligned}|I_2^k| \le& C\|\Lambda^\alpha u\|_{L^r}2^{(1-\alpha+\frac{2}{r})k}\|\Delta_k G\|_{L^r}^{r-1}\Big(\|\Delta_k G\|_{L^r} + \sum_{m\le k-1}2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G\|_{L^r}\\ &+\sum_{m\ge k-1}2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G\|_{L^r}\Big)\\ \le& C2^{(1-\alpha+\frac{2}{r})k}\|\Delta_k G\|_{L^r}^{r-1}\Big(\|\Delta_k G\|_{L^r} + \sum_{m\le k-1}2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G\|_{L^r}\\ &+\sum_{m\ge k-1}2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G\|_{L^r}\Big).\end{aligned} \tag{3.41}$$

Finally, it follows from the Bernstein inequality

$$\begin{aligned}|I_3^k| \le& C\|\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta\|_{L^r}\|\Delta_k G\|_{L^r}^{r-1}\\ \le& C2^{(\beta+1-\alpha)k}\|\Delta_k\theta\|_{L^r}\|\Delta_k G\|_{L^r}^{r-1}\\ \le& C2^{(\beta+1-\alpha)k}\|\Delta_k G\|_{L^r}^{r-1}.\end{aligned} \tag{3.42}$$

Inserting (3.40), (3.41) and (3.42) into (3.38) yields that for $\beta \ge 1-\alpha$

$$\begin{aligned}\frac{d}{dt}\|\Delta_k G(t)\|_{L^r} + c2^{\alpha k}\|\Delta_k G\|_{L^r} \le& C2^{2(1-\alpha)k} + C2^{(\beta+1-\alpha)k} + C2^{(1-\alpha+\frac{2}{r})k}L(t)\\ \le& C2^{(\beta+1-\alpha)k} + C2^{(1-\alpha+\frac{2}{r})k}L(t),\end{aligned} \tag{3.43}$$

where $L(t)$ is given by

$$\begin{aligned}L(t) \triangleq \|\Delta_k G(t)\|_{L^r} &+ \sum_{m\le k-1}2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G(t)\|_{L^r}\\ &+\sum_{m\ge k-1}2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G(t)\|_{L^r}.\end{aligned}$$

Integrating (3.43) in time yields

$$\begin{aligned}\|\Delta_k G(t)\|_{L^r} \leq & e^{-ct2^{\alpha k}}\|\Delta_k G_0\|_{L^r} + C2^{(\beta+1-2\alpha)k} \\ & + C2^{(1-\alpha+\frac{2}{r})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}} L(\tau)\, d\tau. \end{aligned} \tag{3.44}$$

Multiplying (3.44) by 2^{sk} with $s \leq 2\alpha - 1 - \beta$ and taking sup with respect to k, it is obvious to see that

$$\|G(t)\|_{B^s_{r,\infty}} \leq \|G_0\|_{B^s_{r,\infty}} + C + M_1 + M_2 + M_3,$$

where

$$M_1 = C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \|\Delta_k G(\tau)\|_{L^r}\, d\tau\Big),$$

$$M_2 = C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r})(m-k)} \|\Delta_m G(\tau)\|_{L^r}\, d\tau\Big),$$

$$M_3 = C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\geq k-1} 2^{(\alpha-\frac{2}{r})(k-m)} \|\Delta_m G(\tau)\|_{L^r}\, d\tau\Big).$$

Thanks to the condition $r > \frac{2}{2\alpha-1}$, we choose ϵ as

$$0 < \epsilon < \min\left\{2\alpha - 1 - \frac{2}{r},\ s\right\}.$$

By the Bernstein inequality and the convolution Young inequality, we can conclude

$$\begin{aligned} M_1 =& C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} 2^{(s-\epsilon)k} \|\Delta_k G(\tau)\|_{L^r}\, d\tau\Big) \\ \leq & C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \|G(\tau)\|_{B^{s-\epsilon}_{r,\infty}}\, d\tau\Big) \\ \leq & C \sup_{k\geq -1} 2^{-(2\alpha-1-\frac{2}{r}-\epsilon)k} \|G\|_{L^\infty(0,\,t;B^{s-\epsilon}_{r,\infty})} \\ \leq & C \sup_{k\geq -1} 2^{-(2\alpha-1-\frac{2}{r}-\epsilon)k} \|G\|_{L^\infty(0,\,T;B^{s-\epsilon}_{r,\infty})} \\ \leq & C\|G\|_{L^\infty(0,\,T;B^{s-\epsilon}_{r,\infty})}. \end{aligned}$$

Similarly, we may derive

$$\begin{aligned} M_2 =& C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r}-s+\epsilon)(m-k)} 2^{(s-\epsilon)m} \|\Delta_m G(\tau)\|_{L^r}\, d\tau\Big) \\ \leq & C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \Big(\sum_{m\leq k-1} 2^{(1+\frac{2}{r}-s+\epsilon)(m-k)}\Big) \|G(\tau)\|_{B^{s-\epsilon}_{r,\infty}}\, d\tau\Big) \end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\|G(\tau)\|_{B^{s-\epsilon}_{r,\infty}}\,d\tau\Big)\\
&\le C\|G\|_{L^\infty(0,T;B^{s-\epsilon}_{r,\infty})}.
\end{aligned}
$$

In terms of the term M_3, we can deduce that

$$
\begin{aligned}
M_3 =&C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\ge k-1}2^{(\alpha-\frac{2}{r}+s-\epsilon)(k-m)}2^{(s-\epsilon)m}\|\Delta_m G(\tau)\|_{L^r}\,d\tau\Big)\\
\le& C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\ge k-1}2^{(\alpha-\frac{2}{r}+s-\epsilon)(k-m)}\|G(\tau)\|_{B^{s-\epsilon}_{r,\infty}}\,d\tau\Big)\\
\le& C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r}+\epsilon)k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\|G(\tau)\|_{B^{s-\epsilon}_{r,\infty}}\,d\tau\Big)\\
\le& C\|G\|_{L^\infty(0,T;B^{s-\epsilon}_{r,\infty})}.
\end{aligned}
$$

Therefore, it follows that

$$
\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\le C+\|G_0\|_{B^s_{r,\infty}}+C\|G\|_{L^\infty(0,T;B^{s-\epsilon}_{r,\infty})}. \tag{3.45}
$$

Using the Bernstein inequality and (3.1), we have for $0<\epsilon<s$ that

$$
\begin{aligned}
C\|G\|_{B^{s-\epsilon}_{r,\infty}} \le& C\sup_{-1\le j\le L}2^{(s-\epsilon)j}\|\Delta_j G\|_{L^r}+C\sup_{j\ge L+1}2^{(s-\epsilon)j}\|\Delta_j G\|_{L^r}\\
\le& C\sup_{-1\le j\le L}2^{(s-\epsilon)j}2^{2j(\frac{1}{2}-\frac{1}{r})}\|\Delta_j G\|_{L^2}+C\sup_{j\ge L+1}2^{-j\epsilon}2^{js}\|\Delta_j G\|_{L^r}\\
\le& C\sup_{-1\le j\le L}2^{j(1+s-\epsilon-\frac{2}{r})}\|G\|_{L^2}+C\sup_{j\ge L+1}2^{-j\epsilon}\|G\|_{B^s_{r,\infty}}\\
\le& C2^{L(1+s-\epsilon-\frac{2}{r})}\|G\|_{L^2}+C2^{-L\epsilon}\|G\|_{B^s_{r,\infty}}\\
\le& C+\frac{1}{2}\|G\|_{B^s_{r,\infty}},
\end{aligned} \tag{3.46}
$$

where in the last line we have fixed L satisfying

$$
\frac{1}{4}\le C2^{-L\epsilon}\le\frac{1}{2}.
$$

Inserting (3.46) into (3.45) yields

$$
\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\le C+\|G_0\|_{B^s_{r,\infty}}+\frac{1}{2}\|G\|_{L^\infty(0,T;B^s_{r,\infty})}. \tag{3.47}
$$

Therefore, it follows from (3.47) that

$$
\sup_{0\le t\le T}\|G(t)\|_{B^s_{r,\infty}}\le C(T,u_0,\theta_0). \qquad 0<s\le 2\alpha-1-\beta.
$$

This completes the proof of Proposition 3.5. □

We are ready to prove Theorem 1.1.

Proof According to Proposition 3.4, G satisfies

$$\sup_{0\le t\le T} \|G(t)\|_{L^m} \le C(T, u_0, \theta_0),$$

where

$$m < \min\left\{\frac{8}{2-\alpha}, \frac{1}{1-\alpha}, \frac{2(2+\beta)}{(2+\alpha)\beta}, \frac{2(2\beta+3\alpha-2)}{(2-\alpha)\beta}\right\}.$$

This along with Proposition 3.5 implies

$$\sup_{0\le t\le T} \|G(t)\|_{B^{2\alpha-1-\beta}_{\frac{2m}{2-(1-\alpha)m},\infty}} \le C(T, u_0, \theta_0).$$

For $m > \frac{2}{\alpha}$ and $\sigma = 1+\alpha-\beta-\frac{2}{m} > 1-\beta$, we have

$$\begin{aligned}
\|u_G\|_{C^\sigma} &= \|\nabla^\perp \Delta^{-1} G\|_{C^\sigma} \\
&\approx \|\nabla^\perp \Delta^{-1} G\|_{B^\sigma_{\infty,\infty}} \\
&\le C\|G\|_{L^2} + C\|G\|_{B^{\sigma-1}_{\infty,\infty}} \\
&\le C\|G\|_{L^2} + C\|G\|_{B^{2\alpha-1-\beta}_{\frac{2m}{2-(1-\alpha)m},\infty}} \\
&\le C(T, u_0, \theta_0).
\end{aligned}$$

For u_θ, it is not hard to see that

$$\begin{aligned}
\|u_\theta\|_{C^\alpha} &= \|\nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta\|_{C^\alpha} \\
&\approx \|\nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta\|_{B^\alpha_{\infty,\infty}} \\
&\le C\|\theta\|_{L^2} + C\|\theta\|_{B^0_{\infty,\infty}} \\
&\le C\|\theta\|_{L^2} + C\|\theta\|_{L^\infty} \\
&\le C(T, u_0, \theta_0).
\end{aligned}$$

Letting $\gamma = \min\{\sigma, \alpha\} > 1-\beta$, the above two estimates allow us to conclude

$$\|u\|_{C^\gamma} \le \|u_G\|_{C^\gamma} + \|u_\theta\|_{C^\gamma} \le \|u_G\|_{C^\sigma} + \|u_\theta\|_{C^\alpha} \le C(T, u_0, \theta_0).$$

Now applying Lemma 2.6 to the θ-equation $(1.1)_2$, it implies that θ becomes immediately differentiable, namely for some positive constant ζ

$$\|\theta(t)\|_{C^{1,\zeta}(\mathbb{R}^2))} \le C(T, u_0, \theta_0),$$

which of course gives

$$\int_0^T \|\nabla\theta(t)\|_{L^\infty}\,dt \le C(T,\,u_0,\,\theta_0). \tag{3.48}$$

Moreover, we deduce from (1.2) that

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla\theta(\tau)\|_{L^\infty}\,d\tau. \tag{3.49}$$

Then (3.48) and (3.49) imply $(u,\,\theta)$ is the desired classical solution. Actually, the standard energy method allows us to derive

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\frac{\alpha}{2}}u\|_{L^2}^2 + \|\Lambda^{s+\frac{\beta}{2}}\theta\|_{L^2}^2\\ &\le C(1+\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).\end{aligned} \tag{3.50}$$

Recalling the following logarithmic Sobolev embedding inequality

$$\|\nabla u\|_{L^\infty} \le C\Big(1+\|u\|_{L^2}+\|\omega\|_{L^\infty}\ln\big(e+\|\Lambda^s u\|_{L^2}\big)\Big),$$

we deduce from (3.50) that

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\frac{\alpha}{2}}u\|_{L^2}^2 + \|\Lambda^{s+\frac{\beta}{2}}\theta\|_{L^2}^2\\ &\le C(1+\|\omega\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})\ln(e+\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).\end{aligned} \tag{3.51}$$

Keeping in mind (3.48) and (3.49), one obtains by applying the Log-Gronwall inequality to (3.51)

$$\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{s+\frac{\alpha}{2}}u\|_{L^2}^2 + \|\Lambda^{s+\frac{\beta}{2}}\theta\|_{L^2}^2)(\tau)\,d\tau \le C(t,\,u_0,\,\theta_0).$$

This completes the proof of Theorem 1.1. □

4 The proof of Theorem 1.3

The main effort of this section is devoted to the proof of Theorem 1.3. It suffices to consider the case $0<\alpha<\frac{2}{3}$ as the case $\beta\ge\alpha=\frac{2}{3}$ can be handled without using the following combined quantity (see the end of this section). First, we define the combined quantity

$$G_1 = \omega + \mathcal{R}_\beta\theta, \quad \mathcal{R}_\beta \triangleq \partial_{x_1}\Lambda^{-\beta}.$$

Apply the integral operator $\mathcal{R}_\beta$ to θ-equation to obtain

$$\partial_t \mathcal{R}_\beta \theta + (u \cdot \nabla) \mathcal{R}_\beta \theta + \partial_{x_1} \theta = -[\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{4.1}$$

Combining (1.2) and (4.1) yields

$$\partial_t G_1 + (u \cdot \nabla) G_1 + \Lambda^\alpha G_1 = \Lambda^{\alpha-\beta} \partial_{x_1} \theta - [\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{4.2}$$

Note that the singularity of $\Lambda^{\alpha-\beta}\partial_{x_1}\theta$ at the right hand side of (4.2) seems to be higher. In fact, to control the term $\Lambda^{\alpha-\beta}\partial_{x_1}\theta$, it requires $\beta \geq \frac{2+\alpha}{4}$, which is stronger than $\beta > \alpha$ when $\alpha < \frac{2}{3}$. Therefore, we naturally weaken this singularity. Precisely, we need the iterative method. Actually, we apply $\Lambda^{\alpha-2\beta}\partial_{x_1}$ to θ-equation to obtain

$$\partial_t \Lambda^{\alpha-2\beta}\partial_{x_1}\theta + (u \cdot \nabla)\Lambda^{\alpha-2\beta}\partial_{x_1}\theta + \Lambda^{\alpha-\beta}\partial_{x_1}\theta = -[\Lambda^{\alpha-2\beta}\partial_{x_1}, u \cdot \nabla]\theta. \tag{4.3}$$

Setting $G_2 = G_1 + \Lambda^{\alpha-2\beta}\partial_{x_1}\theta$, one deduces from (4.2) and (4.3) that

$$\partial_t G_2 + (u \cdot \nabla) G_2 + \Lambda^\alpha G_2 = \Lambda^{2(\alpha-\beta)}\partial_{x_1}\theta - [\mathcal{R}_\beta, u \cdot \nabla]\theta - [\Lambda^{\alpha-\beta}\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{4.4}$$

Applying $\Lambda^{2\alpha-3\beta}\partial_{x_1}$ to θ-equation, one gets

$$\partial_t \Lambda^{2\alpha-3\beta}\partial_{x_1}\theta + (u \cdot \nabla)\Lambda^{2\alpha-3\beta}\partial_{x_1}\theta + \Lambda^{2(\alpha-\beta)}\partial_{x_1}\theta = -[\Lambda^{2\alpha-3\beta}\partial_{x_1}, u \cdot \nabla]\theta. \tag{4.5}$$

Denoting $G_3 = G_2 + \Lambda^{2\alpha-3\beta}\partial_{x_1}\theta$, we deduce from (4.4) and (4.5) that

$$\begin{aligned}\partial_t G_3 + (u \cdot \nabla) G_3 + \Lambda^\alpha G_3 =& \Lambda^{3(\alpha-\beta)}\partial_{x_1}\theta - [\mathcal{R}_\beta, u \cdot \nabla]\theta - [\Lambda^{\alpha-\beta}\mathcal{R}_\beta, u \cdot \nabla] \\ &- [\Lambda^{2(\alpha-\beta)}\mathcal{R}_\beta, u \cdot \nabla]\theta.\end{aligned}$$

Repeating the same arguments above, we are able to conclude that there exist consequence $\{G_m\}_{m\in\mathbb{N}}$ such that

$$\partial_t G_m + (u \cdot \nabla) G_m + \Lambda^\alpha G_m = \Lambda^{m(\alpha-\beta)}\partial_{x_1}\theta - [\mathcal{R}_\beta, u \cdot \nabla]\theta - f_m, \tag{4.6}$$

where G_m and f_m are given by

$$G_m = \omega + \sum_{l=1}^{m} \Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta, \qquad f_m = \sum_{l=1}^{m-1} [\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta, u \cdot \nabla]\theta.$$

Due to $\beta > \alpha$, we choose the unique integer $k \geq 1$ such that

$$\frac{2-\alpha-2\beta}{2(\beta-\alpha)} < k \leq \frac{2-3\alpha}{2(\beta-\alpha)},$$

which ensures

$$(k-1)\alpha - k\beta + 1 \geq 0, \quad 0 \leq k(\alpha-\beta)+1 < \beta + \frac{\alpha}{2}, \quad (k-1)(\alpha-\beta)+1 \geq \beta + \frac{\alpha}{2}.$$

Now we denote

$$G = \omega + \sum_{l=1}^{k} \Lambda^{(l-1)\alpha - l\beta} \partial_{x_1}\theta, \qquad f = \sum_{l=1}^{k-1} [\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\, u\cdot\nabla]\theta,$$

then it follows from (4.6) that

$$\partial_t G + (u\cdot\nabla)G + \Lambda^\alpha G = \Lambda^{k(\alpha-\beta)}\partial_{x_1}\theta - [\mathcal{R}_\beta,\, u\cdot\nabla]\theta - f. \tag{4.7}$$

Since u is determined by ω via the Biot-Savart law, we have

$$\begin{aligned} u &= \nabla^\perp\Delta^{-1}\omega \\ &= \nabla^\perp\Delta^{-1}\left(G - \sum_{l=1}^{k}\Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta\right) \\ &= \nabla^\perp\Delta^{-1}G - \sum_{l=1}^{k}\nabla^\perp\Delta^{-1}\Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta \\ &\triangleq u_G + \sum_{l=1}^{k} u_\theta^{(l)}. \end{aligned} \tag{4.8}$$

Roughly, the terms at the right hand side of (4.8) can be viewed as

$$u_G \approx \Lambda^{-1}G, \quad u_\theta^{(l)} \approx \Lambda^{l(\alpha-\beta)-\alpha}\theta.$$

Moreover, it holds

$$\|u_\theta^{(l)}\|_{L^p} \leq \|u_\theta^{(1)}\|_{L^p} + \|u_\theta^{(k)}\|_{L^p}, \quad 1 \leq l \leq k.$$

Now we establish the following commutator estimate involving $\mathcal{R}_\beta$.

Lemma 4.1 *Let $r \in [1,\infty]$ and $p,\, p_1,\, p_2 \in (1,\infty)$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Assume that $\epsilon \in [0,1)$ and $\beta \in (0,1)$ satisfy $s \in (-1, \beta-\epsilon)$. If f is a divergence-free vector field, then it holds true*

$$\|[\mathcal{R}_\beta,\, f\cdot\nabla]g\|_{\dot{B}^s_{p,r}} \leq C\|\Lambda^{1-\epsilon}f\|_{L^{p_1}}\|g\|_{\dot{B}^{s+1+\epsilon-\beta}_{p_2,r}}. \tag{4.9}$$

Remark 4.1 Note that one does not necessarily need precisely the form $\mathcal{R}_\beta$ of (4.9). In fact, the estimate applies for any Fourier multiplier Γ such that its symbol $\widehat{\Gamma}(\xi)$ is a homogeneous function of degree $1-\beta$ and $\widehat{\Gamma}(\xi) \in C^\infty(\mathbb{S}^{n-1})$, for example $\Gamma = \Lambda^{1-\beta}$.

Proof According to the Bony decomposition, we have

$$\begin{aligned}\dot\Delta_k[\mathcal{R}_\beta, f\cdot\nabla]g &= \sum_{|j-k|\le 4}\dot\Delta_k\Big([\mathcal{R}_\beta,\ \dot S_{j-1}f\cdot\nabla]\dot\Delta_j g\Big)\\ &\quad+\sum_{|j-k|\le 4}\dot\Delta_k\Big([\mathcal{R}_\beta,\ \dot\Delta_j f\cdot\nabla]\dot S_{j-1}g\Big)\\ &\quad+\sum_{j-k\ge -4}\dot\Delta_k\Big([\mathcal{R}_\beta,\ \dot\Delta_j f\cdot\nabla]\widetilde{\dot\Delta}_j g\Big)\\ &\triangleq \widetilde N_1+\widetilde N_2+\widetilde N_3.\end{aligned}\tag{4.10}$$

Notice that for fixed k, the summation over $|j-k|\le 4$ involves only a finite number of $j's$. For the sake of simplicity, we shall replace the summations by their representative term with $j=k$ in $\widetilde N_1$ and $\widetilde N_2$. Notice that if $\mathcal{Z}$ is an annulus centered at the origin, then for every F with spectrum supported on $2^j\mathcal{Z}$, there exists $\eta\in\mathcal{S}(\mathbb{R}^2)$ whose Fourier transform supported away from the origin, such that

$$\mathcal{R}_\beta F = 2^{j(3-\beta)}\eta(2^j.)\star F.$$

Based on this observation, we deduce from [42, Proposition A.3] and the Bernstein inequality that

$$\begin{aligned}\|\widetilde N_1\|_{L^p} &\le C\|x2^{k(3-\beta)}\eta(2^kx)\|_{L^1}\|\nabla\dot S_{k-1}f\|_{L^{p_1}}\|\dot\Delta_k\nabla g\|_{L^{p_2}}\\ &\le C\|x2^{k(3-\beta)}\eta(2^kx)\|_{L^1}\|\Lambda\dot S_{k-1}f\|_{L^{p_1}}\|\dot\Delta_k\nabla g\|_{L^{p_2}}\\ &\le C\|x2^{k(3-\beta)}\eta(2^kx)\|_{L^1}2^{k\epsilon}\|\dot S_{k-1}\Lambda^{1-\epsilon}f\|_{L^{p_1}}\|\dot\Delta_k\nabla g\|_{L^{p_2}}\\ &\le C2^{k(1-\beta+\epsilon)}\|\Lambda^{1-\epsilon}f\|_{L^{p_1}}\|\dot\Delta_k g\|_{L^{p_2}}.\end{aligned}$$

Similarly, one gets

$$\begin{aligned}\|\widetilde N_2\|_{L^p} &\le C\|x2^{k(3-\beta)}\eta(2^kx)\|_{L^1}\|\dot\Delta_k\nabla f\|_{L^{p_1}}\|\dot S_{k-1}\nabla g\|_{L^{p_2}}\\ &\le C2^{k(\epsilon-\beta)}\|\Lambda^{1-\epsilon}f\|_{L^{p_1}}\sum_{l\le k-2}\|\dot\Delta_l\nabla g\|_{L^{p_2}}\\ &\le C\|\Lambda^{1-\epsilon}f\|_{L^{p_1}}\sum_{l\le k-2}2^{(k-l)(\epsilon-\beta)}2^{l(1+\epsilon-\beta)}\|\dot\Delta_l g\|_{L^{p_2}}.\end{aligned}$$

In view of $\nabla\cdot f=0$, the term $\widetilde N_3$ can be rewritten as

$$\widetilde N_3=\sum_{j-k\ge -4}\dot\Delta_k\nabla\cdot\Big(\mathcal{R}_\beta(\dot\Delta_j f\widetilde{\dot\Delta}_j g)-\Delta_j f\mathcal{R}_\beta\widetilde{\dot\Delta}_j g\Big).$$

We thus derive that

$$\begin{aligned}
\|\widetilde{N}_3\|_{L^p} \leq & C \sum_{j-k\geq -4} 2^k\Big(\Big\|\dot{\Delta}_k\big(\mathcal{R}_\beta(\dot{\Delta}_j f\widetilde{\dot{\Delta}}_j g)\big)\Big\|_{L^p} + \Big\|\dot{\Delta}_k\big(\dot{\Delta}_j f\mathcal{R}_\beta\widetilde{\dot{\Delta}}_j g\big)\Big\|_{L^p}\Big)\\
\leq & C \sum_{j-k\geq -4} 2^k\Big(\Big\|\mathcal{R}_\beta(\dot{\Delta}_j f\widetilde{\dot{\Delta}}_j g)\Big\|_{L^p} + \Big\|\dot{\Delta}_j f\mathcal{R}_\beta\widetilde{\dot{\Delta}}_j g\Big\|_{L^p}\Big)\\
\leq & C \sum_{j-k\geq -4} 2^k 2^{j(\epsilon-\beta)}\|\dot{\Delta}_j\Lambda^{1-\epsilon} f\|_{L^{p_1}}\|\dot{\Delta}_j g\|_{L^{p_2}}\\
\leq & C \sum_{j-k\geq -4} 2^k 2^{j(\epsilon-\beta)}\|\Lambda^{1-\epsilon} f\|_{L^{p_1}}\|\dot{\Delta}_j g\|_{L^{p_2}}.
\end{aligned}$$

Putting all the above estimates into (4.10) and using the definition of $\dot{B}^s_{p,r}$, we are able to show

$$\begin{aligned}
\|[\mathcal{R}_\beta, f\cdot\nabla]g\|_{\dot{B}^s_{p,r}} \leq & \Big\|2^{ks}\|\widetilde{N}_1\|_{L^p}\Big\|_{l^r_k} + \Big\|2^{ks}\|\widetilde{N}_2\|_{L^p}\Big\|_{l^r_k} + \Big\|2^{ks}\|\widetilde{N}_3\|_{L^p}\Big\|_{l^r_k}\\
\leq & C\|\Lambda^{1-\epsilon} f\|_{L^{p_1}}\Big\|2^{k(s+1-\beta+\epsilon)}\|\dot{\Delta}_k g\|_{L^{p_2}}\Big\|_{l^r_k}\\
& + C\|\Lambda^{1-\epsilon} f\|_{L^{p_1}}\Big\|\sum_{l\leq k-2} 2^{(k-l)(s+\epsilon-\beta)}2^{l(s+1+\epsilon-\beta)}\|\dot{\Delta}_l g\|_{L^{p_2}}\Big\|_{l^r_k}\\
& + C\|\Lambda^{1-\epsilon} f\|_{L^{p_1}}\Big\|\sum_{j-k\geq -4} 2^{(k-j)(s+1)}2^{j(s+1+\epsilon-\beta)}\|\dot{\Delta}_j g\|_{L^{p_2}}\Big\|_{l^r_k}\\
\leq & C\|\Lambda^{1-\epsilon} f\|_{L^{p_1}}\|g\|_{\dot{B}^{s+1+\epsilon-\beta}_{p_2,r}},
\end{aligned}$$

where we have used $s\in(-1,\beta-\epsilon)$. Therefore, the desired bound (4.9) holds true. □

With (4.9) in hand, we are now in the position to derive the following estimate involving G and θ.

Lemma 4.2 *If $\alpha,\beta\in(0,1)$ satisfy (1.8), then the following estimate holds*

$$\|G(t)\|^2_{L^2} + \|\Lambda^{\frac{\beta}{2}}\theta(t)\|^2_{L^2} + \int_0^t \big(\|\Lambda^{\frac{\alpha}{2}}G(\tau)\|^2_{L^2} + \|\Lambda^\beta\theta(\tau)\|^2_{L^2}\big)\,d\tau \leq C(t, u_0, \theta_0). \tag{4.11}$$

Proof Recalling (3.3) and (3.5), we may conclude

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{\beta}{2}}\theta(t)\|^2_{L^2} + \|\Lambda^\beta\theta\|^2_{L^2}\\
&\quad = -\int_{\mathbb{R}^2}\Lambda^{\frac{\beta}{2}}\big(u\cdot\nabla\theta\big)\Lambda^{\frac{\beta}{2}}\theta\,dx\\
&\quad = -\int_{\mathbb{R}^2}[\Lambda^{\frac{\beta}{2}}, u\cdot\nabla]\theta\ \Lambda^{\frac{\beta}{2}}\theta\,dx\\
&\quad \leq C\|[\Lambda^{\frac{\beta}{2}}, u\cdot\nabla]\theta\|_{L^{\frac{4}{3}}}\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^4}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla u\|_{L^2}\|\Lambda^{\frac{\beta}{2}}\theta\|_{L^4}^2\\
&\leq C\|\nabla u\|_{L^2}\|\Lambda^{\frac{\beta}{2}}\theta\|_{\dot{B}^{-\frac{\beta}{2}}_{\infty,\infty}}\|\Lambda^{\frac{\beta}{2}}\theta\|_{\dot{B}^{\frac{\beta}{2}}_{2,2}}\\
&\leq C\|\omega\|_2\|\theta\|_{\dot{B}^0_{\infty,\infty}}\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq C\left(\|G\|_2+\sum_{l=1}^{k}\|\Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta\|_2\right)\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq C\left(\|G\|_2+\sum_{l=1}^{k}\|\Lambda^{l(\alpha-\beta)+1-\alpha}\theta\|_2\right)\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq C\left(\|G\|_2+\|\Lambda^{1-\beta}\theta\|_2+\|\Lambda^{k(\alpha-\beta)+1-\alpha}\theta\|_2\right)\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq C\left(\|G\|_2+\|\Lambda^{1-\beta}\theta\|_2+\|\theta\|_2\right)\|\theta\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq C(\|\theta\|_2+\|G\|_{L^2}+\|\theta\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^{\beta}\theta\|_{L^2}^{\frac{1-\beta}{\beta}})\|\Lambda^{\beta}\theta\|_{L^2}\\
&\leq \frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C(1+\|G\|_{L^2}^2),
\end{aligned}
$$

where we have used $\beta > \frac{1}{2}$ due to $\beta > \frac{4-\alpha^2}{4+3\alpha}$ with $0 < \alpha \leq \frac{2}{3}$. As a result, it follows that

$$
\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2+\frac{7}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2 \leq C(1+\|G\|_{L^2}^2). \tag{4.12}
$$

To close (4.12), it suffices to estimate $\|G\|_{L^2}$. To this end, multiplying (4.7) by G and using (4.8), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 &= \int_{\mathbb{R}^2}\Lambda^{k(\alpha-\beta)}\partial_{x_1}\theta\ G\,dx-\int_{\mathbb{R}^2}[\mathcal{R}_\beta,\ u\cdot\nabla]\theta\ G\,dx\\
&\quad-\sum_{l=1}^{k-1}\int_{\mathbb{R}^2}[\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\ u\cdot\nabla]\theta\ G\,dx\\
&=\int_{\mathbb{R}^2}\Lambda^{k(\alpha-\beta)}\partial_{x_1}\theta\ G\,dx-\int_{\mathbb{R}^2}[\mathcal{R}_\beta,\ u_G\cdot\nabla]\theta\ G\,dx\\
&\quad-\sum_{m=1}^{k}\int_{\mathbb{R}^2}[\mathcal{R}_\beta,\ u_\theta^{(m)}\cdot\nabla]\theta\ G\,dx\\
&\quad-\sum_{l=1}^{k-1}\int_{\mathbb{R}^2}[\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\ u_G\cdot\nabla]\theta\ G\,dx\\
&\quad-\sum_{l=1}^{k-1}\sum_{m=1}^{k}\int_{\mathbb{R}^2}[\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\ u_\theta^{(m)}\cdot\nabla]\theta\ G\,dx.
\end{aligned} \tag{4.13}
$$

By means of the interpolation inequality, it yields

$$
\begin{aligned}
\int_{\mathbb{R}^2} \Lambda^{k(\alpha-\beta)}\partial_{x_1}\theta \; G\,dx \le& C\|\Lambda^{k(\alpha-\beta)+1-\frac{\alpha}{2}}\theta\|_{L^2}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\le& C\|\theta\|_{L^2}^{1-\frac{k(\alpha-\beta)+1-\frac{\alpha}{2}}{\beta}}\|\Lambda^{\beta}\theta\|_{L^2}^{\frac{k(\alpha-\beta)+1-\frac{\alpha}{2}}{\beta}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\le& \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\frac{1}{32}\|\Lambda^{\beta}\theta\|_{L^2}^2+C\|\theta\|_{L^2}^2\\
\le& \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\frac{1}{32}\|\Lambda^{\beta}\theta\|_{L^2}^2+C.
\end{aligned}
$$

According to (2.22) and (2.23) of [42], the following estimate is valid as long as $\beta > \frac{4-\alpha^2}{4+3\alpha}$,

$$
-\int_{\mathbb{R}^2}[\mathcal{R}_\beta,\, u_G\cdot\nabla]\theta \; G\,dx \le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C(1+\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}})\|G\|_{L^2}^2,
$$

where $0<\gamma<\frac{1}{2}$ and

$$
\|\Lambda^{\gamma\beta}\theta\|_{L_t^{\frac{1}{\gamma}}L_x^{\frac{1}{\gamma}}} \le C\|\Lambda^{\frac{\beta}{2}}\theta\|_{L_t^2L_x^2}^{2\gamma}\|\theta\|_{L_t^\infty L_x^\infty}^{1-2\gamma} < \infty. \tag{4.14}
$$

We also point out that this is the only place in the proof where we use $\beta > \frac{4-\alpha^2}{4+3\alpha}$. Due to $\beta>\alpha$, modifying the proof of (2.22) and (2.23) of [42], we are able to show

$$
-\sum_{l=1}^{k-1}\int_{\mathbb{R}^2}[\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\, u_G\cdot\nabla]\theta \; G\,dx \le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2 + C(1+\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}})\|G\|_{L^2}^2.
$$

Using (4.9) with $s=-\frac{\alpha}{2},\ \epsilon=\frac{\alpha}{4},\ p=r=2$, one has

$$
\begin{aligned}
-\int_{\mathbb{R}^2}[\mathcal{R}_\beta,\, u_\theta^{(1)}\cdot\nabla]\theta \; G\,dx \le& \|[\mathcal{R}_\beta,\, u_\theta^{(1)}\cdot\nabla]\theta\|_{\dot B_{2,2}^{-\frac{\alpha}{2}}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\le& C\|\Lambda^{1-\frac{\alpha}{4}}u_\theta^{(1)}\|_{L^4}\|\theta\|_{\dot B_{4,2}^{1-\beta-\frac{\alpha}{4}}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\le& C\|\Lambda^{1-\beta-\frac{\alpha}{4}}\theta\|_{L^4}\|\theta\|_{\dot B_{4,2}^{1-\beta-\frac{\alpha}{4}}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\le& C\|\Lambda^{1-\beta-\frac{\alpha}{4}}\theta\|_{\dot B_{4,2}^{0}}\|\theta\|_{\dot B_{4,2}^{1-\beta-\frac{\alpha}{4}}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}\\
\approx& \|\theta\|_{\dot B_{4,2}^{1-\beta-\frac{\alpha}{4}}}^2\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2},
\end{aligned} \tag{4.15}
$$

where we have used the embedding $\dot{B}^0_{4,2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$. We may assume $\beta \le 1-\frac{\alpha}{4}$, otherwise we immediately have

$$\|\theta\|_{\dot{B}^{1-\beta-\frac{\alpha}{4}}_{4,2}} \le C(\|\theta\|_{L^2} + \|\theta\|_{L^4}) \le C.$$

Making use of the fractional type Gagliardo-Nirenberg inequality (see [19]), we are able to derive

$$\begin{aligned}\|\theta\|_{\dot{B}^{1-\beta-\frac{\alpha}{4}}_{4,2}} &\le C\|\theta\|^{\frac{1}{2}}_{\dot{B}^{-\kappa}_{\infty,2}} \|\theta\|^{\frac{1}{2}}_{\dot{B}^{\kappa+2-2\beta-\frac{\alpha}{2}}_{2,2}} \\ &\le C(\|\theta\|_{L^2} + \|\theta\|_{L^\infty})^{\frac{1}{2}} \|\theta\|^{\frac{1-\tau}{2}}_{L^2} \|\Lambda^\beta\theta\|^{\frac{\tau}{2}}_{L^2},\end{aligned} \tag{4.16}$$

where

$$\tau = \frac{4+2\kappa-\alpha-4\beta}{2\beta} \in (0,1)$$

by selecting κ as

$$\max\left\{0,\ \frac{\alpha}{2}+2\beta-2\right\} < \kappa < \min\left\{1,\ \frac{\alpha}{2}+3\beta-2\right\}.$$

To ensure the existence of κ, we need $\beta > \frac{4-\alpha}{6}$. Obviously, we have $\frac{4-\alpha^2}{4+3\alpha} > \frac{4-\alpha}{6}$ when $0 < \alpha \le \frac{2}{3}$. Now inserting (4.16) into (4.15), it ensures

$$\begin{aligned}-\int_{\mathbb{R}^2} [\mathcal{R}_\beta,\ u^{(1)}_\theta \cdot \nabla]\theta\ G\,dx &\le \|\theta\|^2_{\dot{B}^{1-\beta-\frac{\alpha}{4}}_{4,2}} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2} \\ &\le C(\|\theta\|_{L^2} + \|\theta\|_{L^\infty})\|\theta\|^{1-\tau}_{L^2} \|\Lambda^\beta\theta\|^{\tau}_{L^2} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2} \\ &\le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \frac{1}{32}\|\Lambda^\beta\theta\|^2_{L^2} \\ &\quad + C(\|\theta\|_{L^2} + \|\theta\|_{L^\infty})^{\frac{2}{1-\tau}} \|\theta\|^2_{L^2} \\ &\le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \frac{1}{32}\|\Lambda^\beta\theta\|^2_{L^2} + C.\end{aligned} \tag{4.17}$$

Thanks to $\beta > \alpha$, it follows from the same argument in proving (4.17) that

$$\begin{aligned}&-\sum_{m=2}^{k} \int_{\mathbb{R}^2} [\mathcal{R}_\beta,\ u^{(m)}_\theta \cdot \nabla]\theta\ G\,dx \le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \frac{1}{32}\|\Lambda^\beta\theta\|^2_{L^2} + C, \\ &-\sum_{l=1}^{k-1}\sum_{m=1}^{k} \int_{\mathbb{R}^2} [\Lambda^{l(\alpha-\beta)}\mathcal{R}_\beta,\ u^{(m)}_\theta \cdot \nabla]\theta\ G\,dx \le \frac{1}{32}\|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \frac{1}{32}\|\Lambda^\beta\theta\|^2_{L^2} + C.\end{aligned}$$

Inserting the above estimates into (4.13), we infer

$$\frac{1}{2}\frac{d}{dt}\|G(t)\|_{L^2}^2+\frac{5}{8}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2\le\frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2+C(1+\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}})(1+\|G\|_{L^2}^2). \tag{4.18}$$

Summarizing (4.12) and (4.18), one derives

$$\frac{d}{dt}(\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2)+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2+\|\Lambda^{\beta}\theta\|_{L^2}^2\le C(1+\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}})(1+\|G\|_{L^2}^2).$$

It thus follows from the Gronwall inequality and (4.14) that

$$\|G(t)\|_{L^2}^2+\|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^2+\int_0^t\big(\|\Lambda^{\frac{\alpha}{2}}G(\tau)\|_{L^2}^2+\|\Lambda^{\beta}\theta(\tau)\|_{L^2}^2\big)\,d\tau\le C(t,\,u_0,\,\theta_0).$$

We therefore complete the proof of Lemma 4.2. □

With the help of (4.11), we are able to improve the regularity estimate of θ. Here we mention that $\beta\le\frac{2}{3}$ is our main target as the case $\beta>\frac{2}{3}$ was already considered in [42]. In this sense, the regularity of θ in (4.19) is higher than of θ in (4.11). This improved regularity estimate (4.19) is crucial for us to derive the global L^2-bound of the vorticity.

Lemma 4.3 *If* $\alpha,\beta\in(0,1)$ *satisfy (1.8), then the following estimate holds*

$$\|\Lambda^{1-\beta}\theta(t)\|_{L^2}^2+\int_0^t\|\Lambda^{1-\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2\,d\tau\le C(t,\,u_0,\,\theta_0). \tag{4.19}$$

Proof Applying $\Lambda^{1-\beta}$ to $(1.1)_2$ and multiplying the resultant by $\Lambda^{1-\beta}\theta$, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{1-\beta}\theta(t)\|_{L^2}^2+\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2&=-\int_{\mathbb{R}^2}\Lambda^{1-\beta}\big(u\cdot\nabla\theta\big)\Lambda^{1-\beta}\theta\,dx\\&=-\int_{\mathbb{R}^2}[\Lambda^{1-\beta},u\cdot\nabla]\theta\;\Lambda^{1-\beta}\theta\,dx\\&=-\int_{\mathbb{R}^2}[\Lambda^{1-\beta},u_G\cdot\nabla]\theta\;\Lambda^{1-\beta}\theta\,dx\\&\quad-\sum_{m=1}^{k}\int_{\mathbb{R}^2}[\Lambda^{1-\beta},u_\theta^{(m)}\cdot\nabla]\theta\;\Lambda^{1-\beta}\theta\,dx\\&\triangleq M_1+M_2.\end{aligned} \tag{4.20}$$

Making use of (2.3) and (3.6), it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}[\Lambda^{1-\beta}, u_\theta^{(1)}\cdot\nabla]\theta\ \Lambda^{1-\beta}\theta\,dx\right| &\le C\|[\Lambda^{1-\beta}, u_\theta^{(1)}\cdot\nabla]\theta\|_{L^{\frac{3}{2}}}\|\Lambda^{1-\beta}\theta\|_{L^3}\\ &\le C\|\nabla u_\theta^{(1)}\|_{L^3}\|\Lambda^{1-\beta}\theta\|_{L^3}^2\\ &\le C\|\Lambda^{1-\beta}\theta\|_{L^3}^3\\ &\le C\|\Lambda^{1-\beta}\theta\|_{\dot B^{-(1-\beta)}_{\infty,\infty}}\|\Lambda^{1-\beta}\theta\|^2_{\dot B^{\frac{1-\beta}{2}}_{2,2}}\\ &\le C\|\theta\|_{\dot B^0_{\infty,\infty}}\|\Lambda^{\frac{3(1-\beta)}{2}}\theta\|_{L^2}^2\\ &\le C\|\theta\|_{L^\infty}\|\theta\|_{L^2}^{\frac{2(2\beta-1)}{2-\beta}}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^{\frac{6(1-\beta)}{2-\beta}}\\ &\le \frac{1}{8}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^{\frac{2-\beta}{2\beta-1}}\|\theta\|_{L^2}^2\\ &\le \frac{1}{8}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C, \end{aligned}\tag{4.21}$$

where $\beta > \frac{1}{2}$ was used. Similar to (4.21), we also get

$$\left|\sum_{m=2}^{k}\int_{\mathbb{R}^2}[\Lambda^{1-\beta}, u_\theta^{(m)}\cdot\nabla]\theta\ \Lambda^{1-\beta}\theta\,dx\right| \le \frac{1}{8}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C. \tag{4.22}$$

A direct consequence of (4.21) and (4.22) is

$$|M_2| \le \frac{1}{4}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C. \tag{4.23}$$

By (2.3) and (3.6) again, we get

$$\begin{aligned}|M_1| &\le C\|[\Lambda^{1-\beta}, u_G\cdot\nabla]\theta\|_{L^{\frac{2p}{p+1}}}\|\Lambda^{1-\beta}\theta\|_{L^{\frac{2p}{p-1}}}\\ &\le C\|\nabla u_G\|_{L^p}\|\Lambda^{1-\beta}\theta\|^2_{L^{\frac{2p}{p-1}}}\\ &\le C\|G\|_{L^p}\|\Lambda^{1-\beta}\theta\|^2_{L^{\frac{2p}{p-1}}}\\ &\le C\|G\|_{L^2}^{\frac{4-(2-\alpha)p}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{\frac{2(p-2)}{\alpha p}}\|\Lambda^{1-\beta}\theta\|^{\frac{2}{p}}_{\dot B^{-\frac{\beta(p-1)}{2}}_{\infty,\infty}}\|\Lambda^{1-\beta}\theta\|^{\frac{2p-2}{p}}_{\dot B^{\frac{\beta}{2}}_{2,2}}\\ &\le C\|G\|_{L^2}^{\frac{4-(2-\alpha)p}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{\frac{2(p-2)}{\alpha p}}\|\theta\|^{\frac{2}{p}}_{\dot B^{1-\beta-\frac{\beta(p-1)}{2}}_{\infty,\infty}}\|\Lambda^{1-\beta}\theta\|^{\frac{2p-2}{p}}_{\dot B^{\frac{\beta}{2}}_{2,2}}\\ &\le C\|G\|_{L^2}^{\frac{4-(2-\alpha)p}{\alpha p}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{\frac{2(p-2)}{\alpha p}}(\|\theta\|_{L^2}+\|\theta\|_{L^\infty})^{\frac{2}{p}}\|\Lambda^{1-\beta}\theta\|^{\frac{2p-2}{p}}_{\dot B^{\frac{\beta}{2}}_{2,2}}\end{aligned}$$

$$\leq \frac{1}{4}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C\|G\|_{L^2}^{\frac{4-(2-\alpha)p}{\alpha}}\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^{\frac{2(p-2)}{\alpha}}(\|\theta\|_{L^2}+\|\theta\|_{L^\infty})^2$$
$$\leq \frac{1}{4}\|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 + C(1+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2), \tag{4.24}$$

where p satisfies

$$\max\left\{2,\ \frac{2-\beta}{\beta}\right\} \leq p \leq \min\left\{2+\alpha,\ \frac{4-\beta}{\beta}\right\}.$$

In order to ensure the existence of such p, we need

$$\beta \geq \frac{2}{3+\alpha}.$$

Obviously, we have $\frac{4-\alpha^2}{4+3\alpha} > \frac{2}{3+\alpha}$ when $0 < \alpha \leq \frac{2}{3}$. We finally get by putting (4.23) and (4.24) into (4.20)

$$\frac{d}{dt}\|\Lambda^{1-\beta}\theta(t)\|_{L^2}^2 + \|\Lambda^{1-\frac{\beta}{2}}\theta\|_{L^2}^2 \leq C(1+\|\Lambda^{\frac{\alpha}{2}}G\|_{L^2}^2). \tag{4.25}$$

Integrating (4.25) in time and using (4.11), we immediately derive (4.19). Therefore, this completes the proof of Lemma 4.3. □

We now briefly sketch the proof of Theorem 1.3.

Proof Recalling

$$G = \omega + \sum_{l=1}^{k}\Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta,$$

it is clear from (4.11) and (4.19) that

$$\begin{aligned}\|\omega\|_{L^2} &\leq \|G\|_{L^2} + \sum_{l=1}^{k}\|\Lambda^{(l-1)\alpha-l\beta}\partial_{x_1}\theta\|_{L^2}\\ &\leq \|G\|_{L^2} + \sum_{l=1}^{k}\|\Lambda^{l(\alpha-\beta)+1-\alpha}\theta\|_{L^2}\\ &\leq \|G\|_{L^2} + C(\|\Lambda^{1-\beta}\theta\|_{L^2} + \|\Lambda^{k(\alpha-\beta)+1-\alpha}\theta\|_{L^2})\\ &\leq \|G\|_{L^2} + C(\|\Lambda^{1-\beta}\theta\|_{L^2} + \|\theta\|_{L^2})\\ &\leq C(t,\ u_0,\ \theta_0),\end{aligned}$$

which implies for any $2 \leq r < \infty$

$$\|u(t)\|_{L^r} \leq C(r)\|u\|_{L^2}^{\frac{2}{r}}\|\omega\|_{L^2}^{1-\frac{2}{r}} \leq C(t,\ u_0,\ \theta_0). \tag{4.26}$$

Moreover, one may check that

$$\max\left\{\alpha,\ \frac{4-\alpha^2}{4+3\alpha}\right\} \geq \frac{1}{1+\alpha}. \tag{4.27}$$

Keeping in mind (4.26) and (4.27), the following key bound is an easy consequence of [42, Lemma 2.12]

$$\|\nabla\theta(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty} \leq C(t,\ u_0,\ \theta_0). \tag{4.28}$$

With (4.28) in hand, the remainder proof of Theorem 1.3 is the same as the proof for Theorem 1.1. We thus complete the proof of Theorem 1.3.

Finally, let us show the global regularity of (1.1) with $\alpha = \beta \geq \frac{2}{3}$, namely

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda^\alpha u + \nabla p = \theta e_2, \qquad x \in \mathbb{R}^2,\ \ t > 0, \\ \partial_t\theta + (u\cdot\nabla)\theta + \kappa\Lambda^\alpha\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.29}$$

where μ and κ are two positive constants. As stated in introduction, it suffices to consider the case $\alpha = \frac{2}{3}$ as the case $\alpha > \frac{2}{3}$ is more easier to deal with. It should be pointed out that the combined quantity G is not workable for the system (4.29). In fact, one may check that the corresponding combined quantity G obeys the same equation as the vorticity equation, which reads

$$\partial_t\omega + (u\cdot\nabla)\omega + \mu\Lambda^{\frac{2}{3}}\omega = \partial_{x_1}\theta.$$

One thus derives

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \mu\|\Lambda^{\frac{1}{3}}\omega\|_{L^2}^2 = \int_{\mathbb{R}^2}\partial_{x_1}\theta\,\omega\,dx. \tag{4.30}$$

It follows from $(4.29)_2$ that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{3}}\theta(t)\|_{L^2}^2 + \kappa\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2}[\Lambda^{\frac{1}{3}}, u\cdot\nabla]\theta\ \Lambda^{\frac{1}{3}}\theta\,dx,$$

which yields

$$\frac{1}{2}\frac{d}{dt}\eta\|\Lambda^{\frac{1}{3}}\theta(t)\|_{L^2}^2 + \eta\kappa\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2 = -\eta\int_{\mathbb{R}^2}[\Lambda^{\frac{1}{3}}, u\cdot\nabla]\theta\ \Lambda^{\frac{1}{3}}\theta\,dx \tag{4.31}$$

with $\eta \geq \frac{2}{\mu\kappa}$. Summing up (4.30) and (4.31) implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega(t)\|_{L^2}^2+\eta\|\Lambda^{\frac{1}{3}}\theta(t)\|_{L^2}^2\right)+\mu\|\Lambda^{\frac{1}{3}}\omega\|_{L^2}^2+\eta\kappa\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2\\
&=\int_{\mathbb{R}^2}\partial_{x_1}\theta\,\omega\,dx-\eta\int_{\mathbb{R}^2}[\Lambda^{\frac{1}{3}},u\cdot\nabla]\theta\ \Lambda^{\frac{1}{3}}\theta\,dx. \qquad (4.32)
\end{aligned}
$$

By the Young inequality, one has

$$
\begin{aligned}
\left|\int_{\mathbb{R}^2}\partial_{x_1}\theta\,\omega\,dx\right| &\le \|\Lambda^{\frac{1}{3}}\omega\|_{L^2}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}\\
&\le \frac{\mu}{2}\|\Lambda^{\frac{1}{3}}\omega\|_{L^2}^2+\frac{1}{2\mu}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2\\
&\le \frac{\mu}{2}\|\Lambda^{\frac{1}{3}}\omega\|_{L^2}^2+\frac{\eta\kappa}{4}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2, \qquad (4.33)
\end{aligned}
$$

where in the last line we have used the assumption $\eta \ge \frac{2}{\mu\kappa}$. By means of (2.3) and (3.6), we conclude

$$
\begin{aligned}
\left|-\eta\int_{\mathbb{R}^2}[\Lambda^{\frac{1}{3}},u\cdot\nabla]\theta\ \Lambda^{\frac{1}{3}}\theta\,dx\right| &\le C\|[\Lambda^{\frac{1}{3}},u\cdot\nabla]\theta\|_{L^{\frac{4}{3}}}\|\Lambda^{\frac{1}{3}}\theta\|_{L^4}\\
&\le C\|\nabla u\|_{L^2}\|\Lambda^{\frac{1}{3}}\theta\|_{L^4}^2\\
&\le C\|\omega\|_{L^2}\|\Lambda^{\frac{1}{3}}\theta\|_{\dot{B}^{-\frac{1}{3}}_{\infty,\infty}}\|\Lambda^{\frac{1}{3}}\theta\|_{\dot{B}^{\frac{1}{3}}_{2,2}}\\
&\le C\|\omega\|_{L^2}\|\theta\|_{\dot{B}^{0}_{\infty,\infty}}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}\\
&\le C\|\omega\|_{L^2}\|\theta\|_{L^\infty}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}\\
&\le \frac{\eta\kappa}{4}\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|\omega\|_{L^2}^2. \qquad (4.34)
\end{aligned}
$$

Putting (4.33) and (4.34) into (4.32), we have

$$
\frac{d}{dt}\left(\|\omega(t)\|_{L^2}^2+\eta\|\Lambda^{\frac{1}{3}}\theta(t)\|_{L^2}^2\right)+\mu\|\Lambda^{\frac{1}{3}}\omega\|_{L^2}^2+\eta\kappa\|\Lambda^{\frac{2}{3}}\theta\|_{L^2}^2\le C\|\theta\|_{L^\infty}^2\|\omega\|_{L^2}^2,
$$

which implies

$$
\|\omega(t)\|_{L^2}\le C(t,\ u_0,\ \theta_0). \qquad (4.35)
$$

Keeping in mind (4.35) and the proof of Theorem 1.3, the global regularity of (4.29) with $\alpha \ge \frac{2}{3}$ follows immediately. $\square$

Appendix A. The proof of Theorem 1.2

In this appendix, we sketch the proof of Theorem 1.2, which is largely inspired by the proof of Theorem 1.1. Of course, the vorticity obeys

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda^\alpha \omega = \nu \partial_{x_1} \theta.$$

We denote $G = \omega - \nu \mathcal{R}_\alpha \theta$, then G satisfies

$$\partial_t G + (u \cdot \nabla) G + \Lambda^\alpha G = \nu [\mathcal{R}_\alpha,\ u \cdot \nabla]\theta + \nu \Lambda^{\beta-\alpha} \partial_{x_1} \theta.$$

Based on the Biot-Savart law, the velocity u can be divided into two parts

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1}(G + \nu \mathcal{R}_\alpha \theta) = \nabla^\perp \Delta^{-1} G + \nu \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta \triangleq u_G + u_\theta.$$

Roughly, we have

$$u_G \approx \Lambda^{-1} G, \quad u_\theta \approx \nu \Lambda^{-\alpha} \theta.$$

Based on this observation, we are able to prove Theorem 1.2 which is divided into the following several steps.

Step 1: If $\alpha + \beta = 1$ and $\frac{2}{3} < \alpha \leq \frac{10}{13}$, then it holds

$$\|G(t)\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}} \theta(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{\frac{\alpha}{2}} G(\tau)\|_{L^2}^2 + \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 \right) d\tau \leq C(t,\ u_0,\ \theta_0). \tag{A.1}$$

In fact, according to (3.4), we have

$$\frac{1}{2}\frac{d}{dt}(\|G(t)\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}} \theta(t)\|_{L^2}^2) + \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \|\Lambda^\beta \theta\|_{L^2}^2 = N_1 + N_2 + N_3 + N_4. \tag{A.2}$$

The estimates of N_1, N_2, N_3 stated in Proposition 3.1 are still valid for $\alpha + \beta = 1$ with $\alpha > \frac{2}{3}$. Therefore, it suffices to consider the term N_4, which can be bounded by

$$\begin{aligned}
N_4 &\leq C \|[\Lambda^{\frac{\beta}{2}}, u \cdot \nabla]\theta\|_{L^{\frac{4}{3}}} \|\Lambda^{\frac{\beta}{2}} \theta\|_{L^4} \\
&\leq C \|\nabla u\|_{L^2} \|\Lambda^{\frac{\beta}{2}} \theta\|_{L^4}^2 \\
&\leq C \|\nabla u\|_{L^2} \|\Lambda^{\frac{\beta}{2}} \theta\|_{\dot{B}^{-\frac{\beta}{2}}_{\infty,\infty}} \|\Lambda^{\frac{\beta}{2}} \theta\|_{\dot{B}^{\frac{\beta}{2}}_{2,2}} \\
&\leq C \|\omega\|_2 \|\theta\|_{\dot{B}^0_{\infty,\infty}} \|\Lambda^\beta \theta\|_{L^2} \\
&\leq C(\|G\|_{L^2} + |\nu| \|\Lambda^{1-\alpha} \theta\|_{L^2}) \|\theta\|_{L^\infty} \|\Lambda^\beta \theta\|_{L^2} \\
&\leq C(\|G\|_{L^2} + |\nu| \|\Lambda^\beta \theta\|_{L^2}) \|\theta_0\|_{L^\infty} \|\Lambda^\beta \theta\|_{L^2}
\end{aligned}$$

$$\leq \frac{1}{16}\|\Lambda^{\beta}\theta\|_{L^2}^2 + C\|\theta_0\|_{L^\infty}^2\|G\|_{L^2}^2 + C_1|\nu|\|\theta_0\|_{L^\infty}\|\Lambda^{\beta}\theta\|_{L^2}^2$$
$$\leq \frac{1}{8}\|\Lambda^{\beta}\theta\|_{L^2}^2 + C\|G\|_{L^2}^2,$$

where in the last line we have taken ν satisfying

$$|\nu| \leq \frac{1}{16C_1\|\theta_0\|_{L^\infty}}. \tag{A.3}$$

Inserting the above estimates of N_1, N_2, N_3 and N_4 into (A.2), we are able to show the desired bound (A.1).

Step 2: Let $\alpha + \beta = 1$ and $\frac{2}{3} < \alpha \leq \frac{10}{13}$. If it holds

$$\|G(t)\|_{L^{m_k}}^{m_k} + \int_0^t \|G(\tau)\|_{L^{\frac{2m_k}{2-\alpha}}}^{m_k}\, d\tau \leq M,$$

then

$$\|G(t)\|_{L^{m_{k+1}}}^{m_{k+1}} + \int_0^t \|G(\tau)\|_{L^{\frac{2m_{k+1}}{2-\alpha}}}^{m_{k+1}}\, d\tau \leq C(t,\ M,\ u_0,\ \theta_0), \tag{A.4}$$

where

$$m_{k+1} < \frac{8(1-\alpha)m_k}{2(4-5\alpha) + (1-\alpha)(2-\alpha)m_k}.$$

Furthermore, we may restrict

$$2 < m_k,\ m_{k+1} < \min\left\{\frac{8}{2-\alpha},\ \frac{1}{1-\alpha},\ \frac{2(3-\alpha)}{(1-\alpha)(2+\alpha)}\right\} = \frac{1}{1-\alpha}.$$

Actually, according to the proof of Proposition 3.3, it is sufficient to estimate N_6 as the remainder terms are still valid. Invoking again (2.3) and keeping in mind $\alpha+\beta = 1$, it is not hard to check

$$\begin{aligned}
N_6 &\leq C\|[\Lambda^{\delta_k}, u_\theta\cdot\nabla]\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k+\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}} \\
&\leq C\|\nabla u_\theta\|_{L^{\frac{2\delta_k+\beta}{\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}} \\
&\leq C|\nu|\|\Lambda^{1-\alpha}\theta\|_{L^{\frac{2\delta_k+\beta}{\beta}}}\|\Lambda^{\delta_k}\theta\|_{L^{\frac{2\delta_k+\beta}{\delta_k}}}^2 \\
&\leq C|\nu|\left(\|\Lambda^{1-\alpha}\theta\|_{\dot B^{-(1-\alpha)}_{\infty,\infty}}^{\frac{2\delta_k-\beta}{2\delta_k+\beta}}\|\Lambda^{1-\alpha}\theta\|_{\dot B^{\frac{(1-\alpha)(2\delta_k-\beta)}{2\beta}}_{2,2}}^{\frac{2\beta}{2\delta_k+\beta}}\right)\left(\|\Lambda^{\delta_k}\theta\|_{\dot B^{-\delta_k}_{\infty,\infty}}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k}\theta\|_{\dot B^{\frac{\beta}{2}}_{2,2}}^{\frac{4\delta_k}{2\delta_k+\beta}}\right) \\
&\leq C|\nu|\left(\|\theta\|_{L^\infty}^{\frac{2\delta_k-\beta}{2\delta_k+\beta}}\|\Lambda^{\frac{(1-\alpha)(2\delta_k+\beta)}{2\beta}}\theta\|_{L^2}^{\frac{2\beta}{2\delta_k+\beta}}\right)\left(\|\theta\|_{L^\infty}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq C|\nu|\|\theta\|_{L^\infty}\|\Lambda^{\frac{(1-\alpha)(2\delta_k+\beta)}{2\beta}}\theta\|_{L^2}^{\frac{2\beta}{2\delta_k+\beta}}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^{\frac{4\delta_k}{2\delta_k+\beta}}\\
&\leq C_2|\nu|\|\theta_0\|_{L^\infty}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2\\
&\leq \frac{1}{4}\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2,
\end{aligned} \tag{A.5}$$

where in the last line we have taken $\epsilon > 0$ satisfying

$$|\nu| \leq \frac{1}{4C_2\|\theta_0\|_{L^\infty}}. \tag{A.6}$$

As a result, (A.5) can be absorbed by the left quantity $\|\Lambda^{\delta_k+\frac{\beta}{2}}\theta\|_{L^2}^2$. Therefore, (A.4) holds true. Moreover, combining (A.3) and (A.6), the C_0 of (1.7) can be fixed as

$$C_0 = \min\left\{\frac{1}{16C_1},\ \frac{1}{4C_2}\right\}.$$

Step 3: If $\alpha + \beta = 1$ and $\frac{2}{3} < \alpha \leq \frac{10}{13}$, then it holds

$$\|G(t)\|_{L^m}^m + \int_0^t \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m\, d\tau \leq C(t,\ u_0,\ \theta_0), \tag{A.7}$$

where m satisfies

$$2 \leq m < \frac{1}{1-\alpha}.$$

In fact, recalling (A.4) and (3.1), we have for $k = 0, 1, 2, \cdots$

$$\|G(t)\|_{L^{m_{k+1}}}^{m_{k+1}} + \int_0^t \|G(\tau)\|_{L^{\frac{2m_{k+1}}{2-\alpha}}}^{m_{k+1}}\, d\tau \leq C(t,\ u_0,\ \theta_0),$$

where $m_1 = 2$ and

$$m_{k+1} < \frac{8(1-\alpha)m_k}{2(4-5\alpha)+(1-\alpha)(2-\alpha)m_k}.$$

We take m_{k+1} as

$$m_{k+1} = \frac{8(1-\alpha)m_k}{2(4-5\alpha+\epsilon)+(1-\alpha)(2-\alpha)m_k}$$

with arbitrarily small $\epsilon > 0$ to be fixed later. By means of the direct computations, m_k can be solved as

$$m_k = \frac{2(\alpha - \epsilon)}{(1-\alpha)(2-\alpha) - (\alpha^2 - 4\alpha + 2 + \epsilon)\left(\frac{4-5\alpha+\epsilon}{4(1-\alpha)}\right)^{k-1}}, \quad k \geq 1.$$

If we fix $\epsilon > 0$ as

$$0 < \epsilon < -(\alpha^2 - 4\alpha + 2),$$

then the sequence $\{m_k\}_{k\in\mathbb{N}}$ is increasing. Moreover, it holds

$$\lim_{k\to\infty} m_k = \frac{2(\alpha - \epsilon)}{(1-\alpha)(2-\alpha)}.$$

Due to the arbitrariness of $\epsilon > 0$, (A.7) holds true when m further satisfies

$$2 \leq m < \frac{2\alpha}{(1-\alpha)(2-\alpha)}.$$

Furthermore, due to $\alpha > \frac{2}{3}$, we have

$$\frac{2\alpha}{(1-\alpha)(2-\alpha)} > \frac{1}{1-\alpha} > \frac{2}{\alpha}.$$

In summary, m should be satisfied

$$2 \leq m < \min\left\{\frac{1}{1-\alpha}, \frac{2\alpha}{(1-\alpha)(2-\alpha)}\right\} = \frac{1}{1-\alpha}.$$

Step 4: It follows from Proposition 3.5 that: If $\alpha + \beta = 1$ with $\frac{2}{3} < \alpha \leq \frac{10}{13}$,and G satisfies

$$\sup_{0\leq t\leq T} \|G(t)\|_{L^q} < \infty, \qquad q > \frac{2}{\alpha} \quad (\text{we may assume } q < \frac{2}{1-\alpha}),$$

then it holds

$$\sup_{0\leq t\leq T} \|G(t)\|_{B^{3\alpha-2}_{r,\infty}} < \infty, \tag{A.8}$$

where r obeys

$$\frac{2}{2\alpha - 1} < r \leq \frac{2q}{2-(1-\alpha)q}.$$

With (A.8) in hand, we are able to complete the proof of Theorem 1.2. Thanks to (A.7), we can check that G satisfies

$$\sup_{0\leq t\leq T} \|G(t)\|_{L^m} \leq C(T, u_0, \theta_0)$$

where $m \geq 2$ satisfies

$$m < \frac{1}{1-\alpha}.$$

This together with (A.8) implies

$$\sup_{0\leq t\leq T} \|G(t)\|_{B^{3\alpha-2}_{\frac{2m}{2-(1-\alpha)m},\infty}} \leq C(T,\ u_0,\ \theta_0).$$

Due to $\alpha > \frac{2}{3}$, we have $m > \frac{2}{\alpha}$, which yields $\widetilde{\gamma} = 2\alpha - \frac{2}{m} > \alpha = 1-\beta$. As a result, we are able to show

$$\begin{aligned}
\|u_G\|_{C^{\widetilde{\gamma}}} &= \|\nabla^{\perp}\Delta^{-1}G\|_{C^{\widetilde{\gamma}}} \\
&\approx \|\nabla^{\perp}\Delta^{-1}G\|_{B^{\widetilde{\gamma}}_{\infty,\infty}} \\
&\leq C\|G\|_{L^2} + C\|G\|_{B^{\widetilde{\gamma}-1}_{\infty,\infty}} \\
&\leq C\|G\|_{L^2} + C\|G\|_{B^{3\alpha-2}_{\frac{2m}{2-(1-\alpha)m},\infty}} \\
&\leq C(T,\ u_0,\ \theta_0).
\end{aligned}$$

Moreover, one also obtains

$$\begin{aligned}
\|u_\theta\|_{C^\alpha} &= |\nu|\|\nabla^{\perp}\Delta^{-1}\mathcal{R}_\alpha\theta\|_{C^\alpha} \\
&\approx |\nu|\|\nabla^{\perp}\Delta^{-1}\mathcal{R}_\alpha\theta\|_{B^{\alpha}_{\infty,\infty}} \\
&\leq C\|\theta\|_{L^2} + C\|\theta\|_{B^{0}_{\infty,\infty}} \\
&\leq C\|\theta\|_{L^2} + C\|\theta\|_{L^\infty} \\
&\leq C(T,\ u_0,\ \theta_0).
\end{aligned}$$

The above estimates imply

$$\|u\|_{C^\alpha} \leq \|u_G\|_{C^\alpha} + \|u_\theta\|_{C^\alpha} \leq \|u_G\|_{C^{\widetilde{\gamma}}} + \|u_\theta\|_{C^\alpha} \leq C(T,\ u_0,\ \theta_0). \tag{A.9}$$

Noticing (A.9) and applying Lemma 2.5 to the θ-equation $(1.1)_2$, we are able to show that θ is Hölder continuous, namely $\|\theta\|_{C^\eta} < \infty$ for some $\eta > 0$, which of course implies

$$\|u_\theta\|_{C^{\alpha+\eta}} = |\nu|\|\nabla^{\perp}\Lambda^{-2-\alpha}\partial_{x_1}\theta\|_{C^{\alpha+\eta}} \leq C\|\theta\|_{L^2} + C\|\theta\|_{C^\eta} \leq C(T,\ u_0,\ \theta_0).$$

Letting $\gamma = \min\{\widetilde{\gamma}, \alpha+\eta\} > \alpha = 1-\beta$, it gives

$$\|u\|_{C^\gamma} \leq \|u_G\|_{C^\gamma} + \|u_\theta\|_{C^\gamma} \leq \|u_G\|_{C^{\widetilde{\gamma}}} + \|u_\theta\|_{C^{\alpha+\eta}} \leq C(T,\ u_0,\ \theta_0). \tag{A.10}$$

Thanks to (A.10) and Lemma 2.6, it implies that θ becomes immediately differentiable, namely for some positive constant ζ

$$\|\theta(t)\|_{C^{1,\zeta}(\mathbb{R}^2))} \leq C(T, u_0, \theta_0),$$

which immediately gives

$$\int_0^T \|\nabla\theta(t)\|_{L^\infty} \, dt \leq C(T, u_0, \theta_0). \tag{A.11}$$

Moreover, we deduce from (1.2) that

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla\theta(\tau)\|_{L^\infty} \, d\tau \leq C(T, u_0, \theta_0). \tag{A.12}$$

With (A.11) and (A.12) in hand, it is not hard to finish the proof of Theorem 1.2 (see the proof of Theorem 1.1).

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Data Availability Since this work is of abstract theoretical nature, no data sets are generated or analyzed. One can obtain the relevant materials from the reference list.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Bahouri, H., Chemin, J.-Y., Danchin, R.: *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, 343, Springer (2011)
2. Caffarelli, L., Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Ann. Math. **171**, 1903–1930 (2010)
3. Cannon, J., DiBenedetto, E.: The initial value problem for the Boussinesq equation with data in L^p. Lecture Notes in Mathematics, vol. 771. Springer, Berlin (1980)
4. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent, II. Geophys. J. Roy. Astronomical Soc. **13**, 529–539 (1967)
5. Chae, D., Nam, H.: Local existence and blow-up criterion for the boussinesq equations. Proc. Roy. Soc. Edinburgh Sect. A **127**, 935–946 (1997)
6. Córdoba, A., Córdoba, D.: A maximum principle applied to quasi-geostrophic equations. Comm. Math. Phys. **249**, 511–528 (2004)
7. Chae, D.: Global regularity for the 2D Boussinesq equations with partial viscosity terms. Adv. Math. **203**, 497–513 (2006)
8. Chen, J., Hou, T.Y.: *Stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth data I: Analysis*, arXiv:2210.07191v3 [math.AP]
9. Chen, Q., Miao, C., Zhang, Z.: A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation. Commun. Math. Phys. **271**, 821–838 (2007)

10. Constantin, P.: Euler equations, Navier-Stokes equations and turbulence, *Mathematical foundation of turbulent viscous flows*, 1-43, Lecture Notes in Math., **1871**, Springer, Berlin, (2006)
11. Constantin, P., Iyer, G., Wu, J.: Global regularity for a modified critical dissipative quasi-geostrophic equation. Indiana Univ. Math. J. **57**, 2681–2692 (2008)
12. Constantin, P., Majda, A., Tabak, E.: Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. Nonlinearity **7**, 1495–1533 (1994)
13. Constantin, P., Vicol, V.: Nonlinear maximum principles for dissipative linear nonlocal operators and applications. Geom. Funct. Anal. **22**, 1289–1321 (2012)
14. Córdoba, A., Córdoba, D.: A maximum princple applied to quasi-geostroohhic equations. Comm. Math. Phys. **249**, 511–528 (2004)
15. Danchin, R., Paicu, M.: Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data. Commun. Math. Phys. **290**, 1–14 (2009)
16. Elgindi, T., Jeong, I.: *Finite-time singularity formation for strong solutions to the Boussinesq system*, Ann. PDE **6** (2020), no. 1, Paper No. 5, 50 pp
17. Elgindi, T., Pasqualotto, F.: *From instability to singularity formation in incompressible fluids*, arXiv:2310.19780v1 [math.AP]
18. Gill, A.: Atmosphere-ocean dynamics, International Geophysics Series 30. Academic Press, New York (1982)
19. Hajaiej, H., Molinet, L., Ozawa, T., Wang, B.: *Sufficient and necessary conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized Boson equations*, in: Harmonic Analysis and Nonlinear Partial Differential Equations, in: RIMS Kokyuroku Bessatsu, vol. B26, Res. Inst. Math. Sci. (RIMS), Kyoto, 159–175 (2011)
20. Hmidi, T., Keraani, S.: On the global well-posedness of the Boussinesq system with zero viscosity. Indiana Univ. Math. J. **58**, 1591–1618 (2009)
21. Hmidi, T., Keraani, S., Rousset, F.: Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation. J. Differential Equations **249**, 2147–2174 (2010)
22. Hmidi, T., Keraani, S., Rousset, F.: Global well-posedness for Euler-Boussinesq system with critical dissipation. Comm. Partial Differential Equations **36**, 420–445 (2011)
23. Hmidi, T., Zerguine, M.: On the global well-posedness of the Euler-Boussinesq system with fractional dissipation. Physica D **239**, 1387–1401 (2010)
24. Hou, T.Y., Li, C.: Global well-posedness of the viscous Boussinesq equations. Discrete Contin. Dyn. Syst. **12**, 1–12 (2005)
25. Jiu, Q., Miao, C., Wu, J., Zhang, Z.: The 2D incompressible Boussinesq equations with general critical dissipation. SIAM J. Math. Anal. **46**, 3426–3454 (2014)
26. Kiselev, A., Nazarov, F., Volberg, A.: Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. Invent. Math. **167**, 445–453 (2007)
27. Lenzmann, E., Schikorra, A.: *Sharp commutator estimates via harmonic extensions*, Nonlinear Anal. **193**, 111375, 37 (2020)
28. Li, D.: On Kato-Ponce and fractional Leibniz. Rev. Mat. Iberoam. **35**, 23–100 (2019)
29. Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge (2001)
30. Miao, C., Xue, L.: On the global well-posedness of a class of Boussinesq-Navier-Stokes systems. NoDEA Nonlinear Differential Equations Appl. **18**, 707–735 (2011)
31. Pedlosky, J.: Geophysical Fluid Dynamics. Springer, New York (1987)
32. Silvestre, L.: *Hölder estimates for advection fractional-diffusion equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11**, 843–855 (2012)
33. Silvestre, L.: On the differentiability of the solution to an equation with drift and fractional diffusion. Indiana Univ. Math. J. **61**, 557–584 (2012)
34. Stefanov, A., Wu, J.: A global regularity result for the 2D Boussinesq equations with critical dissipation. J. Anal. Math. **137**, 269–290 (2019)
35. Triebel, H.: *Theory of function spaces, II*, Monographs in Math. 84, Birkhäuser, Basel, (1992)
36. Wu, J., Xu, X., Xue, L., Ye, Z.: Regularity results for the 2D Boussinesq equations with critical or supercritical dissipation. Commun. Math. Sci. **14**, 1963–1997 (2016)
37. Xu, X.: Global regularity of solutions of 2D Boussinesq equations with fractional diffusion. Nonlinear Anal. **72**, 677–681 (2010)
38. Xue, L., Ye, Z.: On the differentiability issue of the drift-diffusion equation with nonlocal Lévy-type diffusion. Pacific J. Math. **293**, 471–510 (2018)

39. Yang, W., Jiu, Q., Wu, J.: Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation. J. Differential Equations **257**, 4188–4213 (2014)
40. Ye, Z.: Global smooth solution to the 2D Boussinesq equations with fractional dissipation. Math. Methods Appl. Sci. **40**, 4595–4612 (2017)
41. Ye, Z.: *An alternative approach to global regularity for the 2D Euler-Boussinesq equations with critical dissipation*, Nonlinear Anal. **190** (2020), 111591, 5
42. Ye, Z., Xu, X.: Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation. J. Differential Equations **260**, 6716–6744 (2016)
43. Zhou, D., Li, Z., Shang, H., Wu, J., Yuan, B., Zhao, J.: Global well-posedness for the 2D fractional Boussinesq equations in the subcritical case. Pacific J. Math. **298**, 233–255 (2019)