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Stability and enhanced decay rate for 3D anisotropic Boussinesq equations near the hydrostatic balance

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Abstract

In this paper, we study the stability and optimal decay for the 3D incompressible Boussinesq system with only horizontal dissipation. Due to the lack of dissipation in vertical direction, many important techniques such as Schonbek's Fourier splitting method can not be directly used to obtain the decay rates. For the whole space $\mathbb{R}^3$, we first establish the global stability of solutions in H^1-norm. More importantly, we represent the solution of Boussinesq system in an integral form which reflects the enhanced regularity and decay rates in the vertical components. Moreover, we remark that such enhanced decay rate for the third components is caused by the interplay between the divergence free condition of the velocity field and the horizontal Laplacian in the anisotropic Boussinesq system.

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1. Introduction

The aim of this paper is to investigate the stability of three-dimensional (3D) Boussinesq equations and provide optimal decay estimates on large-time behavior of perturbations near the hydrostatic equilibrium. The Boussniesq equations are the most frequently used models for buoyancy-driven fluids such as many large-scale geophysical flows and the Rayleigh-Bénard convection (see [14]). This Boussinesq equations studied here are for anisotropic fluids and involve only horizontal dissipation,

$$\begin{cases} \partial_t U + U\cdot\nabla U = -\nabla P + \nu\Delta_h U + \Theta e_3, \quad x\in\mathbb{R}^3,\ t>0,\\ \partial_t\Theta + U\cdot\nabla\Theta = \eta\Delta_h\Theta,\\ \nabla\cdot U = 0,\\ (U,\Theta)(x,t)|_{t=0} = (U_0,\Theta_0)(x), \end{cases} \tag{1.1}$$

where $U(x,t)=(U_1,U_2,U_3)$ denotes the fluid velocity, $\Theta(x,t)$ the temperature, $P(x,t)$ the scalar pressure and $e_3=(0,0,1)$. The coefficients $\nu>0$ and $\eta>0$ are the kinematic viscosity and the thermal diffusivity, respectively. Here $\Delta_h=\partial_{11}+\partial_{22}$ stands for the horizontal Laplacian. For notational convenience, we have written ∂_i for the partial derivatives ∂_{x_i} with $i=1,2,3$ and shall use $\nabla_h:=(\partial_1,\partial_2)$ for the horizontal gradient. The Boussinesq equations (1.1) arise naturally in the modeling of anisotropic fluids such as the rotating fluids in Ekman layers. A standard reference is in Chapter 4 of Pedlosky's book [26].

The Boussinesq equations have attracted considerable interests recently. In physical applications. The Boussinesq equations are the most frequently used models for atmospheric and oceanographic flows (see [2,4]). From the mathematical point of view the Boussinesq equations are also important because they retain essential structural features of the 3D Navier-Stokes equations with only horizontal dissipation,

$$\begin{cases} \partial_t U + U\cdot\nabla U = -\nabla P + \nu\Delta_h U, \quad x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot U = 0,\\ U(x,t)|_{t=0} = U_0(x). \end{cases} \tag{1.2}$$

The global stability and large-time behavior of the 3D Navier-Stokes equations (1.2) are challenging problem. The major difficulty lies in the fact to control all nonlinear terms in the whole space $\mathbb{R}^3$. In recent work of [8], Ji, Wu and Yang employed the integral representation and used a continuous argument to prove that if the initial velocity is small in the Sobolev space $H^4(\mathbb{R}^3)\cap H_h^{-\sigma}(\mathbb{R}^3)$ with $\frac{3}{4}\le\sigma<1$, the anisotropic Navier-Stokes equations (1.2) have a unique global solution and its first-order derivatives all decay at the optimal rates. Here $H_h^{-\sigma}(\mathbb{R}^3)$ denotes a Sobolev space of distribution f with negative horizontal index with $\sigma>0$ satisfying

$$\|\Lambda_h^{-\sigma} f\|_{L^2}^2 := \int_{\mathbb{R}^3} |\xi_h|^{-2\sigma}|\widehat{f}(\xi)|^2 d\xi < \infty$$

and the fractional Laplacian operation $\Lambda_h^{-\sigma}$ is defined via the Fourier transform

$$\widehat{\Lambda_h^{-\sigma} f}(\xi) = |\xi_h|^{-\sigma}\widehat{f}(\xi).$$

The regularity condition on initial data was soon relaxed by Xu and Zhang in [25]. Ji, Tian and Wu [6] later considered the solutions of Navier-Stokes system in $H^m(\mathbb{T}^2 \times \mathbb{R})$ with $m \geq 2$, and further demonstrated H^{m-1}-norm of the oscillation part $\tilde{u}$ of u decays exponentially in time. However, the evolution of temperature and its influence upon the velocity make asymptotic analysis for the Boussinesq system more involved.

The hydrostatic balance given by

$$U_{he} \equiv (0,0,0), \quad \Theta_{he} = x_3, \quad P_{he} = \frac{1}{2}x_3^2$$

is a very special steady-state solution of (1.1) with great geophysical and astrophysical importance. To investigate the stability and large-time behavior of the anisotropic systems, we consider the equations governing perturbation (u, θ, p) with

$$u = U - U_{he}, \quad \theta = \Theta - \Theta_{he} \quad \text{and} \quad p = P - P_{he}.$$

In view of (1.1), one can check that (u, θ, p) satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta_h u + \theta e_3, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_3 = \eta \Delta_h \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3}$$

Many efforts have been devoted to understanding two fundamental problems concerning the anisotropic Boussinesq systems on perturbation near the hydrostatic balance in (1.3). The first is the well-posedness problem for small initial data. In the pioneering work [5], Doering, Wu, Zhao and Zheng initiated the rigorous study on the stability and global large-time behavior of the hydrostatic equilibrium to 2D Boussinesq equations with only kinematic dissipation (without thermal diffusion). In addition, extensive numerical simulations are performed in [5] to corroborate the analytical results and predict some phenomena that are not proven. The follow up work [29], Tao, Wu, Zhao and Zheng resolved several important issues left open in [5]. In particular, [29] established the large-time behavior and the eventual temperature profile. The work of [3], Castro, Córdoba and Lear later successfully established the stability and large-time behavior on 2D Boussinesq equations with velocity damping instead of dissipation. More recent work on the hydrostatic equilibrium can be found in (see, e.g., [7,10,12,15,16,20,28,31]). There is a growing literature devoted to these Boussinesq systems so that it is almost impossible to be exhaustive in this introduction, but instead considering the second problem: the large-time behavior of solutions to the Boussinesq equations (1.3).

Recently, there are some new works to establish the large-time behavior of solutions to the Boussinesq equations. Wu and Zhang [19] established the stability of Boussinesq equation with horizontal dissipation and vertical diffusion in $\mathbb{R}^2 \times \mathbb{T}$ and also obtained the exponent decay of oscillation part of the velocity field and the thermal. In the spirit of optimal decay rate for heat equation. Shang and Xu [23] examined the stability of two Boussinesq equation with dissipation and thermal diffusion in two directions as well as the decay of corresponding linearized systems. However the decay result for nonlinear systems is open in $\mathbb{R}^3$. In addition, the energy method and other classical tools such as Schonbek's Fourier-splitting scheme introduced by Wiegner in [13] can not be applied to investigate the decay rates for global small solutions of nonlinear

Boussinesq equation considered here. Ji, Yan and Wu [9] later found the stability and large-time behavior of perturbations near the hydrostatic balance for 3D nonlinear Boussniesq equation involving only horizontal dissipation. More precisely, as long as the initial data in $H^4(\mathbb{R}^3) \cap H_h^{-\sigma}(\mathbb{R}^3)$ with $\frac{3}{4} \leq \sigma < 1$, the optimal decay rates of solutions presented here are the same as 3D anisotropic heat equation. It is worth mentioning the work of [24], Shang, Wu and Zhang implemented iterative procedure to improve the decay rates of $\|(\nabla_h u, \nabla_h b)(t)\|_{L^2(\mathbb{R}^3)}$ to 3D anisotropic MHD equations instead of applying the bootstrap argument to integral representation of nonlinear system in [9]. The purpose of this paper is to refine the results obtained by [9,23,24] and to clarify the effect of anisotropy on the large-time behavior of solution in terms of L^2 decay rates.

Now, we are ready to state our main results presenting the stability and optimal decay rates for perturbations near the hydrostatic balance. Our first result of this paper concerning the small data global well-posedness problem on nonlinear system in H^k-framework with $k \geq 1$, which is formulated in the following theorem.

Theorem 1.1. *Consider the nonlinear system in (1.3) with $\nu > 0$ and $\eta > 0$. Assume $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ satisfies $\nabla \cdot u_0 = 0$. Then there exists $\epsilon = \epsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^1} + \|\theta_0\|_{H^1} \leq \epsilon,$$

then (1.3) has a unique global solution (u, θ) on $\mathbb{R}^3 \times [0, \infty)$ satisfying,

$$\|u(t)\|_{H^1} + \|\theta(t)\|_{H^1} \leq C_0 \epsilon,$$

where the constant $C_0 = C_0(\nu, \eta) > 0$. Furthermore, if $(u_0, \theta_0) \in H^k(\mathbb{R}^3)$ with $k \geq 1$, then the following global $priori$ bound hold for the solution (u, θ):

$$\|u(t)\|_{H^k} + \|\theta(t)\|_{H^k} \leq C \|(u_0, \theta_0)\|_{H^k},$$

where $C > 0$ is a constant proportional to the initial norm $\|(u_0, \theta_0)\|_{H^k}$.

Remark 1.1. In fact, this theorem gives the global existence and stability for any initial data with H^1-norm by an iterative process. On the other hand, we establish the uniformly bounded of any H^k-norm of the solution u while the initial H^k-norm is not assumed to be small for $k \geq 2$.

Before introducing the large-time behavior of global solution in Theorem 1.2, we first give a definition:

Definition 1.1. For any $s \in \mathbb{R}$ and $s' \in \mathbb{R}$, the anisotropic Sobolev space $\dot{H}^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution f such that

$$\|f\|^2_{\dot{H}^{s,s'}} := \int |\xi_h|^{2s} |\xi_3|^{2s'} |\widehat{f}(\xi)|^2 d\xi < \infty \quad with \quad \xi_h = (\xi_1, \xi_2).$$

For simplicity of notation, we denote by

$$\|f\|_{L^p} := \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_{W^{k,p}} := \|f\|_{W^{k,p}(\mathbb{R}^3)},$$

and $\langle f, g\rangle$ the L^2 inner product of f and g.

Our second main result of this paper is concerned with the optimal decay rates for perturbations near the hydrostatic balance.

Theorem 1.2. *In addition to the conditions of Theorem 1.1. Consider the nonlinear system in (1.3) with $\nu > 0$ and $\eta > 0$. Let $\frac{1}{2} < \sigma < 1$ and $m \geq 3$. Assume (u_0, θ_0) satisfies $\nabla \cdot u_0 = 0$ and*

$$(u_0, \theta_0) \in \big(\dot{H}^{-\sigma,0} \cap \dot{H}^{-\sigma,-\frac{\sigma}{2}} \cap H^m\big)(\mathbb{R}^3), \quad (\partial_3 u_0, \partial_3\theta_0) \in \dot{H}^{-\sigma,0}(\mathbb{R}^3),$$

then the global solution (u, θ) of (1.3) satisfies, for any $t \geq 0$,

$$\begin{aligned}
\|u(t)\|_{H^m} + \|\theta(t)\|_{H^m} &\leq C,\\
\|(u,\theta)(t)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u, \partial_3\theta)(t)\|_{\dot{H}^{-\sigma,0}} &\leq C,\\
\||(u,\theta)(t)\|_{L^2} + \|(\partial_3 u, \partial_3\theta)(t)\|_{L^2} &\leq C(1+t)^{-\frac{\sigma}{2}},\\
\|\nabla_h u(t)\|_{L^2} + \|\nabla_h\theta(t)\|_{L^2} &\leq C(1+t)^{-(\frac{\sigma}{2}+\frac{1}{2})},\\
\|u_3(t)\|_{L^2} &\leq C(1+t)^{-\frac{3\sigma}{4}}, \qquad (1.4)\\
\|\nabla_h u_3(t)\|_{L^2} &\leq C(1+t)^{-(\frac{3\sigma}{4}+\frac{1}{2})}, \qquad (1.5)
\end{aligned}$$

where $C > 0$ is a constant proportional to the initial norm $\|(u_0, \theta_0)\|_{H^k}$.

Remark 1.2. To established the sharp decay rates of global solutions in (1.3). In previous work [9], Ji, Yan and Wu assumed that the initial data (u_0, θ_0) obeys $\nabla \cdot u_0 = 0$ and for $\frac{3}{4} \leq \sigma < 1$,

$$\|(u_0, \theta_0)\|_{H^4} + \|(u_0, \theta_0)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u_0, \partial_3\theta_0)\|_{\dot{H}^{-\sigma,0}} \leq \epsilon,$$

for some sufficiently small $\epsilon > 0$, then the corresponding solution (u, θ) to (1.3) has following decay rates:

$$\|(u,\theta)(t)\|_{L^2} + \|(\partial_3 u, \partial_3\theta)(t)\|_{L^2} \leq C\epsilon(1+t)^{-\frac{\sigma}{2}},$$

and

$$\|(\nabla_h u, \nabla_h\theta)(t)\|_{L^2} \leq C\epsilon(1+t)^{-\frac{1+\sigma}{2}}.$$

Thus, our result extends the work [9] to the lower regularity assumptions of the initial data (u_0, θ_0).

Remark 1.3. The enhanced decay rates in the vertical components $u_3(t)$ of the velocity $u(t)$: We remark (1.4) that the vertical components $u_3(t)$ of $u(t)$ decays faster than the horizontal components $u_h(t)$, while the horizontal derivatives of $u_3(t)$ in (1.5) increase the decay rate by $-1/2$. It was first remarkably observed by Xu and Zhang in [25] with the initial data was settled in Besov space $B^{0,\frac{1}{2}} \cap \dot{H}^{-s,-\frac{s}{2}-\frac{1}{4}}$ with $s \in (\frac{1+3s_1}{10(s_1-1)}, 1)$ and $s_1 > 0$, they proved the horizontal components of velocity field decay like the solutions of 2D classical Navier-Stokes equations, while the third component of the velocity field decay as the solutions of 3D Navier-Stokes equations. In fact, the enhanced decaying rate of the third component $u_3(t)$ is caused by the interplay between divergence free condition $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$ which transports the horizontal regularity of $u_h(t)$ to the vertical regularity of $u_3(t)$. That is the reason why they used the word "enhanced dissipation" in [1,17].

Next, we briefly sketch the main ideas of proofs in Theorem 1.1 and Theorem 1.2. Since the local (in time) well-posedness of (1.3) can be shown via a standard method (see [13]). To prove Theorem 1.1, we first define the energy functional $E(t)$ by

$$E(t) := \sup_{\tau\in[0,t]} \|(u,\theta)(\tau)\|_{H^1}^2 + \nu \int_0^t \|\nabla_h u(\tau)\|_{H^1}^2 \, d\tau + \eta \int_0^t \|\nabla_h \theta(\tau)\|_{H^1}^2 \, d\tau.$$

The core part of bootstrap argument is to derive the following energy estimate

$$E(t) \le E(0) + C E(t)^{\frac{3}{2}}. \tag{1.6}$$

Moreover, through an inductive process, we get that any H^k-norm with $k \ge 1$ is uniformly bounded. In particular, Theorem 1.1 is legitimate to study its precise large-time behavior in Theorem 1.2. The framework of proof in Theorem 1.2 is to apply bootstrap argument, we assume the initial data (u_0, θ_0) satisfies the assumptions

$$\|u_0\|_{H^1} + \|\theta_0\|_{H^1} \le \epsilon,$$

and we write

$$\epsilon_0 := \|(u_0,\theta_0)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u_0, \partial_3\theta_0)\|_{\dot{H}^{-\sigma,0}} < \infty,$$

then we make the ansatz that the solution (u,θ) satisfies, for $t \in [0,T]$ with $T > 0$,

$$\|(u,\theta)(t)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u, \partial_3\theta)(t)\|_{\dot{H}^{-\sigma,0}} \le 3\epsilon_0.$$

The initial time $T > 0$ exists by local well-posedness. We then show that the corresponding (u,θ) actually satisfies

$$\|(u,\theta)(t)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u, \partial_3\theta)(t)\|_{\dot{H}^{-\sigma,0}} \le 2\epsilon_0.$$

Then the bootstrap argument implies that $T = \infty$.

Our main efforts are devoted to proving the estimates of (1.4) and (1.5). Various anisotropic inequalities are invoked by fully making use of the anisotropic dissipation in the system (1.3). In order to obtain suitable upper bounds, we have to exploit the structure of the kernel function. To explain this point, we take the representation of $u_3(t)$ in (3.1) as an example,

$$\big(W_2 + |\xi_h|^2 W_1\big)(t-\tau)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3(\tau). \tag{1.7}$$

The kernel function in (1.7) can obtain a bound of the form $e^{-|\xi_h|^2 t}$, which is the symbol of heat operator associated with the horizontal Laplacian as follows,

$$\int_0^t \int \big(W_2 + |\xi_h|^2 W_1\big)(t-\tau)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3(\tau)\cdot \widehat{u}_3(t)\,dx d\tau. \tag{1.8}$$

While integrating by parts in (1.8), we need to generate a factor ξ_h to get the decay rate of $\|\nabla_h u\|_{L^2}$. By applying the definition of Helmholz-Leray projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ and the divergence-free condition $\nabla \cdot u = 0$, we find the following identity

$$\begin{aligned}(\mathbb{P}\widehat{(u\cdot\nabla u)})_3 = & \widehat{\nabla_h\cdot(u_h u_3)} - \nabla_h\cdot\partial_3\widehat{\nabla\cdot(u}\otimes u_h) \\ & - \nabla_h\cdot\widehat{\Delta^{-1}\partial_{33}}(u_h u_3) + \Delta^{-1}\widehat{\Delta_h\partial_3(}u_3u_3).\end{aligned}$$

Then the Fourier transform of the right-hand side involves ξ_h, which allows us to get the decay rate of $\|\nabla_h u\|_{L^2}$. We refer to Section 3 for more technical details.

The rest of this paper is organized as follows. In Section 2, we will establish the H^k-stability by an inductive process with the initial H^1-norm is assumed to be small while the initial H^k-norm with $k \geq 2$ not need to be small. The optimal decay rates, our main results stated in Theorem 1.2 are established in Section 3. For the sake of clarity. Section 3 is further divided into five steps.

2. Nonlinear stability and proof of Theorem 1.1

In this section, we first introduce a useful mathematical tool and then establish the global well-posedness result, i.e. Theorem 1.1.

2.1. An elementary inequality

In this subsection, we will recall an elementary inequality and result which be used frequently later. This lemma provides an anisotropic upper bound for the integral of a triple product. It is a very powerful tool in dealing with anisotropic equations. A simple proof of this lemma can be found in [19].

Lemma 2.1. *The following estimates hold when the right-hand sides are all bounded*

$$\int_{\mathbb{R}^3} |fgh|\,dx \le C\|f\|_{L^2}^{\frac12}\|\partial_1 f\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}\|\partial_2 g\|_{L^2}^{\frac12}\|h\|_{L^2}^{\frac12}\|\partial_3 h\|_{L^2}^{\frac12},$$

and

$$\int_{\mathbb{R}^3} |fgh|\,dx \le C\|f\|_{L^2}^{\frac14}\|\partial_1 f\|_{L^2}^{\frac14}\|\partial_2 f\|_{L^2}^{\frac14}\|\partial_1\partial_2 f\|_{L^2}^{\frac14}\|g\|_{L^2}^{\frac12}\|\partial_3 g\|_{L^2}^{\frac12}\|h\|_{L^2}.$$

2.2. H^1-stability

In this subsection, we prove the H^1-stability of Theorem 1.1. This theorem serves as a preparation of our main result on optimal decay proven in the next section.

Proof. Since the local well-posedness of (1.3) can be obtained by a standard approach such as Friedrichs' method of cutoff in Fourier space (see [13]), our attention is focused on the global bound of (u,θ). Due to the equivalence of $\|(u,\theta)\|_{H^1}$ with $\|(u,\theta)\|_{L^2}+\|(u,\theta)\|_{\dot H^1}$, it suffices to bound the L^2-norm and the $\dot H^1$-norm of (u,θ). First of all, based on a simple energy estimate and $\nabla\cdot u=0$, we have the following global L^2-norm of (u,θ) obeys, for any $0\le t\le T$,

$$\frac{1}{2}\frac{d}{dt}\|(u,\theta)\|_{L^2}^2+\nu\|\nabla_h u\|_{L^2}^2+\eta\|\nabla_h\theta\|_{L^2}^2=0. \tag{2.1}$$

Integrating in time, yields that

$$\|(u,\theta)(t)\|_{L^2}^2+2\nu\int_0^t\|\nabla_h u(\tau)\|_{L^2}^2\,d\tau+2\eta\int_0^t\|\nabla_h\theta(\tau)\|_{L^2}^2\,d\tau=\|(u_0,\theta_0)\|_{L^2}^2.$$

Applying ∂_i to the equations in (1.3) and then dotting by $\langle\partial_i u,\partial_i\theta\rangle$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3\|(\partial_i u,\partial_i\theta)\|_{L^2}^2+\nu\|\partial_i\nabla_h u\|_{L^2}^2+\eta\|\partial_i\nabla_h\theta\|_{L^2}^2:=I_1+I_2+I_3 \tag{2.2}$$

where

$$I_1=\sum_{i=1}^3\int\partial_i(\theta e_3)\cdot\partial_3 u-\partial_i u_3\cdot\partial_3\theta\,dx,$$

$$I_2=-\sum_{i=1}^3\int\partial_i(u\cdot\nabla u)\cdot\partial_i u\,dx,$$

$$I_3=-\sum_{i=1}^3\int\partial_i(u\cdot\nabla\theta)\cdot\partial_i\theta\,dx.$$

We are now in a position of estimating each term on the right-hand side of (2.2). First, it is easy to check the first term $I_1=0$. To bound the second term I_2, we further decompose it into two pieces,

$$\begin{aligned}I_2&=-\sum_{i=1}^3\int\partial_i u\cdot\nabla u\cdot\partial_i u\,dx\\&=-\sum_{i=1}^3\int\partial_i u_h\cdot\nabla_h u\cdot\partial_i u\,dx-\sum_{i=1}^3\int\partial_i u_3\partial_3 u\cdot\partial_i u\,dx\end{aligned}$$

$$:= I_{21} + I_{22}.$$

The term I_{21} follows from Lemma 2.1 that,

$$\begin{aligned} I_{21} &\leq C \sum_{i=1}^{3} \|\partial_i u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i u_h\|_{L^2}^{\frac{1}{2}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|u\|_{H^1} \|\nabla_h u\|_{H^1}^2. \end{aligned}$$

For the term I_{22}, in view of $\nabla \cdot u = 0$ or $\partial_3 u_3 = -\nabla_h \cdot u_h$ and Lemma 2.1, one has

$$\begin{aligned} I_{22} &= -\sum_{i=1}^{2} \int \partial_i u_3 \partial_3 u \cdot \partial_i u \, dx - \int \partial_3 u_3 \partial_3 u \cdot \partial_3 u \, dx \\ &= -\sum_{i=1}^{2} \int \partial_i u_3 \partial_3 u \cdot \partial_i u \, dx + \int \nabla_h \cdot u_h \partial_3 u \cdot \partial_3 u \, dx \\ &\leq C \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u_3\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u_h\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|u\|_{H^1} \|\nabla_h u\|_{H^1}^2. \end{aligned}$$

It follows from the estimates of I_{21} and I_{22} that,

$$I_2 \leq C \|u\|_{H^1} \|\nabla_h u\|_{H^1}^2.$$

Similar to the estimate of I_{22}, we have

$$\begin{aligned} I_3 &= -\sum_{i=1}^{3} \int \partial_i u_h \cdot \nabla_h \theta \cdot \partial_i \theta \, dx - \sum_{i=1}^{3} \int \partial_i u_3 \partial_3 \theta \cdot \partial_i \theta \, dx \\ &\leq C \sum_{i=1}^{3} \|\partial_i u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i u_h\|_{L^2}^{\frac{1}{2}} \|\partial_i \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i \theta\|_{L^2}^{\frac{1}{2}} \|\nabla_h \theta\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h \theta\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \sum_{i=1}^{3} \|\partial_3 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_i \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i \theta\|_{L^2}^{\frac{1}{2}} \|\partial_i u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i u_3\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|(u, \theta)\|_{H^1} \|(\nabla_h u, \nabla_h \theta)\|_{H^1}^2. \end{aligned}$$

Combining the estimates above, we have

$$\frac{1}{2} \frac{d}{dt} \|(u, \theta)\|_{\dot{H}^1}^2 + \nu \|\nabla_h u\|_{\dot{H}^1}^2 + \eta \|\nabla_h \theta\|_{\dot{H}^1}^2 \leq C \|(u, \theta)\|_{H^1} \|(\nabla_h u, \nabla_h \theta)\|_{H^1}^2. \tag{2.3}$$

Adding (2.1) and (2.3) together yields that

$$\frac{d}{dt}\|(u,\theta)\|_{H^1}+2\nu\|\nabla_h u\|_{H^1}^2+2\eta\|\nabla_h\theta\|_{H^1}^2\le C\|(u,\theta)\|_{H^1}\|(\nabla_h u,\nabla_h\theta)\|_{H^1}^2. \tag{2.4}$$

Integrating (2.4) in time over [0,t], and invoking the equivalence, we find

$$\|(u,\theta)(t)\|_{H^1}^2+2\nu\int_0^t\|\nabla_h u(\tau)\|_{H^1}^2\,d\tau+2\eta\int_0^t\|\nabla_h\theta(\tau)\|_{H^1}^2\,d\tau$$
$$\le\|(u_0,\theta_0)\|_{H^1}^2+C\int_0^t\|(u,\theta)(\tau)\|_{H^1}\|(\nabla_h u,\nabla_h\theta)(\tau)\|_{H^1}^2\,d\tau, \tag{2.5}$$

and this can obtain the desired inequality in (1.6). When the initial data (u_0,θ_0) is taken to be sufficiently small, namely

$$\|u_0\|_{H^1}+\|\theta_0\|_{H^1}\le\epsilon,$$

for some sufficiently small $\epsilon>0$, one can use bootstrap argument and (2.5) to show that

$$\|(u,\theta)(t)\|_{H^1}^2+\nu\int_0^t\|\nabla_h u(\tau)\|_{H^1}^2\,d\tau+\eta\int_0^t\|\nabla_h\theta(\tau)\|_{H^1}^2\,d\tau\le\|(u_0,\theta_0)\|_{H^1}^2. \tag{2.6}$$

This yields the desired global H^1-stability for $\|(u,\theta)\|_{H^1}$.

2.3. *A priori bounds for* $\|(u,\theta)\|_{H^k}$

Next we establish the following H^k-norm of global a *priori* bound for (u,θ) by induction on $k\ge1$ that

$$\|(u,\theta)(t)\|_{H^k}^2+\nu\int_0^t\|\nabla_h u(\tau)\|_{H^k}^2\,d\tau+\eta\int_0^t\|\nabla_h\theta(\tau)\|_{H^k}^2\,d\tau\le C\|(u_0,\theta_0)\|_{H^k}^2. \tag{2.7}$$

Clearly, the bound (2.7) holds for $k=1$ in (2.6). Then we assume that for any integer $k\ge2$, we have

$$\|(u,\theta)(t)\|_{H^{k-1}}^2+\nu\int_0^t\|\nabla_h u(\tau)\|_{H^{k-1}}^2\,d\tau+\eta\int_0^t\|\nabla_h\theta(\tau)\|_{H^{k-1}}^2\,d\tau\le C\|(u_0,\theta_0)\|_{H^{k-1}}^2. \tag{2.8}$$

Applying ∂_i^k with $i=1,2,3$ to the first two equations in (1.3), dotting the results by $\partial_i^k u$ and $\partial_i^k\theta$ in L^2, respectively. Integrating over $\mathbb{R}^3$ and adding them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\|(\partial_i^k u,\partial_i^k\theta)\|_{L^2}^2+\nu\|\partial_i^k\nabla_h u\|_{L^2}^2+\eta\|\partial_i^k\nabla_h\theta\|_{L^2}^2\\
&=-\sum_{i=1}^{3}\int\partial_i^k(u\cdot\nabla u)\cdot\partial_i^k u\,dx-\sum_{i=1}^{3}\int\partial_i^k(u\cdot\nabla\theta)\cdot\partial_i^k\theta\,dx\\
&:=J_1+J_2.
\end{aligned}$$

By the Leibniz Formula and the facts

$$\int(u\cdot\nabla\partial_i^k u)\cdot\partial_i^k u\,dx=0,\ \ i=1,2,3,$$

we then decompose J_1 into two pieces,

$$\begin{aligned}
J_1&=-\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j u\cdot\partial_i^{k-j}\nabla u\cdot\partial_i^k u\,dx\\
&=-\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j u_h\cdot\partial_i^{k-j}\nabla_h u\cdot\partial_i^k u\,dx-\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j u_3\partial_i^{k-j}\partial_3 u\cdot\partial_i^k u\,dx\\
&:=J_{11}+J_{12},
\end{aligned}$$

where C_k^j denotes the combinatorial number

$$C_k^j=\frac{k!}{j!(k-j)!}.$$

J_{11} follows from Lemma 2.1 and can be bounded easily that

$$\begin{aligned}
J_{11}&\le C\sum_{i=1}^{3}\sum_{j=1}^{k}\|\partial_i^j u_h\|_{L^2}^{\frac12}\|\partial_1\partial_i^j u_h\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_2\partial_i^k u\|_{L^2}^{\frac12}\|\partial_i^{k-j}\nabla_h u\|_{L^2}^{\frac12}\|\partial_3\partial_i^{k-j}\nabla_h u\|_{L^2}^{\frac12}\\
&\le C\|\nabla_h u\|_{H^k}\|u\|_{H^k}\|\nabla_h u\|_{H^{k-1}}\\
&\le\frac{\nu}{6}\|\nabla_h u\|_{H^k}^2+C\|u\|_{H^k}^2\|\nabla_h u\|_{H^{k-1}}^2.
\end{aligned}$$

Applying Lemma 2.1 and Young inequality,

$$\begin{aligned}
J_{12}&\le C\sum_{i=1}^{3}\sum_{j=1}^{k}\|\partial_i^{k-j}\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_i^{k-j}\partial_3 u\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_2\partial_i^k u\|_{L^2}^{\frac12}\|\partial_i^j u_3\|_{L^2}^{\frac12}\|\partial_3\partial_i^j u_3\|_{L^2}^{\frac12}\\
&\le C\|\nabla_h u\|_{H^k}\|u\|_{H^k}\|\nabla_h u\|_{H^{k-1}}\\
&\le\frac{\nu}{6}\|\nabla_h u\|_{H^k}^2+C\|u\|_{H^k}^2\|\nabla_h u\|_{H^{k-1}}^2,
\end{aligned}$$

here we have used

$$\sum_{i=1}^{3} \|\partial_i^j u_3\|_{L^2} \le C(\|\nabla_h^j u_3\|_{L^2} + \|\partial_3^{j-1}\nabla_h \cdot u_h\|_{L^2}).$$

Similarly, by the Leibniz Formula and

$$\int (u \cdot \nabla \partial_i^k \theta) \cdot \partial_i^k \theta dx = 0,\ i = 1, 2, 3,$$

we have the estimate of J_2 that

$$\begin{aligned}
J_2 =& -\sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u \cdot \partial_i^{k-j} \nabla\theta \cdot \partial_i^k \theta \, dx, \\
=& -\sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_h \cdot \partial_i^{k-j} \nabla_h\theta \cdot \partial_i^k \theta \, dx - \sum_{i=1}^{3}\sum_{j=1}^{k} C_k^j \int \partial_i^j u_3 \partial_i^{k-j} \partial_3\theta \cdot \partial_i^k \theta \, dx \\
\le& C\sum_{i=1}^{3}\sum_{j=1}^{k} \|\partial_i^j u_h\|_{L^2}^{\frac12} \|\partial_1\partial_i^j u_h\|_{L^2}^{\frac12} \|\partial_i^k\theta\|_{L^2}^{\frac12} \|\partial_2\partial_i^k\theta\|_{L^2}^{\frac12} \|\partial_i^{k-j}\nabla_h\theta\|_{L^2}^{\frac12} \|\partial_3\partial_i^{k-j}\nabla_h\theta\|_{L^2}^{\frac12} \\
&+ C\sum_{i=1}^{3}\sum_{j=1}^{k} \|\partial_i^{k-j}\partial_3\theta\|_{L^2}^{\frac12} \|\partial_1\partial_i^{k-j}\partial_3\theta\|_{L^2}^{\frac12} \|\partial_i^k\theta\|_{L^2}^{\frac12} \|\partial_2\partial_i^k\theta\|_{L^2}^{\frac12} \|\partial_i^j u_3\|_{L^2}^{\frac12} \|\partial_3\partial_i^j u_3\|_{L^2}^{\frac12} \\
\le& \frac{\nu}{6}\|\nabla_h u\|_{H^k}^2 + \frac{\eta}{2}\|\nabla_h\theta\|_{H^k}^2 + C\|(u,\theta)\|_{H^k}^2 \|(\nabla_h u, \nabla_h\theta)\|_{H^{k-1}}^2.
\end{aligned}$$

Combining the estimates above and the L^2-bound in (2.1), we obtain

$$\frac{d}{dt}\|(u,\theta)\|_{H^k}^2 + \nu\|\nabla_h u\|_{H^k}^2 + \eta\|\nabla_h\theta\|_{H^k}^2 \le C\|(u,\theta)\|_{H^k}^2 \|(\nabla_h u, \nabla_h\theta)\|_{H^{k-1}}^2.$$

By Grönwall inequality and (2.8), we deduce

$$\begin{aligned}
&\|(u,\theta)(t)\|_{H^k}^2 + \nu\int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 \, d\tau + \eta\int_0^t \|\nabla_h\theta(\tau)\|_{H^k}^2 \, d\tau \\
&\le C\|(u_0,\theta_0)\|_{H^k}^2 e^{\nu\int_0^t \|\nabla_h u\|_{H^{k-1}}^2 \, d\tau + \eta\int_0^t \|\nabla_h\theta\|_{H^{k-1}}^2 \, d\tau} \\
&\le C\|(u_0,\theta_0)\|_{H^k}^2 e^{C\|(u_0,\theta_0)\|_{H^{k-1}}^2} \\
&\le C\|(u_0,\theta_0)\|_{H^k}^2,
\end{aligned}$$

which yields the desired global uniform bound for $\|(u,\theta)\|_{H^k}$ with $k \ge 1$. This finishes the proof of global bound in Theorem 1.1. □

3. Decay rates and proof of Theorem 1.2

This section serves two purposes. The first is to provide several tools in the following lemmas. The second is to prove Theorem 1.2 by applying the bootstrap argument to the improved inequality in (3.3).

3.1. Mathematical tools for the decay estimates

In this subsection, we state some useful inequalities which play important roles in the derivations of decay rates. We begin with an upper bound for the L^p-norm of a one-dimensional function, which serves as a basic ingredient for anisotropic upper bounds. A proof can be found in [30].

Lemma 3.1. *Let $2 \le p \le \infty$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$ be the Zygmund operator. Then, there exists a positive constant $C = C(d, p, s)$ such that, for any d-dimensional function $f \in H^s(\mathbb{R}^d)$ with $s > d(\frac{1}{2} - \frac{1}{p})$,*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{s}(\frac{1}{2}-\frac{1}{p})} \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{s}(\frac{1}{2}-\frac{1}{p})}.$$

In particular, if $p = \infty$, $s = 1$ and $d = 1$, then any $f = f(x_3) \in H^1(\mathbb{R})$ satisfies

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

We then recall Minkowski inequality (see, e.g., [11]). It is an elementary tool that allows us to estimate the Lebesgue norm with larger index first followed by the Lebesgue norm with a smaller index.

Lemma 3.2. *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces. Let f be a nonnegative measurable function over $X_1 \times X_2$. For all $1 \le p \le q \le \infty$, we have*

$$\Big\| \|f(\cdot, x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \Big\| \|f(x_1, \cdot)\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

In particular, for a nonnegative measurable function f over $\mathbb{R}^m \times \mathbb{R}^n$ and for $1 \le p \le q \le \infty$,

$$\Big\| \|f\|_{L^p(\mathbb{R}^m)} \Big\|_{L^q(\mathbb{R}^n)} \le \Big\| \|f\|_{L^q(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m)}.$$

Additionally, we introduce the notation

$$L_h^q := L_{x_1,x_2}^q, \quad \|f\|_{L_h^p L_{x_3}^q} := \Big\| \|f\|_{L_{x_3}^q} \Big\|_{L_h^p},$$

which can be frequently used in the context.

The third lemma states an exact decay estimates for the heat operator associated with the fractional Laplacian. A proof can be found in [21] and [22].

Lemma 3.3. *Let $\alpha \geq 0$, $\beta > 0$, $\nu > 0$, $1 \leq q \leq p \leq \infty$ and $f \in L^q(\mathbb{R}^d)$. There holds*

$$\|\Lambda^\alpha e^{-\nu(-\Delta)^\beta t} f\|_{L^p(\mathbb{R}^d)} \leq C t^{-\frac{\alpha}{2\beta}-\frac{d}{2\beta}(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q(\mathbb{R}^d)}, \quad \forall t > 0.$$

Moreover, if $\|\Lambda^\alpha f\|_{L^p(\mathbb{R}^d)} \leq C < \infty$. Then, for any $t \geq 0$,

$$\|\Lambda^\alpha e^{-\nu(-\Delta)^\beta t} f\|_{L^p(\mathbb{R}^d)} \leq C(1+t)^{-\frac{\alpha}{2\beta}-\frac{d}{2\beta}(\frac{1}{q}-\frac{1}{p})} (\|f\|_{L^q(\mathbb{R}^d)} + \|\Lambda^\alpha f\|_{L^p(\mathbb{R}^d)}).$$

The fourth lemma provides an upper bound on a convolution integral, which can be proved similarly to that of Lemma 2.4 in [18].

Lemma 3.4. *For any $0 < s_1 < 1$ and $s_2 > 0$, there exists a positive constant C, depending only on s_1 and s_2 such that, for any $t \geq 0$,*

$$\int_0^t (t-\tau)^{-s_1}(1+\tau)^{-s_2}\, d\tau \leq \begin{cases} C(1+t)^{-s_1}, & s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t), & s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & s_2 < 1. \end{cases}$$

Remark 3.1. The condition $0 < s_1 < 1$ is necessary, because when $s_1 \geq 1$ and $0 < t < 1$, the convolution integral is unbounded. But it is not required when replacing $(t-\tau)^{-s_1}$ with $(1+t-\tau)^{-s_1}$. More precisely, if $0 < s_1 \leq s_2$, then

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\, d\tau \leq \begin{cases} C(1+t)^{-s_1}, & s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t), & s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & s_2 < 1. \end{cases}$$

The last lemma represents (1.3) in an integral from via Duhamel's principle. A simple proof of this lemma can be found in [9].

Lemma 3.5. *The system in (1.3) can be converted into the following integral form*

$$\begin{aligned}
\widehat{u}_h(\xi,t) =& e^{\lambda_1 t}\widehat{u}_{0h} + (\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1 t} + \frac{\xi_h\xi_3}{|\xi_h|^2}W_2 + \xi_h\xi_3 W_1)\widehat{u}_{03} - \frac{\xi_h\xi_3}{|\xi|^2}W_1\widehat{\theta}_0 \\
&- \int_0^t e^{\lambda_1(t-\tau)}\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_h(\tau)\, d\tau \\
&- \int_0^t \left(\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1(t-\tau)} + \frac{\xi_h\xi_3}{|\xi_h|^2}W_2 + \xi_h\xi_3 W_1\right)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3(\tau)\, d\tau \\
&+ \int_0^t \frac{\xi_h\xi_3}{|\xi|^2}W_1(t-\tau)\widehat{(u\cdot\nabla\theta)}(\tau)\, d\tau,
\end{aligned}$$

$$\begin{aligned}\widehat{u}_3(\xi,t) =& -(W_2+|\xi_h|^2W_1)\widehat{u}_{03}+\frac{|\xi_h|^2}{|\xi|^2}W_1\widehat{\theta}_0\\&+\int_0^t \big(W_2+|\xi_h|^2W_1\big)(t-\tau)\big(\mathbb{P}\,\widehat{(u\cdot\nabla u)}\big)_3(\tau)\,d\tau\\&-\int_0^t \frac{|\xi_h|^2}{|\xi|^2}W_1(t-\tau)\widehat{(u\cdot\nabla\theta)}(\tau)\,d\tau,\\ \widehat{\theta}(\xi,t) =& -W_1\widehat{u}_{03}+(W_3+|\xi_h|^2W_1)\widehat{\theta}_0+\int_0^t W_1(t-\tau)\big(\mathbb{P}\,\widehat{(u\cdot\nabla u)}\big)_3(\tau)\,d\tau\\&-\int_0^t \big(W_3+|\xi_h|^2W_1\big)(t-\tau)\widehat{(u\cdot\nabla\theta)}(\tau)\,d\tau,\end{aligned} \tag{3.1}$$

where we have written

$$\begin{aligned}W_1(\xi,t)&=\frac{e^{\lambda_4 t}-e^{\lambda_3 t}}{\lambda_4-\lambda_3}=e^{-|\xi_h|^2t}\left(\frac{|\xi_h|}{|\xi|}\right)^{-1}\sin\frac{|\xi_h|}{|\xi|}t,\\ W_2(\xi,t)&=\frac{\lambda_3e^{\lambda_4 t}-\lambda_4e^{\lambda_3 t}}{\lambda_4-\lambda_3}=\lambda_3W_1-e^{\lambda_3 t},\\ W_3(\xi,t)&=\frac{\lambda_4e^{\lambda_4 t}-\lambda_3e^{\lambda_3 t}}{\lambda_4-\lambda_3}=\lambda_3W_1+e^{\lambda_4 t},\end{aligned}$$

with $\lambda_1, \lambda_2, \lambda_3$ and λ_4 given by

$$\lambda_1=\lambda_2=-|\xi_h|^2,\ \ \lambda_3=-|\xi_h|^2-\frac{|\xi_h|}{|\xi|}i,\ \ \lambda_4=-|\xi_h|^2+\frac{|\xi_h|}{|\xi|}i.$$

3.2. *Optimal decays for nonlinear terms*

We are now ready to prove the decay estimates in Theorem 1.2, which rely on bootstrap argument and Theorem 1.1.

Proof of Theorem 1.2. The framework of the proof is bootstrap argument. A very useful abstract version of the bootstrap principle and some simple examples can be found in T. Tao's book (see [27]). The H^1-norm of initial data (u_0,θ_0) is assumed to be small, namely,

$$\|u_0\|_{H^1}+\|\theta_0\|_{H^1}\le\epsilon$$

for some sufficiently small $\epsilon>0$, and we write

$$\epsilon_0:=\|(u_0,\theta_0)\|_{\dot{H}^{-\sigma,0}}+\|(\partial_3u_0,\partial_3\theta_0)\|_{\dot{H}^{-\sigma,0}}<\infty.$$

Note that, ϵ_0 here is not assumed to be small. We make the ansatz that, for $t \in [0, T]$ with $T > 0$,

$$\|(u,\theta)(t)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u, \partial_3\theta)(t)\|_{\dot{H}^{-\sigma,0}} \le 3\epsilon_0. \tag{3.2}$$

The initial time $T > 0$ exists by local well-posedness. We then show via (3.2) that (u,θ) actually satisfies the following improved inequalities, for all $t \in [0, T]$,

$$\|(u,\theta)(t)\|_{\dot{H}^{-\sigma,0}} + \|(\partial_3 u, \partial_3\theta)(t)\|_{\dot{H}^{-\sigma,0}} \le 2\epsilon_0. \tag{3.3}$$

Then the bootstrap argument implies that the maximal time T with this property is given by $T = \infty$. Thus, the inequality (3.3) indeed holds for all $t < \infty$. The rest of proof is devoted to showing (3.3) and the proof is divided into five parts as follows.

Step 1. Decay rate of $\|(\nabla u, \nabla\theta)\|_{L^2}$. We first estimate the decay rate of $\|(u, \theta, \partial_3 u, \partial_3\theta)\|_{L^2}$. Applying ∂_3 to the first two equations of (1.3), and taking the inner product with $\langle \partial_3 u, \partial_3 b\rangle$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3 u, \partial_3\theta)\|_{L^2}^2 + \nu\|\nabla_h \partial_3 u\|_{L^2}^2 + \eta\|\nabla_h\partial_3\theta\|_{L^2}^2\\
&= -\int \partial_3(u\cdot\nabla u)\cdot\partial_3 u\, dx - \int \partial_3(u\cdot\nabla\theta)\cdot\partial_3\theta\, dx\\
&:= M_1 + M_2,
\end{aligned}$$

where we have used the fact that

$$\int \partial_3(\theta e_3)\cdot\partial_3 u dx - \int \partial_3 u_3\cdot\partial_3\theta dx = 0.$$

To bound M_1, we decompose M_1 as

$$\begin{aligned}
M_1 &= -\int \partial_3 u_h\cdot\nabla_h u\cdot\partial_3 u\, dx - \int \partial_3 u_3\partial_3 u\cdot\partial_3 u\, dx\\
&:= M_{11} + M_{12}.
\end{aligned}$$

M_{11} involves the favorable partial derivatives in x_1 and x_2, respectively. According to Lemma 2.1,

$$\begin{aligned}
M_{11} &\le C\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_{13}u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_{23}u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}}\\
&\le C\|\partial_3 u\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 u\|_{L^2}^{\frac{3}{2}}\\
&\le C\|\partial_3 u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\partial_3 u\|_{L^2}^2).
\end{aligned}$$

In term of M_{12}, due to the lack of dissipation in the vertical direction, we use $\nabla\cdot u = 0$ and Lemma 2.1,

$$M_{12} = \int \nabla_h\cdot u_h\partial_3 u\cdot\partial_3 u\, dx \le C\|\partial_3 u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h\partial_3 u\|_{L^2}^2).$$

Thus

$$M_1 \le C\|\partial_3 u\|_{L^2}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2).$$

We proceed to estimate M_2, similarly to the estimate of M_1 and applying Lemma 2.1,

$$\begin{aligned}
M_2 =& -\int \partial_3 u_h \cdot \nabla_h \theta \cdot \partial_3 \theta \, dx - \int \partial_3 u_3 \partial_3 \theta \cdot \partial_3 \theta \, dx \\
\le& C\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\partial_{13} u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{23}\theta\|_{L^2}^{\frac{1}{2}} \|\nabla_h \theta\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h \theta\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_3 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{13}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_3 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{23}\theta\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|\partial_{33} u_3\|_{L^2}^{\frac{1}{2}} \\
\le& C\|(\partial_3 u, \partial_3 \theta)\|_{L^2} \|(\nabla_h u, \nabla_h \theta, \nabla_h \partial_3 u, \nabla_h \partial_3 \theta)\|_{L^2}^2.
\end{aligned}$$

Combining the estimates above yield that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3 u, \partial_3 \theta)\|_{L^2}^2 + \nu\|\nabla_h \partial_3 u\|_{L^2}^2 + \eta\|\nabla_h \partial_3 \theta\|_{L^2}^2 \\
&\le C\|(\partial_3 u, \partial_3 \theta)\|_{L^2} \|(\nabla_h u, \nabla_h \theta, \nabla_h \partial_3 u, \nabla_h \partial_3 \theta)\|_{L^2}^2.
\end{aligned}$$

Together this with (2.1), we arrive that,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u, \theta, \partial_3 u, \partial_3 \theta)\|_{L^2}^2 + \nu\|(\nabla_h u, \nabla_h \partial_3 u)\|_{L^2}^2 + \eta\|(\nabla_h \theta, \nabla_h \partial_3 \theta)\|_{L^2}^2 \\
&\le C\|(\partial_3 u, \partial_3 \theta)\|_{L^2} \|(\nabla_h u, \nabla_h \theta, \nabla_h \partial_3 u, \nabla_h \partial_3 \theta)\|_{L^2}^2.
\end{aligned} \tag{3.4}$$

Then, for sufficiently small ϵ with $\|u\|_{H^1} + \|\theta\|_{H^1} \le \frac{1}{2}\min\{\nu, \eta\}$, we get

$$\frac{d}{dt}\|(u, \theta, \partial_3 u, \partial_3 \theta)\|_{L^2}^2 + C_0\|(\nabla_h u, \nabla_h \theta, \nabla_h \partial_3 u, \nabla_h \partial_3 \theta)\|_{L^2}^2 \le 0.$$

Applying Hölder inequality in the frequency space and using (3.2), we find

$$\|u\|_{L^2} \le \|u\|_{\dot{H}^{-\sigma,0}}^{\frac{1}{1+\sigma}} \|\nabla_h u\|_{L^2}^{\frac{\sigma}{1+\sigma}} \le C\|\nabla_h u\|_{L^2}^{\frac{\sigma}{1+\sigma}}.$$

Similarly, we also obtain

$$\begin{aligned}
\|\theta\|_{L^2} &\le C\|\nabla_h \theta\|_{L^2}^{\frac{\sigma}{1+\sigma}}, \\
\|\partial_3 u\|_{L^2} &\le C\|\nabla_h \partial_3 u\|_{L^2}^{\frac{\sigma}{1+\sigma}}, \\
\|\partial_3 \theta\|_{L^2} &\le C\|\nabla_h \partial_3 \theta\|_{L^2}^{\frac{\sigma}{1+\sigma}}.
\end{aligned}$$

Inserting these estimates in (3.4), there exists, for a positive constant $C_0 > 0$,

$$\frac{d}{dt}\|(u,\theta,\partial_3 u,\partial_3\theta)\|_{L^2}^2+C_0(\|(u,\theta,\partial_3 u,\partial_3\theta)\|_{L^2}^2)^{\frac{1+\sigma}{\sigma}}\le 0.$$

Integrating in time yields

$$\|(u,\theta,\partial_3 u,\partial_3\theta)\|_{L^2}\le C(1+t)^{-\frac{\sigma}{2}}. \tag{3.5}$$

Then, similar as the estimate of M_1 and M_2, applying ∇_h to the first two equations, and dotting with $\langle\nabla_h u,\nabla_h\theta\rangle$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla_h u,\nabla_h\theta)\|_{L^2}^2+\nu\|\nabla_h^2 u\|_{L^2}^2+\eta\|\nabla_h^2\theta\|_{L^2}^2\\&=-\int\nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,dx-\int\nabla_h(u\cdot\nabla\theta)\cdot\nabla_h\theta\,dx\\&:=N_1+N_2.\end{aligned}$$

Similar to the estimate of the terms M_1 and M_2, one has

$$\begin{aligned}N_1=&-\int\nabla_h u_h\cdot\nabla_h u\cdot\nabla_h u\,dx-\int\nabla_h u_3\partial_3 u\cdot\nabla_h u\,dx\\&\le C\|\nabla_h u\|_{L^2}^{\frac{3}{2}}\|\nabla_h^2 u_h\|_{L^2}\|\nabla_h\partial_3 u\|_{L^2}^{\frac{1}{2}}+\|\nabla_h u\|_{L^2}\|\nabla_h^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 u\|_{L^2}\\&\le C\|\nabla u\|_{L^2}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2).\end{aligned}$$

In terms of N_2, we obtain in a similar manner as the derivation of M_2 that

$$\begin{aligned}N_2=&-\int\nabla_h u_h\cdot\nabla_h\theta\cdot\nabla_h\theta\,dx-\int\nabla_h u_3\partial_3\theta\cdot\nabla_h\theta\,dx\\&\le C\|(\nabla u,\nabla\theta)\|_{L^2}\|(\nabla_h u,\nabla_h\theta,\nabla_h^2 u,\nabla_h^2\theta,\nabla_h\partial_3 u,\nabla_h\partial_3\theta)\|_{L^2}^2.\end{aligned}$$

The estimates of the terms N_1 and N_2 imply that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla_h u,\nabla_h\theta)\|_{L^2}^2+\nu\|\nabla_h^2 u\|_{L^2}^2+\eta\|\nabla_h^2\theta\|_{L^2}^2\\&\le C\|(\nabla u,\nabla\theta)\|_{L^2}\|(\nabla_h u,\nabla_h\theta,\nabla_h^2 u,\nabla_h^2\theta,\nabla_h\partial_3 u,\nabla_h\partial_3\theta)\|_{L^2}^2.\end{aligned}$$

Adding this to (3.4) and using $\|u\|_{H^1}+\|\theta\|_{H^1}\le\frac{1}{2}\min\{\nu,\eta\}$, we have

$$\frac{d}{dt}\|(u,\theta,\nabla u,\nabla\theta)\|_{L^2}^2+C_0\|(\nabla_h u,\nabla_h\theta,\nabla_h\nabla u,\nabla_h\nabla\theta)\|_{L^2}^2\le 0.$$

It suffices to obtain the decay rate of $\|(\nabla_h u,\nabla_h\theta)\|_{L^2}$, we then apply the Gagliardo-Nirenberg inequality and together with (3.2), we have

$$\|\nabla_h u\|_{L^2}\le\|u\|_{\dot H^{-\sigma,0}}^{\frac{1}{1+\sigma}}\|\nabla_h^2 u\|_{L^2}^{\frac{1+\sigma}{2+\sigma}}\le C\|\nabla_h^2 u\|_{L^2}^{\frac{\sigma}{1+\sigma}},$$

here we have used the fact $\frac{1+\sigma}{2+\sigma} > \frac{\sigma}{1+\sigma}$ and $\|\nabla_h^2 u\|_{L^2} \le C$ in the last inequality. Combining these estimates yield that

$$\frac{d}{dt}\|(u,\theta,\nabla u,\nabla\theta)\|_{L^2}^2 + C_0\big(\|(u,\theta,\nabla u,\nabla\theta)\|_{L^2}^2\big)^{\frac{1+\sigma}{\sigma}} \le 0.$$

Integrating with respect to t, we have

$$\|(\nabla_h u, \nabla_h\theta)\|_{L^2} \le C(1+t)^{-\frac{\sigma}{2}}. \tag{3.6}$$

Step 2. Improved decay rate for $\|(\nabla_h u, \nabla_h\theta)\|_{L^2}$. This subsection provides an improved upper bound for $\|\nabla_h u\|_{L^2}$ and $\|\nabla_h\theta\|_{L^2}$. To estimate $\|\nabla_h u\|_{L^2}$, we treat $\|\nabla_h u_h\|_{L^2}$ and $\|\nabla_h u_3\|_{L^2}$, respectively. Since $\|\nabla_h u_3\|_{L^2}$ has the same bound as $\|\nabla_h u\|_{L^2}$, here we only deal with $\|\nabla_h u_h\|_{L^2}$. Applying ξ_h to (3.1) and taking the L^2-norm,

$$\begin{aligned}
\|\nabla_h u_h\|_{L^2} =& \|\widehat{\nabla_h u_h}\|_{L^2} = \|\xi_h \widehat{u}_h\|_{L^2} \\
\le& \Big\| e^{\lambda_1 t}\widehat{\nabla_h u_{0h}} + \Big(\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1 t} + \frac{\xi_h\xi_3}{|\xi_h|^2}W_2 + \xi_h\xi_3 W_1\Big)\widehat{\nabla_h u_{03}} - \frac{\xi_h\xi_3}{|\xi|^2}W_1\widehat{\nabla_h\theta_0}\Big\|_{L^2} \\
&+ \int_0^t \Big\| e^{\lambda_1(t-\tau)}\big(\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}\big)(\tau)\Big\|_{L^2} d\tau \\
&+ \int_0^t \Big\|\Big(\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1(t-\tau)} + \frac{\xi_h\xi_3}{|\xi_h|^2}W_2 + \xi_h\xi_3 W_1\Big)\big(\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}\big)_3(\tau)\Big\|_{L^2} d\tau \\
&+ \int_0^t \Big\|\frac{\xi_h\xi_3}{|\xi|^2}W_1(t-\tau)(\nabla_h\widehat{(u\cdot\nabla\theta)})(\tau)\Big\|_{L^2} d\tau \\
:=& L_1 + L_2 + L_3 + L_4,
\end{aligned}$$

for any $t \ge 0$, we first establish the estimate of the term L_1,

$$\begin{aligned}
L_1 \le& \|e^{\lambda_1 t}\widehat{\nabla_h u_{0h}}\|_{L^2} + \Big\|\big(\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1 t} + \frac{\xi_h\xi_3}{|\xi_h|^2}W_2 + \xi_h\xi_3 W_1\big)\widehat{\nabla_h u_{03}}\Big\|_{L^2} + \Big\|\frac{\xi_h\xi_3}{|\xi|^2}W_1\widehat{\nabla_h\theta_0}\Big\|_{L^2} \\
:=& L_{11} + L_{12} + L_{13}.
\end{aligned}$$

It follows from Lemma 3.3 that

$$L_{11} \le C(1+t)^{-\frac{\sigma+1}{2}}(\|\Lambda_h^{-\sigma}u_{0h}\|_{L^2} + \|u_{0h}\|_{L^2}).$$

In order to estimate the terms of L_{12} and L_{13}, we provide upper bounds for $W_1 - W_3$. For any $\xi \in \mathbb{R}^3$ and $t \ge 0$. The upper bound for W_1 follows directly from the definition of W_1 in Lemma 3.5 and the simple fact that $|\sin y| \le |y|$ for any real number y. Then

$$|W_1(\xi,t)| = \left|e^{-|\xi_h|^2 t}(\frac{|\xi_h|}{|\xi|})^{-1}\sin\frac{|\xi_h|}{|\xi|}t\right| \le te^{-|\xi_h|^2 t}, \tag{3.7}$$

and for a constant $C_1 > 0$,

$$|\xi_h|^2|W_1(\xi,t)| \le |\xi_h|^2 te^{-|\xi_h|^2 t} \le C_1 e^{-c_1|\xi_h|^2 t}, \tag{3.8}$$

where we have used the fact that $xe^{-x} \le e^{-c_1 x}$ for $x \ge 0$ and some $c_1 > 0$.

$$|W_2(\xi,t)| = |\lambda_3 W_1 - e^{\lambda_3 t}| \le |\xi_h|^2 te^{-|\xi_h|^2 t} + 2e^{-|\xi_h|^2 t} \le C_1 e^{-c_1|\xi_h|^2 t}. \tag{3.9}$$

Similarly, W_3 can be bound that

$$|W_3(\xi,t)| = |\lambda_3 W_1 + e^{\lambda_4 t}| \le C_1 e^{-c_1|\xi_h|^2 t}.$$

Here, we elucidate that we would only give the estimates for $t \ge 1$ in the rest of the proof. By $\xi_3\widehat{u}_{03} = -\xi_h \cdot \widehat{u}_{0h}$ and Lemma 3.3,

$$\begin{aligned}
L_{12} &= \left\|\xi_h(\frac{\xi_h}{|\xi_h|^2}e^{\lambda_1 t} + \frac{\xi_h}{|\xi_h|^2}W_2 + \xi_h W_1)\xi_3\widehat{u}_{03}\right\|_{L^2} \\
&= \left\|(e^{\lambda_1 t} + W_2 + |\xi_h|^2 W_1)\xi_h \cdot \widehat{u}_{0h}\right\|_{L^2} \\
&\le C\left\|e^{-c_1|\xi_h|^2 t}\xi_h \cdot \widehat{u}_{0h}\right\|_{L^2} \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}\|\Lambda_h^{-\sigma}u_{0h}\|_{L^2} \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}.
\end{aligned}$$

Then,

$$L_{12} \le C(1+t)^{-\frac{\sigma+1}{2}}(\|\Lambda_h^{-\sigma}u_{0h}\|_{L^2} + \|u_{0h}\|_{L^2})$$

As for L_{13}, we first bound the kernel. Recalling the definition of $W_1(\xi,t)$,

$$|\frac{\xi_h\xi_3}{|\xi|^2}W_1(\xi,t)| \le |\frac{\xi_h\xi_3}{|\xi|^2}e^{-|\xi_h|^2 t}(\frac{\xi_h}{|\xi|})^{-1}\sin\frac{|\xi_h|}{|\xi|}t| \le e^{-|\xi_h|^2 t}, \tag{3.10}$$

we use Lemma 3.3 and the bound of (3.10), we have

$$\begin{aligned}
\left\|\frac{\xi_h\xi_3}{|\xi|^2}W_1\widehat{\nabla_h\theta_0}\right\|_{L^2} &\le \left\|\xi_h e^{-|\xi_h|^2 t}\widehat{\theta_0}\right\|_{L^2} \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}(\|\Lambda_h^{-\sigma}\theta_0\|_{L^2} + \|\theta_0\|_{L^2}) \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}.
\end{aligned}$$

Combining the above estimates, it follows that

$$L_1 \le C(1+t)^{-\frac{\sigma+1}{2}}.$$

To bound L_{12}, we further write it into two terms,

$$
\begin{aligned}
L_2 &\le \int_0^t \left\| \nabla_h e^{\Delta_h (t-\tau)} \big(u \cdot \nabla u\big)_h(\tau) \right\|_{L^2} d\tau \\
&= \int_0^t \left\| \nabla_h e^{\Delta_h (t-\tau)} \big(u_h \cdot \nabla_h u_h\big)(\tau) \right\|_{L^2} d\tau + \int_0^t \left\| \nabla_h e^{\Delta_h (t-\tau)} \big(u_3 \partial_3 u_h\big)(\tau) \right\|_{L^2} d\tau \\
&:= L_{21} + L_{22},
\end{aligned}
$$

where we have used the boundedness of Riesz transform on L^2-norm

$$
\|\nabla \Delta^{-1} \nabla \cdot \varphi\|_{L^2} \le C \|\varphi\|_{L^2}.
$$

For $\sigma < \delta < 1$, we use Lemma 3.1−Lemma 3.4 to provide a suitable bound for L_{21} that,

$$
\begin{aligned}
L_{21} &= \int_0^t \left\| \|\nabla_h e^{\Delta_h (t-\tau)} \big(u_h \cdot \nabla_h u_h\big)(\tau)\|_{L_h^2} \right\|_{L_{x_3}^2} d\tau. \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|\big(u_h \cdot \nabla_h u_h\big)(\tau)\|_{L_h^{\frac{2}{1+\delta}}} \right\|_{L_{x_3}^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|\big(u_h \cdot \nabla_h u_h\big)(\tau)\|_{L_{x_3}^2} \right\|_{L_h^{\frac{2}{1+\delta}}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|u_h(\tau)\|_{L_{x_3}^\infty} \|\nabla_h u_h\big(\tau\big)\|_{L_{x_3}^2} \right\|_{L_h^{\frac{2}{1+\delta}}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| u_h(\tau) \right\|_{L_h^{\frac{2}{\delta}} L_{x_3}^\infty} \left\| \nabla_h u_h(\tau) \right\|_{L^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|u_h(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 u_h(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \right\|_{L_h^{\frac{2}{\delta}}} \left\| \nabla_h u_h(\tau) \right\|_{L^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|u_h(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \right\|_{L_h^{\frac{4}{2\delta-1}}} \left\| \|\partial_3 u_h(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \right\|_{L_h^4} \left\| \nabla_h u_h(\tau) \right\|_{L^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \left\| \|u_h(\tau)\|_{L_h^{\frac{2}{2\delta-1}}} \right\|_{L_{x_3}^2}^{\frac{1}{2}} \left\| \partial_3 u_h(\tau) \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla_h u_h(\tau) \right\|_{L^2} d\tau
\end{aligned}
$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \big\| \|u_h(\tau)\|_{L_h^2}^{2\delta-1} \|\nabla_h u_h(\tau)\|_{L_h^2}^{2-2\delta} \big\|_{L_{x_3}^2}^{\frac{1}{2}} \big\|\partial_3 u_h(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\nabla_h u_h(\tau)\big\|_{L^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u_h(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \big\|\partial_3 u_h(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\nabla_h u_h(\tau)\big\|_{L^2}^{2-\delta} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \big\|\partial_3 u(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\nabla_h u(\tau)\big\|_{L^2}^{2-\delta} d\tau.$$

By virtue of the incompressible condition $\partial_3 u_3 = -\nabla_h \cdot u_h$ and Lemma 3.1−Lemma 3.3, it follows from the similar estimate of L_{21},

$$L_{22} = \int_0^t \big\| \|\nabla_h e^{\Delta_h(t-\tau)}\big(u_3\partial_3 u_h\big)(\tau)\|_{L_h^2}\big\|_{L_{x_3}^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \big\| \|\big(u_3\partial_3 u_h\big)(\tau)\|_{L_h^{\frac{2}{1+\sigma}}}\big\|_{L_{x_3}^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \big\| \|u_3(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \|_{L_h^{\frac{4}{2\delta-1}}} \big\| \|\partial_3 u_3(\tau)\|_{L_{x_3}^2}^{\frac{1}{2}} \big\|_{L_h^4} \big\|\partial_3 u_h(\tau)\big\|_{L^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \big\| \|u_3(\tau)\|_{L_h^{\frac{2}{2\delta-1}}} \big\|_{L_{x_3}^2}^{\frac{1}{2}} \big\|\nabla_h \cdot u_h(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\partial_3 u_h(\tau)\big\|_{L^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \big\| \|u_3(\tau)\|_{L_h^2}^{2\delta-1} \|\nabla_h u_3(\tau)\|_{L_h^2}^{2-2\delta} \big\|_{L_{x_3}^2}^{\frac{1}{2}} \big\|\nabla_h u_h(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\partial_3 u_h(\tau)\big\|_{L^2} d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\big\|_{L^2}^{\frac{3}{2}-\delta} \big\|\partial_3 u(\tau)\big\|_{L^2} d\tau.$$

Incorporating these upper bounds, we obtain

$$L_2 \leq C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \big\|\partial_3 u(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|\nabla_h u(\tau)\big\|_{L^2}^{2-\delta} d\tau$$
$$+ C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\big\|_{L^2}^{\frac{3}{2}-\delta} \big\|\partial_3 u(\tau)\big\|_{L^2} d\tau.$$

For the third term L_3, in order to bound

$$\left|\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1(t-\tau)}+\frac{\xi_h\xi_3}{|\xi_h|^2}W_2+\xi_h\xi_3W_1\right|,$$

we need to get the factor ξ_h from $(\mathbb{P}(u\cdot\nabla u))_3$. By means of the definition of $\mathbb{P}=I-\nabla\Delta^{-1}\nabla\cdot$, we obtain that

$$\begin{aligned}(\mathbb{P}(u\cdot\nabla u))_3=&u\cdot\nabla u_3-\partial_3\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\\=&\partial_1(u_1u_3)+\partial_2(u_2u_3)+\partial_3(u_3u_3)\\&-\partial_3\Delta^{-1}(\partial_1(u\cdot\nabla u_1)+\partial_2(u\cdot\nabla u_2)+\partial_3(u\cdot\nabla u_3))\\=&\partial_1(u_1u_3)+\partial_2(u_2u_3)-\partial_3\Delta^{-1}(\partial_1\nabla\cdot(uu_1)+\partial_2\nabla\cdot(uu_2))\\&+\partial_3(u_3u_3)-\partial_3\Delta^{-1}\partial_{31}(u_1u_3)-\partial_3\Delta^{-1}\partial_{32}(u_2u_3)-\partial_3\Delta^{-1}\partial_{33}(u_3u_3)\\=&\partial_1(u_1u_3)+\partial_2(u_2u_3)-\partial_3\Delta^{-1}(\partial_1\nabla\cdot(uu_1)+\partial_2\nabla\cdot(uu_2))\\&-\partial_3\Delta^{-1}\partial_{31}(u_1u_3)-\partial_3\Delta^{-1}\partial_{32}(u_2u_3)+\Delta^{-1}(\Delta\partial_3(u_3u_3)-\partial_{333}(u_3u_3))\\=&\partial_1(u_1u_3)+\partial_2(u_2u_3)-\partial_3\Delta^{-1}(\partial_1\nabla\cdot(uu_1)+\partial_2\nabla\cdot(uu_2))\\&-\partial_3\Delta^{-1}\partial_{31}(u_1u_3)-\partial_3\Delta^{-1}\partial_{32}(u_2u_3)+\Delta^{-1}\Delta_h\partial_3(u_3u_3)\\:=&F_1+F_2+F_3+F_4,\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}&F_1=\nabla_h\cdot(u_hu_3),\\&F_2=-\nabla_h\cdot\partial_3\Delta^{-1}\nabla\cdot(u\otimes u_h),\\&F_3=-\nabla_h\cdot\Delta^{-1}\partial_{33}(u_hu_3),\\&F_4=\Delta^{-1}\Delta_h\partial_3(u_3u_3),\end{aligned}$$

and we also have used the fact that

$$\partial_3(u_3u_3)-\partial_3\Delta^{-1}\partial_{33}(u_3u_3)=\partial_3\Delta^{-1}\Delta_h(u_3u_3).$$

According to the estimates of $W_1(\xi,t)$ and $W_2(\xi,t)$ in (3.7) – (3.9), it follows that

$$\begin{aligned}&\left|\Big(\frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1(t-\tau)}+\frac{\xi_h\xi_3}{|\xi_h|^2}W_2+\xi_h\xi_3W_1\Big)\big(\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}\big)_3\right|\\=&\left|\xi_h\Big(\frac{\xi_h}{|\xi_h|^2}e^{\lambda_1(t-\tau)}+\frac{\xi_h}{|\xi_h|^2}W_2+\xi_hW_1\Big)\big(\xi_3\widehat{\mathbb{P}(u\cdot\nabla u)}\big)_3\right|\\\le&C|\xi_h|e^{-c_1|\xi_h|^2(t-\tau)}\big(|\widehat{\partial_3(u_hu_3)}|+|\widehat{\nabla\cdot(uu_h)}|+|\widehat{\nabla_h\cdot(u_3u_3)}|\big).\end{aligned}$$

Thus, we have

$$
\begin{aligned}
L_3 \le & C \int_0^t \|\nabla_h e^{c_1\Delta_h(t-\tau)} \partial_3(u_h u_3)\|_{L^2}\, d\tau + C \int_0^t \|\nabla_h e^{c_1\Delta_h(t-\tau)} \nabla \cdot (u u_h)\|_{L^2}\, d\tau \\
& + C \int_0^t \|\nabla_h e^{c_1\Delta_h(t-\tau)} \nabla_h \cdot (u_3 u_3)\|_{L^2}\, d\tau.
\end{aligned}
$$

Similarly, L_3 can be bounded as L_2 that

$$
\begin{aligned}
L_3 \le & C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \left\|\nabla_h u(\tau)\right\|_{L^2}^{2-\delta} \left\|\partial_3 u(\tau)\right\|_{L^2}^{\frac{1}{2}}\, d\tau \\
& + C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{3}{2}-\delta} \left\|\partial_3 u(\tau)\right\|_{L^2}\, d\tau.
\end{aligned}
$$

By virtue of the inequality (3.10) and the similar estimate of L_2, we have

$$
\begin{aligned}
L_4 \le & C \int_0^t \left\|\nabla_h e^{\Delta_h(t-\tau)} (u_h \cdot \nabla_h \theta + u_3 \partial_3 \theta)(\tau)\right\|_{L^2}\, d\tau \\
\le & C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \left\|\partial_3 u(\tau)\right\|_{L^2}^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{1-\delta} \left\|\nabla_h \theta(\tau)\right\|_{L^2}\, d\tau \\
& + C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{3}{2}-\delta} \left\|\partial_3 \theta(\tau)\right\|_{L^2}\, d\tau.
\end{aligned}
$$

The estimate of $\|\nabla_h u_3\|_{L^2}$ is similarly as $\|\nabla_h u_h\|_{L^2}$, we omitted here, then

$$
\begin{aligned}
\|\nabla_h u\|_{L^2} \le & C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \left\|\partial_3 u(\tau)\right\|_{L^2}^{\frac{1}{2}} \|(\nabla_h u, \nabla_h \theta)(\tau)\|_{L^2}^{2-\delta}\, d\tau \\
& + C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{3}{2}-\delta} \left\|(\partial_3 u, \partial_3 \theta)(\tau)\right\|_{L^2}\, d\tau \\
& + C(1+t)^{-\frac{1+\sigma}{2}}.
\end{aligned}
$$

Next, we estimate the decay of $\|\nabla_h \theta\|_{L^2}$. By Plancherel formula, one has

$$
\|\nabla_h \theta\|_{L^2} \le \|\xi_h W_1 \widehat{u}_{03}\|_{L^2} + \|\xi_h (W_3 + |\xi_h|^2 W_1) \widehat{\theta_0}\|_{L^2}
$$

$$
\begin{aligned}
&+\int_0^t \|\xi_h W_1(t-\tau)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3(\tau)\|_{L^2}\,d\tau \\
&+\int_0^t \|\xi_h (W_3+|\xi_h|^2 W_1)(t-\tau)\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}\,d\tau \\
:=&G_1+G_2+G_3+G_4.
\end{aligned}
$$

The estimate of first term for $\|\nabla_h\theta\|_{L^2}$ needs more attention. By $\xi_3\widehat{u}_{03}=-\xi_h\cdot\widehat{u}_{0h}$ and Lemma 3.3, we deduce

$$
\begin{aligned}
G_1 &\le \|\xi_h e^{-|\xi_h|^2 t}\big(\frac{|\xi_h|}{|\xi|}\big)^{-1}\sin\frac{|\xi_h|t}{|\xi|}\widehat{u}_{03}\|_{L^2} \\
&\le \|\xi_h e^{-|\xi_h|^2 t}\frac{|\xi|}{|\xi_h|}\widehat{u}_{03}\|_{L^2} \\
&\le \|\xi_h e^{-|\xi_h|^2 t}\frac{|\xi_h|+|\xi_3|}{|\xi_h|}\widehat{u}_{03}\|_{L^2} \\
&\le \|\xi_h e^{-|\xi_h|^2 t}\widehat{u}_{03}\|_{L^2}+\|e^{-|\xi_h|^2 t}\xi_h\cdot\widehat{u}_{0h}\|_{L^2} \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}(\|\Lambda_h^{-\sigma}u_0\|_{L^2}+\|u_0\|_{L^2}) \\
&\le C(1+t)^{-\frac{\sigma+1}{2}}.
\end{aligned}
$$

For G_2, which is similar to the estimate of L_1, we have

$$
G_2\le C(1+t)^{-\frac{\sigma+1}{2}}(\|\Lambda_h^{-\sigma}\theta_0\|_{L^2}+\|\theta_0\|_{L^2})\le C(1+t)^{-\frac{\sigma+1}{2}}.
$$

To bound G_3, we notice that

$$
\begin{aligned}
&\Big|\xi_h W_1(t-\tau)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3\Big| \\
\le&\Big|\xi_h e^{-|\xi_h|^2(t-\tau)}\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3\Big|+\Big|\xi_h e^{-|\xi_h|^2(t-\tau)}\frac{|\xi_3|}{|\xi_h|}\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3\Big|,
\end{aligned}
$$

which likes the term in L_3, and then

$$
\begin{aligned}
G_3\le&C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}}\|u(\tau)\|_{L^2}^{\delta-\frac12}\big\|\partial_3 u(\tau)\big\|_{L^2}^{\frac12}\|\nabla_h u(\tau)\big\|_{L^2}^{2-\delta}\,d\tau \\
&+C\int_0^t (t-\tau)^{-\frac{1+\delta}{2}}\|u(\tau)\|_{L^2}^{\delta-\frac12}\|\nabla_h u(\tau)\big\|_{L^2}^{\frac32-\delta}\big\|\partial_3 u(\tau)\big\|_{L^2}\,d\tau.
\end{aligned}
$$

With help of the estimates of W_1 and W_2 in (3.7)–(3.9), G_4 has a same estimate as L_4, namely

$$G_4 \leq C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{1-\delta} \|\nabla_h \theta(\tau)\|_{L^2} d\tau$$
$$+ C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{3}{2}-\delta} \|\partial_3 \theta(\tau)\|_{L^2} d\tau.$$

Adding the decay estimates of $\|(\nabla_h u, \nabla_h \theta)\|_{L^2}$ above give

$$\|(\nabla_h u, \nabla_h \theta)\|_{L^2} \leq C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}} \|(\nabla_h u, \nabla_h \theta)(\tau)\|_{L^2}^{2-\delta} d\tau$$
$$+ C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} \|u(\tau)\|_{L^2}^{\delta-\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{3}{2}-\delta} \|(\partial_3 u, \partial_3 \theta)(\tau)\|_{L^2} d\tau$$
$$+ C(1+t)^{-\frac{1+\sigma}{2}}. \tag{3.12}$$

Using the decay estimate of (3.5) and (3.6), combining these above estimates, for $\sigma < \delta < 1$, we can obtain from Lemma 3.4 that

$$\begin{aligned}\|(\nabla_h u, \nabla_h \theta)\|_{L^2} &\leq C(1+t)^{-\frac{1+\sigma}{2}} + C \int_0^t (t-\tau)^{-\frac{1+\delta}{2}} (1+\tau)^{-\sigma} d\tau \\ &\leq C(1+t)^{-\frac{1+\sigma}{2}} + C(1+t)^{-(\frac{\delta}{2}+\sigma-\frac{1}{2})} \\ &\leq C(1+t)^{-(\frac{\delta}{2}+\sigma-\frac{1}{2})}.\end{aligned} \tag{3.13}$$

From (3.13), we observe that the decay estimate of $\|(\nabla_h u, \nabla_h \theta)\|_{L^2}$ can be improved by an iterative procedure. Inserting (3.13) in (3.12) and using the decay rate of $\|(u, \theta, \partial_3 u, \partial_3 \theta)\|_{L^2}$ in (3.5). Combining the above estimates. Repeating this procedure n times, we have

$$\|(\nabla_h u, \nabla_h \theta)\|_{L^2} \leq C(1+t)^{-\frac{1+\sigma}{2}} + C(1+t)^{-\min\{\alpha_n, \frac{1+\sigma}{2}\}},$$

where

$$\alpha_n = \alpha_0 + (\alpha_{n-1} - \frac{\sigma}{2})(\frac{3}{2} - \delta),$$

and

$$\alpha_0 = \frac{\delta}{2} + \sigma - \frac{1}{2}.$$

We claim that, by choosing $n > 1$ sufficiently large and $\delta > \sigma$ close to σ,

$$\alpha_n > \frac{1+\sigma}{2}.$$

In fact, by the iterative formula,

$$\begin{aligned}\alpha_n &= \alpha_0 + (\alpha_0 - \frac{\sigma}{2})((\frac{3}{2}-\delta) + (\frac{3}{2}-\delta)^2 + \ldots + (\frac{3}{2}-\delta)^n)\\ &= \alpha_0 + (\frac{\delta}{2} + \frac{\sigma}{2} - \frac{1}{2})(\frac{3}{2}-\delta)\frac{1-(\frac{3}{2}-\delta)^n}{\delta - \frac{1}{2}}.\end{aligned}$$

Since $\frac{1}{2} < \sigma < \delta < 1$, we have

$$0 < \frac{3}{2} - \delta < 1.$$

Therefore, as $n \to \infty$,

$$\alpha_n \to \alpha(\delta) = \alpha_0 + (\frac{\delta}{2} + \frac{\sigma}{2} - \frac{1}{2})(\frac{3}{2}-\delta)(\delta - \frac{1}{2})^{-1}.$$

Note that $\alpha(\sigma) = 1 + \frac{\sigma}{2} > \frac{1+\sigma}{2}$. Therefore, by limit-preserving property, we get $\alpha_n > \frac{1+\sigma}{2}$ with sufficiently large n. Then, we have

$$\|(\nabla_h u, \nabla_h \theta)\|_{L^2} \le C(1+t)^{-\frac{1+\sigma}{2}}.$$

Step 3. Decay rate of $\|u_3\|_{L^2}$. This subsection proves the enhanced dissipation for u_3. Multiplying (3.1) by $\widehat{u}_3$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\|\widehat{u}_3\|_{L^2}^2 \le& \int (W_2 + |\xi_h|^2 W_1)\widehat{u}_{03} \cdot \widehat{u}_3(t)\,dx + \int \frac{|\xi_h|^2}{|\xi|^2} W_1 \widehat{\theta}_0 \cdot \widehat{u}_3(t)\,dx\\ &+ \int_0^t \int \big(W_2 + |\xi_h|^2 W_1\big)(t-\tau)\big(\mathbb{P}\widehat{(u\cdot\nabla u)}\big)_3(\tau)\cdot \widehat{u}_3(t)\,dxd\tau\\ &+ \int_0^t \int \frac{|\xi_h|^2}{|\xi|^2} W_1(t-\tau)\widehat{(u\cdot\nabla\theta)}(\tau)\cdot\widehat{u}_3(t)\,dxd\tau\\ :=& K_1 + K_2 + K_3 + K_4.\end{aligned}$$

We estimate K_1 through K_4. Firstly, K_1 follows from Plancherel formula, Young inequality and the estimates of (3.7) – (3.9), we apply Lemma 3.3 to get that

$$\begin{aligned}K_1 \le& C\|(W_2 + |\xi_h|^2 W_1)\widehat{u}_{03}\|_{L^2}^2\\ \le& C\|e^{-c_1|\xi_h|^2 t}\widehat{u}_{03}\|_{L^2}^2\\ =& C\int_{|\xi_3|\le|\xi_h|} e^{-2c_1|\xi_h|^2 t}|\xi_h|^{2\sigma}|\xi_3|^{\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{u}_{03}(\xi)|^2\,d\xi\end{aligned}$$

$$
\begin{aligned}
&+C\int_{|\xi_3|>|\xi_h|} e^{-2c_1|\xi_h|^2t}|\xi_h|^{2\sigma+2}|\xi_3|^{\sigma}|\xi_3|^{-2}|\xi_h|^{-2\sigma-2}|\xi_3|^{-\sigma}|\xi_3\widehat{u}_{03}(\xi)|^2\,d\xi\\
\le& C\int_{|\xi_3|\le|\xi_h|} e^{-2c_1|\xi_h|^2t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{u}_{03}(\xi)|^2\,d\xi\\
&+C\int_{|\xi_3|>|\xi_h|} e^{-2c_1|\xi_h|^2t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{u}_{0h}(\xi)|^2\,d\xi\\
\le& C(1+t)^{-\frac{3\sigma}{2}}\int_{|\xi_3|\le|\xi_h|}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{u}_{03}(\xi)|^2\,d\xi\\
&+C(1+t)^{-\frac{3\sigma}{2}}\int_{|\xi_3|>|\xi_h|}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{u}_{0h}(\xi)|^2\,d\xi\\
\le& C(1+t)^{-\frac{3\sigma}{2}}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}}\widehat{u}_0\|_{L^2}^2,
\end{aligned}
$$

where we have used the divergence free condition $\xi_3\widehat{u}_{03}=-\xi_h\cdot\widehat{u}_{0h}$ and the fact that for any $\alpha>0$,

$$
\left\|e^{-|\xi_h|^2(t-\tau)}|\xi_h|^{2\alpha}\right\|_{L^\infty_{\xi_h}}\le C(t-\tau)^{\alpha}. \tag{3.14}
$$

To bound K_2, using Lemma 3.3 and (3.14), for $\frac{1}{2}<\sigma<1$, one has

$$
\begin{aligned}
K_2\le& C\left\|\frac{|\xi_h|^2}{|\xi|^2}W_1\widehat{\theta}_0\right\|_{L^2}^2\\
\le& C\|e^{-|\xi_h|^2t}|\xi_h||\xi|^{-1}\widehat{\theta}_0\|_{L^2}^2\\
=& C\int_{|\xi_3|\le|\xi_h|} e^{-2|\xi_h|^2t}|\xi_h|^2|\xi|^{-2}|\xi_h|^{2\sigma}|\xi_3|^{\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta}_0|^2\,d\xi\\
&+C\int_{|\xi_3|>|\xi_h|} e^{-2|\xi_h|^2t}|\xi_h|^2|\xi|^{-2}|\xi_h|^{2\sigma}|\xi_3|^{\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta}_0|^2\,d\xi\\
\le& C\int_{|\xi_3|\le|\xi_h|} e^{-2|\xi_h|^2t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta}_0|^2\,d\xi\\
&+C\int_{|\xi_3|>|\xi_h|} e^{-2|\xi_h|^2t}|\xi_h|^{2\sigma+2}|\xi_3|^{\sigma-2}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta}_0|^2\,d\xi\\
\le& C\int_{|\xi_3|\le|\xi_h|} e^{-2|\xi_h|^2t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta}_0|^2\,d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+C\int\limits_{|\xi_3|>|\xi_h|} e^{-2|\xi_h|^2 t}|\xi_h|^{3\sigma}|\xi_h|^{-2\sigma}|\xi_3|^{-\sigma}|\widehat{\theta_0}|^2\,d\xi\\
&\le C(1+t)^{-\frac{3\sigma}{2}}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}}\widehat{\theta_0}\|_{L^2}^2,
\end{aligned}
$$

where we have used the fact that

$$
\Big|\frac{|\xi_h|^2}{|\xi|^2}W_1\Big| = \Big|\frac{|\xi_h|^2}{|\xi|^2}e^{-|\xi_h|^2 t}\Big(\frac{|\xi_h|}{|\xi|}\Big)^{-1}\sin\frac{|\xi_h|t}{|\xi|}\Big| \le \Big||\xi_h||\xi|^{-1}e^{-|\xi_h|^2 t}\Big|.
$$

It is clear that $(\mathbb{P}(u\cdot\nabla u))_3$ contains ∂_1 or ∂_2 to $F_1 - F_4$ in (3.11). That is, its Fourier transform has the desired factor ξ_h.

$$
\begin{aligned}
K_3 &\le \int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)\sum_{i=1}^4 \widehat{F_i}(\tau)\cdot\widehat{u_3}(t)\,dxd\tau\\
&=\int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)\sum_{j=1}^2 \xi_j(\widehat{u_ju_3})(\tau)\cdot\widehat{u_3}(t)\,dxd\tau\\
&\quad+\int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)\xi_3|\xi|^{-2}\sum_{j=1}^2\sum_{k=1}^3 \xi_j\xi_k(\widehat{u_ku_j})(\tau)\cdot\widehat{u_3}(t)\,dxd\tau\\
&\quad+\int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)|\xi|^{-2}|\xi_3|^2\sum_{j=1}^2 \xi_j(\widehat{u_ju_3})(\tau)\cdot\widehat{u_3}(t)\,dxd\tau\\
&\quad+\int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)|\xi_h|^2|\xi|^{-2}\xi_3(\widehat{u_3u_3})(\tau)\cdot\widehat{u_3}(t)\,dxd\tau\\
&:=K_{31}+K_{32}+K_{33}+K_{34}.
\end{aligned}
$$

Keeping the estimates of W_1 and W_2 in (3.7)−(3.9) in mind, together with Lemma 3.1−Lemma 3.4, we have

$$
\begin{aligned}
K_{31} &\le \int_0^t \big\|\big(W_2+|\xi_h|^2W_1\big)(t-\tau)\widehat{u_hu_3}\big\|_{L^2}\,d\tau\,\|\nabla_h u(t)\|_{L^2}\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t \|e^{-|\xi_h|^2(t-\tau)}\widehat{u_hu_3}\|_{L^2}\,d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t (t-\tau)^{-\frac12}\big\|\|u_hu_3\|_{L_h^1}\big\|_{L_{x_3}^2}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \left\| \|u_h\|_{L_h^2} \right\|_{L_{x_3}^2} \left\| \|u_3\|_{L_h^2} \right\|_{L_{x_3}^\infty} d\tau \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_h\|_{L^2} \left\| \|u_3\|_{L_{x_3}^\infty} \right\|_{L_h^2} d\tau \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_h\|_{L^2} \left\| \|u_3\|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 u_3\|_{L_{x_3}^2}^{\frac{1}{2}} \right\|_{L_h^2} d\tau \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_h\|_{L^2} \|u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} d\tau \\
&= C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_h\|_{L^2} \|u_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_h\|_{L^2}^{\frac{1}{2}} d\tau \\
&\le C(1+t)^{-\frac{1+\sigma}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-(\sigma-\frac{1}{4})} d\tau \\
&\le C(1+t)^{-\frac{1+2\sigma}{2}} \\
&\le C(1+t)^{-\frac{3\sigma}{2}}.
\end{aligned}
$$

Integrating by part and using the divergence-free condition $\xi_3 \widehat{u}_3 = -\xi_h \cdot \widehat{u}_h$ again, we deduce from Hölder inequality that

$$
\begin{aligned}
K_{32} &= -\int_0^t \int \big(W_2 + |\xi_h|^2 W_1\big)(t-\tau)|\xi|^{-2} \sum_{j=1}^{2}\sum_{k=1}^{3} \xi_j \xi_k (\widehat{u_k u}_j)(\tau) \cdot \xi_3 \widehat{u}_3(t)\, dx d\tau \\
&\le C \int_0^t \|\big(W_2 + |\xi_h|^2 W_1\big)(t-\tau)|\xi_h||\xi|^{-1} \widehat{u u_h}\|_{L^2} d\tau \|\nabla_h u_h(t)\|_{L^2} \\
&\le C \int_0^t \|e^{-|\xi_h|^2(t-\tau)}|\xi_h||\xi|^{-1} \widehat{u u_h}\|_{L^2} d\tau \|\nabla_h u(t)\|_{L^2}.
\end{aligned}
$$

It follows from Lemma 3.1 and Lemma 3.3, we infer from (3.14), for $2\sigma - \frac{1}{2} < \gamma < \frac{3}{2}$,

$$
\int_0^t \|e^{-|\xi_h|^2(t-\tau)}|\xi_h||\xi|^{-1} \widehat{u u_h}\|_{L^2}\, d\tau
$$

$$
\begin{aligned}
&\le C\int_0^t \Big\| \|e^{-|\xi_h|^2(t-\tau)}|\xi_h||\xi|^{-1}\widehat{uu_h}\|_{L^2_{\xi_3}}\Big\|_{L^2_h} d\tau \\
&\le C\int_0^t \Big\| \||\xi|^{-1}\|_{L^2_{\xi_3}} \|e^{-|\xi_h|^2(t-\tau)}|\xi_h|\widehat{uu_h}\|_{L^\infty_{\xi_3}}\Big\|_{L^2_h} d\tau \\
&\le C\int_0^t \Big\| \|e^{-|\xi_h|^2(t-\tau)}|\xi_h|^{\frac{1}{2}}\widehat{uu_h}\|_{L^\infty_{\xi_3}}\Big\|_{L^2_h} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \Big\| |\xi_h|^{-(\gamma-\frac{1}{2})}\|\widehat{uu_h}\|_{L^\infty_{\xi_3}}\Big\|_{L^2_h} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \Big\| \|\Lambda_h^{-(\gamma-\frac{1}{2})}(uu_h)\|_{L^2_h}\Big\|_{L^1_{x_3}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \Big\| \|(uu_h)\|_{L_h^{\frac{4}{1+2\gamma}}}\Big\|_{L^1_{x_3}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \Big\| \|(uu_h)\|^2_{L_h^{\frac{8}{1+2\gamma}}}\Big\|_{L^1_{x_3}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \Big\| \|u\|_{L^2_h}^{\frac{1+2\gamma}{4}} \|\nabla_h u\|_{L^2_h}^{\frac{3-2\gamma}{4}}\Big\|^2_{L^2_{x_3}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|u\|_{L^2}^{\frac{1+2\gamma}{2}} \|\nabla_h u\|_{L^2}^{\frac{3-2\gamma}{2}} d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{\gamma}{2}} (1+\tau)^{-\frac{\sigma(1+2\gamma)}{4}}(1+\tau)^{-\frac{(1+\sigma)(3-2\gamma)}{4}} d\tau \\
&\le C(1+\tau)^{-(\sigma-\frac{1}{4})},
\end{aligned}
$$

where we observe that

$$
\big\||\xi|^{-1}\big\|_{L^2_{\xi_3}} = \Big(\int_{\mathbb{R}} \frac{1}{|\xi_h|^2+|\xi_3|^2} d\xi_3\Big)^{\frac{1}{2}} \le C|\xi_h|^{-\frac{1}{2}}.
$$

Thus, K_{32} has the estimate as follows that

$$K_{32} \le C(1+t)^{-(\frac{3\sigma}{2}+\frac{1}{4})} \le C(1+t)^{-\frac{3\sigma}{2}}.$$

Similarly to the estimate of K_{32}, we deduce from Hölder inequality and Lemma 3.3 that

$$\begin{aligned}
K_{33} \le& C\int_0^t \|\big(W_2+|\xi_h|^2W_1\big)(t-\tau)|\xi_h||\xi|^{-1}\widehat{u_hu_3}\|_{L^2}\,d\tau\,\|\widehat{\nabla_h u_h}(t)\|_{L^2}\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t \|e^{-|\xi_h|^2(t-\tau)}|\xi_h||\xi|^{-1}\widehat{u_hu_3}\|_{L^2}\,d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\big\|\|(u_hu_3)\|_{L_h^{\frac{2}{1+\gamma}}}\big\|_{L^2_{x_3}}\,d\tau\\
&\le C(1+t)^{-\frac{1+\sigma}{2}}\int_0^t (t-\tau)^{-\frac{\gamma}{2}}\|u_3\|_{L^2}^{\gamma-\frac{1}{2}}\|\nabla_h u_3\|_{L^2}^{1-\gamma}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}\,d\tau\\
&\le C(1+\tau)^{-(\frac{3\sigma}{2}+\frac{1}{4})}\\
&\le C(1+t)^{-\frac{3\sigma}{2}}.
\end{aligned}$$

The decay estimate of K_{34} is similar as K_{33}, one has

$$K_{34} \le C(1+t)^{-(\frac{3\sigma}{2}+\frac{1}{4})} \le C(1+t)^{-\frac{3\sigma}{2}}.$$

Inserting the bounds of K_{31} through K_{34}, we get

$$K_3 \le C(1+t)^{-\frac{3\sigma}{2}}.$$

For K_4, using Hölder inequality and integrating by parts, we have

$$\begin{aligned}
K_4 =& -2\int_0^t\int |\xi_h||\xi|^{-1}W_1(t-\tau)\sum_{j=1}^2 \xi_j\widehat{u_j\theta}(\tau)\cdot\widehat{u}_3(t)\,dxd\tau\\
&-2\int_0^t\int |\xi_h||\xi|^{-1}W_1(t-\tau)\xi_3\widehat{u_3\theta}(\tau)\cdot\widehat{u}_3(t)\,dxd\tau\\
\le& C\int_0^t \big\||\xi_h||\xi|^{-1}W_1(t-\tau)\widehat{u_h\theta}\big\|_{L^2}d\tau\,\|\nabla_h u_3\|_{L^2}\\
&+C\int_0^t \big\||\xi_h||\xi|^{-1}W_1(t-\tau)\widehat{u_3\theta}\big\|_{L^2}d\tau\,\|\nabla_h u_h\|_{L^2}.
\end{aligned}$$

Analogously, the first term of K_4 admits the same estimate of K_{32}. Thus, for $2\sigma - \frac{1}{2} < \gamma < \frac{3}{2}$,

$$
\begin{aligned}
&\int_0^t \left\| |\xi_h||\xi|^{-1} W_1(t-\tau)\widehat{u_h\theta} \right\|_{L^2} d\tau \\
&\le C \int_0^t \| e^{-|\xi_h|^2(t-\tau)} |\xi_h||\xi|^{-1} \widehat{(u_h\theta)} \|_{L^2} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \left\| \|(u_h\theta)\|_{L_h^{\frac{4}{1+2\gamma}}} \right\|_{L^1_{x_3}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \left\| \|u_h\|_{L_h^{\frac{8}{1+2\gamma}}} \|\theta\|_{L_h^{\frac{8}{1+2\gamma}}} \right\|_{L^1_{x_3}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \left\| \|u_h\|_{L_h^2}^{\frac{1+2\gamma}{4}} \|\nabla_h u_h\|_{L_h^2}^{\frac{3-2\gamma}{4}} \|\theta\|_{L_h^2}^{\frac{1+2\gamma}{4}} \|\nabla_h \theta\|_{L_h^2}^{\frac{3-2\gamma}{4}} \right\|_{L^1_{x_3}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|u\|_{L^2}^{\frac{1+2\gamma}{4}} \|\nabla_h u\|_{L^2}^{\frac{3-2\gamma}{4}} \|\theta\|_{L^2}^{\frac{1+2\gamma}{4}} \|\nabla_h \theta\|_{L^2}^{\frac{3-2\gamma}{4}} d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{\gamma}{2}} (1+\tau)^{-\frac{\sigma(1+2\gamma)}{4}} (1+\tau)^{-\frac{(1+\sigma)(3-2\gamma)}{4}} d\tau \\
&\le C(1+t)^{-(\sigma-\frac{1}{4})}.
\end{aligned} \tag{3.15}
$$

The decay estimate of the second term for K_4 can be achieved in the estimate of K_{33}, we deduce from Lemma 3.4, for any $\frac{1}{2} < \gamma < 1$,

$$
\begin{aligned}
&\int_0^t \left\| |\xi_h||\xi|^{-1} W_1(t-\tau)\widehat{u_3\theta} \right\|_{L^2} d\tau \\
&\le \int_0^t \| e^{-|\xi_h|^2(t-\tau)} |\xi_h||\xi|^{-1} \widehat{u_3\theta} \|_{L^2} d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \left\| \|u_3\theta\|_{L_h^{\frac{2}{1+\gamma}}} \right\|_{L^2_{x_3}} d\tau \\
&\le \int_0^t (t-\tau)^{-\frac{\gamma}{2}} \|u_3\|_{L^2}^{\gamma-\frac{1}{2}} \|\nabla_h u\|_{L^2}^{\frac{3}{2}-\gamma} \|\theta\|_{L^2} d\tau
\end{aligned}
$$

$$\leq \int_0^t (t-\tau)^{-\frac{\gamma}{2}}(1+\tau)^{-\frac{\sigma}{2}(\frac{1}{2}+\gamma)}(1+\tau)^{-\frac{(1+\sigma)(\frac{3}{2}-\gamma)}{2}}d\tau$$
$$\leq C(1+t)^{-(\sigma-\frac{1}{4})}. \tag{3.16}$$

As a result, together with (3.15) and (3.16) give

$$K_4 \leq C(1+t)^{-\frac{3\sigma}{2}}.$$

Inserting the bounds for K_1 through K_4, we obtain

$$\|u_3\|_{L^2} \leq C(1+t)^{-\frac{3\sigma}{4}}.$$

Step 4. Decay estimate of $\|\nabla_h u_3\|_{L^2}$. Applying ξ_h to the integral form of u_3, and multiply (3.1) by $\xi_h\widehat{u}_3$ and integrating over $\mathbb{R}^3$, we have

$$\|\nabla_h u_3\|_{L^2}^2 \leq \int (W_2+|\xi_h|^2W_1)\widehat{\nabla_h u_{03}}(\tau)\cdot\widehat{\nabla_h u_3}(t)\,dx + \int \frac{|\xi_h|^2}{|\xi|^2}W_1\widehat{\nabla_h\theta_0}(\tau)\cdot\widehat{\nabla_h u_3}(t)\,dx$$
$$+ \int_0^t\int \big(W_2+|\xi_h|^2W_1\big)(t-\tau)\big(\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}\big)_3(\tau)\cdot\widehat{\nabla_h u_3}(t)\,dxd\tau$$
$$+ \int_0^t\int \frac{|\xi_h|^2}{|\xi|^2}W_1(t-\tau)\big(\nabla_h\widehat{(u\cdot\nabla\theta)}\big)(\tau)\cdot\widehat{\nabla_h u_3}(t)\,dxd\tau$$
$$:= P_1+P_2+P_3+P_4.$$

As in the estimate of K_1, using Young inequality, we obtain from Lemma 3.3 that

$$P_1 \leq C\|(W_2+|\xi_h|^2W_1)\widehat{\nabla_h u_{03}}\|_{L^2}^2 \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}}\widehat{u}_0\|_{L^2}^2.$$

We proceed to estimate P_2. In a similar manner as that used in the derivation of K_2, we find

$$P_2 \leq C\|\frac{|\xi_h|^2}{|\xi|^2}W_1\widehat{\nabla_h\theta_0}\|_{L^2}^2 \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}\|\Lambda_h^{-\sigma}\Lambda_3^{-\frac{\sigma}{2}}\widehat{\theta}_0\|_{L^2}^2.$$

Similar to the estimate of K_3. We use Young inequality and further decompose P_3 into four pieces

$$P_3 \leq C\int_0^t \|\big(W_2+|\xi_h|^2W_1\big)(t-\tau)\big(\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}\big)_3(\tau)\|_{L^2}\,d\tau\,\|\widehat{\nabla_h u_3}(t)\|_{L^2}$$
$$= C\int_0^t \|\big(W_2+|\xi_h|^2W_1\big)(t-\tau)\sum_{i=1}^4\widehat{\nabla_h F_i}(\tau)\|_{L^2}\,d\tau\,\|\nabla_h u(t)\|_{L^2}$$

$$:= P_{31} + P_{32} + P_{33} + P_{34}.$$

Analogously to the treatment of L_{21} and K_{31},

$$\begin{aligned}
P_{31} \leq & C(1+t)^{-\frac{1+\sigma}{2}} \int_0^{\frac{t}{2}} (t-\tau)^{-1} \left\| \|\widehat{u_h u_3}\|_{L_h^1} \right\|_{L_{x_3}^2} d\tau \\
& + C(1+t)^{-\frac{1+\sigma}{2}} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} \left\| \|u_h \cdot \nabla_h u_3\|_{L_h^{\frac{2}{1+\sigma}}} \right\|_{L_{x_3}^2} d\tau \\
\leq & C(1+t)^{-\frac{1+\sigma}{2}} \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_3\|_{L^2} d\tau \\
& + C(1+t)^{-\frac{1+\sigma}{2}} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} \|u_h\|_{L^2}^{\sigma-\frac{1}{2}} \|\nabla_h u_h\|_{L^2}^{1-\sigma} \|\partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h u\|_{L^2} d\tau \\
\leq & C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}$$

Similarly to that in Step 3, we deduce from K_{32} and L_{21} that

$$\begin{aligned}
P_{32} \leq & C(1+t)^{-\frac{1+\sigma}{2}} \int_0^{\frac{t}{2}} \|e^{-|\xi_h|^2(t-\tau)} |\xi|^{-1} |\xi_h|^3 \widehat{u u_h}\|_{L^2} d\tau \\
& + C(1+t)^{-\frac{1+\sigma}{2}} \int_{\frac{t}{2}}^{t} \|e^{-|\xi_h|^2(t-\tau)} |\xi|^{-1} |\xi_h|^3 \widehat{u u_h}\|_{L^2} d\tau \\
:= & C(1+t)^{-\frac{1+\sigma}{2}} (P_{321} + P_{322}).
\end{aligned} \tag{3.17}$$

Clearly, it suffices to consider the first term of P_{32}, similar as the estimate of K_{32} with Lemma 3.4, for $2\sigma - \frac{1}{2} < \gamma < 1$,

$$\begin{aligned}
P_{321} \leq & C \int_0^{\frac{t}{2}} \left\| \|e^{-|\xi_h|^2(t-\tau)} |\xi|^{-1} |\xi_h|^3 \widehat{u u_h}\|_{L_{\xi_3}^2} \right\|_{L_h^2} d\tau \\
\leq & C \int_0^{\frac{t}{2}} \left\| \||\xi|^{-1}\|_{L_{\xi_3}^2} \|e^{-|\xi_h|^2(t-\tau)} |\xi_h|^3 \widehat{u u_h}\|_{L_{\xi_3}^\infty} \right\|_{L_h^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{\frac{t}{2}} \big\| \|e^{-|\xi_h|^2(t-\tau)}|\xi_h|^{\frac{5}{2}}\widehat{uu_h}\|_{L^\infty_{\xi_3}}\big\|_{L^2_h}\,d\tau \\
&\le C\int_0^{\frac{t}{2}} (t-\tau)^{-(1+\frac{\gamma}{2})}\big\| |\xi_h|^{-(\gamma-\frac{1}{2})}\widehat{uu_h}\|_{L^\infty_{\xi_3}}\big\|_{L^2_h}\,d\tau \\
&\le C\int_0^{\frac{t}{2}} (t-\tau)^{-(1+\frac{\gamma}{2})}\|u\|_{L^2}^{\frac{1+2\gamma}{2}}\|\nabla_h u\|_{L^2}^{\frac{3-2\gamma}{2}}\,d\tau \\
&\le C(1+t)^{-(\sigma+\frac{3}{4})}.
\end{aligned} \tag{3.18}
$$

Analogously to the estimate of L_{21}, it is easy to see that

$$
\begin{aligned}
P_{322} &\le C\int_{\frac{t}{2}}^{t} \|e^{-|\xi_h|^2(t-\tau)}|\xi|^{-1}|\xi_h|^3\widehat{uu_h}\|_{L^2_h}\,d\tau \\
&\le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}}\big\| \|u\cdot\nabla_h u_h\|_{L_h^{\frac{2}{1+\sigma}}}\big\|_{L^2_{x_3}}\,d\tau \\
&\le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}}\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\,d\tau \\
&\le C(1+t)^{-(\sigma+\frac{3}{4})}.
\end{aligned} \tag{3.19}
$$

Which, together with (3.17)-(3.19), yield that

$$
P_{32} \le C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})}.
$$

Recalling the definition of W_1 and W_2 in Lemma 3.5, we obtain from Plancherel formula that,

$$
\begin{aligned}
P_{33} \le\, & C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}} \|e^{-|\xi_h|^2(t-\tau)}|\xi|^{-2}|\xi_h|^2|\xi_3|^2\widehat{u_hu_3}\|_{L^2_{\xi_3}}\,d\tau \\
&+ C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac{t}{2}}^{t} \|e^{-|\xi_h|^2(t-\tau)}|\xi|^{-2}|\xi_h|^2|\xi_3|^2\widehat{u_hu_3}\|_{L^2_{\xi_3}}\,d\tau \\
:=\, & P_{331}+P_{332},
\end{aligned}
$$

then, the decay estimate of P_{331} can be achieved in a similar manner in K_{31} that

$$
\begin{aligned}
P_{331} &\leq C(1+t)^{-\frac{1+\sigma}{2}} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}} \left\| \|u_h u_3\|_{L_h^1} \right\|_{L_{x_3}^2} d\tau \\
&\leq C(1+t)^{-\frac{1+\sigma}{2}} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2} d\tau \\
&\leq C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})},
\end{aligned}
$$

and in view of the estimate in L_{21}, we have that

$$
\begin{aligned}
P_{332} &\leq C(1+t)^{-\frac{1+\sigma}{2}} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} \left\| \|(u_h \cdot \nabla_h u_3)\|_{L_h^{\frac{2}{1+\sigma}}} \right\|_{L_{x_3}^2} d\tau \\
&\leq C(1+t)^{-\frac{1+\sigma}{2}} \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} \|u_h\|_{L^2}^{\sigma-\frac{1}{2}} \|\nabla_h u_h\|_{L^2}^{1-\sigma} \|\partial_3 u_h\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_3\|_{L^2} d\tau \\
&\leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_{33} &\leq C(1+t)^{-(\frac{3\sigma}{2}+\frac{5}{4})} + C(1+t)^{-(\frac{3\sigma}{2}+1)} \\
&\leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}
$$

The same estimate also holds for P_{34} that

$$
P_{34} \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.
$$

Collecting the estimates above together imply that

$$
P_3 \leq C(1+t)^{-(\frac{3\sigma}{2}+1)}.
$$

For P_3, similar as the estimate of P_2, we have

$$
\begin{aligned}
P_4 \leq& \int_0^t \int \frac{|\xi_h|^2}{|\xi|^2} W_1(t-\tau)(\nabla_h \widehat{(u\cdot\nabla\theta)})(\tau) \cdot \widehat{\nabla_h u_3}(t)\, dx d\tau, \\
\leq& C \int_0^{\frac{t}{2}} \|\nabla_h e^{\Delta_h(t-\tau)}(u_h \cdot \nabla_h \theta + u_3 \partial_3 \theta)(\tau)\|_{L^2} d\tau \|\nabla_h u\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\frac{t}{2}}^{t}\|\nabla_h e^{\Delta_h(t-\tau)}(u_h\cdot\nabla_h\theta+u_3\partial_3\theta)(\tau)\|_{L^2}d\tau\|\nabla_h u\|_{L^2}\\
\leq &C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-1}\|u_h\|_{L^2}^{\frac12}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h\theta\|_{L^2}\,d\tau\\
&+C(1+t)^{-\frac{1+\sigma}{2}}\int_0^{\frac{t}{2}}(t-\tau)^{-1}\|u_3\|_{L^2}^{\frac12}\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_3\theta\|_{L^2}\,d\tau\\
&+C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}\|u_h\|_{L^2}^{\sigma-\frac12}\|\nabla_h u_h\|_{L^2}^{1-\sigma}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h\theta\|_{L^2}\,d\tau\\
&+C(1+t)^{-\frac{1+\sigma}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}\|u_3\|_{L^2}^{\sigma-\frac12}\|\nabla_h u_3\|_{L^2}^{1-\sigma}\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_3\theta\|_{L^2}\,d\tau\\
\leq &C(1+t)^{-(\frac{3\sigma}{2}+1)}.
\end{aligned}
$$

Combining these estimates of P_1-P_4, we have

$$
\|\nabla_h u_3\|_{L^2}\leq C(1+t)^{-(\frac{3\sigma}{4}+\frac12)}.
$$

Step 5. Closing of the bootstrap argument. We first give the estimate of $\|(u,\theta)\|_{\dot H^{-\sigma,0}}$. Applying $\Lambda_h^{-\sigma}$ to (1.3) and dotting with $\langle\Lambda_h^{-\sigma}u,\Lambda_h^{-\sigma}\theta\rangle$, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|(u,\theta)\|_{\dot H^{-\sigma,0}}^2+\nu\|\nabla_h u\|_{\dot H^{1-\sigma,0}}^2+\eta\|\nabla_h\theta\|_{\dot H^{1-\sigma,0}}^2\\
&=-\int\Lambda_h^{-\sigma}(u\cdot\nabla u)\cdot\Lambda_h^{-\sigma}u\,dx-\int\Lambda_h^{-\sigma}(u\cdot\nabla\theta)\cdot\Lambda_h^{-\sigma}\theta\,dx\\
&:=M_1+M_2.
\end{aligned}
$$

Using Hölder inequality and the Hardy-Littlewood-Sobolev inequality,

$$
\begin{aligned}
M_1&\leq\|u\cdot\nabla u\|_{\dot H^{-\sigma,0}}\|u\|_{\dot H^{-\sigma,0}}\\
&\leq C(\big\|\|u_h\cdot\nabla_h u\|_{L_h^{\frac{2}{1+\sigma}}}+\|u_3\partial_3 u\|_{L_h^{\frac{2}{1+\sigma}}}\big\|_{L^2_{x_3}})\|u\|_{\dot H^{-\sigma,0}}\\
&:=C(M_{11}+M_{12})\|u\|_{\dot H^{-\sigma,0}}.
\end{aligned}
$$

As the estimate of K_{33}, for $\frac12<\sigma<1$, we have

$$M_{11} \leq C\|u_h\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u_h\|_{L^2}^{1-\sigma}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}$$
$$\leq C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}},$$

and using the divergence free condition $\nabla \cdot u = 0$, one has

$$M_{12} \leq C\|u_3\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u_3\|_{L^2}^{1-\sigma}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}$$
$$\leq C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 u\|_{L^2}.$$

Thus, together with M_{11} and M_{12}, we have

$$M_1 \leq C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{2-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|u\|_{\dot{H}^{-\sigma,0}}$$
$$+ C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 u\|_{L^2}\|u\|_{\dot{H}^{-\sigma,0}}.$$

M_2 an be similarly bounded as M_1, which we obtain

$$M_2 \leq C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \theta\|_{L^2}\|\theta\|_{\dot{H}^{-\sigma,0}}$$
$$+ C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|\partial_3 \theta\|_{L^2}\|\theta\|_{\dot{H}^{-\sigma,0}}.$$

Inserting the bounds for M_1 and M_2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u,\theta)\|_{\dot{H}^{-\sigma,0}}^2 + \nu\|\nabla_h u\|_{\dot{H}^{1-\sigma,0}}^2 + \eta\|\nabla_h \theta\|_{\dot{H}^{1-\sigma,0}}^2 \\
\leq &C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|(\nabla_h u, \nabla_h \theta)\|_{L^2}\|(u,\theta)\|_{\dot{H}^{-\sigma,0}} \\
&+ C\|u\|_{L^2}^{\sigma-\frac{1}{2}}\|\partial_h u\|_{L^2}^{\frac{3}{2}-\sigma}\|(\partial_3 u, \partial_3 \theta)\|_{L^2}\|(u,\theta)\|_{\dot{H}^{-\sigma,0}}.
\end{aligned} \tag{3.20}$$

Secondly, we have the estimates of $\|(\partial_3 u, \partial_3 \theta)\|_{\dot{H}^{-\sigma,0}}$. Applying $\Lambda_h^{-\sigma}\partial_3$ to the first two equations of (1.3), and taking the L^2-inner products with $\Lambda_h^{-\sigma}\partial_3 u$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\partial_3 u, \partial_3 \theta)\|_{\dot{H}^{-\sigma,0}}^2 + \nu\|\partial_3 u\|_{\dot{H}^{1-\sigma,0}}^2 + \eta\|\partial_3 \theta\|_{\dot{H}^{1-\sigma,0}}^2 \\
&= -\int \Lambda_h^{-\sigma}\partial_3(u\cdot\nabla u)\cdot\Lambda_h^{-\sigma}\partial_3 u\,dx - \int \Lambda_h^{-\sigma}\partial_3(u\cdot\nabla\theta)\cdot\Lambda_h^{-\sigma}\partial_3\theta\,dx \\
&:= H_1 + H_2.
\end{aligned} \tag{3.21}$$

Similar as the estimate of M_1,

$$\begin{aligned}H_1 \le& C\|\partial_3 u\|_{L^2}^{\sigma-\frac{1}{2s_1}}\|\nabla_h u\|_{L^2}^{1+\frac{(s_1-1)(1-\sigma)}{s_1}}\|\nabla_h\partial_3^{s_1}u\|_{L^2}^{\frac{1-\sigma}{s_1}}\|\partial_3^{s_1+1}u\|_{L^2}^{\frac{1}{2s_1}}\|\partial_3 u\|_{\dot H^{-\sigma,0}}\\&+C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{2-\sigma-\frac{1}{s_1}}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^{s_1}u\|_{L^2}^{\frac{1}{s_1}}\|\partial_3 u\|_{\dot H^{-\sigma,0}}\\&+C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{\frac52-\sigma-\frac{1}{s_1}}\|\partial_3^{s_1+1}u\|_{L^2}^{\frac{1}{s_1}}\|\partial_3 u\|_{\dot H^{-\sigma,0}},\end{aligned}\tag{3.22}$$

where we have used the interpolation inequalities, for $s_1 > 1$,

$$\|\nabla_h\partial_3 u_h\|_{L^2} \le C\|\nabla_h u_h\|_{L^2}^{\frac{s_1-1}{s_1}}\|\nabla_h\partial_3^{s_1}b_h\|_{L^2}^{\frac{1}{s_1}},$$

$$\|\partial_3^2 u_h\|_{L^2} \le C\|\partial_3 u_h\|_{L^2}^{\frac{s_1-1}{s_1}}\|\partial_3^{s_1+1}u_h\|_{L^2}^{\frac{1}{s_1}}.$$

Analogously to the estimate of H_1 in (3.22), for $s_1 > 1$, we have

$$\begin{aligned}H_2 \le& C\|\partial_3 u\|_{L^2}^{\sigma-\frac{1}{2s_1}}\|\nabla_h u\|_{L^2}^{\frac{(s_1-1)(1-\sigma)}{s_1}}\|\nabla_h\theta\|_{L^2}\|\nabla_h\partial_3^{s_1}\theta\|_{L^2}^{\frac{1-\sigma}{s_1}}\|\partial_3^{s_1+1}\theta\|_{L^2}^{\frac{1}{2s_1}}\|\partial_3\theta\|_{\dot H^{-\sigma,0}}\\&+C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{1-\sigma}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\theta\|_{L^2}^{\frac{s_1-1}{s_1}}\|\nabla_h\partial_3^{s_1}\theta\|_{L^2}^{\frac{1}{s_1}}\|\partial_3\theta\|_{\dot H^{-\sigma,0}}\\&+C\|u\|_{L^2}^{\sigma-\frac12}\|\nabla_h u\|_{L^2}^{\frac32}\|\nabla_h\theta\|_{L^2}^{\frac{s_1-1}{s_1}}\|\partial_3^{s_1+1}\theta\|_{L^2}^{\frac{1}{s_1}}\|\partial_3\theta\|_{\dot H^{-\sigma,0}}.\end{aligned}\tag{3.23}$$

Finally, we complete the bootstrap argument. Integrating (3.20) in time over $[0,t]$ with $0 < t \le T$, we get

$$\begin{aligned}\|(u,\theta)(t)\|_{\dot H^{-\sigma,0}}^2 \le& \|(u_0,\theta_0)\|_{\dot H^{-\sigma,0}}^2 + C\epsilon^{\delta_0}\sup_{0\le\tau\le t}\|(u,\theta)(\tau)\|_{\dot H^{-\sigma,0}}^2\\&\times\int_0^t \big(1+\tau)^{-\frac{\sigma}{2}(\sigma-\delta_0)-\frac{(1+\sigma)(2-\sigma)}{2}} + (1+\tau)^{-\frac{\sigma}{2}(\sigma-\frac12-\delta_0)-\frac{(1+\sigma)(\frac32-\sigma)}{2}}\,d\tau\\ \le& \|(u_0,\theta_0)\|_{\dot H^{-\sigma,0}}^2 + C\epsilon^{\delta_0}\sup_{0\le\tau\le t}\|(u,\theta)(\tau)\|_{\dot H^{-\sigma,0}}^2\int_0^t(1+\tau)^{-(\frac{3+2\sigma}{4}-\delta_0)}\,d\tau\\ \le& \|(u_0,\theta_0)\|_{\dot H^{-\sigma,0}}^2 + C\epsilon^{\delta_0}\sup_{0\le\tau\le t}\|(u,\theta)(\tau)\|_{\dot H^{-\sigma,0}}^2,\end{aligned}\tag{3.24}$$

where we have used the conclusion in Theorem 1.1 that

$$\|u(t)\|_{H^1} + \|\theta(t)\|_{H^1} \le C_0\epsilon,$$

and the assumption $\frac12 < \sigma < 1$, choosing δ_0 small enough such that

$$\frac{3+2\sigma}{4} - \delta_0 > 1.$$

Similarly, integrating (3.21)–(3.23) over $[0,t]$ with $0 < t \le T$, for $s_1 > 1$, we have

$$\begin{aligned}\|(\partial_3 u, \partial_3\theta)(t)\|^2_{\dot{H}^{-\sigma,0}} \le& \|(\partial_3 u_0, \partial_3\theta_0)\|^2_{\dot{H}^{-\sigma,0}} + C\epsilon^{\delta_0} \sup_{0\le\tau\le t} \|(\partial_3 u, \partial_3\theta)(\tau)\|^2_{\dot{H}^{-\sigma,0}} \\ &\times \int_0^t (1+\tau)^{-(\frac{\sigma(\sigma-\delta_0)}{2} - \frac{\sigma}{4s_1} + \frac{\sigma+1}{2}(1+\frac{(s_1-1)(1+\sigma)}{s_1}))}\,d\tau \\ \le& \|(\partial_3 u_0, \partial_3\theta_0)\|^2_{\dot{H}^{-\sigma,0}} + C\epsilon^{\delta_0} \sup_{0\le\tau\le t} \|(\partial_3 u, \partial_3\theta)(\tau)\|^2_{\dot{H}^{-\sigma,0}}, \end{aligned} \tag{3.25}$$

where $\delta_0 > 0$ is chosen small enough such that

$$\frac{\sigma(\sigma-\delta_0)}{2} - \frac{\sigma}{4s_1} + \frac{\sigma+1}{2}(1+\frac{(s_1-1)(1+\sigma)}{s_1}) > 1.$$

Together with (3.24), (3.25) and (3.2), we arrive that

$$\|(u, \theta, \partial_3 u, \partial_3\theta)(t)\|^2_{\dot{H}^{-\sigma,0}} \le \epsilon_0 + C\epsilon^{\delta_0} \sup_{0\le\tau\le t} \|(u, \theta, \partial_3 u, \partial_3\theta)(\tau)\|^2_{\dot{H}^{-\sigma,0}}.$$

By choosing ϵ sufficiently small such that $C\epsilon^{\delta_0} < \min\{\frac{1}{3}, \frac{1}{3}\epsilon_0\}$. This inequality together with (3.2) yields (3.3) for all $t \in [0, T]$. Then the bootstrap argument implies that $T = \infty$. This finishes the proof of Theorem 1.2.

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Data availability

No data was used for the research described in the article.

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