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Stability on a 3D incompressible Oldroyd-B model with mixed partial dissipation

Hongxia Lin[1,*], Jiahong Wu[2] & Nicki Boardman[3]

[1] *Geomathematics Key Laboratory of Sichuan Province, College of Mathematics and Physics, Chengdu University of Technology, Chengdu* 610059*, China;*
[2] *Department of Mathematics, University of Notre Dame, Notre Dame, IN* 46556*, USA;*
[3] *Department of Mathematics, Oklahoma State University, Stillwater, OK* 74078*, USA*

Email: linhongxia18@126.com, jwu29@nd.edu, nicki.boardman@okstate.edu

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Abstract The Oldroyd-B equations model viscoelastic fluids. Attention here is focused on a three-dimensional (3D) Oldroyd-B system with mixed dissipation, horizontal velocity dissipation and vertical diffusion for the non-Newtonian stress tensor τ. The equation of τ has no damping, a setup relevant to high Weissenberg viscoelastic flows. In this paper, we solve the small-data global well-posedness and the stability problem in the Sobolev space $H^2(\mathbb{R}^3)$. The lack of the horizontal dissipation or damping in the equation of τ makes the problem almost impossible. This paper discovers that the coupling and interaction of the fluid velocity u and τ generates extra smoothing and stabilization. Mathematically, u and $\mathbb{P}\nabla\cdot\tau$ satisfy a system of wave equations, which provides the desired enhanced dissipation. Here, $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denotes the projection onto divergence-free vector fields. In addition, time-weighted energy functionals are introduced to control low-regularity terms. The second major result of this paper establishes optimal decay rates by making use of the aforementioned enhanced dissipation and the integral representation.

Keywords Oldroyd-B model, mixed partial dissipation, stability, decay rate

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1 Introduction

This paper focuses on a special three-dimensional (3D) Oldroyd-B model with anisotropic dissipation. The Oldroyd-B model, originally derived by Oldroyd [30], governs the motion of viscoelastic fluids such as a solvent with particles suspended in it (see [2]). Mathematically the Oldroyd-B model considered here is given by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla P = \mu\,\Delta_h u + \nu_1\nabla\cdot\tau, \quad x\in\mathbb{R}^3, \quad t>0,\\ \partial_t\tau + (u\cdot\nabla)\tau + Q(\tau,\nabla u) = \eta\,\partial_3^2\tau + \nu_2 D(u),\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x), \quad \tau(x,0) = \tau_0(x), \end{cases} \tag{1.1}$$

* Corresponding author

where $u = (u_1, u_2, u_3)$ and P denote the fluid velocity and pressure, respectively, and τ is the non-Newtonian stress tensor represented by a symmetric matrix. The parameters μ, η, ν_1 and ν_2 are all real and positive. Here, $\Delta_h = \partial_1^2 + \partial_2^2$ denotes the horizontal Laplacian, $D(u)$ the deformation tensor, i.e.,

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top),$$

and $Q(\tau, \nabla u)$ is bilinear and given by

$$Q = \tau\Omega(u) - \Omega(u)\tau + b(D(u)\tau + \tau D(u))$$

with $\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^\top)$ being the vorticity tensor and $b \in [-1, 1]$ a constant. When $b = 0$, the system is called corotational. The term $Q(\tau, \nabla u)$ appears in typical models of viscoelastic fluids.

In this special Oldroyd-B model (1.1), the velocity obeys the anisotropic Navier-Stokes equation forced with the non-Newtonian stress tensor. The anisotropic Navier-Stokes equation models anisotropic fluids such as turbulent flows in Ekman layers [31]. The equation of non-Newtonian stress tensor τ involves no damping but only vertical diffusion. When the Weissenberg number We is high, the damping term with the coefficient $1/We$ can be ignored [32]. In the circumstance when the vertical diffusivity in the equation of τ is far bigger than the horizontal one, (1.1) becomes relevant [29]. The anisotropic Oldroyd-B system considered here can be derived from general viscoelastic models for applications with fluids that are anisotropic in elasticity as well as in dissipation (see, e.g., [5, 33]).

Our goal here is to solve the global well-posedness and stability problem of (1.1). In addition, we are also interested in the precise large-time behavior of the solutions. The motivation for this study is two-fold. The first one is physical. Understanding the stability properties and large-time behavior is essential in modeling physical or biological processes. It is hoped that this investigation will help better predict the behavior of the system in the modeling of real-world phenomena. The second is mathematical. We intend to develop new approaches that are effective for the stability problems concerning anisotropic systems. Most existing methods have been designed for fully dissipative systems and appear to fail on the system studied here. Indeed, the equation of τ in (1.1) has no damping and the dissipation is in a single direction, and classical energy methods would have immediate difficulties. We exploit the extra smoothing and stabilizing effect due to the coupling and interaction within the system. As described later, this paper discovers the hidden wave structure and develops an effective approach on how to incorporate the associated regularization in solving the intended stability problem.

Our main results are stated in the following two theorems. The first one asserts the global existence, regularity and stability of (1.1) in the Sobolev space H^2 while the second presents the optimal decay rates of the solutions when we assume that the initial data is also in L^1. The notation $\mathbb{P}$ denotes the projection onto divergence-free vector fields, i.e., $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$.

Theorem 1.1. *Consider* (1.1) *with* $\mu > 0, \eta > 0$, $\nu_1 > 0$ *and* $\nu_2 > 0$. *Assume that* $(u_0, \tau_0) \in H^2(\mathbb{R}^3)$ *satisfies* $\nabla \cdot u_0 = 0$ *and* $(\tau_0)_{ij} = (\tau_0)_{ji}$ *for* $i, j = 1, 2, 3$. *Then there exists a sufficiently small constant* $\delta > 0$ *such that if*

$$\|u_0\|_{H^2(\mathbb{R}^3)} + \|b_0\|_{H^2(\mathbb{R}^3)} \leqslant \delta, \tag{1.2}$$

then (1.1) *has a unique global solution* $(u, b) \in C([0, \infty); H^2(\mathbb{R}^3))$. *In addition,* (u, b) *obeys the upper bound*

$$\begin{aligned}\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|\tau(t)\|_{H^2(\mathbb{R}^3)}^2 + \int_0^t (\|\nabla u(s)\|_{H^1(\mathbb{R}^3)}^2 + \|\nabla_h \nabla^2 u(s)\|_{L^2(\mathbb{R}^3)}^2 \\ + \|\partial_3 \tau(s)\|_{H^2(\mathbb{R}^3)}^2 + \|\mathbb{P}\nabla \cdot \tau(s)\|_{H^1(\mathbb{R}^3)}^2) ds \leqslant C\delta^2\end{aligned} \tag{1.3}$$

for some uniform constant C *and for any* $t > 0$. *Moreover, the following decay rate holds*: *For a constant* $C > 0$,

$$\|\nabla u(t)\|_{H^1(\mathbb{R}^3)} + \|\mathbb{P}\nabla \cdot \tau(t)\|_{H^1(\mathbb{R}^3)} \leqslant C(1 + t)^{-\frac{1}{2}}. \tag{1.4}$$

(1.3) and (1.4) reveal the extra smoothing and stabilizing effect that does not originate from the dissipation in the system. The velocity equation in (1.1) only has horizontal dissipation, but the time integrability of $\|\nabla u\|_{H^1(\mathbb{R}^3)}^2$ in (1.3) clearly implies the vertical dissipative effect. In addition, (1.3) and (1.4) indicate that the special quantity $\mathbb{P}\nabla\cdot\tau$ enjoys regularized properties that the vertical dissipation of τ cannot provide.

(1.4) provides the decay rates for $\|\nabla u(t)\|_{H^1(\mathbb{R}^3)}$ and $\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1(\mathbb{R}^3)}$. However, without additional conditions on the initial data, it appears impossible to assess the large-time behavior of the solution itself and other derivatives. Our second main result establishes the optimal decay rates when we assume that the initial data is also in L^1.

Theorem 1.2. *Let* $(u_0,\tau_0)\in H^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$ *with* $\nabla\cdot u_0=0$ *and* $(\tau_0)_{ij}=(\tau_0)_{ji}$ *for* $i,j=1,2,3$. *Assume that for some sufficiently small constant* $\delta>0$,

$$\|(u_0,\tau_0)\|_{H^2(\mathbb{R}^3)}+\|(u_0,\tau_0)\|_{L^1(\mathbb{R}^3)}\leqslant\delta. \tag{1.5}$$

Let (u,τ) *be the corresponding global solution of* (1.1) *obtained in Theorem* 1.1. *Then* (u,τ) *obeys the following decay estimates*:

$$\begin{aligned}
&\|u(t)\|_{L^2(\mathbb{R}^3)}+\|\partial_3^2u(t)\|_{L^2(\mathbb{R}^3)}+\|\partial_3\mathbb{P}(\nabla\cdot\tau)(t)\|_{L^2(\mathbb{R}^3)}\leqslant C\delta\,(1+t)^{-\frac34},\\
&\|\nabla u(t)\|_{L^2(\mathbb{R}^3)}+\|\nabla_h\nabla u(t)\|_{L^2(\mathbb{R}^3)}\leqslant C\delta\,(1+t)^{-\frac54},\\
&\|\partial_3\tau(t)\|_{H^1(\mathbb{R}^3)}\leqslant C\delta\,(1+t)^{-\frac12},\quad \|\mathbb{P}(\nabla\cdot\tau)(t)\|_{L^2(\mathbb{R}^3)}\leqslant C\delta(1+t)^{-1},
\end{aligned}$$

where C *is an absolute constant independent of* δ *and* t.

The decay rates for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$ are the same as the corresponding ones for the solution of the heat equation with full Laplacian dissipation in $\mathbb{R}^3$,

$$\partial_t u=\nu\Delta u,\quad u(x,0)=u_0\in L^1\cap H^1.$$

Therefore, they are optimal and reflect the extra smoothing effect resulting from the interaction of u and τ.

We explain the proofs of Theorems 1.1 and 1.2. Since the local (in time) well-posedness can be established via standard procedures (see, e.g., [28]), the proof of Theorem 1.1 reduces to showing the global *a priori* bound on the solution (u,τ) in H^2. We make use of the bootstrapping argument to serve this purpose. The main task is to control the nonlinearity in terms of the anisotropic dissipation. The horizontal velocity dissipation $\Delta_h u$ is sufficient for the Navier-Stokes nonlinearity $(u\cdot\nabla)u$, but the one-directional vertical dissipation $\partial_3^2\tau$ is clearly not enough to bound the nonlinear terms $(u\cdot\nabla)\tau$ and $Q(\nabla u,\tau)$. The idea here is to seek the extra smoothing and stabilizing effect from the interaction between u and τ. Applying the Leray projection operator $\mathbb{P}=I-\nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.1) (to eliminate the pressure) and the operator $\mathbb{P}\nabla\cdot$ to the equation of τ in (1.1), we obtain

$$\begin{cases}\partial_t u+\mathbb{P}(u\cdot\nabla)u=\mu\Delta_h u+\nu_1\mathbb{P}\nabla\cdot\tau,\\[2mm] \mathbb{P}\nabla\cdot\partial_t\tau+\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau+\mathbb{P}\nabla\cdot Q(\tau,\nabla u)=\eta\partial_3^2\mathbb{P}\nabla\cdot\tau+\dfrac12\nu_2\Delta u,\end{cases} \tag{1.6}$$

where we have used $\nabla\cdot D(u)=\frac12\Delta u$. Setting $\mathcal{A}=\mathbb{P}\nabla\cdot\tau$, differentiating (1.6) in t and making several substitutions, we discover that $(u,\mathcal{A})$ satisfies the wave equations

$$\begin{cases}\partial_{tt}u-(\mu\Delta_h+\eta\partial_3^2)\partial_t u-\dfrac12\nu_1\,\nu_2\,\Delta u+\mu\eta\partial_3^2\Delta_h u=R_1,\\[2mm] \partial_{tt}\mathcal{A}-(\mu\Delta_h+\eta\partial_3^2)\partial_t\mathcal{A}-\dfrac12\nu_1\,\nu_2\,\Delta\mathcal{A}+\mu\eta\partial_3^2\Delta_h\mathcal{A}=R_2,\end{cases} \tag{1.7}$$

where R_1 and R_2 represent the nonlinear terms, i.e.,

$$R_1=(-\partial_t+\eta\partial_3^2)\mathbb{P}(u\cdot\nabla u)-\nu_1\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau-\nu_1\mathbb{P}\nabla\cdot Q,$$

$$R_2 = -\frac{1}{2}\nu_2\Delta\mathbb{P}((u\cdot\nabla)u) + (-\partial_t + \mu\Delta_h)(\mathbb{P}\nabla\cdot(u\cdot\nabla)\tau + \mathbb{P}\nabla\cdot Q).$$

That means both u and $\mathcal{A}$ actually satisfy a system of damped wave equations with exactly the same linear parts. In comparison with the original system (1.1), the new system (1.7) exhibits much more smoothing and stabilization properties. It is this wave structure that makes the stability and large-time behavior problem of (1.1) plausible. The extra smoothing properties revealed in (1.1) will be incorporated into the construction of the energy functional.

Naturally, the energy functional should include the H^2-norm of (u,τ) together with the time integral pieces resulting from the dissipative terms $\Delta_h u$ and $\partial_3^2\tau$, i.e.,

$$E_0^{(1)}(t) = \sup_{0\leqslant s\leqslant t}\|(u(s),\tau(s))\|_{H^2}^2 + \int_0^t(\|\nabla_h u(s)\|_{H^2}^2 + \|\partial_3\tau(s)\|_{H^2}^2)ds.$$

In addition, we also take into account the extra regularizing properties revealed by the wave equations (1.7). The terms $\frac{1}{2}\nu_1\,\nu_2\,\Delta u$ and $\frac{1}{2}\nu_1\,\nu_2\,\Delta\mathcal{A}$ in (1.7) yield weaker dissipation that is one-derivative-order lower than the Laplacian dissipation. Therefore, the energy functional associated with the weaker dissipation is given by

$$E_0^{(2)}(t) = \int_0^t(\|\nabla u(s)\|_{H^1}^2 + \|\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2)ds.$$

Putting $E_0^{(1)}(t)$ and $E_0^{(2)}(t)$ together leads to

$$\begin{aligned}E_0(t) &:= E_0^{(1)}(t) + E_0^{(2)}(t)\\ &= \sup_{0\leqslant s\leqslant t}\|(u(s),\tau(s))\|_{H^2}^2 + \int_0^t(\|\nabla_h u(s)\|_{H^2}^2 + \|\partial_3 u(s)\|_{H^1}^2 + \|\partial_3\tau(s)\|_{H^2}^2 + \|\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2)ds.\end{aligned}$$

There appear to be additional difficulties that cannot be overcome by the smoothing due to the wave structure (1.7). One term from the L^2-estimate of τ, i.e.,

$$\int Q(\nabla u,\tau)\cdot\tau dx \leqslant C\int|\nabla u||\tau||\tau|dx,$$

cannot be bounded in terms of $E_0(t)$. Due to the anisotropic dissipation, we naturally use the anisotropic upper bounds for this triple product (see (2.2)), i.e.,

$$\int Q(\nabla u,\tau)\cdot\tau dx \leqslant C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}\|\tau\|_{L^2}. \tag{1.8}$$

However, the time integral of the right-hand side above cannot be bounded in terms of $E_0(t)$ because of the lack of the horizontal dissipation in τ. To overcome this difficulty, we include a time-weighted energy functional

$$\begin{aligned}E_1(t) = &\sup_{0\leqslant s\leqslant t}(1+s)(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2)\\ &+\int_0^t(1+s)(\|\nabla\nabla_h u(s)\|_{H^1}^2 + \|\nabla\partial_3 u(s)\|_{L^2}^2\\ &+\|\partial_3\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2 + \|\nabla_h\mathbb{P}(\nabla\cdot\tau)(s)\|_{L^2}^2)ds.\end{aligned}$$

Therefore, the total energy functional $E(t)$ is defined to be

$$E(t) := E_0(t) + E_1(t). \tag{1.9}$$

The inclusion of $E_1(t)$ helps bound the time integral of the right-hand side of (1.8). In fact, we have

$$\left|\int_0^t\int Q\cdot\tau dxds\right| \leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{8}}\|\nabla u(s)\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{3}{2}}$$

$$
\begin{aligned}
&\times \int_0^t (1+s)^{\frac{3}{8}} \|\nabla_h \nabla u\|_{H^1}^{\frac{3}{4}} \|\partial_3 \tau\|_{L^2}^{\frac{1}{2}} (1+s)^{-\frac{1}{2}} ds \\
&\leqslant C \sup_{0\leqslant s\leqslant t} (1+s)^{\frac{1}{8}} \|\nabla u(s)\|_{L^2}^{\frac{1}{4}} \|\tau\|_{L^2}^{\frac{3}{2}} \left[\int_0^t (1+s)\|\nabla_h \nabla u\|_{H^1}^2 ds\right]^{\frac{3}{8}} \\
&\times \left[\int_0^t \|\partial_3 \tau\|_{L^2}^2 ds\right]^{\frac{1}{4}} \left[\int_0^t (1+s)^{-\frac{4}{3}} ds\right]^{\frac{3}{8}} \\
&\leqslant C E_0(t) E_1(t)^{\frac{1}{2}}.
\end{aligned}
$$

These estimates also illustrate the power of the time-weighted functionals when we deal with anisotropic systems. The rest of our task is to verify that (1.9) is sufficient and allows us to show

$$
\begin{aligned}
E(t) &\leqslant C\|(u_0, \tau_0)\|_{H^2}^2 + C E_0(t)^{\frac{3}{2}} + C E_0(t) E_1(t)^{\frac{1}{2}} + C E_0(t)^{\frac{1}{2}} E_1(t) \\
&\leqslant C\|(u_0, \tau_0)\|_{H^2}^2 + C\, E(t)^{\frac{3}{2}}. \qquad (1.10)
\end{aligned}
$$

Once (1.10) is established, a direct application of the bootstrapping argument would lead to the desired global bound for $\|(u(t), \tau(t))\|_{H^2}$ when the initial norm $\|(u_0, \tau_0)\|_{H^2}$ is taken to be sufficiently small. Our main efforts are devoted to proving (1.10). This is a long process, which fully exploits the smoothing effect of the wave structure in (1.7), and makes use of various anisotropic inequalities and some special identities. The technical details are provided in Section 3.

The proof of Theorem 1.2 relies on the integral representation of (1.6). Naturally, our first step is to convert (1.6) or (1.7) into an integral form. To do so, we take the Fourier transform of (1.6), solve the linearized system and then apply Duhamel's principle to represent the nonlinear system. This representation involves four Fourier multiplier operators. These multipliers are anisotropic and their bounds play an essential role in the decay rates. To obtain their optimal bounds, we decompose the frequency space into subdomains and pinpoint the exact upper bounds for these multipliers in each subdomain (see Proposition 4.1).

With the integral representation and the optimal bounds for the multipliers at our disposal, the proof of the decay rates in Theorem 1.2 is then divided into three parts. In the first part, we apply the bootstrapping argument to establishing the optimal decay rates for $\|u\|_{L^2}$ and $\|\nabla u\|_{L^2}$. In the second part, we prove the decay rates for $\|\nabla\nabla_h u\|_{L^2}, \|\partial_3 \nabla u\|_{L^2}$ and $\|\partial_3 \tau\|_{H^1}$. To facilitate the proof, we introduce three time-weighted energy functionals and invoke the bootstrapping argument. The last part is devoted to the decay rate for $\mathbb{P}\cdot\nabla\tau$, which is obtained by taking the L^2-norm of its integral representation, making use of the anisotropic inequalities and invoking the decay rates from the first two parts. The details are long and presented in Section 4.

Finally, we describe some related works. There are extensive mathematical studies on various models of viscoelastic fluids and significant progress has been made on many fundamental issues (see, e.g., [9, 19–21, 24, 25]). Due to its special structure and the intriguing behavior of its solutions, the Oldroyd-B model has attracted considerable interest. There are substantial developments and understanding of the well-posedness and related problems (see, e.g., [1, 6, 8, 10, 11, 14–18, 22, 23, 27, 35, 36, 41, 42, 46]). More recent investigations focus on the Oldroyd-B models with various partial dissipation. Zhu [45] successfully solved the small-data global well-posedness problem on the 3D Oldroyd-B model with only kinematic dissipation and without damping or diffusion in the equation of the non-Newtonian stress. In a very recent work [12], Constantin et al. were able to obtain the global existence and stability of classical solutions to the general d-dimensional Oldroyd-B model with no damping and only fractional stress-tensor diffusion $-(-\Delta)^{\beta}\tau$ $(1/2 \leqslant \beta \leqslant 1)$. The authors maximally exploited the structure of the system to gain extra regularization for the velocity field, which is governed by the 3D Euler equation. Sharp decay estimates for this Oldroyd-B system have been established by Wang et al. [37]. The global solutions obtained in the aforementioned three papers are in various Sobolev spaces. Several recent papers [7, 39, 43, 44] were able to reestablish their global existence and uniqueness in critical Besov spaces, which helped reduce the regularity requirements on the initial data. This current paper appears to be the first one that focuses on an anisotropic Oldroyd-B model.

We introduce some notation to be used in the rest of this paper. Without loss of generality, we set $\nu_1 = \nu_2 = 1$. The letter C denotes a generic absolute positive constant which may change from line to line. In addition, we use the following abbreviated notation:

$$\begin{aligned}
&\nabla_h = (\partial_1, \partial_2), \quad \Delta_h = \partial_1^2 + \partial_2^2, \quad v_h = (v_1,\ v_2) \quad \text{for } v = (v_1, v_2, v_3),\\
&\|v\|_{L^r_{x_i} L^q_{x_j} L^p_{x_k}} = \|\|\|v\|_{L^p_{x_k}}\|_{L^q_{x_j}}\|_{L^r_{x_i}} \quad \text{for } i \neq j \neq k,\\
&\|(f,g)\|_{H^s}^2 = \|f\|_{H^s}^2 + \|g\|_{H^s}^2 \quad \text{for } s \geqslant 0.
\end{aligned}$$

The rest of the paper is organized as follows. Section 2 presents some anisotropic inequalities and an identity on $\mathbb{P}\nabla \cdot (u \cdot \nabla \tau)$ to be used in the proofs of Theorems 1.1 and 1.2. In Section 3, we prove Theorem 1.1 while Section 4 focuses on the proof of Theorem 1.2.

2 Preliminaries

This section serves as a preparation for the proofs of Theorems 1.1 and 1.2. First, we state and prove several anisotropic inequalities for triple products. Such triple products appear naturally in the estimates of the nonlinear terms. The Oldroyd-B model considered here has anisotropic dissipation and these anisotropic inequalities are indispensable tools in the study of such systems. Attention here is focused on the 3D functions and the inequalities for the 2D ones can be found in several references (see, e.g., [4,26]).

Lemma 2.1. *For some constants $C > 0$, $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, we have*

$$\int |fgh|dx \leqslant C\, \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_2 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_3 h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, \tag{2.1}$$

$$\begin{aligned}
\int |fgh|dx \leqslant{}& C\, \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_i f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_i \partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\\
&\times \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_k g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|h\|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{2.2}$$

$$\|fg\|_{L^2(\mathbb{R}^3)} \leqslant C\, \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_3 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}. \tag{2.3}$$

Proof. The first two inequalities have been stated and proven in [40]. Here, we focus on the proof of (2.3). We first recall the basic 1D inequality

$$\|f\|_{L^\infty_{x_l}(\mathbb{R})} \leqslant \sqrt{2} \|f\|_{L^2_{x_l}(\mathbb{R})}^{\frac{1}{2}} \|\partial_l f\|_{L^2_{x_l}(\mathbb{R})}^{\frac{1}{2}}, \quad l = 1, 2, 3. \tag{2.4}$$

As a consequence,

$$\|fg\|_{L^2(\mathbb{R}^3)} \leqslant \|f\|_{L^\infty_{x_1 x_2} L^2_{x_3}} \|g\|_{L^2_{x_1 x_2} L^\infty_{x_3}} \leqslant C \|f\|_{L^\infty_{x_1 x_2} L^2_{x_3}} \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_3 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.$$

Applying the Minkowski inequality, (2.4) and Hölder's inequality yields

$$\begin{aligned}
\|f\|_{L^\infty_{x_1 x_2} L^2_{x_3}} &\leqslant \|\|\|f\|_{L^\infty_{x_1}}\|_{L^2_{x_3} L^\infty_{x_2}} \leqslant C \|\|\|f\|_{L^2_{x_1}}^{\frac{1}{2}} \|\partial_1 f\|_{L^2_{x_1}}^{\frac{1}{2}}\|_{L^2_{x_3} L^\infty_{x_2}}\\
&\leqslant C \|\|\|\|f\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}^{\frac{1}{2}} \|\|\partial_1 f\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}^{\frac{1}{2}}\|_{L^2_{x_3}}\\
&\leqslant C \|\|\|f\|_{L^\infty_{x_2}}\|_{L^2_{x_1 x_3}}^{\frac{1}{2}} \|\|\partial_1 f\|_{L^\infty_{x_2}}\|_{L^2_{x_1 x_3}}^{\frac{1}{2}}\\
&\leqslant C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}.
\end{aligned}$$

Therefore,

$$\|fg\|_{L^2(\mathbb{R}^3)} \leqslant C \|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\partial_3 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.1. □

Next, we state an identity of Zhu [45] as a lemma. This identity decomposes a special nonlinear term into several terms that can be dealt with more easily. The proof of this identity can be found in [45].

Lemma 2.2. *For any smooth u and τ, the following decomposition holds*:

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)=\mathbb{P}((u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau))+\mathbb{P}(\nabla u\cdot\nabla\tau)-\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau). \tag{2.5}$$

To avoid any confusion on the vectors on the right-hand side, we write their k-th components as follows:

$$\begin{aligned}
&[(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)]_k=\sum_i u_i\,\partial_i[\mathbb{P}(\nabla\cdot\tau)]_k,\\
&[\nabla u\cdot\nabla\tau]_k=\sum_{i,j}\partial_j u_i\,\partial_i\tau_{kj},\\
&[\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau]_k=\sum_{i,j,m}\partial_k u_i\,\partial_i\Delta^{-1}\,\partial_m\partial_j\tau_{mj}.
\end{aligned}$$

3 Proof of Theorem 1.1

This section proves Theorem 1.1. Since the local existence and uniqueness follows from a rather routine process (see, e.g., [28]), our attention is focused on the global *a priori* bound on $\|(u,\tau)\|_{H^2}$. As aforementioned in the introduction, this is accomplished by proving the inequality

$$E_0(t)+E_1(t)\leqslant C\|(u_0,\tau_0)\|_{H^2}^2+CE_0(t)^{\frac{3}{2}}+CE_0(t)E_1(t)^{\frac{1}{2}}+CE_0(t)^{\frac{1}{2}}E_1(t), \tag{3.1}$$

and then applying the bootstrapping argument (see, e.g., [34, p. 21]).

We first assume (3.1) and prove Theorem 1.1. We then come back to prove (3.1).

Proof of Theorem 1.1. We set $E(t)=E_0(t)+E_1(t)$. Then (3.1) implies

$$E(t)\leqslant C_1\|(u_0,\tau_0)\|_{H^2}^2+C_2E(t)^{\frac{3}{2}} \tag{3.2}$$

for two positive constants C_1 and C_2. We show that if δ in (1.2) is taken to be sufficiently small, then the global uniform bound (1.3) in Theorem 1.1 holds for all time $t>0$. To apply the bootstrapping argument, we start with the ansatz that $E(t)\leqslant M:=(\frac{1}{2C_2})^2$. Then it follows from (3.2) and (1.2) that

$$E(t)\leqslant C_1\|(u_0,\tau_0)\|_{H^2}^2+\frac{1}{2}E(t),$$

or

$$E(t)\leqslant 2C_1\,\|(u_0,\tau_0)\|_{H^2}^2\leqslant 2C_1\delta^2. \tag{3.3}$$

If δ satisfies $\delta^2\leqslant\frac{1}{16C_1C_2^2}$, then $E(t)\leqslant\frac{M}{2}$. The bootstrapping argument then implies that (3.3) holds for all $t\geqslant 0$, i.e.,

$$\begin{aligned}
&\|(u(t),\tau(t))\|_{H^2}^2+\int_0^t(\|\nabla_h u(s)\|_{H^2}^2+\|\partial_3 u(s)\|_{H^1}^2\\
&\quad+\|\partial_3\tau(s)\|_{H^2}^2+\|\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2)ds\leqslant C\delta^2, \qquad (3.4)\\
&(1+t)(\|\nabla u(s)\|_{H^1}^2+2\|\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2)+\int_0^t(1+s)(\|\nabla\nabla_h u(s)\|_{H^1}^2+\|\nabla\partial_3 u(s)\|_{L^2}^2\\
&\quad+\|\partial_3\mathbb{P}(\nabla\cdot\tau)(s)\|_{H^1}^2+\|\nabla_h\mathbb{P}(\nabla\cdot\tau)(s)\|_{L^2}^2)ds\leqslant C\delta^2, \qquad (3.5)
\end{aligned}$$

which gives (1.3) and also implies the decay estimate (1.4). □

We now return to prove the estimate (3.1). The proof is split into two parts. The first part proves the bound for $E_0(t)$ while the second shows the estimate for $E_1(t)$. Then (3.1) follows immediately from (3.6) in Proposition 3.1 and (3.21) in Proposition 3.5.

3.1 Upper bound for $E_0(t)$

Proposition 3.1. *Let* (u,τ) *be the solution of* (1.1) *with initial data* (u_0,τ_0) *satisfying* $\mathrm{div} u_0 = 0$ *and* $(\tau_0)_{ij} = (\tau_0)_{ji}$. *Then, for a constant* $C > 0$, *we have*

$$E_0(t) \leqslant C(\|(u_0,\tau_0)\|_{H^2}^2 + E_0(t)^{\frac{3}{2}} + E_0(t)E_1(t)^{\frac{1}{2}}). \tag{3.6}$$

The proof of (3.6) is long, so we divide it into three lemmas.

Lemma 3.2. *For a constant* $C > 0$, *we have*

$$\begin{aligned}
&\|(u(t),\tau(t))\|_{H^2}^2 + 2\int_0^t (\mu\|\nabla_h u(s)\|_{H^2}^2 + \eta\|\partial_3\tau(s)\|_{H^2}^2)ds \\
&\quad \leqslant C\|(u_0,\tau_0)\|_{H^2}^2 + CE_0(t)E_1(t)^{\frac{1}{2}} + CE_0(t)^{\frac{3}{2}}.
\end{aligned} \tag{3.7}$$

Proof. Due to the equivalence of $\|(u,\tau)\|_{H^2}$ with $\|(u,\tau)\|_{L^2} + \|(u,\tau)\|_{\dot{H}^2}$, it suffices to establish the estimates for $\|(u,\tau)\|_{L^2}$ and $\|(u,\tau)\|_{\dot{H}^2}$. Taking the L^2-inner product of (1.1) with (u,τ), we have

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\tau(t)\|_{L^2}^2) + (\mu\|\nabla_h u(t)\|_{L^2}^2 + \eta\|\partial_3\tau(t)\|_{L^2}^2) = -\int Q\cdot\tau dx.$$

Here, we have used

$$\int(\nabla\cdot\tau)\cdot u dx + \int D(u)\cdot\tau dx = 0, \tag{3.8}$$

which can be verified by integration by parts and the symmetry of τ,

$$\begin{aligned}
&\int(\nabla\cdot\tau)\cdot u dx + \int D(u)\cdot\tau dx \\
&\quad = \sum_{i,j}\int \partial_j\tau_{ij}\, u_i dx + \sum_{i,j}\frac{1}{2}\int(\partial_j u_i + \partial_i u_j)\tau_{ij} dx \\
&\quad = -\sum_{i,j}\int \tau_{ij}\,\partial_j u_i dx + \sum_{i,j}\frac{1}{2}\int(\partial_j u_i\,\tau_{ij} + \partial_i u_j\,\tau_{ji}) dx \\
&\quad = -\sum_{i,j}\int \tau_{ij}\,\partial_j u_i dx + \sum_{i,j}\int \tau_{ij}\,\partial_j u_i dx = 0.
\end{aligned} \tag{3.9}$$

We note that even in the case where ν_1 and ν_2 are different, we can still obtain (3.8) by taking the inner product of the equation for u in (1.1) with $\nu_2 u$ and the equation for τ with $\nu_1\tau$. Clearly, multiplying by extra constants in the L^2-estimate would not affect any other terms. By the anisotropic inequality (2.2), we obtain

$$\begin{aligned}
-\int Q\cdot\tau dx &\leqslant C\int|\nabla u||\tau||\tau|dx \\
&\leqslant C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}\|\tau\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla_h\nabla u\|_{H^1}^{\frac{3}{4}}\|\tau\|_{L^2}^{\frac{3}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Then, by virtues of Hölder's inequality and the definitions of E_0 and E_1, we have

$$\begin{aligned}
-\int_0^t\int Q\cdot\tau dxds &\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{8}}\|\nabla u(s)\|_{L^2}^{\frac{1}{4}}\|\tau(s)\|_{L^2}^{\frac{3}{2}} \\
&\qquad\times\int_0^t(1+s)^{\frac{3}{8}}\|\nabla_h\nabla u\|_{H^1}^{\frac{3}{4}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds \\
&\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{8}}\|\nabla u(s)\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{3}{2}}\left[\int_0^t(1+s)\|\nabla_h\nabla u\|_{H^1}^2 ds\right]^{\frac{3}{8}}
\end{aligned}$$

$$\times\left[\int_0^t\|\partial_3\tau\|_{L^2}^2ds\right]^{\frac14}\left[\int_0^t(1+s)^{-\frac43}ds\right]^{\frac38}$$
$$\leqslant CE_0(t)E_1(t)^{\frac12}.$$

Therefore,

$$\begin{aligned}&(\|u(t)\|_{L^2}^2+\|\tau(t)\|_{L^2}^2)+2\int_0^t(\mu\|\nabla_hu(s)\|_{L^2}^2+\eta\|\partial_3\tau(s)\|_{L^2}^2)ds\\&\quad\leqslant\|(u_0,\tau_0)\|_{L^2}^2+CE_0(t)E_1(t)^{\frac12}.\end{aligned}\tag{3.10}$$

Next, we estimate $\|(u(t),\tau(t))\|_{\dot H^2}$. Applying the operator Δ to (1.1) and taking the L^2-inner product of the resulting equations with $(\Delta u,\Delta\tau)$, we have

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\Delta u(t)\|_{L^2}^2+\|\Delta\tau(t)\|_{L^2}^2)+(\mu\|\nabla_h\Delta u(t)\|_{L^2}^2+\eta\|\partial_3\Delta\tau(t)\|_{L^2}^2)\\&\quad=-\int\Delta(u\cdot\nabla u)\cdot\Delta udx-\int\Delta(u\cdot\nabla\tau)\cdot\Delta\tau dx-\int\Delta Q\cdot\Delta\tau dx\\&\quad=:I_1+I_2+I_3.\end{aligned}\tag{3.11}$$

Here, we have used

$$\int\Delta(\nabla\cdot\tau)\cdot\Delta udx+\int\Delta D(u)\cdot\Delta\tau dx=0,$$

which can be verified similarly as (3.9). By integration by parts and the anisotropic inequality (2.1),

$$\begin{aligned}I_1&=-\int\Delta u\cdot\nabla u\cdot\Delta udx-2\int(\nabla u\cdot\nabla)\nabla u\cdot\Delta udx\leqslant3\int|\nabla u||\nabla^2u||\Delta u|dx\\&\leqslant C\|\nabla u\|_{L^2}^{\frac12}\|\partial_3\nabla u\|_{L^2}^{\frac12}\|\nabla^2u\|_{L^2}^{\frac12}\|\partial_1\nabla^2u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}^{\frac12}\|\partial_2\Delta u\|_{L^2}^{\frac12}\\&\leqslant C\|\nabla u\|_{H^1}(\|\nabla^2u\|_{L^2}^2+\|\nabla_h\nabla^2u\|_{L^2}^2).\end{aligned}$$

Then,

$$\int_0^tI_1(s)ds\leqslant C\sup_{0\leqslant s\leqslant t}\|\nabla u(s)\|_{H^1}\int_0^t(\|\nabla^2u(s)\|_{L^2}^2+\|\nabla_h\nabla^2u(s)\|_{L^2}^2)ds\leqslant CE_0(t)^{\frac32}.\tag{3.12}$$

For I_2, by (2.1) and (2.2),

$$\begin{aligned}I_2&=-\int\Delta u\cdot\nabla\tau\cdot\Delta\tau dx-2\int(\nabla u\cdot\nabla)\nabla\tau\cdot\Delta\tau dx\\&\leqslant C\|\Delta u\|_{L^2}^{\frac12}\|\partial_1\Delta u\|_{L^2}^{\frac12}\|\nabla\tau\|_{L^2}^{\frac12}\|\partial_2\nabla\tau\|_{L^2}^{\frac12}\|\Delta\tau\|_{L^2}^{\frac12}\|\partial_3\Delta\tau\|_{L^2}^{\frac12}\\&\quad+C\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\|\nabla^2\tau\|_{L^2}^{\frac12}\|\partial_3\nabla^2\tau\|_{L^2}^{\frac12}\|\Delta\tau\|_{L^2}\\&\leqslant C\|\nabla\tau\|_{H^1}^{\frac32}\|\Delta u\|_{L^2}^{\frac12}\|\partial_1\Delta u\|_{L^2}^{\frac12}\|\partial_3\Delta\tau\|_{L^2}^{\frac12}+C\|\nabla^2\tau\|_{L^2}^{\frac32}\|\nabla u\|_{L^2}^{\frac14}\|\nabla_h\nabla u\|_{H^1}^{\frac34}\|\partial_3\Delta\tau\|_{L^2}^{\frac12}.\end{aligned}$$

Integrating in time and applying Hölder's inequality, we have

$$\begin{aligned}\int_0^tI_2(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}\|\nabla\tau(s)\|_{H^1}^{\frac32}\int_0^t(1+s)^{\frac14}\|\Delta u(s)\|_{L^2}^{\frac12}\\&\qquad\times(1+s)^{\frac14}\|\partial_1\Delta u(s)\|_{L^2}^{\frac12}\|\partial_3\Delta\tau(s)\|_{L^2}^{\frac12}(1+s)^{-\frac12}ds\\&\qquad+C\sup_{0\leqslant s\leqslant t}\|\nabla^2\tau(s)\|_{L^2}^{\frac32}(1+s)^{\frac18}\|\nabla u(s)\|_{L^2}^{\frac14}\\&\qquad\times\int_0^t(1+s)^{\frac38}\|\nabla_h\nabla u(s)\|_{H^1}^{\frac34}\|\partial_3\Delta\tau(s)\|_{L^2}^{\frac12}(1+s)^{-\frac12}ds\\&\leqslant CE_0(t)E_1(t)^{\frac12}.\end{aligned}\tag{3.13}$$

By integration by parts,

$$
\begin{aligned}
I_3 &\leqslant C\int|\nabla_h\nabla^2u||\tau||\Delta\tau|dx+C\int|\partial_3^2u|(|\partial_3\tau||\partial_3^2\tau|+|\tau||\partial_3^3\tau|)dx\\
&\quad+C\int|\nabla^2u||\nabla\tau||\Delta\tau|dx+C\int|\nabla u||\Delta\tau||\Delta\tau|dx\\
&\leqslant C\int|\nabla_h\nabla^2u||\tau||\Delta\tau|dx+C\int|\partial_3^2u||\tau||\partial_3^3\tau|dx\\
&\quad+C\int|\nabla^2u||\nabla\tau||\Delta\tau|dx+C\int|\nabla u||\Delta\tau||\Delta\tau|dx\\
&=: I_{31}+I_{32}+I_{33}+I_{34}.
\end{aligned}\tag{3.14}
$$

By (2.2) and Sobolev's inequality, I_{31} and I_{32} can be bounded by

$$
\begin{aligned}
I_{31}+I_{32}&\leqslant C\|\nabla_h\nabla^2u\|_{L^2}\|\tau\|_{L^2}^{\frac14}\|\partial_1\tau\|_{L^2}^{\frac14}\|\partial_2\tau\|_{L^2}^{\frac14}\|\partial_1\partial_2\tau\|_{L^2}^{\frac14}\|\Delta\tau\|_{L^2}^{\frac12}\|\Delta\partial_3\tau\|_{L^2}^{\frac12}\\
&\quad+C\|\tau\|_\infty\|\partial_3^2u\|_{L^2}\|\partial_3^3\tau\|_{L^2}\\
&\leqslant C\|\nabla_h\nabla^2u\|_{L^2}\|\tau\|_{H^2}^{\frac32}\|\partial_3\Delta\tau\|_{L^2}^{\frac12}+\|\tau\|_{H^2}(\|\partial_3^2u\|_{L^2}^2+\|\partial_3^3\tau\|_{L^2}^2).
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^t(I_{31}(s)+I_{32}(s))ds&\leqslant C\sup_{0\leqslant s\leqslant t}\|\tau(s)\|_{H^2}^{\frac32}\int_0^t(1+s)^{\frac12}\|\nabla_h\nabla^2u(s)\|_{L^2}\|\partial_3\Delta\tau(s)\|_{L^2}^{\frac12}(1+s)^{-\frac12}ds\\
&\quad+C\sup_{0\leqslant s\leqslant t}\|\tau(s)\|_{H^2}\int_0^t(\|\partial_3^2u(s)\|_{L^2}^2+\|\partial_3^3\tau(s)\|_{L^2}^2)ds\\
&\leqslant CE_0(t)E_1(t)^{\frac12}+CE_0(t)^{\frac32}.
\end{aligned}
$$

For I_{33} and I_{34}, invoking the estimate as in I_2, we have

$$
\int_0^t(I_{33}(s)+I_{34}(s))ds\leqslant CE_0(t)E_1(t)^{\frac12}.
$$

Therefore,

$$
\int_0^tI_3(s)ds\leqslant CE_0(t)E_1(t)^{\frac12}.\tag{3.15}
$$

Integrating (3.11) over $[0,t]$ and using (3.12), (3.13) and (3.15), we get

$$
\begin{aligned}
&\|\Delta u(t)\|_{L^2}^2+\|\Delta\tau(t)\|_{L^2}^2+2\int_0^t(\mu\|\nabla_h\Delta u(s)\|_{L^2}^2+\eta\|\partial_3\Delta\tau(s)\|_{L^2}^2)ds\\
&\quad\leqslant\|(\Delta u_0,\Delta\tau_0)\|_{L^2}^2+CE_0(t)^{\frac32}+CE_0(t)E_1(t)^{\frac12},
\end{aligned}
$$

which, together with (3.10), implies the desired estimate (3.7). This completes the proof of Lemma 3.2. □

The next two lemmas establish the desired upper bounds for the last two time integrals in $E_0(t)$, i.e., $\int_0^t\|\partial_3u(s)\|_{H^1}^2ds$ and $\int_0^t\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2ds$. The time integrability of these two terms is a consequence of the regularization revealed by the wave structure (1.7). We extract the time integrability by considering the time evolution of two cross inner products $(\Lambda^{-1}\partial_3u,\Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot\tau)_{H^1}$ and $(u(t),\mathbb{P}\nabla\cdot\tau(t))_{H^1}$. Here, $(F,G)_{H^1}$ denotes the H^1-inner product of F and G.

Lemma 3.3. *For a constant* $C>0$,

$$
\begin{aligned}
&(\Lambda^{-1}\partial_3u,\Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot\tau)_{H^1}+\int_0^t\left[\frac14\|\partial_3u(s)\|_{H^1}^2-\frac{\mu}{2}\|\Delta_hu(s)\|_{H^1}^2-\left(1+\frac{\mu}{2}+\eta^2\right)\|\partial_3\tau(s)\|_{H^2}^2\right]ds\\
&\quad\leqslant\frac12\|(u_0,\nabla\tau_0)\|_{H^1}^2+CE_0(t)^{\frac32}.
\end{aligned}\tag{3.16}
$$

Proof. By (1.6), a simple calculation yields

$$\begin{aligned}
&\frac{d}{dt}(\Lambda^{-1}\partial_3 u, \Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot\tau)_{H^1} + \frac{1}{2}\|\partial_3 u\|_{H^1}^2 - \|\partial_3\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 \\
&\quad = -(\Lambda^{-1}\partial_3\mathbb{P}(u\cdot\nabla u), \Lambda^{-1}\partial_3\mathbb{P}(\nabla\cdot\tau))_{H^1} - (\Lambda^{-1}\partial_3 u, \Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot(u\cdot\nabla\tau))_{H^1} \\
&\qquad - (\Lambda^{-1}\partial_3 u, \Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot Q)_{H^1} + \mu(\Lambda^{-1}\partial_3\Delta_h u, \Lambda^{-1}\partial_3\mathbb{P}(\nabla\cdot\tau))_{H^1} \\
&\qquad + \eta(\Lambda^{-1}\partial_3 u, \Lambda^{-1}\partial_3^3\mathbb{P}(\nabla\cdot\tau))_{H^1} \\
&\quad =: I_4 + \cdots + I_8,
\end{aligned}$$

where we have used the fact that $\|\Lambda^{-1}\nabla\partial_3 u\|_{H^1} = \|\partial_3 u\|_{H^1}$ due to Plancherel's theorem. Invoking $\|\mathbb{P}v\|_{L^2} \leqslant C\|v\|_{L^2}$ and applying Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
I_4 &\leqslant C\|\Lambda^{-1}\partial_3\mathbb{P}(u\cdot\nabla u)\|_{H^1}\|\Lambda^{-1}\partial_3\mathbb{P}(\nabla\cdot\tau)\|_{H^1} \\
&\leqslant \|u\cdot\nabla u\|_{H^1}\|\mathbb{P}(\nabla\cdot\tau)\|_{H^1} \\
&\leqslant (\|u\|_{L^\infty}\|\nabla u\|_{H^1} + \|\nabla u\|_{L^4}^2)\|\mathbb{P}(\nabla\cdot\tau)\|_{H^1} \\
&\leqslant C\|u\|_{H^2}(\|\nabla u\|_{H^1}^2 + \|\mathbb{P}(\nabla\cdot\tau)\|_{H^1}^2),
\end{aligned}$$

where we have used the property that the Riesz operator $\mathcal{R}_i = \Lambda^{-1}\partial_i$ is L^2-bounded. To bound I_5, we integrate by parts and apply Hölder's inequality and Sobolev's inequality to get

$$\begin{aligned}
I_5 &= (\Lambda^{-1}\partial_3^2 u, \Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau))_{H^1} \leqslant C\|\partial_3 u\|_{H^1}\|u\cdot\nabla\tau\|_{H^1} \\
&\leqslant C\|\nabla u\|_{H^1}(\|u\|_{L^\infty}\|\nabla\tau\|_{H^1} + \|\nabla u\|_{L^4}\|\nabla\tau\|_{L^4}) \\
&\leqslant C\|\nabla u\|_{H^1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{H^1} + \|\nabla\tau\|_{H^1}\|\nabla u\|_{H^1}^2 \\
&\leqslant C\|\nabla\tau\|_{H^1}\|\nabla u\|_{H^1}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_6 &= \sum_{i,j}(\Lambda^{-1}\partial_3\partial_j u, \Lambda^{-1}\partial_3 Q_{ij})_{H^1} \leqslant C\|\nabla u\|_{H^1}\|Q\|_{H^1} \\
&\leqslant C\|\nabla u\|_{H^1}(\|\tau\|_{L^\infty}\|\nabla u\|_{H^1} + \|\nabla u\|_{L^4}\|\nabla\tau\|_{L^4}) \leqslant C\|\nabla u\|_{H^1}^2\|\tau\|_{H^2}.
\end{aligned}$$

By integration by parts and Hölder's inequality,

$$\begin{aligned}
I_7 + I_8 &= \mu(\Lambda^{-1}\partial_3\Delta_h u, \Lambda^{-1}\partial_3\mathbb{P}(\nabla\cdot\tau))_{H^1} - \eta(\Lambda^{-1}\partial_3^2 u, \Lambda^{-1}\partial_3^2\mathbb{P}(\nabla\cdot\tau))_{H^1} \\
&\leqslant \frac{\mu}{2}(\|\Delta_h u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2) + \frac{1}{4}\|\partial_3 u\|_{H^1}^2 + \eta^2\|\partial_3\nabla\tau\|_{H^1}^2.
\end{aligned}$$

Collecting all estimates for I_4 through I_8 above, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\Lambda^{-1}\partial_3 u, \Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot\tau)_{H^1} + \frac{1}{4}\|\partial_3 u\|_{H^1}^2 - \frac{\mu}{2}\|\Delta_h u\|_{H^1}^2 - \left(\frac{\mu}{2}+1\right)\|\partial_3\tau\|_{H^1}^2 - \eta^2\|\partial_3\nabla\tau\|_{H^1}^2 \\
&\quad \leqslant C(\|u\|_{H^2} + \|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2 + \|\mathbb{P}(\nabla\cdot\tau)\|_{H^1}^2). \qquad (3.17)
\end{aligned}$$

Integrating (3.17) in time derives (3.16). This completes the proof of Lemma 3.3. □

Lemma 3.4. *For a constant $C > 0$, we have*

$$\begin{aligned}
&(u(t), \mathbb{P}\nabla\cdot\tau(t))_{H^1} + \int_0^t \left[\frac{1}{2}\|\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2 - \left(\frac{1}{2}+\frac{\eta}{2}\right)\|\nabla u(s)\|_{H^1}^2 - \frac{\mu^2}{2}\|\Delta_h u(s)\|_{H^1}^2 - \frac{\eta}{2}\|\partial_3\nabla\tau(s)\|_{H^1}^2\right] ds \\
&\quad \leqslant \frac{1}{2}\|(u_0, \nabla\tau_0)\|_{H^1}^2 + CE_0(t)^{\frac{3}{2}}. \qquad (3.18)
\end{aligned}$$

Proof. As in the proof of Lemma 3.3, we have

$$-\frac{d}{dt}(u, \mathbb{P}\nabla\cdot\tau)_{H^1} + \|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2 - \frac{1}{2}\|\nabla u\|_{H^1}^2$$

$$
\begin{aligned}
&= (\mathbb{P}(u\cdot\nabla u),\mathbb{P}(\nabla\cdot\tau))_{H^1}+(u,\mathbb{P}\nabla\cdot(u\cdot\nabla\tau))_{H^1}+(u,\mathbb{P}\nabla\cdot Q)_{H^1}\\
&\quad-\mu(\Delta_h u,\mathbb{P}(\nabla\cdot\tau))_{H^1}-\eta(u,\partial_3^2\mathbb{P}(\nabla\cdot\tau))_{H^1}\\
&=: I_9+\cdots+I_{13}.
\end{aligned}
$$

Invoking the estimates for I_4 through I_6 yields

$$
\begin{aligned}
I_9+I_{10}+I_{11}&\leqslant \|u\cdot\nabla u\|_{H^1}\|\mathbb{P}\nabla\cdot\tau\|_{H^1}+\|\nabla u\|_{H^1}\|u\cdot\nabla\tau\|_{H^1}+\|\nabla u\|_{H^1}\|Q\|_{H^1}\\
&\leqslant C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2+\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2).
\end{aligned}
$$

Clearly, for I_{12} and I_{13}, we have

$$
\begin{aligned}
I_{12}+I_{13}&=-\mu(\Delta_h u,\mathbb{P}(\nabla\cdot\tau))_{H^1}+\eta(\partial_3 u,\partial_3\mathbb{P}(\nabla\cdot\tau))_{H^1}\\
&\leqslant \frac{\mu^2}{2}\|\Delta_h u\|_{H^1}^2+\frac{1}{2}\|\mathbb{P}(\nabla\cdot\tau)\|_{H^1}^2+\frac{\eta}{2}(\|\partial_3 u\|_{H^1}^2+\|\partial_3\nabla\tau\|_{H^1}^2).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&-\frac{d}{dt}(u,\mathbb{P}\nabla\cdot\tau)_{H^1}+\frac{1}{2}\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2-\left(\frac{1}{2}+\frac{\eta}{2}\right)\|\nabla u\|_{H^1}^2-\frac{\mu^2}{2}\|\Delta_h u\|_{H^1}^2-\frac{\eta}{2}\|\partial_3\nabla\tau\|_{H^1}^2\\
&\leqslant C(\|u\|_{H^2}+\|\tau\|_{H^2})(\|\nabla u\|_{H^1}^2+\|\mathbb{P}\nabla\cdot\tau\|_{H^1}^2). \qquad (3.19)
\end{aligned}
$$

Integrating (3.19) in time yields the desired estimate (3.18). This completes the proof of Lemma 3.4. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Multiplying (3.16) by λ_1 and then adding it to (3.7), and combining

$$
(\Lambda^{-1}\partial_3 u,\Lambda^{-1}\partial_3\mathbb{P}\nabla\cdot\tau)_{H^1}\geqslant-\frac{1}{2}(\|u\|_{H^1}^2+\|\nabla\tau\|_{H^1}^2),
$$

we see that

$$
\begin{aligned}
&\|(u(t),\tau(t))\|_{H^2}^2+\int_0^t(\|\nabla_h u(s)\|_{H^2}^2+\|\partial_3 u\|_{H^1}^2+\|\partial_3\tau(s)\|_{H^2}^2)ds\\
&\leqslant C\|(u_0,\tau_0)\|_{H^2}^2+CE_0(t)E_1(t)^{\frac{1}{2}}+CE_0(t)^{\frac{3}{2}}, \qquad (3.20)
\end{aligned}
$$

provided that λ_1 is sufficiently small. Furthermore, we make a similar calculation (3.18) $\times\,\lambda_2$ + (3.20) and select λ_2 to be sufficiently small to derive (3.6). This completes the proof of Proposition 3.1. □

3.2 Estimates for $E_1(t)$

Proposition 3.5. *Let* (u,τ) *be the solution of* (1.1) *associated with the initial data* (u_0,τ_0) *satisfying* $\operatorname{div} u_0=0$ *and* $(\tau_0)_{ij}=(\tau_0)_{ji}$. *Then, for a constant* $C>0$, *we have*

$$
E_1(t)\leqslant CE_0(t)+C\|(u_0,\tau_0)\|_{H^2}^2+CE_0(t)^{\frac{1}{2}}E_1(t). \qquad (3.21)
$$

The proof of Proposition 3.5 is divided into two parts, which will be presented in two lemmas.

Lemma 3.6. *For some constant* $C>0$, *we have*

$$
\begin{aligned}
&(1+t)(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2)+\int_0^t(1+s)(\mu\|\nabla_h\nabla u(s)\|_{H^1}^2+\eta\|\partial_3\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2)ds\\
&\leqslant 2E_0(t)+2\|(\nabla u_0,\nabla\tau_0)\|_{H^1}^2+CE_0(t)^{\frac{1}{2}}E_1(t). \qquad (3.22)
\end{aligned}
$$

Proof. Taking the H^1-inner product of (1.6) with $(\Delta u,\mathbb{P}\nabla\cdot\tau)$ and then multiplying the resulting equations by the time weight $(1+t)$, we get

$$
\frac{1}{2}\frac{d}{dt}(1+t)(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2)+(1+t)(\mu\|\nabla_h\nabla u(t)\|_{H^1}^2+2\eta\|\partial_3\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2)
$$

$$
\begin{aligned}
&= \frac{1}{2}(\|\nabla u(t)\|_{H^1}^2 + 2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2) + (1+t)(\mathbb{P}(u\cdot\nabla u), \Delta u)_{H^1} \\
&\quad - 2(1+t)(\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \mathbb{P}\nabla\cdot\tau)_{H^1} - 2(1+t)(\mathbb{P}(\nabla\cdot Q), \mathbb{P}\nabla\cdot\tau)_{H^1} \\
&=: J_1 + J_2 + J_3 + J_4. \qquad (3.23)
\end{aligned}
$$

By integration by parts, Hölder's inequality, Gagliardo-Nirenberg's inequality and the anisotropic inequality (2.1), we have

$$
\begin{aligned}
J_2 &= (1+t)\int u\cdot\nabla u\cdot\Delta u dx - (1+t)\int(\Delta u\cdot\nabla u\cdot\Delta u + 2\nabla u\cdot\nabla(\nabla u)\cdot\Delta u)dx \\
&\leqslant (1+t)\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} \\
&\quad + C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2 u\|_{L^2}^{\frac{1}{2}} \\
&\leqslant C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} + C(1+t)\|\nabla u\|_{H^1}\|\nabla^2 u\|_{L^2}\|\nabla_h\nabla^2 u\|_{L^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_0^t J_2(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{2}}\|\nabla u(s)\|_{L^2}\int_0^s\|\nabla u(s)\|_{H^1}(1+s)^{\frac{1}{2}}\|\Delta u(s)\|_{L^2}ds \\
&\quad + C\sup_{0\leqslant s\leqslant t}\|\nabla u(s)\|_{H^1}\int_0^t(1+s)(\|\nabla^2 u(s)\|_{L^2}^2 + \|\nabla_h\nabla^2 u(s)\|_{L^2}^2)ds \\
&\leqslant CE_0(t)^{\frac{1}{2}}E_1(t). \qquad (3.24)
\end{aligned}
$$

The estimate for J_3 is more subtle. We first rewrite it as follows:

$$
\begin{aligned}
J_3 &= -2(1+t)\int\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\mathbb{P}\nabla\cdot\tau dx - 2(1+t)\int\nabla\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\nabla\mathbb{P}\nabla\cdot\tau dx \\
&= J_{31} + J_{32}.
\end{aligned}
$$

By (2.5), $\mathbb{P}\mathbb{P}v = \mathbb{P}v$, integration by parts and $\nabla\cdot u = 0$, we further split J_{31} into two parts

$$
J_{31} = -2(1+t)\bigg(\int\mathbb{P}(\nabla u\cdot\nabla\tau)\cdot\mathbb{P}\nabla\cdot\tau dx - \int\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau)\cdot\mathbb{P}\nabla\cdot\tau dx\bigg).
$$

Applying integration by parts and (2.2) to the first term of J_{31} yields

$$
\begin{aligned}
&-2(1+t)\int\mathbb{P}(\nabla u\cdot\nabla\tau)\cdot\mathbb{P}\nabla\cdot\tau dx \\
&\quad = 2(1+t)\sum_{i,j,k}\int\partial_k u_i\,\tau_{jk}\,\partial_i[\mathbb{P}\nabla\cdot\tau]_j dx \\
&\quad \leqslant C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2} \\
&\quad \leqslant C(1+t)\|\tau\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{H^1}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2},
\end{aligned}
$$

where we have used $\mathbb{P}\mathbb{P}v = \mathbb{P}v$. Due to the fact that $\|\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2} \leqslant \|\tau\|_{L^2}$, the second term of J_{31} admits the same bound as the first one. Therefore, we derive

$$
\begin{aligned}
\int_0^t J_{31}(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}\|\tau(s)\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{1}{4}}\|\nabla u(s)\|_{H^1}^{\frac{1}{2}}\int_0^t(1+s)^{\frac{1}{4}}\|\nabla_h\nabla u(s)\|_{H^1}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}} \\
&\quad \times(1+s)^{\frac{1}{2}}\|\nabla\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}
$$

For J_{32}, similar to J_{31}, we first write

$$
J_{32} = -2(1+t)\bigg(\int\nabla\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau)\cdot\nabla\mathbb{P}\nabla\cdot\tau dx + \int\nabla\mathbb{P}(\nabla u\cdot\nabla\tau)\cdot\nabla\mathbb{P}\nabla\cdot\tau dx
$$

$$
\begin{aligned}
&\quad - \int \nabla \mathbb{P}(\nabla u \cdot \nabla \Delta^{-1} \nabla \cdot \nabla \cdot \tau) \cdot \nabla \mathbb{P} \nabla \cdot \tau dx \Big) \\
&=: J_{32,1} + J_{32,2} + J_{32,3}.
\end{aligned}
$$

By $\mathbb{P}\mathbb{P}v = \mathbb{P}v$, integration by parts and (2.2), we have

$$
\begin{aligned}
J_{32,1} &= -2(1+t) \int \nabla u \cdot \nabla \mathbb{P} \nabla \cdot \tau \cdot \nabla \mathbb{P} \nabla \cdot \tau dx \\
&\leqslant C(1+t) \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 \nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2} \\
&\leqslant C(1+t) \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{H^1}^{\frac{1}{2}} \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_0^t J_{32,1}(s) ds &\leqslant C \sup_{0 \leqslant s \leqslant t} \|\nabla u(s)\|_{H^1}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \int_0^t (1+s)(\|\partial_2 \nabla u(s)\|_{H^1}^2 + \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2}^2 + \|\nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2}^2) ds \\
&\leqslant C E_0(t)^{\frac{1}{2}} E_1(t).
\end{aligned}
$$

For $J_{32,2}$, by means of (2.1) and (2.2), we can similarly obtain

$$
\begin{aligned}
J_{32,2} &\leqslant 2(1+t) \int (|\nabla^2 u||\nabla \tau| + |\nabla u||\nabla^2 \tau|) |\nabla \mathbb{P} \nabla \cdot \tau| dx \\
&\leqslant C(1+t) \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla \tau\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \\
&\quad + C(1+t) \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla \partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 \nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla^2 \tau\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla^2 \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2} \\
&\leqslant C(1+t) \|\nabla \tau\|_{H^1} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}^{\frac{1}{2}} \\
&\quad + C(1+t) \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\nabla^2 \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_3 \nabla^2 \tau\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau\|_{L^2}.
\end{aligned}
$$

Thus, integrating $J_{32,2}$ in t yields

$$
\begin{aligned}
\int_0^t J_{32,2}(s) ds &\leqslant C \sup_{0 \leqslant s \leqslant t} \|\nabla \tau(s)\|_{H^1} \|\nabla^2 u(s)\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \int_0^t (1+s)(\|\partial_1 \nabla^2 u(s)\|_{L^2}^2 + \|\nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2}^2 + \|\partial_3 \nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2}^2) ds \\
&\quad + C \sup_{0 \leqslant s \leqslant t} (1+s)^{\frac{1}{4}} \|\nabla u(s)\|_{H^1}^{\frac{1}{2}} \|\nabla^2 \tau(s)\|_{L^2}^{\frac{1}{2}} \int_0^t (1+s)^{\frac{1}{4}} \|\nabla \partial_2 u(s)\|_{H^1}^{\frac{1}{2}} \|\partial_3 \nabla^2 \tau(s)\|_{L^2}^{\frac{1}{2}} \\
&\quad \times (1+s)^{\frac{1}{2}} \|\nabla \mathbb{P} \nabla \cdot \tau(s)\|_{L^2} ds \\
&\leqslant C E_0(t)^{\frac{1}{2}} E_1(t).
\end{aligned}
$$

Similarly,

$$
\int_0^t J_{32,3}(s) ds \leqslant C E_0(t)^{\frac{1}{2}} E_1(t).
$$

Combining all the estimates above, we obtain

$$
\int_0^t J_{32}(s) ds \leqslant C E_0(t)^{\frac{1}{2}} E_1(t).
$$

Consequently,

$$
\int_0^t J_3(s) ds \leqslant C E_0(t)^{\frac{1}{2}} E_1(t). \tag{3.25}
$$

We finally bound J_4. By integration by parts, we rewrite it as

$$J_4 = 2(1+t)\sum_{i,j}\int Q_{ij}\,\partial_j[\mathbb{P}\nabla\cdot\tau]_i dx - 2(1+t)\int \nabla(\nabla\cdot Q)\cdot\nabla\mathbb{P}\nabla\cdot\tau dx =: J_{41}+J_{42}.$$

It follows from the estimate of J_{31} that

$$\int_0^t J_{41}(s)ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t). \tag{3.26}$$

For J_{42}, invoking (3.14), with a similar argument, we first have

$$\begin{aligned}
J_{42} &\leqslant C(1+t)\int|\nabla_h\nabla^2 u||\tau||\nabla\mathbb{P}\nabla\cdot\tau|dx + C(1+t)\int|\partial_3^2 u||\tau||\partial_3^2\mathbb{P}\nabla\cdot\tau|dx\\
&\quad + C(1+t)\int|\nabla^2 u||\nabla\tau||\nabla\mathbb{P}\nabla\cdot\tau|dx + C(1+t)\int|\nabla u||\nabla^2\tau||\nabla\mathbb{P}\nabla\cdot\tau|dx\\
&=: J_{42,1}+J_{42,2}+J_{42,3}+J_{42,4}.
\end{aligned}$$

For the first two terms, by Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
J_{42,1}+J_{42,2} &\leqslant C(1+t)\|\tau\|_\infty(\|\nabla_h\nabla^2 u\|_{L^2}\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}+\|\nabla^2 u\|_{L^2}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2})\\
&\leqslant C(1+t)\|\tau\|_{H^2}(\|\nabla_h\nabla^2 u\|_{L^2}^2+\|\nabla\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}^2).
\end{aligned}$$

Hence,

$$\int_0^t (J_{42,1}(s)+J_{42,2}(s))ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t).$$

Invoking the estimates of $J_{32,2}$ yields

$$\int_0^t (J_{42,3}+J_{42,4})(s)ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t).$$

Thus,

$$\int_0^t J_{42}(s)ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t),$$

which, together with (3.26), gives

$$\int_0^t J_4(s)ds \leqslant CE_0(t)^{\frac{1}{2}}E_1(t). \tag{3.27}$$

Inserting (3.24), (3.25) and (3.27) into (3.23) and then integrating in time, we conclude

$$\begin{aligned}
&(1+t)(\|\nabla u(t)\|_{H^1}^2+2\|\mathbb{P}\nabla\cdot\tau(t)\|_{H^1}^2)+\int_0^t(1+s)(\mu\|\nabla_h\nabla u(s)\|_{H^1}^2+\eta\|\partial_3\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2)ds\\
&\quad\leqslant 2E_0(t)+2\|(\nabla u_0,\nabla\tau_0)\|_{H^1}^2+CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}$$

This completes the proof of Lemma 3.6. □

Next, we make use of the wave structure in (1.7) to gain the time integrability

$$\int_0^t(1+s)(\|\partial_3\nabla u(s)\|_{L^2}^2+\|\nabla_h\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2)ds.$$

Lemma 3.7. *For some constant $C>0$, we have*

$$(1+t)[(\partial_3 u(t),\partial_3\mathbb{P}\nabla\cdot\tau(t))+(\nabla_h u(t),\nabla_h\mathbb{P}\nabla\cdot\tau(t))]$$

$$
\begin{aligned}
&+\frac{1}{4}\int_0^t(1+s)(\|\partial_3\nabla u(s)\|_{L^2}^2+2\|\nabla_h\mathbb{P}\nabla\cdot\tau(s)\|_{L^2}^2)ds\\
&-\tilde{C}_1\int_0^t(1+s)(\|\nabla_h\nabla u(s)\|_{H^1}^2+\|\partial_3\mathbb{P}\nabla\cdot\tau(s)\|_{H^1}^2)ds\\
&\leqslant E_0(t)+\|(\nabla u_0,\nabla^2\tau_0)\|_{L^2}^2+CE_0(t)^{\frac{1}{2}}E_1(t),
\end{aligned}\tag{3.28}
$$

where $\tilde{C}_1=\max\{\frac{1}{2}(1+\mu+\mu^2+\eta),1+\frac{\mu}{2}+\eta^2+\frac{\eta}{2}\}$.

Proof. The proof is divided into two steps. The first step establishes the bound for $\int_0^t(1+s)\|\partial_3\nabla u\|_{L^2}^2ds$ while the second is devoted to $\int_0^t(1+s)\|\nabla_h\mathbb{P}\nabla\cdot\tau\|_{L^2}^2ds$.

Step 1. The bound for $\int_0^t(1+s)\|\partial_3\nabla u\|_{L^2}^2ds$. We consider $\frac{d}{dt}(1+t)(\partial_3u,\partial_3\mathbb{P}\nabla\cdot\tau)$. By (1.6), we have

$$
\begin{aligned}
&\frac{d}{dt}(1+t)(\partial_3u,\partial_3\mathbb{P}\nabla\cdot\tau)+\frac{1}{2}(1+t)\|\partial_3\nabla u\|_{L^2}^2-(1+t)\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\
&\quad=(\partial_3u,\partial_3\mathbb{P}\nabla\cdot\tau)-(1+t)(\partial_3\mathbb{P}(u\cdot\nabla u),\partial_3\mathbb{P}(\nabla\cdot\tau))\\
&\qquad-(1+t)(\partial_3u,\partial_3\mathbb{P}\nabla\cdot(u\cdot\nabla\tau))-(1+t)(\partial_3u,\partial_3\mathbb{P}\nabla\cdot Q)\\
&\qquad+\mu(1+t)(\partial_3\Delta_hu,\partial_3\mathbb{P}(\nabla\cdot\tau))+\eta(1+t)(\partial_3u,\partial_3^3\mathbb{P}(\nabla\cdot\tau))\\
&\quad=:(\partial_3u,\partial_3\mathbb{P}\nabla\cdot\tau)+J_5+\cdots+J_9,
\end{aligned}\tag{3.29}
$$

where (F,G) denotes the L^2-inner product of F and G. By integration by parts, Hölder's inequality and Gagliardo-Nirenberg's inequality, we have

$$
\begin{aligned}
J_5&=(1+t)\int u\cdot\nabla u\cdot\partial_3^2\mathbb{P}(\nabla\cdot\tau)dx\leqslant C(1+t)\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\partial_3^2\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\\
&\leqslant C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\partial_3^2\mathbb{P}(\nabla\cdot\tau)\|_{L^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^tJ_5(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{1/2}\|\nabla u(s)\|_{H^1}\int_0^t\|\nabla u(s)\|_{L^2}(1+s)^{1/2}\|\partial_3^2\mathbb{P}(\nabla\cdot\tau)(s)\|_{L^2}ds\\
&\leqslant CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}
$$

By integration by parts and (2.5),

$$
\begin{aligned}
J_6&=(1+t)\int\partial_3^2u\cdot\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)dx\\
&=(1+t)\int\partial_3^2u\cdot(\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau)+\mathbb{P}(\nabla u\cdot\nabla\tau)-\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau))dx\\
&=(1+t)\int\partial_3^2u\cdot(u\cdot\nabla\mathbb{P}\nabla\cdot\tau+\nabla u\cdot\nabla\tau-\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau)dx,
\end{aligned}
$$

where we have used $(\mathbb{P}u,v)=(u,\mathbb{P}v)$. Hölder's inequality, Sobolev's inequality and (2.1) yield

$$
\begin{aligned}
J_6&\leqslant C(1+t)(\|u\|_{L^\infty}\|\partial_3^2u\|_{L^2}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\\
&\quad+\|\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{2}}\\
&\quad+\|\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2}^{\frac{1}{2}})\\
&\leqslant C(1+t)\|u\|_{H^2}\|\nabla^2u\|_{L^2}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\\
&\quad+C(1+t)\|\nabla\tau\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}\|\partial_1\partial_3^2u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$
\int_0^tJ_6(s)ds\leqslant C\sup_{0\leqslant s\leqslant t}\|u(s)\|_{H^2}\int_0^t(1+s)(\|\nabla^2u(s)\|_{L^2}^2+\|\nabla\mathbb{P}(\nabla\cdot\tau)(s)\|_{L^2}^2)ds
$$

$$
\begin{aligned}
&+ C \sup_{0\leqslant s\leqslant t} \|\nabla\tau(s)\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{1}{4}}\|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \\
&\times \int_0^t (1+s)^{\frac{1}{2}}\|\nabla^2 u(s)\|_{L^2}(1+s)^{\frac{1}{4}}\|\partial_1\partial_3^2 u(s)\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau(s)\|_{L^2}^{\frac{1}{2}} ds \\
\leqslant\, & CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}
$$

J_7 can be bounded by

$$
J_7 \leqslant C(1+t)\int |\partial_3^2 u|(|\nabla^2 u||\tau| + |\nabla u||\nabla\tau|)dx.
$$

As in the estimates of the first two terms of J_6, we have

$$
\begin{aligned}
\int_0^t J_7(s)ds &\leqslant C\int_0^t (1+s)(\|\tau\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2 + \|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{2}})ds \\
&\leqslant CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
J_8 + J_9 &= \mu(1+t)(\partial_3\Delta_h u, \partial_3\mathbb{P}(\nabla\cdot\tau)) - \eta(1+t)(\partial_3^2 u, \partial_3^2\mathbb{P}(\nabla\cdot\tau)) \\
&\leqslant (1+t)\left[\frac{\mu}{2}(\|\partial_3\Delta_h u\|_{L^2}^2 + \|\partial_3\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2) + \frac{1}{4}\|\partial_3^2 u\|_{L^2}^2 + \eta^2\|\partial_3^2\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2\right].
\end{aligned}
$$

Integrating (3.29) over $[0,t]$ and combining all the estimates above, we see that

$$
\begin{aligned}
&(1+t)(\partial_3 u, \partial_3\mathbb{P}\nabla\cdot\tau) + \int_0^t (1+s)\left[\frac{1}{4}\|\partial_3\nabla u\|_{L^2}^2 - \frac{\mu}{2}\|\partial_3\Delta_h u\|_{L^2}^2 - \left(\frac{\mu}{2}+1\right)\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\right. \\
&\quad\left. - \eta^2\|\partial_3^2\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2\right]ds \\
&\quad\leqslant \frac{1}{2}\int_0^t (\|\partial_3 u\|_{L^2}^2 + \|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)ds + \frac{1}{2}(\|\partial_3 u_0\|_{L^2}^2 + \|\partial_3\mathbb{P}\nabla\cdot\tau_0\|_{L^2}^2) + CE_0(t)^{\frac{1}{2}}E_1(t) \\
&\quad\leqslant \frac{1}{2}E_0(t) + \frac{1}{2}(\|\nabla u_0\|_{L^2} + \|\nabla^2\tau_0\|_{L^2}^2) + CE_0(t)^{\frac{1}{2}}E_1(t).
\end{aligned}
\tag{3.30}
$$

Step 2. The bound for $\int_0^t (1+s)\|\nabla_h\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 ds$. We calculate $\frac{d}{dt}(1+t)(\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot\tau)$ to get

$$
\begin{aligned}
&-\frac{d}{dt}(1+t)(\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot\tau) + (1+t)\|\nabla_h\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 - \frac{1}{2}(1+t)\|\nabla_h\nabla u\|_{L^2}^2 \\
&= (\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot\tau) + (1+t)(\nabla_h\mathbb{P}(u\cdot\nabla u), \nabla_h\mathbb{P}(\nabla\cdot\tau)) \\
&\quad + (1+t)(\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)) + (1+t)(\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot Q) \\
&\quad - \mu(1+t)(\nabla_h\Delta_h u, \nabla_h\mathbb{P}(\nabla\cdot\tau)) - \eta(1+t)(\nabla_h u, \nabla_h\partial_3^2\mathbb{P}(\nabla\cdot\tau)) \\
&=: (\nabla_h u, \nabla_h\mathbb{P}\nabla\cdot\tau) + J_{10} + \cdots + J_{14}.
\end{aligned}
\tag{3.31}
$$

J_{10} through J_{14} can be similarly estimated as the corresponding terms in J_5 through J_9. Clearly, the linear integral can be bounded by

$$
\begin{aligned}
&J_{13} + J_{14} \\
&\quad\leqslant (1+t)\left[\frac{\mu^2}{2}\|\nabla_h\Delta_h u\|_{L^2}^2 + \frac{1}{2}\|\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2 + \frac{\eta}{2}\|\partial_3\nabla_h u\|_{L^2}^2 + \frac{\eta}{2}\|\partial_3\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2\right].
\end{aligned}
$$

By Hölder's and Sobolev's inequalities,

$$
J_{10} = (1+t)\int (u\cdot\nabla\nabla_h u + \nabla_h u\cdot\nabla u)\cdot\nabla_h\mathbb{P}(\nabla\cdot\tau)dx
$$

$$
\begin{aligned}
&\leqslant C(1+t)(\|u\|_{L^\infty}\|\nabla\nabla_h u\|_{L^2}\|\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}+\|\nabla u\|_{L^4}^2\|\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2})\\
&\leqslant C(1+t)\|u\|_{H^2}\|\nabla\nabla_h u\|_{L^2}\|\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}+\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac32}\|\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_0^t J_{10}(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}\|u(s)\|_{H^2}\int_0^t(1+s)(\|\nabla^2u(s)\|_{L^2}^2+\|\nabla_h\mathbb{P}(\nabla\cdot\tau)(s)\|_{L^2}^2)ds\\
&\leqslant CE_0(t)^{\frac12}E_1(t).
\end{aligned}
$$

As in J_6, we have

$$
\int_0^t J_{11}(s)ds=-\int_0^t(1+s)\int\Delta_h u\cdot\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)dxds\leqslant CE_0(t)^{\frac12}E_1(t).
$$

For J_{12}, we first bound it by

$$
J_{12}\leqslant C(1+t)\int|\Delta_h u|(|\nabla^2u||\tau|+|\nabla u||\nabla\tau|)dx
$$

and then invoke the estimate of J_7 to obtain

$$
\int_0^t J_{12}(s)ds\leqslant CE_0(t)^{\frac12}E_1(t).
$$

Consequently, integrating (3.31) in time and incorporating the estimates for J_{10} through J_{14}, we obtain

$$
\begin{aligned}
&-(1+t)(\nabla_h u,\nabla_h\mathbb{P}\nabla\cdot\tau)+\int_0^t(1+s)\bigg[\frac12\|\nabla_h\mathbb{P}\nabla\cdot\tau\|_{L^2}^2-\left(\frac12+\frac{\eta}{2}\right)\|\nabla_h\nabla u\|_{L^2}^2\\
&-\frac{\mu^2}{2}\|\nabla_h\Delta_h u\|_{L^2}^2-\frac{\eta}{2}\|\partial_3\nabla_h\mathbb{P}(\nabla\cdot\tau)\|_{L^2}^2\bigg]ds\\
&\leqslant\frac12E_0(t)+\frac12(\|\nabla u_0\|_{L^2}^2+\|\nabla^2\tau_0\|_{L^2}^2)+CE_0(t)^{\frac12}E_1(t),
\end{aligned}
$$

which, together with (3.30), gives the desired estimate. This completes the proof of Lemma 3.7. □

We combine Lemmas 3.6 and 3.7 to complete the proof of Proposition 3.5.

Proof of Proposition 3.5. According to Lemmas 3.6 and 3.7, for a sufficiently small λ_3, a direct calculation of (3.22) + λ_3(3.28) yields the desired *a prior* estimate

$$
E_1(t)\leqslant CE_0(t)+C\|(\nabla u_0,\nabla\tau_0)\|_{H^1}^2+CE_0(t)^{\frac12}E_1(t).
$$

This completes the proof of Proposition 3.5. □

4 Proof of Theorem 1.2

The section is devoted to proving Theorem 1.2. The proof makes use of the integral representations of $(u,\mathbb{P}\cdot\nabla\tau)$. For the sake of clarity, we divide this section into four subsections. The first subsection derives the integral representation of $(u,\mathbb{P}\nabla\cdot\tau)$ from the differential equations (1.6) and establishes sharp upper bounds for the Fourier multiplier operators involved. The multipliers are anisotropic and the frequency space is divided into subdomains. Optimal bounds are obtained for them in each subdomain. The second subsection makes use of the integral representation and applies the bootstrapping argument to obtain the optimal decay rates for $\|u\|_{L^2}$ and $\|\nabla u\|_{L^2}$. The third subsection extracts the decay rates for $\|\nabla\nabla_h u\|_{L^2},\|\partial_3\nabla u\|_{L^2}$ and $\|\partial_3\tau\|_{H^1}$. To facilitate the proof, we introduce three time-weighted energy functionals and prove their boundedness via the bootstrapping argument. The last subsection computes the decay rate for $\|\mathbb{P}\cdot\nabla\tau\|_{L^2}$.

4.1 Integral representation and sharp bounds for the kernels

We first derive the integral representation for $(u, \mathbb{P}\nabla\cdot\tau)$ satisfying (1.6). Taking the Fourier transform of the equations (1.6) yields

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{\mathbb{P}\nabla\cdot\tau} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{\mathbb{P}\nabla\cdot\tau} \end{pmatrix} + \begin{pmatrix} \widehat{N}_1 \\ \widehat{N}_2 + \widehat{N}_3 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\mu|\xi_h|^2 & 1 \\ -\frac{1}{2}|\xi|^2 & -\eta\xi_3^2 \end{pmatrix}, \quad N_1 = -\mathbb{P}(u\cdot\nabla u),$$
$$N_2 = -\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \quad N_3 = -\mathbb{P}(\nabla\cdot Q)$$

with $|\xi_h|^2 = \xi_1^2 + \xi_2^2$. The eigenvalues of the matrix A are given by

$$\lambda_1 = \frac{-(\mu|\xi_h|^2 + \eta\xi_3^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\mu|\xi_h|^2 + \eta\xi_3^2) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma = (\mu|\xi_h|^2 + \eta\xi_3^2)^2 - 4\left(\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2\right).$$

The corresponding eigenvectors are

$$\rho_1 = \begin{pmatrix} 1 \\ \lambda_1 + \mu|\xi_h|^2 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 \\ \lambda_2 + \mu|\xi_h|^2 \end{pmatrix}.$$

Therefore, e^{At} can be explicitly written as

$$\mathrm{e}^{At} = (\rho_1, \rho_2)\begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{pmatrix}(\rho_1, \rho_2)^{-1} = \begin{pmatrix} G_3 - \mu|\xi_h|^2 G_1 & -G_1 \\ \frac{|\xi|^2}{2} G_1 & G_2 + \mu|\xi_h|^2 G_1 \end{pmatrix},$$

where

$$G_1(t) = \frac{\mathrm{e}^{\lambda_2 t} - \mathrm{e}^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad G_2(t) = \frac{\lambda_2\mathrm{e}^{\lambda_2 t} - \lambda_1\mathrm{e}^{\lambda_1 t}}{\lambda_2 - \lambda_1} = \mathrm{e}^{\lambda_2 t} + \lambda_1 G_1(t),$$
$$G_3(t) = \frac{\lambda_2\mathrm{e}^{\lambda_1 t} - \lambda_1\mathrm{e}^{\lambda_2 t}}{\lambda_2 - \lambda_1} = \mathrm{e}^{\lambda_1 t} - \lambda_1 G_1(t).$$

By Duhamel's principle,

$$\begin{aligned} \widehat{u}(\xi,t) &= \widehat{Q}_1(t)\widehat{u}_0 + \widehat{Q}_2(t)\widehat{\mathbb{P}\nabla\cdot\tau_0} \\ &\quad + \int_0^t (\widehat{Q}_1(t-s)\widehat{N}_1(s) + \widehat{Q}_2(t-s)\widehat{N}_2(s) + \widehat{Q}_2(t-s)\widehat{N}_3(s))ds, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \widehat{\mathbb{P}\nabla\cdot\tau}(\xi,t) &= \widehat{Q}_3(t)\widehat{u}_0 + \widehat{Q}_4(t)\widehat{\mathbb{P}\nabla\cdot\tau_0} \\ &\quad + \int_0^t (\widehat{Q}_3(t-s)\widehat{N}_1(s) + \widehat{Q}_4(t-s)\widehat{N}_2(s) + \widehat{Q}_4(t-s)\widehat{N}_3(s))ds, \end{aligned} \tag{4.2}$$

where

$$\widehat{Q}_1(t) = G_3(t) - \mu|\xi_h|^2 G_1(t), \quad \widehat{Q}_2(t) = -G_1(t),$$
$$\widehat{Q}_3(t) = \frac{|\xi|^2}{2}G_1, \quad \widehat{Q}_4(t) = G_2 + \mu|\xi_h|^2 G_1.$$

We remark that when $\lambda_1 = \lambda_2$, the representation in (4.1) and (4.2) remains valid if we replace G_1 by its limiting form

$$G_1(t) = \lim_{\lambda_2 \to \lambda_1} \frac{\mathrm{e}^{\lambda_2 t} - \mathrm{e}^{\lambda_1 t}}{\lambda_2 - \lambda_1} = t\mathrm{e}^{\lambda_1 t}.$$

One can refer to [3] for a detailed explanation of a similar situation.

The decay rates depend crucially on the upper bounds of the Fourier multiplier operators. Since the multipliers are anisotropic, we divide the frequency space into subdomains and seek optimal upper bounds for the multipliers in each domain.

Proposition 4.1. *The domain* $\mathbb{R}^3$ *is split into two subdomains,* $\mathbb{R}^3 = A_1 \cup A_2$ *with*

$$A_1 := \left\{ \xi \in \mathbb{R}^3 : \Gamma \leqslant \frac{(\mu|\xi_h|^2 + \eta\xi_3^2)^2}{4} \ \text{or}\ 3(\mu|\xi_h|^2 + \eta\xi_3^2)^2 \leqslant 16\left(\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2\right) \right\},$$
$$A_2 := \left\{ \xi \in \mathbb{R}^3 : \Gamma > \frac{(\mu|\xi_h|^2 + \eta\xi_3^2)^2}{4} \ \text{or}\ 3(\mu|\xi_h|^2 + \eta\xi_3^2)^2 > 16\left(\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2\right) \right\}.$$

(1) *There exist two constants* $C > 0$ *and* $c_0 > 0$ *such that for any* $\xi \in A_1$,

$$\mathrm{Re}\lambda_1 \leqslant -\frac{\mu|\xi_h|^2 + \eta\xi_3^2}{2}, \quad \mathrm{Re}\lambda_2 \leqslant -\frac{\mu|\xi_h|^2 + \eta\xi_3^2}{4},$$
$$|G_1(t)| \leqslant t\mathrm{e}^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t}, \quad |\widehat{Q}_i(t)| \leqslant C\mathrm{e}^{-c_0|\xi|^2 t}, \quad i = 1, 4,$$
$$|\widehat{Q}_2(t)| \leqslant C|\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2 t}, \quad |\widehat{Q}_3(t)| \leqslant C|\xi|\mathrm{e}^{-c_0|\xi|^2 t},$$

where $\mathrm{Re}f$ *denotes the real part of* f.

(2) *There are two constants* $C > 0$ *and* $c_1 > 0$ *such that for any* $\xi \in A_2$,

$$\lambda_1 < -\frac{3(\mu|\xi_h|^2 + \eta\xi_3^2)}{4}, \quad \lambda_2 < -\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2},$$
$$|\widehat{Q}_2(t)| = |G_1(t)| < \frac{2}{\mu|\xi_h|^2 + \eta\xi_3^2}(\mathrm{e}^{-\frac{3}{4}(\mu|\xi_h|^2+\eta\xi_3^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2+\frac{1}{2}|\xi|^2}{\mu|\xi_h|^2+\eta\xi_3^2}t}) < C|\xi|^{-2}\mathrm{e}^{-c_1 t},$$
$$|\widehat{Q}_i(t)| < C(\mathrm{e}^{-\frac{3}{4}(\mu|\xi_h|^2+\eta\xi_3^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2+\frac{1}{2}|\xi|^2}{\mu|\xi_h|^2+\eta\xi_3^2}t}) < C\mathrm{e}^{-c_1 t}, \quad i = 1, 3, 4.$$

If we further divide A_2 *into three subdomains* A_{21}, A_{22} *and* A_{23} *with*

$$A_{21} = \{\xi \in A_2,\ \mu|\xi_h|^2 \sim \eta\xi_3^2\},$$
$$A_{22} = \{\xi \in A_2,\ \mu|\xi_h|^2 \gg \eta\xi_3^2\},$$
$$A_{23} = \{\xi \in A_2,\ \mu|\xi_h|^2 \ll \eta\xi_3^2\},$$

then for three constants $C > 0$, $c_2 > 0$ *and* $c_3 > 0$,

$$|\widehat{Q}_i(t)| < C\mathrm{e}^{-c_2|\xi|^2 t - c_3 t}, \quad i = 1, 3, 4, \quad |\widehat{Q}_2(t)| < C|\xi|^{-2}\mathrm{e}^{-c_2|\xi|^2 t - c_3 t}, \quad \text{if } \xi \in A_{21},$$
$$|\widehat{Q}_i(t)| < C\mathrm{e}^{-c_2\xi_3^2 t - c_3 t}, \quad i = 1, 3, 4, \quad |\widehat{Q}_2(t)| < C|\xi|^{-2}\mathrm{e}^{-c_2\xi_3^2 t - c_3 t}, \quad \text{if } \xi \in A_{22},$$
$$|\widehat{Q}_i(t)| < C\mathrm{e}^{-c_2|\xi_h|^2 t - c_3 t}, \quad i = 1, 3, 4, \quad |\widehat{Q}_2(t)| < C|\xi|^{-2}\mathrm{e}^{-c_2|\xi_h|^2 t - c_3 t}, \quad \text{if } \xi \in A_{23}.$$

Proof. (i) For $\xi \in A_1$, we have $\Gamma \leqslant \frac{(\mu|\xi_h|^2+\eta\xi_3^2)^2}{4}$. By the definitions of λ_1 and λ_2 and the mean-value theorem applied to G_1,

$$-\frac{3(\mu|\xi_h|^2 + \eta\xi_3^2)}{4} \leqslant \mathrm{Re}\lambda_1 \leqslant -\frac{\mu|\xi_h|^2 + \eta\xi_3^2}{2},$$
$$\mathrm{Re}\lambda_2 \leqslant -\frac{\mu|\xi_h|^2 + \eta\xi_3^2}{4}, \quad |G_1(t)| \leqslant t\mathrm{e}^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t}.$$

Then, by the simple fact that $xe^{-x} \leqslant C$ for $x \geqslant 0$, we have

$$\begin{aligned}|\widehat{Q}_1(t)| &= |e^{\lambda_1 t} - \lambda_1 G_1(t) - \mu|\xi_h|^2 G_1(t)| \\ &\leqslant e^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{2}t} + (\mu|\xi_h|^2+\eta\xi_3^2)te^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t} + |\lambda_1 G_1(t)| \\ &\leqslant Ce^{-c_0|\xi|^2 t} + |\lambda_1 G_1(t)| \end{aligned} \tag{4.3}$$

for some constant $c_0 > 0$. Next, we bound $|\lambda_1 G_1|$ in two cases: $\Gamma \geqslant 0$ and $\Gamma < 0$. In the case $\Gamma \geqslant 0$, it is easily seen that

$$|\lambda_1 G_1(t)| \leqslant (\mu|\xi_h|^2+\eta\xi_3^2)te^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t} \leqslant Ce^{-c_0|\xi|^2 t}. \tag{4.4}$$

In the case $\Gamma < 0$, λ_1 is an imaginary number and we have

$$|\lambda_1|^2 = \mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2, \quad |\Gamma| = 4|\lambda_1|^2 - (\mu|\xi_h|^2+\eta\xi_3^2)^2.$$

We further divide the consideration into two subcases: $|\lambda_1| \leqslant |\sqrt{\Gamma}\,|$ and $|\lambda_1| \geqslant |\sqrt{\Gamma}\,|$. If $|\lambda_1| \leqslant |\sqrt{\Gamma}|$, according to the definition of G_1, we can infer

$$|\lambda_1 G_1(t)| = \frac{|\lambda_1|}{|\sqrt{\Gamma}|}|e^{\lambda_1 t} - e^{\lambda_2 t}| \leqslant Ce^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t}. \tag{4.5}$$

If $|\lambda_1| \geqslant |\sqrt{\Gamma}\,|$, then

$$|\lambda_1|^2 \geqslant 4|\lambda_1|^2 - (\mu|\xi_h|^2+\eta\xi_3^2)^2$$

or

$$\sqrt{3}|\lambda_1| \leqslant \mu|\xi_h|^2 + \eta\xi_3^2.$$

Hence,

$$|\lambda_1 G_1(t)| \leqslant \frac{1}{\sqrt{3}}(\mu|\xi_h|^2+\eta\xi_3^2)|G_1| \leqslant \frac{1}{\sqrt{3}}(\mu|\xi_h|^2+\eta\xi_3^2)te^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t} \leqslant Ce^{-c_0|\xi|^2 t}. \tag{4.6}$$

As a consequence of (4.4)–(4.6),

$$|\lambda_1 G_1(t)| \leqslant Ce^{-c_0|\xi|^2 t},$$

which, together with (4.3), yields that for $\xi \in A_1$,

$$|\widehat{Q}_1(t)| \leqslant Ce^{-c_0|\xi|^2 t}.$$

Similarly,

$$|\widehat{Q}_4(t)| = |e^{\lambda_2 t} + \lambda_1 G_1(t) + \mu|\xi_h|^2 G_1(t)| \leqslant Ce^{-c_0|\xi_h|^2 t}.$$

Now we proceed to bound $\widehat{Q}_2(t)$. Again the consideration is divided into two cases: $\Gamma \geqslant 0$ and $\Gamma < 0$. When $\Gamma \geqslant 0$,

$$(\mu|\xi_h|^2+\eta\xi_3^2)^2 \geqslant 4\left(\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2\right),$$

which implies $\sqrt{2}|\xi| \leqslant \mu|\xi_h|^2 + \eta\xi_3^2$. Thus

$$|\widehat{Q}_2(t)| = |G_1(t)| \leqslant te^{-\frac{\mu|\xi_h|^2+\eta\xi_3^2}{4}t} \leqslant C\frac{1}{\mu|\xi_h|^2+\eta\xi_3^2}e^{-C(\mu|\xi_h|^2+\eta\xi_3^2)t} \leqslant C\frac{1}{|\xi|}e^{-c_0|\xi|^2 t},$$

where we have used the fact that $xe^{-x} \leqslant C$ for $x \geqslant 0$. When $\Gamma < 0$, we further divide the consideration into subcases: $|\sqrt{\Gamma}| \geqslant |\xi|$ and $|\sqrt{\Gamma}| < |\xi|$. If $|\sqrt{\Gamma}| \geqslant |\xi|$, the definition of $\widehat{Q}_2(t)$ implies

$$|\widehat{Q}_2(t)| = \left|\frac{1}{\sqrt{\Gamma}}\right| |\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}| \leqslant \frac{1}{|\xi|} \mathrm{e}^{-c_0|\xi|^2 t}.$$

If $|\sqrt{\Gamma}| < |\xi|$ or $|\Gamma| < |\xi|^2$, we have

$$4\left(\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2\right) - (\mu|\xi_h|^2 + \eta\xi_3^2)^2 < |\xi|^2$$

or

$$4\mu\eta|\xi_h|^2\xi_3^2 + |\xi|^2 < (\mu|\xi_h|^2 + \eta\xi_3^2)^2,$$

which implies $|\xi| < \mu|\xi_h|^2 + \eta\xi_3^2$. Then,

$$|\widehat{Q}_2(t)| = |G_1(t)| < C\frac{1}{\mu|\xi_h|^2 + \eta\xi_3^2} \mathrm{e}^{-c_0(\mu|\xi_h|^2 + \eta\xi_3^2)t} \leqslant C\frac{1}{|\xi|}\mathrm{e}^{-c_0|\xi_h|^2 t}.$$

Therefore, for all the cases, $\widehat{Q}_2(t)$ is bounded by

$$|\widehat{Q}_2(t)| \leqslant C|\xi|^{-1}\mathrm{e}^{-c_0|\xi_h|^2 t}.$$

For $\widehat{Q}_3(t)$, we also have

$$|\widehat{Q}_3(t)| \leqslant \frac{|\xi|^2}{2}|G_1(t)| \leqslant C|\xi|\mathrm{e}^{-c_0|\xi|^2 t}.$$

(ii) For $\xi \in A_2$, we have

$$\frac{\mu|\xi_h|^2 + \eta\xi_3^2}{2} < \sqrt{\Gamma} \leqslant \mu|\xi_h|^2 + \eta\xi_3^2.$$

It is then easy to check that

$$\begin{aligned} &-(\mu|\xi_h|^2 + \eta\xi_3^2) \leqslant \lambda_1 < -\frac{3}{4}(\mu|\xi_h|^2 + \eta\xi_3^2), \\ &\lambda_2 = \frac{\Gamma - (\mu|\xi_h|^2 + \eta\xi_3^2)^2}{2(\mu|\xi_h|^2 + \eta\xi_3^2 + \sqrt{\Gamma})} \leqslant -\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2}. \end{aligned}$$

For $\xi \in A_2$, we have $|\xi| > C$ for some $C > 0$. Therefore,

$$\begin{aligned} |\widehat{Q}_2(t)| = |G_1(t)| &\leqslant \frac{1}{\lambda_2 - \lambda_1}(\mathrm{e}^{\lambda_1 t} + \mathrm{e}^{\lambda_2 t}) \\ &< \frac{2}{\mu|\xi_h|^2 + \eta\xi_3^2}(\mathrm{e}^{-\frac{3}{4}(\mu\xi_1^2 + \eta|\xi_h|^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2}}) \leqslant C|\xi|^{-2}\mathrm{e}^{-c_1 t} \end{aligned}$$

for some constant $c_1 > 0$. Furthermore,

$$\begin{aligned} |\widehat{Q}_1(t)| &= |\mathrm{e}^{\lambda_1 t} - \lambda_1 G_1(t) - \mu|\xi_h|^2 G_1(t)| \leqslant \mathrm{e}^{\lambda_1 t} + 2(\mu|\xi_h|^2 + \eta\xi_3^2)|G_1(t)| \\ &< C(\mathrm{e}^{-\frac{3}{4}(\mu|\xi_h|^2 + \eta\xi_3^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2}t}) \leqslant C\mathrm{e}^{-c_1 t}. \end{aligned}$$

Similarly,

$$|\widehat{Q}_4(t)| = |\mu|\xi_h|^2 G_1(t) + \lambda_1 G_1(t) + \mathrm{e}^{\lambda_2 t}| < C(\mathrm{e}^{-\frac{3}{4}(\mu|\xi_h|^2 + \eta\xi_3^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2}t}) \leqslant C\mathrm{e}^{-c_1 t}.$$

Also,

$$|\widehat{Q}_3(t)| = \frac{|\xi|^2}{2}|G_1(t)| < C(\mathrm{e}^{-\frac{3}{4}(\mu|\xi_h|^2 + \eta\xi_3^2)t} + \mathrm{e}^{-\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2}t}) \leqslant C\mathrm{e}^{-c_1 t}.$$

We finally present more definite upper bounds by dividing A_2 into A_{21}, A_{22} and A_{23}. For $\xi \in A_{21}$ and $\mu|\xi_h|^2 \sim \eta\xi_3^2$, we have

$$\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2} \sim |\xi|^2 + 1.$$

For $\xi \in A_{22}$ and $\mu|\xi_h|^2 \gg \eta\xi_3^2$, there exists sufficiently small $c_2 > 0$ and $c_3 > 0$ such that

$$\frac{\mu\eta|\xi_h|^2\xi_3^2 + \frac{1}{2}|\xi|^2}{\mu|\xi_h|^2 + \eta\xi_3^2} \geqslant c_2\xi_3^2 + c_3.$$

The case $\xi \in A_{23}$ can be similarly handled. This completes the proof of Proposition 4.1. □

Next, we present two important tools to be used in the proof of the decay rates in the next few subsections. The first one provides an exact decay rate for the general heat operator associated with a fractional Laplacian operator $\Lambda^\beta := (-\Delta)^{\beta/2}$. Here, the fractional Laplacian operator can be defined through the Fourier transform, for any $\beta \in \mathbb{R}$,

$$\widehat{\Lambda^\beta f}(\xi) = |\xi|^\beta \widehat{f}(\xi).$$

The proof of the following lemma can be found in many references (see, e.g., [13, 38]).

Lemma 4.2. *Assume that $\alpha \geqslant 0$ and $\beta > 0$ are real numbers. Let $1 \leqslant p \leqslant q \leqslant \infty$. Then there exists a constant $C > 0$ such that for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-\frac{\alpha}{\beta} - \frac{d}{\beta}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.7}$$

The second lemma gives sharp upper bounds for a convolution type integral.

Lemma 4.3. *Assume $0 < s_1 \leqslant s_2$. Then, for some constant $C > 0$,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2} d\tau \leqslant \begin{cases} C(1+t)^{-s_1}, & \text{if } s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t), & \text{if } s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & \text{if } s_2 < 1. \end{cases} \tag{4.8}$$

Now we start proving Theorem 1.2. The decay rates are shown in the following three subsections. Throughout the proof, without loss of generality, we assume $t > 1$.

4.2 The decay for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$

This subsection is devoted to the proof of the decay rates for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$. We apply the bootstrapping argument to establish the desired decay estimates. We start with the ansatz that for $t \leqslant T$,

$$(1+t)^{\frac{3}{4}}\|u(t)\|_{L^2} + (1+t)^{\frac{5}{4}}\|\nabla u(t)\|_{L^2} \leqslant C_0\,\delta, \tag{4.9}$$

where C_0 is a suitably selected pure constant and will be specified in the proof. Our goal is to show

$$(1+t)^{\frac{3}{4}}\|u(t)\|_{L^2} + (1+t)^{\frac{5}{4}}\|\nabla u(t)\|_{L^2} \leqslant \frac{C_0}{2}\,\delta \tag{4.10}$$

by using the ansatz (4.9) and the integral representation (4.1) for u. Then the bootstrap argument verifies that the desired global bound (4.10) actually holds for any time $t > 0$.

Proof. By the integral representation (4.1) and Plancherel's theorem,

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^3)} &= \|\widehat{u}(t)\|_{L^2(\mathbb{R}^3)} \\ &\leqslant \|\widehat{Q}_1(t)\widehat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_2(t)\widehat{\mathbb{P}\nabla\cdot\tau_0}\|_{L^2(\mathbb{R}^3)} + \int_0^t \|\widehat{Q}_1(t-s)\widehat{N}_1(s)\|_{L^2(\mathbb{R}^3)}ds \\ &\quad + \int_0^t \|\widehat{Q}_2(t-s)\widehat{N}_2(s)\|_{L^2(\mathbb{R}^3)}ds + \int_0^t \|\widehat{Q}_2(t-s)\widehat{N}_3(s)\|_{L^2(\mathbb{R}^3)}ds \end{aligned}$$

$$=: K_1 + \cdots + K_5. \tag{4.11}$$

Next, we use the ansatz (4.9) to bound the terms in (4.11) one by one. Recalling the upper bounds on $\widehat{Q}_1(t)$, Proposition 4.1, (4.7) and (1.5), we have

$$\begin{aligned} K_1 &\leqslant C\|\mathrm{e}^{-c_0|\xi|^2 t}\widehat{u}_0\|_{L^2(A_1)} + C\|\mathrm{e}^{-c_1 t}\widehat{u}_0\|_{L^2(A_2)} \\ &\leqslant C(1+t)^{-\frac{3}{4}}\|u_0\|_{L^1} + (1+t)^{-\frac{3}{4}}\|u_0\|_{L^2} \leqslant C\delta(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.12}$$

where we have used the fact that $\mathrm{e}^{-c_1 t}(1+t)^m \leqslant C(c_1, m)$ for any $m \geqslant 0$ and $t > 0$. Similarly,

$$\begin{aligned} K_2 &\leqslant C\||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2 t}\widehat{\mathbb{P}\nabla\cdot\tau}_0\|_{L^2(A_1)} + C\||\xi|^{-2}\mathrm{e}^{-c_1 t}\widehat{\mathbb{P}\nabla\cdot\tau}_0\|_{L^2(A_2)} \\ &\leqslant C(1+t)^{-\frac{3}{4}}(\|\tau_0\|_{L^1} + \|\tau_0\|_{L^2}) \leqslant C\delta(1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.13}$$

where we have used the fact $|\xi| \geqslant C$ for $\xi \in A_2$. For K_3, by Proposition 4.1 and the fact that the projection operator $\mathbb{P}$ is bounded in L^2,

$$\begin{aligned} K_3 &\leqslant C\int_0^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)}ds + C\int_0^t \|\mathrm{e}^{-c_1(t-s)}\widehat{N}_1\|_{L^2(A_2)}ds \\ &= C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)}ds + C\int_{t-1}^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)}ds \\ &\quad + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|N_1\|_{L^2(A_2)}ds \\ &\leqslant C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{u\cdot\nabla u}\|_{L^2(\mathbb{R}^3)}ds + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|u\cdot\nabla u\|_{L^2(\mathbb{R}^3)}ds, \end{aligned}$$

where, due to the simple fact that $\mathrm{e}^{-c_1(t-s)} \geqslant \mathrm{e}^{-c_1}$ for $s \in [t-1, t]$, we have used

$$\int_{t-1}^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)}ds \leqslant \mathrm{e}^{c_1}\int_{t-1}^t \mathrm{e}^{-c_1(t-s)}\|\widehat{N}_1\|_{L^2(A_1)}ds.$$

Then, applying Hölder's inequality, Sobolev's inequality, (4.9) and (4.8), and combining $\|u\|_{H^2} \leqslant C\delta$, we find that for $m > 2$,

$$\begin{aligned} K_3 &\leqslant C\int_0^{t-1}(1+t-s)^{-\frac{3}{4}}\|u\cdot\nabla u\|_{L^1}ds + C\int_0^t(1+t-s)^{-m}\|u\cdot\nabla u\|_{L^2}ds \\ &\leqslant C\int_0^t(1+t-s)^{-\frac{3}{4}}\|u\|_{L^2}\|\nabla u\|_{L^2}ds + C\int_0^t(1+t-s)^{-m}\|u\|_{L^\infty}\|\nabla u\|_{L^2}ds \\ &\leqslant C\int_0^t(1+t-s)^{-\frac{3}{4}}\|u\|_{L^2}\|\nabla u\|_{L^2}ds + C\int_0^t(1+t-s)^{-m}\|u\|_{H^2}\|\nabla u\|_{L^2}ds \\ &\leqslant CC_0\delta^2\int_0^t(1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}}ds + CC_0\delta^2\int_0^t(1+t-s)^{-m}(1+s)^{-\frac{5}{4}}ds \\ &\leqslant CC_0\delta^2(1+t)^{-\frac{3}{4}}. \end{aligned} \tag{4.14}$$

Similar to K_3, K_4 can be first bounded by

$$\begin{aligned} K_4 &\leqslant C\int_0^t \||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds \\ &\quad + C\int_0^t \||\xi|^{-2}\mathrm{e}^{-c_1(t-s)}\widehat{\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_2)}ds \\ &\leqslant C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{u\cdot\nabla\tau}\|_{L^2(\mathbb{R}^3)}ds + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|u\cdot\nabla\tau\|_{L^2(\mathbb{R}^3)}ds, \end{aligned}$$

where we have used the fact that $|\xi| > C$ for $\xi \in A_2$. By Lemmas 4.2 and 4.3, Hölder's inequality and Gagliardo-Nirenberg's inequality,

$$\begin{aligned}
K_4 &\leqslant C\sum_{i,j,k}\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|\widehat{u_i\,\tau_{jk}}\|_{L^2(\mathbb{R}^3)}ds + C\int_0^t (1+t-s)^{-m}\|u\|_{L^\infty}\|\nabla\tau\|_{L^2}ds \\
&\leqslant C\sum_{i,j,k}\int_0^t (1+t-s)^{-\frac{5}{4}}\|u_i\tau_{jk}\|_{L^1}ds + C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{L^2}ds \\
&\leqslant C\int_0^t (1+t-s)^{-\frac{5}{4}}\|u\|_{L^2}\|\tau\|_{L^2}ds + C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{L^2}ds \\
&\leqslant CC_0\delta^2\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{4}}ds + CC_0^{\frac{1}{2}}\delta^2\int_0^t (1+t-s)^{-m}(1+s)^{-\frac{7}{8}}ds \\
&\leqslant CC_0\delta^2(1+t)^{-\frac{3}{4}} + CC_0^{\frac{1}{2}}\delta^2(1+t)^{-\frac{7}{8}} \leqslant C(C_0+C_0^{\frac{1}{2}})\delta^2(1+t)^{-\frac{3}{4}},
\end{aligned} \tag{4.15}$$

where $m > 1$ and we have used $\|\nabla^2 u\|_{L^2} \leqslant C\delta(1+t)^{-\frac{1}{2}}$ and $\|\tau\|_{H^2} \leqslant C\delta$. Finally, with a similar argument as in the estimate of K_4, for $m > 2$, we have

$$\begin{aligned}
K_5 &\leqslant C\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{Q}\|_{L^2(\mathbb{R}^3)}ds + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|Q\|_{L^2(\mathbb{R}^3)}ds \\
&\leqslant C\int_0^t (1+t-s)^{-\frac{3}{4}}\|Q\|_{L^1}ds + C\int_0^t (1+t-s)^{-m}\|Q\|_{L^2}ds \\
&\leqslant C\int_0^t (1+t-s)^{-\frac{3}{4}}\|\nabla u\|_{L^2}\|\tau\|_{L^2}ds + C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}\|\tau\|_{L^\infty}ds \\
&\leqslant CC_0\delta^2\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}}ds + CC_0\delta^2\int_0^t (1+t-s)^{-m}(1+s)^{-\frac{5}{4}}ds \\
&\leqslant CC_0\delta^2(1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{4.16}$$

Combining the bounds (4.12)–(4.16), we conclude that

$$\|u\|_{L^2} \leqslant C\delta(1+t)^{-\frac{3}{4}} + C(C_0+C_0^{\frac{1}{2}})\delta^2(1+t)^{-\frac{3}{4}}. \tag{4.17}$$

Next, we prove the decay rates for $\|\nabla u\|_{L^2}$. For $k = 1,2,3$, we have

$$\begin{aligned}
\widehat{\partial_k u}(\xi,t) &= \widehat{Q}_1(t)\widehat{\partial_k u_0} + \widehat{Q}_2(t)\partial_k\widehat{\mathbb{P}\nabla\cdot}\,\tau_0 \\
&\quad + \int_0^t (\widehat{Q}_1(t-s)\widehat{\partial_k N}_1(s) + \widehat{Q}_2(t-s)(\widehat{\partial_k N}_2(s) + \widehat{\partial_k N}_3(s)))ds.
\end{aligned}$$

We take the $L^2(\mathbb{R}^3)$-norm to obtain

$$\begin{aligned}
\|\nabla u\|_{L^2} &= \|\widehat{\nabla u}\|_{L^2} \\
&\leqslant \|\widehat{Q}_1(t)\widehat{\nabla u_0}\|_{L^2} + \|\widehat{Q}_2(t)\widehat{\nabla\mathbb{P}\nabla\cdot}\,\tau_0\|_{L^2} + \int_0^t \|\widehat{Q}_1(t-s)\widehat{\nabla N}_1(s)\|_{L^2}ds \\
&\quad + \int_0^t \|\widehat{Q}_2(t-s)\widehat{\nabla N}_2(s)\|_{L^2}ds + \int_0^t \|\widehat{Q}_2(t-s)\widehat{\nabla N}_3(s)\|ds \\
&=: K_6 + K_7 + K_8 + K_9 + K_{10}.
\end{aligned}$$

According to Proposition 4.1 and Lemma 4.2,

$$\begin{aligned}
K_6 &\leqslant C\|\mathrm{e}^{-c_0|\xi|^2 t}\widehat{\nabla u}_0\|_{L^2(A_1)} + C\|\mathrm{e}^{-c_1 t}\widehat{\nabla u}_0\|_{L^2(A_2)} \\
&\leqslant C\|\mathrm{e}^{-c_0|\xi|^2 t}|\xi|\widehat{u}_0\|_{L^2(A_1)} + C\mathrm{e}^{-c_1 t}\|\nabla u_0\|_{L^2(A_2)} \\
&\leqslant C(1+t)^{-\frac{5}{4}}\|u_0\|_{L^1(\mathbb{R}^3)} + C(1+t)^{-\frac{5}{4}}\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leqslant C\delta(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{4.18}$$

Similarly,

$$K_7 \leqslant C(1+t)^{-\frac{5}{4}}(\|\tau_0\|_{L^1} + \|\tau_0\|_{L^2}) \leqslant C\delta(1+t)^{-\frac{5}{4}}. \tag{4.19}$$

As in the estimate of K_3, K_8 can be first bounded by

$$\begin{aligned} K_8 &\leqslant C\int_0^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla N_1}\|_{L^2(A_1)}ds + C\int_0^t \|\mathrm{e}^{-c_1(t-s)}\widehat{\nabla N_1}\|_{L^2(A_2)}ds \\ &\leqslant C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\nabla\widehat{(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^3)}ds + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|\nabla(u\cdot\nabla u)\|_{L^2(\mathbb{R}^3)}ds. \end{aligned}$$

Then, by Hölder's inequality and Gagliardo-Nirenberg's inequality, for $m > 1$,

$$\begin{aligned} K_8 &\leqslant C\int_0^{t-1}(1+t-s)^{-\frac{5}{4}}\|u\cdot\nabla u\|_{L^1}ds \\ &\quad + C\int_0^t (1+t-s)^{-m}(\|\nabla u\|_{L^4}^2 + \|u\|_{L^\infty}\|\nabla^2 u\|_{L^2})ds \\ &\leqslant C\int_0^t (1+t-s)^{-\frac{5}{4}}\|u\|_{L^2}\|\nabla u\|_{L^2}ds + C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}}ds \\ &\leqslant CC_0\delta^2\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}ds + CC_0^{\frac{1}{2}}\delta^2\int_0^t (1+t-s)^{-m}(1+s)^{-\frac{11}{8}}ds \\ &\leqslant C(C_0 + C_0^{\frac{1}{2}})\delta^2(1+t)^{-\frac{5}{4}}, \end{aligned} \tag{4.20}$$

where we have used $\|\nabla^2 u\|_{L^2} \leqslant C\delta(1+t)^{-\frac{1}{2}}$. The estimate for K_9 is more elaborate. We first bound it by

$$\begin{aligned} K_9 &\leqslant C\int_0^t \||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla N_2}\|_{L^2(A_1)}ds + C\int_0^t \||\xi|^{-2}\mathrm{e}^{-c_1(t-s)}\widehat{\nabla N_2}\|_{L^2(A_2)}ds \\ &= C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\mathbb{P}\nabla\cdot\widehat{(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds + C\int_{t-1}^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\mathbb{P}\nabla\cdot\widehat{(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds \\ &\quad + C\int_0^t \mathrm{e}^{-c_1(t-s)}\||\xi|^{-1}\mathbb{P}\nabla\cdot\widehat{(u\cdot\nabla\tau)}\|_{L^2(A_2)}ds \\ &=: K_{91} + K_{92} + K_{93}. \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} \mathbb{P}\nabla\cdot(u\cdot\nabla\tau) &= \mathbb{P}((u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)) + \mathbb{P}(\nabla u\cdot\nabla\tau) + \mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau) \\ &= \mathbb{P}\partial_j(u_j\,\mathbb{P}\nabla\cdot\tau) + \mathbb{P}\partial_j(\nabla u_j\,\tau) + \mathbb{P}\partial_j(\nabla u_j\,\Delta^{-1}\nabla\cdot\nabla\cdot\tau). \end{aligned}$$

Therefore,

$$\begin{aligned} K_{91} &\leqslant C\sum_j\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|(\widehat{u_j\,\mathbb{P}\nabla\cdot\tau} + \widehat{\nabla u_j\,\tau} + (\nabla u_j\,\widehat{\Delta^{-1}\nabla\cdot}\nabla\cdot\tau))\|_{L^2}ds \\ &\leqslant C\sum_j\int_0^t (1+t-s)^{-\frac{5}{4}}(\|u_j\mathbb{P}\nabla\cdot\tau\|_{L^1} + \|\nabla u_j\,\tau\|_{L^1} + \|\nabla u_j\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^1})ds \\ &\leqslant C\int_0^t (1+t-s)^{-\frac{5}{4}}(\|u\|_{L^2}\|\mathbb{P}\nabla\cdot\tau\|_{L^2} + \|\nabla u\|_{L^2}\,\|\tau\|_{L^2} + \|\nabla u\|_{L^2}\|\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2})ds \\ &\leqslant CC_0\delta^2\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}ds \leqslant CC_0\delta^2(1+s)^{-\frac{5}{4}}, \end{aligned} \tag{4.21}$$

where we have used the ansatz (4.10) and $\|\mathbb{P}\nabla\cdot\tau\|_{L^2} \leqslant C\delta(1+t)^{-\frac{1}{2}}$. For K_{92}, applying (2.5) again, we further split it into three terms as follows:

$$K_{92} \leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}\|(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)\|_{L^2(A_1)}ds + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla u\cdot\nabla\tau}\|_{L^2(A_1)}ds$$

$$
\begin{aligned}
&+ C\int_0^t \mathrm{e}^{-c_1(t-s)}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau}\|_{L^2(A_1)}ds\\
&=: K_{92,1}+K_{92,2}+K_{92,3}.
\end{aligned}
$$

For $K_{92,1}$, a direct estimate leads to

$$
\begin{aligned}
K_{92,1} &\leqslant C\int_0^t (1+t-s)^{-m}\|u\|_{L^\infty}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}ds\\
&\leqslant C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}ds\\
&\leqslant CC_0^{\frac12}\delta^2\int_0^t (1+t-s)^{-m}(1+s)^{-\frac{11}{8}}ds \leqslant CC_0^{\frac12}\delta^2(1+t)^{-\frac{11}{8}}.
\end{aligned}
$$

Recalling the inequality (4.7) and the fact that $\mathrm{e}^{-c_1(t-s)}\geqslant \mathrm{e}^{-c_1}$ for $s\in[t-1,t]$, $m>2$ and $\frac1p+\frac1q=1$ with $1<p<2$, we have

$$
\begin{aligned}
K_{92,2} &\leqslant C\sum_j\int_0^t \mathrm{e}^{-c_1(t-s)}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\partial_j(\widehat{\nabla u_j}\cdot\tau)\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac12}\|\nabla u\cdot\tau\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac12}\|\nabla u\|_{L^2}\|\tau\|_{L^\infty}ds\\
&\leqslant CC_0\delta^2\int_0^t \mathrm{e}^{-\frac{c_1}{2}(t-s)}(t-s)^{-\frac12}(1+t-s)^{-m}(1+s)^{-\frac54}ds\\
&\leqslant CC_0\delta^2\left[\int_0^t \mathrm{e}^{-\frac{c_1p}{2}(t-s)}(t-s)^{-\frac p2}ds\right]^{\frac1p}\left[\int_0^t (1+t-s)^{-mq}(1+s)^{-\frac54 q}ds\right]^{\frac1q}\\
&\leqslant CC_0\delta^2(1+t)^{-\frac54},
\end{aligned}
$$

where we have used $\mathrm{e}^{-\frac{c_1}{2}(t-s)}\leqslant C(1+t-s)^{-m}$ for any $m>0$ and the fact that the integral $\int_0^\infty x^{s-1}\mathrm{e}^{-x}dx$ $(s>0)$ converges to $\Gamma(s)$. Similarly,

$$
K_{92,3}\leqslant CC_0\delta^2(1+t)^{-\frac54}.
$$

Consequently,

$$
K_{92}\leqslant C(C_0+C_0^{\frac12})\delta^2(1+t)^{-\frac54}. \tag{4.22}
$$

Similarly,

$$
\begin{aligned}
K_{93} &\leqslant C\sum_j\int_0^t \mathrm{e}^{-c_1(t-s)}(\|u_j\,\mathbb{P}(\nabla\cdot\tau)\|_{L^2}+\|\nabla u_j\,\tau\|_{L^2}+\|\nabla u_j\,\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2})ds\\
&\leqslant C(C_0+C_0^{\frac12})\delta^2(1+t)^{-\frac54}.
\end{aligned}
$$

Combining the bounds for J_{91}, J_{92} and J_{93} yields

$$
K_9\leqslant C(C_0+C_0^{\frac12})\delta^2(1+t)^{-\frac54}. \tag{4.23}
$$

We now bound the last term K_{10}. First, we have

$$
\begin{aligned}
K_{10} &\leqslant C\int_0^t \||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla\mathbb{P}\nabla\cdot Q}\|_{L^2(A_1)}ds + C\int_0^t \||\xi|^{-2}\mathrm{e}^{-c_1(t-s)}\widehat{\nabla\mathbb{P}\nabla\cdot Q}\|_{L^2(A_2)}ds\\
&\leqslant C\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_1)}ds + C\int_{t-1}^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_1)}ds
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t \mathrm{e}^{-c_1(t-s)}\|Q\|_{L^2(A_2)}ds\\
&=: K_{10,1}+K_{10,2}+K_{10,3}.
\end{aligned}
$$

By (4.7), (4.8) and the ansatz (4.10),

$$
\begin{aligned}
K_{10,1} &\leqslant C\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|\widehat{Q}\|_{L^2}ds\\
&\leqslant C\int_0^t (1+t-s)^{-\frac{5}{4}}\|Q\|_{L^1}ds \leqslant C\int_0^t (1+t-s)^{-\frac{5}{4}}\|\nabla u\|_{L^2}\|\tau\|_{L^2}ds\\
&\leqslant CC_0\delta^2\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}ds \leqslant CC_0\delta^2(1+t)^{-\frac{5}{4}}. \qquad (4.24)
\end{aligned}
$$

As in $K_{92,2}$, we have

$$
\begin{aligned}
K_{10,2} &\leqslant C\int_{t-1}^t \mathrm{e}^{-c_1(t-s)}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|\widehat{Q}\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|Q\|_{L^2}ds \leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|\nabla u\|_{L^2}\|\tau\|_{H^2}ds\\
&\leqslant CC_0\delta^2\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{5}{4}}ds \leqslant CC_0\delta^2(1+t)^{-\frac{5}{4}}. \qquad (4.25)
\end{aligned}
$$

A simple estimate leads to, for $m>2$,

$$
K_{10,3} \leqslant C\int_0^t (1+t-s)^{-m}\|\nabla u\|_{L^2}\|\tau\|_{H^2}ds \leqslant CC_0\delta^2(1+t)^{-\frac{5}{4}}.
$$

Therefore,

$$
K_{10} \leqslant CC_0\delta^2(1+t)^{-\frac{5}{4}}. \qquad (4.26)
$$

Combining (4.18)–(4.20), (4.23) and (4.26) leads to

$$
\|\nabla u\|_{L^2} \leqslant C\delta(1+t)^{-\frac{5}{4}} + C(C_0^{\frac{1}{2}}+C_0)\delta^2(1+t)^{-\frac{5}{4}},
$$

which, together with (4.17), gives

$$
(1+t)^{\frac{3}{4}}\|u\|_{L^2}+(1+t)^{\frac{5}{4}}\|\nabla u\|_{L^2} \leqslant C_1\delta + C_2(C_0+1)\delta^2.
$$

Then, if we select C_0 and δ satisfying

$$
C_1 \leqslant \frac{C_0}{4}, \quad C_2(C_0+1)\delta \leqslant \frac{C_0}{4},
$$

we obtain the desired estimate

$$
(1+t)^{\frac{3}{4}}\|u\|_{L^2}+(1+t)^{\frac{5}{4}}\|\nabla u\|_{L^2} \leqslant \frac{C_0}{2}\delta.
$$

This completes the proof of (4.10). □

4.3 The decay rates for $\|\nabla^2 u\|_{L^2}$ and $\|\partial_3\tau\|_{H^1}$

This subsection proves the decay rates for $\|\nabla^2 u\|_{L^2}$ and $\|\partial_3\tau\|_{H^1}$. For clarity and notational convenience, we introduce three functionals, i.e.,

$$
E_3(t) = (1+t)^{\frac{5}{2}}\|\nabla_h\nabla u\|_{L^2}^2,
$$

$$E_4(t)=\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\|(\partial_3\nabla u,\partial_3\mathbb{P}\nabla\cdot\tau)\|_{L^2}^2$$
$$+\int_0^t(1+s)^{\frac{3}{2}}(\mu\|\nabla_h\partial_3\nabla u\|_{L^2}^2+\eta\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)ds,$$
$$E_5(t)=\sup_{0\leqslant s\leqslant t}(1+s)\|(\partial_3 u,\partial_3\tau)\|_{H^1}^2+\int_0^t(1+s)(\mu\|\nabla_h\partial_3 u\|_{H^1}^2+\eta\|\partial_3^2\tau\|_{H^1}^2)ds.$$

We use the bootstrapping argument to show their boundedness and start with the ansatz for $t\leqslant T$,

$$E_3(t)+E_4(t)+E_5(t)\leqslant C_3\delta^2, \tag{4.27}$$

where C_3 will be determined later. We show that under the ansatz, $E_3(t),E_4(t)$ and $E_5(t)$ actually satisfy

$$E_3(t)+E_4(t)+E_5(t)\leqslant\frac{C_3}{2}\delta^2. \tag{4.28}$$

Then the bootstrapping argument implies that (4.28) holds for any time $t>0$. The desired decay rates for $\|\nabla\nabla_h u\|_{L^2},\|\nabla\partial_3 u\|_{L^2}$ and $\|\partial_3\tau\|_{H^1}$ then follow. The detailed estimates for $E_3(t),E_4(t)$ and $E_5(t)$ are presented in the three lemmas below.

Lemma 4.4. *For a constant $C>0$,*

$$E_3(t)\leqslant C\delta^2+C(1+C_3)\delta^4. \tag{4.29}$$

Proof. First of all, we have that for $i,j=1,2,3$,

$$\widehat{\partial_i\partial_j u}(\xi,t)=\widehat{Q}_1(t)\widehat{\partial_i\partial_j u_0}+\widehat{Q}_2(t)\partial_i\widehat{\partial_j\mathbb{P}\nabla}\cdot\tau_0+\int_0^t(\widehat{Q}_1(t-\tau)\widehat{\partial_i\partial_j N}_1(s)$$
$$+\widehat{Q}_2(t-s)(\widehat{\partial_i\partial_j N}_2(s)+\widehat{\partial_i\partial_j N}_3(s)))ds.$$

Then

$$\begin{aligned}\|\nabla\nabla_h u\|_{L^2(\mathbb{R}^3)}&=\|\widehat{\nabla\nabla_h u}\|_{L^2(\mathbb{R}^3)}\\&\leqslant\|\widehat{Q}_1(t)\widehat{\nabla\nabla_h u_0}\|_{L^2(\mathbb{R}^3)}+\|\widehat{Q}_2(t)\widehat{\nabla\nabla_h\mathbb{P}\nabla}\cdot\tau_0\|_{L^2(\mathbb{R}^3)}\\&\quad+\int_0^t\|\widehat{Q}_1(t-s)\widehat{\nabla\nabla_h N}_1(s)\|_{L^2(\mathbb{R}^3)}ds+\int_0^t\|\widehat{Q}_2(t-s)\widehat{\nabla\nabla_h N}_2(s)\|_{L^2(\mathbb{R}^3)}ds\\&\quad+\int_0^t\|\widehat{Q}_2(t-s)\widehat{\nabla\nabla_h N}_3(s)\|_{L^2(\mathbb{R}^3)}ds\\&=:L_1+L_2+L_3+L_4+L_5.\end{aligned}$$

Clearly,

$$\begin{aligned}L_1&\leqslant C\|e^{-c_0|\xi|^2t}\widehat{\nabla\nabla_h u}_0\|_{L^2(A_1)}+C\|e^{-c_1t}\widehat{\nabla_h\nabla u}_0\|_{L^2(A_2)}\\&\leqslant C\|e^{-c_0|\xi|^2t}|\xi|^2\widehat{u}_0\|_{L^2(A_1)}+Ce^{-c_1t}\|\nabla\nabla_h u_0\|_{L^2(A_2)}\\&\leqslant C(1+t)^{-\frac{7}{4}}\|u_0\|_{L^1}+C(1+t)^{-\frac{7}{4}}\|\nabla\nabla_h u_0\|_{L^2}\leqslant C\delta(1+t)^{-\frac{7}{4}}.\end{aligned}$$

Similarly,

$$L_2\leqslant C(1+t)^{-\frac{7}{4}}(\|\tau_0\|_{L^1}+\|\nabla\tau_0\|_{L^2})\leqslant C\delta(1+t)^{-\frac{7}{4}}.$$

As in the estimate of K_3, L_3 can first be bounded by

$$\begin{aligned}L_3&\leqslant C\int_0^{t-1}\|e^{-c_0|\xi|^2(t-s)}\nabla\widehat{\nabla_h(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^3)}ds+C\int_0^t e^{-c_1(t-s)}\|\nabla\nabla_h(u\cdot\nabla u)\|_{L^2(\mathbb{R}^3)}ds\\&=:L_{31}+L_{32}.\end{aligned}$$

For L_{31}, we have

$$
\begin{aligned}
L_{31} &\leqslant C\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|^2\widehat{u\cdot\nabla u}\|_{L^2}ds \leqslant C\int_0^{t-1}(1+t-s)^{-\frac{7}{4}}\|u\cdot\nabla u\|_{L^1}ds\\
&\leqslant C\int_0^{t}(1+t-s)^{-\frac{7}{4}}\|u\|_{L^2}\|\nabla u\|_{L^2}ds \leqslant C\delta^2\int_0^{t}(1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{5}{4}}ds\\
&\leqslant C\delta^2(1+t)^{-\frac{5}{4}},
\end{aligned}
$$

where we have used $\|\nabla u\|_{L^2}\leqslant C\delta(1+t)^{-\frac{5}{4}}$. By Hölder's inequality and Gagliardo-Nirenberg's inequality,

$$
\begin{aligned}
\|\nabla_h\nabla(u\cdot\nabla u)\|_{L^2} &\leqslant \|\nabla\nabla_h u\|_{L^4}\|\nabla u\|_{L^4}+\|\nabla_h u\|_{L^\infty}\|\nabla^2 u\|_{L^2}+\|u\|_{L^\infty}\|\nabla_h\nabla^2 u\|_{L^2}\\
&\leqslant C\|\nabla\nabla_h u\|_{L^2}^{\frac{1}{4}}\|\nabla^2\nabla_h u\|_{L^2}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}^{\frac{3}{4}}+C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}\\
&\quad+C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla^2 u\|_{L^2}.
\end{aligned}\tag{4.30}
$$

Inserting (4.30) in L_{32} and using (4.27), (4.10), (3.5) and Hölder's inequality, we find

$$
\begin{aligned}
L_{32} &\leqslant C\int_0^t(1+t-s)^{-m}\|\nabla_h\nabla(u\cdot\nabla u)\|_{L^2}ds\\
&\leqslant C\delta^{\frac{5}{4}}C_3^{\frac{1}{8}}\int_0^s(1+t-s)^{-m}(1+s)^{-\frac{1}{2}\cdot\frac{5}{4}-\frac{3}{4}}(1+s)^{\frac{3}{8}}\|\nabla^2\nabla_h u\|_{L^2}^{\frac{3}{4}}ds\\
&\quad+C\delta^2\int_0^t(1+t-s)^{-m}(1+s)^{-\frac{1}{2}\cdot\frac{5}{4}-\frac{3}{4}}ds\\
&\quad+C\delta\int_0^t(1+t-s)^{-m}(1+s)^{-\frac{1}{2}\cdot\frac{5}{4}-\frac{3}{4}}(1+s)^{\frac{1}{2}}\|\nabla^2\nabla_h u\|_{L^2}ds\\
&\leqslant C(C_3^{\frac{1}{8}}+1)\delta^2(1+t)^{-\frac{11}{8}}.
\end{aligned}
$$

Thus,

$$
L_3\leqslant C(C_3^{\frac{1}{8}}+1)\delta^2(1+t)^{-\frac{11}{8}}.
$$

As in the estimate of K_9, L_4 can first be bounded by

$$
\begin{aligned}
L_4 &\leqslant C\int_0^t\||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla\nabla_h N_2}\|_{L^2(A_1)}ds+C\int_0^t\||\xi|^{-2}\mathrm{e}^{-c_1(t-s)}\widehat{\nabla\nabla_h N_2}\|_{L^2(A_2)}ds\\
&=C\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\nabla_h\mathbb{P}\widehat{\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds\\
&\quad+C\int_{t-1}^t\|\mathrm{e}^{-c_0|\xi|^2(t-s)}\nabla_h\mathbb{P}\widehat{\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds+C\int_0^t\mathrm{e}^{-c_1(t-s)}\|\mathbb{P}\widehat{\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_2)}ds\\
&=: L_{41}+L_{42}+L_{43}.
\end{aligned}
$$

It follows from (4.7) and the anisotropic inequality (2.3) that

$$
\begin{aligned}
L_{41} &\leqslant C\sum_{i,j,k}\int_0^{t-1}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|^3\widehat{u_j\tau_{ik}}\|_{L^2(\mathbb{R}^3)}ds\leqslant C\sum_{i,j,k}\int_0^t(1+t-s)^{-\frac{3}{2}}\|u_j\tau_{ik}\|_{L^2}ds\\
&\leqslant C\int_0^t(1+t-s)^{-\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}}ds\\
&\leqslant C\delta^2C_3^{\frac{3}{8}}\int_0^s(1+t-s)^{-m}(1+s)^{-\frac{1}{4}\cdot\frac{3}{4}-\frac{3}{4}\cdot\frac{5}{4}-\frac{1}{2}\cdot\frac{1}{2}}ds\leqslant C\delta^2C_3^{\frac{3}{8}}(1+t)^{-\frac{11}{8}}.
\end{aligned}
$$

Substituting (2.5) into L_{42} and using (2.2) yield

$$
L_{42}\leqslant C\int_{t-1}^t\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|
$$

$$
\begin{aligned}
&\times (\mathbb{P}(\widehat{u\cdot\nabla\mathbb{P}\nabla}\cdot\tau)+\mathbb{P}(\widehat{\nabla u\cdot\nabla}\tau)+\mathbb{P}(\nabla u\cdot\widehat{\nabla\Delta^{-1}}\nabla\cdot\nabla\cdot\tau))\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\\
&\quad\times (\|(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)\|_{L^2}+\|\nabla u\cdot\nabla\tau\|_{L^2}+\|\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2})ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\\
&\quad+\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{2}})ds,
\end{aligned}
$$

where we have used $\|\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2}\leqslant C\|\nabla\tau\|_{L^2}$ in the last inequality. Then, by (4.27), (4.10) and (3.5),

$$
\begin{aligned}
L_{42}&\leqslant C\delta^2\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}\cdot\frac{5}{4}-\frac{3}{4}}ds\\
&\quad+C\delta^{\frac{7}{4}}C_3^{\frac{1}{2}}\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{3}{4}\cdot\frac{5}{4}-\frac{3}{4}\cdot\frac{1}{2}}(1+s)^{\frac{1}{8}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}ds\\
&\leqslant C\delta^2(1+t)^{-\frac{11}{8}}+C\delta^2C_3^{\frac{1}{2}}(1+t)^{-\frac{21}{16}}\leqslant C\delta^2(1+C_3^{\frac{1}{2}})(1+t)^{-\frac{21}{16}},
\end{aligned}
$$

where we have used a similar argument as in $K_{92,2}$ in the second inequality. Similarly,

$$
\begin{aligned}
L_{43}&\leqslant C\int_0^t (1+t-s)^{-m}(\|(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)\|_{L^2}+\|\nabla u\cdot\nabla\tau\|_{L^2}+\|\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{L^2})ds\\
&\leqslant C\delta^2(1+C_3^{\frac{1}{2}})(1+t)^{-\frac{21}{16}}. \qquad (4.31)
\end{aligned}
$$

Therefore,

$$
L_4\leqslant C\delta^2(1+C_3^{\frac{3}{8}}+C_3^{\frac{1}{2}})(1+t)^{-\frac{21}{16}}.
$$

Finally, we bound L_5. First, we have

$$
\begin{aligned}
L_5&\leqslant C\int_0^t \||\xi|^{-1}\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla\nabla_h N_3}\|_{L^2(A_1)}ds+C\int_0^t \||\xi|^{-2}\mathrm{e}^{-c_1(t-s)}\widehat{\nabla\nabla_h N_3}\|_{L^2(A_2)}ds\\
&\leqslant C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla_h\nabla\cdot Q}\|_{L^2(A_1)}ds+C\int_{t-1}^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\nabla_h\nabla\cdot Q}\|_{L^2(A_1)}ds\\
&\quad+C\int_0^t \mathrm{e}^{-c_1(t-s)}\|\nabla_h Q\|_{L^2(A_2)}ds\\
&=:L_{51}+L_{52}+L_{53}. \qquad (4.32)
\end{aligned}
$$

The terms on the right-hand side of (4.32) can be estimated similarly as K_{10}. Clearly,

$$
\begin{aligned}
L_{51}&\leqslant C\int_0^{t-1} \|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|^2\widehat{Q}\|_{L^2}ds\\
&\leqslant C\int_0^t (1+t-s)^{-\frac{7}{4}}\|Q\|_{L^1}ds\leqslant C\int_0^t (1+t-s)^{-\frac{7}{4}}\|\nabla u\|_{L^2}\|\tau\|_{L^2}ds\\
&\leqslant C\delta^2\int_0^t (1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{5}{4}}ds\leqslant C\delta^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Applying (2.3) and combining (3.5), (4.10) and the ansatz (4.27) yield

$$
\begin{aligned}
L_{52}&\leqslant C\int_{t-1}^t \mathrm{e}^{-c_1(t-s)}\|\mathrm{e}^{-c_0|\xi|^2(t-s)}|\xi|\widehat{\nabla_h Q}\|_{L^2}ds\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|\nabla_h Q\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(\|\nabla_h\nabla u\|_{L^2}\|\tau\|_{H^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla_h\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h\tau\|_{L^2}^{\frac{1}{2}})ds\\
&\leqslant C\delta^2C_3^{\frac{1}{2}}\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{5}{4}}ds\\
&\quad+C\delta^{\frac{7}{4}}C_3^{\frac{1}{2}}\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{21}{16}}(1+s)^{\frac{1}{8}}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}ds\\
&\leqslant C\delta^2C_3^{\frac{1}{2}}(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Similarly,

$$
L_{53}\leqslant C\int_0^t(1+t-s)^{-m}\|\nabla_hQ\|_{L^2}ds\leqslant C\delta^2C_3^{\frac{1}{2}}(1+t)^{-\frac{5}{4}}.
$$

Consequently,

$$
L_5\leqslant C\delta^2(1+C_3^{\frac{1}{2}})(1+t)^{-\frac{5}{4}}.
$$

Collecting all the estimates of L_1 through L_5 yields

$$
\|\nabla\nabla_hu\|_{L^2}\leqslant C\delta(1+t)^{-\frac{5}{4}}+C\delta^2(1+C_3^{\frac{1}{8}}+C_3^{\frac{3}{8}}+C_3^{\frac{1}{2}})(1+t)^{-\frac{5}{4}}.
$$

This completes the proof of Lemma 4.4. □

Lemma 4.5. *For a constant $C>0$,*

$$
E_4(t)\leqslant C\delta^2+C(1+C_3)\delta^3. \tag{4.33}
$$

Proof. Differentiating (1.6) with respect to x_3, taking the L^2-inner product of the resulting equations with $(\partial_3\Delta u,\partial_3\mathbb{P}\nabla\cdot\tau)$ and multiplying the time weight $(1+t)^{\frac{3}{2}}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{\frac{3}{2}}(\|\partial_3\nabla u\|_{L^2}^2+2\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)+(1+t)^{\frac{3}{2}}(\mu\|\partial_3\nabla_h\nabla u\|_{L^2}^2+\eta\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)\\
&\quad=\frac{3}{4}(1+t)^{\frac{1}{2}}(\|\partial_3\nabla u\|_{L^2}^2+2\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)-(1+t)^{\frac{3}{2}}\int\partial_3\nabla\mathbb{P}(u\cdot\nabla u)\cdot\partial_3\nabla udx\\
&\qquad-2(1+t)^{\frac{3}{2}}\int\partial_3\mathbb{P}\nabla\cdot(u\cdot\nabla\tau)\cdot\partial_3\mathbb{P}\nabla\cdot\tau dx-2(1+t)^{\frac{3}{2}}\int\partial_3\mathbb{P}\nabla\cdot Q\cdot\partial_3\mathbb{P}\nabla\cdot\tau dx\\
&\quad=:\frac{3}{4}(1+t)^{\frac{1}{2}}(\|\partial_3\nabla u\|_{L^2}^2+2\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)+L_6+L_7+L_8.
\end{aligned}
\tag{4.34}
$$

It follows from (2.1) that

$$
\begin{aligned}
L_6&=-(1+t)^{\frac{3}{2}}\int[\partial_3\nabla u\cdot\nabla u\cdot\partial_3\nabla u+(\partial_3u\cdot\nabla)\nabla u\cdot\partial_3\nabla u+(\nabla u\cdot\nabla)\partial_3u\cdot\partial_3\nabla u]dx\\
&\leqslant C(1+t)^{\frac{3}{2}}\int|\nabla u||\nabla^2u||\nabla^2u|dx\\
&\leqslant C(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2u\|_{L^2}^{\frac{1}{2}}\\
&\leqslant C(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}(\|\nabla^2u\|_{L^2}^2+\|\nabla^2\nabla_hu\|_{L^2}^2).
\end{aligned}
$$

Then, by (3.4), (3.5) and (4.10), we deduce

$$
\begin{aligned}
\int_0^tL_6(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{8}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\int_0^t(1+s)(\|\nabla^2u\|_{L^2}^2+\|\nabla^2\nabla_hu\|_{L^2}^2)ds\\
&\leqslant C\delta^3.
\end{aligned}
$$

By integration by parts and inserting (2.5) in L_7, we have

$$
L_7=2(1+t)^{\frac{3}{2}}\bigg(\int(u\cdot\nabla)\mathbb{P}(\nabla\cdot\tau)\cdot\partial_3^2\mathbb{P}\nabla\cdot\tau dx+\int(\nabla u\cdot\nabla\tau)\cdot\partial_3^2\mathbb{P}\nabla\cdot\tau dx
$$

$$+\int(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau)\cdot\partial_3^2\mathbb{P}\nabla\cdot\tau dx\Big)=:L_{71}+L_{72}+L_{73},$$

where we have used $\mathbb{P}\mathbb{P}v=\mathbb{P}v$. By Hölder's inequality and Gagliardo-Nirenberg's inequality, L_{71} can be bounded by

$$\begin{aligned}L_{71}&\leqslant C(1+t)^{\frac{3}{2}}\|u\|_{L^\infty}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}\\&\leqslant C(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}.\end{aligned}$$

By (3.4), (3.5), (4.10), the ansatz (4.27) and Hölder's inequality,

$$\begin{aligned}\int_0^t L_{71}(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{8}}\|\nabla u\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{1}{4}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\\&\quad\times\int_0^t\|\nabla\mathbb{P}(\nabla\cdot\tau)\|_{L^2}(1+s)^{\frac{3}{4}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}ds\\&\leqslant C\delta^3C_3^{\frac{1}{2}}.\end{aligned}$$

Recalling (2.2), we have

$$\begin{aligned}L_{72}&\leqslant C(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla\tau\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3\nabla\tau\|_{L^2}^{\frac{1}{4}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}\\&\leqslant C(1+t)^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{H^1}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{H^1}^{\frac{1}{2}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_0^t L_{72}(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{8}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tau\|_{H^1}^{\frac{1}{2}}\\&\quad\times\int_0^t(1+s)^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{H^1}^{\frac{1}{2}}(1+s)^{\frac{3}{4}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}ds\\&\leqslant C\delta^3C_3^{\frac{1}{2}}.\end{aligned}$$

Similarly,

$$\int_0^t L_{73}(s)ds\leqslant C\delta^3C_3^{\frac{1}{2}}.$$

As a consequence,

$$\int_0^t L_7(s)ds\leqslant C\delta^3C_3^{\frac{1}{2}}.$$

We proceed to estimate L_8. L_8 is first bounded by

$$L_8\leqslant C(1+t)^{\frac{3}{2}}\bigg(\int|\nabla^2u||\tau||\partial_3^2\mathbb{P}\nabla\cdot\tau|dx+\int|\nabla u||\nabla\tau||\partial_3^2\mathbb{P}\nabla\cdot\tau|dx\bigg)=:L_{81}+L_{82}.$$

By (2.2),

$$L_{81}\leqslant C(1+t)^{\frac{3}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\tau\|_{L^2}^{\frac{1}{4}}\|\partial_2\tau\|_{L^2}^{\frac{1}{4}}\|\partial_3\tau\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3\tau\|_{L^2}^{\frac{1}{4}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}.$$

Then

$$\begin{aligned}\int_0^t L_{81}(s)ds&\leqslant C\sup_{0\leqslant s\leqslant t}\|\tau\|_{H^1}^{\frac{1}{2}}(1+s)^{\frac{1}{4}}\|\partial_3\tau\|_{H^1}^{\frac{1}{2}}\\&\quad\times\int_0^t(1+s)^{\frac{1}{4}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{1}{4}}\|\partial_1\nabla^2u\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{3}{4}}\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}ds\end{aligned}$$

$$\leqslant C\delta^3 C_3^{\frac{3}{4}}.$$

Invoking the estimate for L_{72}, we have

$$\int_0^t L_{82}(s)ds \leqslant C\delta^3 C_3^{\frac{1}{2}}.$$

Thus,

$$\int_0^t L_8(s)ds \leqslant C(C_3^{\frac{3}{4}} + C_3^{\frac{1}{2}})\delta^3.$$

Integrating (4.34) in time and combining all the estimates above and (3.5), we conclude

$$\begin{aligned}
&(1+t)^{\frac{3}{2}}(\|\partial_3\nabla u\|_{L^2}^2 + 2\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2) + \int_0^t (1+s)^{\frac{3}{2}}(\mu\|\nabla_h\partial_3\nabla u\|_{L^2}^2 + \eta\|\partial_3^2\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)ds\\
&\quad\leqslant \frac{3}{4}\int_0^t (1+s)^{\frac{1}{2}}(\|\partial_3\nabla u\|_{L^2}^2 + 2\|\partial_3\mathbb{P}\nabla\cdot\tau\|_{L^2}^2)ds\\
&\qquad + \frac{1}{2}(\|\partial_3\nabla u_0\|_{L^2}^2 + 2\|\partial_3\mathbb{P}\nabla\cdot\tau_0\|_{L^2}^2) + C(1+C_3)\delta^3\\
&\quad\leqslant C\delta^2 + C(1+C_3)\delta^3.
\end{aligned}$$

This completes the proof of Lemma 4.5. □

Lemma 4.6. *For a constant $C>0$,*

$$E_5(t) \leqslant C\delta^2 + C(1+C_3)\delta^3. \tag{4.35}$$

Proof. Applying the derivative ∂_3 to (1.1), and then taking the H^1-inner product of the resulting equations with $(\partial_3 u, \partial_3\tau)$, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)(\|\partial_3 u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2) + (1+t)(\mu\|\nabla_h\partial_3 u\|_{H^1}^2 + \eta\|\partial_3^2\tau\|_{H^1}^2)\\
&\quad= \frac{1}{2}(\|\partial_3 u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2) - (1+t)(\partial_3(u\cdot\nabla u), \partial_3 u)_{H^1}\\
&\qquad - (1+t)(\partial_3(u\cdot\nabla\tau), \partial_3\tau)_{H^1} - (1+t)(\partial_3 Q, \partial_3\tau)_{H^1}\\
&\quad=: \frac{1}{2}(\|\partial_3 u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2) + L_9 + L_{10} + L_{11}.
\end{aligned}\tag{4.36}$$

By Hölder's inequality, Gagliardo-Nirenberg's inequality and (2.1),

$$\begin{aligned}
L_9 &\leqslant C(1+t)\int(|\nabla u|^3 + |\nabla u||\nabla^2 u||\nabla^2 u|)dx\\
&\leqslant C(1+t)\|\nabla u\|_{L^4}^2\|\nabla u\|_{L^2}\\
&\quad + C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&\leqslant C(1+t)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2} + C(1+t)\|\nabla u\|_{H^1}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla_h\nabla^2 u\|_{L^2}^2).
\end{aligned}$$

Thus, invoking (3.5), we have

$$\begin{aligned}
\int_0^t L_9(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{2}}\|\nabla u\|_{L^2}\int_0^t \|\nabla u\|_{L^2}^{\frac{1}{2}}(1+s)^{\frac{3}{4}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}}ds\\
&\quad + C\sup_{0\leqslant s\leqslant t}\|\nabla u\|_{H^1}\int_0^t (1+s)(\|\nabla^2 u\|_{L^2}^2 + \|\partial_1\nabla^2 u\|_{L^2}^2)ds\\
&\leqslant C\delta^3.
\end{aligned}$$

To bound L_{10}, we first split it into three terms and apply (2.2) to get

$$
\begin{aligned}
L_{10} &= (1+t)\int u\cdot\nabla\tau\cdot\partial_3^2\tau dx + (1+t)\int \nabla u\cdot\nabla\tau\cdot\partial_3^2\nabla\tau dx + (1+t)\int u\cdot\nabla^2\tau\cdot\partial_3^2\nabla\tau dx\\
&\leqslant C(1+t)\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1\partial_2 u\|_{L^2}^{\frac14}\\
&\quad\times(\|\nabla\tau\|_{L^2}^{\frac12}\|\partial_3\nabla\tau\|_{L^2}^{\frac12}\|\partial_3^2\tau\|_{L^2} + \|\nabla^2\tau\|_{L^2}^{\frac12}\|\partial_3\nabla^2\tau\|_{L^2}^{\frac12}\|\partial_3^2\nabla\tau\|_{L^2})\\
&\quad + C(1+t)\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\|\nabla\tau\|_{L^2}^{\frac12}\|\partial_3\nabla\tau\|_{L^2}^{\frac12}\|\partial_3^2\nabla\tau\|_{L^2}\\
&\leqslant C(1+t)\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{H^1}^{\frac12}\|\nabla\tau\|_{H^1}^{\frac12}\|\partial_3\nabla\tau\|_{H^1}^{\frac12}\|\partial_3^2\tau\|_{H^1}\\
&\quad + C(1+t)\|\nabla u\|_{H^1}^{\frac12}\|\nabla\nabla_h u\|_{H^1}^{\frac12}\|\nabla\tau\|_{L^2}^{\frac12}\|\partial_3\nabla\tau\|_{L^2}^{\frac12}\|\partial_3^2\nabla\tau\|_{L^2}.
\end{aligned}
$$

Then, by (4.10), (3.5) and (4.27),

$$
\begin{aligned}
\int_0^t L_{10}(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}\|\nabla\tau\|_{H^1}^{\frac12}(1+s)^{\frac{3}{16}}\|u\|_{L^2}^{\frac14}(1+s)^{\frac{5}{16}}\|\partial_1 u\|_{L^2}^{\frac14}\\
&\quad\times\int_0^t\|\partial_2 u\|_{H^1}^{\frac12}\|\partial_3\nabla\tau\|_{H^1}^{\frac12}(1+s)^{\frac12}\|\partial_3^2\tau\|_{H^1}ds\\
&\quad + C\sup_{0\leqslant s\leqslant t}(1+s)^{\frac14}\|\nabla u\|_{H^1}^{\frac12}\|\nabla\tau\|_{L^2}^{\frac12}\\
&\quad\times\int_0^t(1+s)^{\frac14}\|\nabla\nabla_h u\|_{H^1}^{\frac12}\|\partial_3\nabla\tau\|_{L^2}^{\frac12}(1+s)^{\frac12}\|\partial_3^2\nabla\tau\|_{L^2}ds\\
&\leqslant CC_3^{\frac12}\delta^3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_{11} &= (1+t)\int Q\cdot\partial_3^2\tau dx + (1+t)\int\nabla Q\cdot\partial_3^2\nabla\tau dx\\
&\leqslant C(1+t)\int(|\nabla u||\tau||\partial_3^2\tau| + |\nabla^2 u||\tau||\partial_3^2\nabla\tau| + |\nabla u||\nabla\tau||\partial_3^2\nabla\tau|)dx\\
&\leqslant C(1+t)\|\nabla u\|_{L^2}^{\frac14}\|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_1\partial_2\nabla u\|_{L^2}^{\frac14}\\
&\quad\times(\|\tau\|_{L^2}^{\frac12}\|\partial_3\tau\|_{L^2}^{\frac12}\|\partial_3^2\tau\|_{L^2} + \|\nabla\tau\|_{L^2}^{\frac12}\|\partial_3\nabla\tau\|_{L^2}^{\frac12}\|\partial_3^2\nabla\tau\|_{L^2})\\
&\quad + C(1+t)\|\nabla^2 u\|_{L^2}\|\tau\|_{L^\infty}\|\partial_3^2\nabla\tau\|_{L^2}\\
&\leqslant C(1+t)\|\nabla u\|_{H^1}^{\frac12}\|\partial_2\nabla u\|_{H^1}^{\frac12}\|\tau\|_{H^1}^{\frac12}\|\partial_3\tau\|_{H^1}^{\frac12}\|\partial_3^2\tau\|_{H^1} + C(1+t)\|\nabla^2 u\|_{L^2}\|\tau\|_{H^2}\|\partial_3^2\nabla\tau\|_{L^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^t L_{11}(s)ds &\leqslant C\sup_{0\leqslant s\leqslant t}\|\tau\|_{H^1}^{\frac12}(1+s)^{\frac14}\|\nabla u\|_{H^1}^{\frac12}\\
&\quad\times\int_0^t(1+s)^{\frac14}\|\partial_2\nabla u\|_{H^1}^{\frac12}\|\partial_3\tau\|_{H^1}^{\frac12}(1+s)^{\frac12}\|\partial_3^2\tau\|_{H^1}ds\\
&\quad + C\sup_{0\leqslant s\leqslant t}\|\tau\|_{H^2}\int_0^t(1+s)(\|\nabla^2 u\|_{L^2}^2 + \|\partial_3^2\nabla\tau\|_{L^2}^2)ds\\
&\leqslant C\delta^3(C_3^{\frac12}+1).
\end{aligned}
$$

As a consequence, integrating (4.36) over $[0,t]$ leads to

$$
\begin{aligned}
&(1+t)(\|\partial_3 u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2) + 2\int_0^t(1+s)(\mu\|\nabla_h\partial_3 u\|_{H^1}^2 + \eta\|\partial_3^2\tau\|_{H^1}^2)ds\\
&\quad\leqslant\int_0^t(\|\partial_3 u\|_{H^1}^2 + \|\partial_3\tau\|_{H^1}^2)ds + (\|\partial_3 u_0\|_{H^1}^2 + \|\partial_3\tau_0\|_{H^1}^2) + C(1+C_3^{\frac12})\delta^3
\end{aligned}
$$

$$\leqslant C\delta^2 + C(1+C_3)\delta^3.$$

This completes the proof of Lemma 4.6. □

Now we are ready to prove (4.28).

Proof of (4.28). It follows from (4.29), (4.33) and (4.35) that

$$E_3(t) + E_4(t) + E_5(t) \leqslant C_4\delta^2 + C_5(1+C_3)\delta^3 + C_6(1+C_3)\delta^4$$

for some constants $C_4, C_5, C_6 > 0$. We choose C_3 to satisfy $C_4 \leqslant \frac{C_3}{6}$ and choose δ sufficiently small such that

$$C_5(1+C_3)\delta \leqslant \frac{C_3}{6}, \quad C_6(1+C_3)\delta^2 \leqslant \frac{C_3}{6}.$$

Then $E_3(t) + E_4(t) + E_5(t) \leqslant \frac{C_3}{2}\delta^2$. This completes the proof of (4.28). □

4.4 The decay rate for $\|\mathbb{P}\nabla\cdot\boldsymbol{\tau}\|_{L^2}$

This subsection is devoted to the decay rate for $\|\mathbb{P}\nabla\cdot\tau\|_{L^2}$. We make use of the decay rates obtained in the previous subsections.

Proof of the decay rate for $\|\mathbb{P}\nabla\cdot\tau\|_{L^2}$. Recalling the integral representation (4.2), we have

$$\begin{aligned}
\|\mathbb{P}\nabla\cdot\tau(t)\|_{L^2(\mathbb{R}^3)} &\leqslant \|\widehat{Q}_3(t)\widehat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_4(t)\widehat{\mathbb{P}\nabla\cdot\tau_0}\|_{L^2(\mathbb{R}^3)} \\
&\quad + \int_0^t \|\widehat{Q}_3(t-s)\widehat{N}_1(s)\|_{L^2(\mathbb{R}^3)} ds \\
&\quad + \int_0^t \|\widehat{Q}_4(t-s)\widehat{N}_2(s)\|_{L^2(\mathbb{R}^3)} ds + \int_0^t \|\widehat{Q}_4(t-s)\widehat{N}_3(s)\|_{L^2(\mathbb{R}^3)} ds \\
&=: M_1 + \cdots + M_5.
\end{aligned}$$

According to Proposition 4.1 and Lemma 4.2,

$$\begin{aligned}
M_1 &\leqslant C\||\xi|\mathrm{e}^{-c_0|\xi|^2t}\widehat{u}_0\|_{L^2(A_1)} + C\mathrm{e}^{-c_1t}\|u_0\|_{L^2(A_2)} \\
&\leqslant C(1+t)^{-\frac{5}{4}}\|u_0\|_{L^1} + C(1+t)^{-\frac{5}{4}}\|u_0\|_{L^2} \leqslant C\delta(1+t)^{-\frac{5}{4}}.
\end{aligned}$$

Similarly, we have

$$M_2 \leqslant C(1+t)^{-\frac{5}{4}}\|\tau_0\|_{L^1} + C(1+t)^{-\frac{3}{4}}\|\nabla\tau_0\|_{L^2} \leqslant C\delta(1+t)^{-\frac{5}{4}}.$$

For M_3, we first decompose it as

$$\begin{aligned}
M_3 &\leqslant C\int_0^t \||\xi|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)} ds + C\int_0^t \|\mathrm{e}^{-c_1(t-s)}\widehat{N}_1\|_{L^2(A_2)} ds \\
&= C\int_0^{t-1} \||\xi|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)} ds + \int_{t-1}^t \||\xi|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{N}_1\|_{L^2(A_1)} ds \\
&\quad + C\int_0^t \mathrm{e}^{-c_1(t-s)}\|N_1\|_{L^2(A_2)} ds \\
&=: M_{31} + M_{32} + M_{33}.
\end{aligned}$$

It follows from the estimates of the first term in (4.20) and the second term in (4.14) that

$$M_{31} + M_{33} \leqslant C\delta^2(1+t)^{-\frac{5}{4}}.$$

By Hölder's inequality and Sobolev's inequality,

$$M_{32} \leqslant C\int_{t-1}^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|u\cdot\nabla u\|_{L^2(\mathbb{R}^3)} ds$$

$$
\begin{aligned}
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|u\|_{L^\infty}\|\nabla u\|_{L^2}ds\\
&\leqslant C\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}\|u\|_{H^2}\|\nabla u\|_{L^2}ds\\
&\leqslant C\delta^2\int_0^t \mathrm{e}^{-c_1(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{5}{4}}ds \leqslant C\delta^2(1+t)^{-\frac{5}{4}},
\end{aligned}
$$

where the last inequality was obtained by using a similar argument as in $K_{92,2}$. Thus, $M_3 \leqslant C\delta^2(1+t)^{-\frac{5}{4}}$. Combining the estimates (4.21), (4.22) and (4.31), we obtain

$$
\begin{aligned}
M_4 &\leqslant C\int_0^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\mathbb{P}\widehat{\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_1)}ds + C\int_0^t \|\mathrm{e}^{-c_1(t-s)}\mathbb{P}\widehat{\nabla\cdot(u\cdot\nabla\tau)}\|_{L^2(A_2)}ds\\
&\leqslant C\delta^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

Next, we bound M_5. First, we have

$$
M_5 \leqslant C\int_0^t \|\mathrm{e}^{-c_0|\xi|^2(t-s)}\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_1)}ds + C\int_0^t \|\widehat{Q_4}\,\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_2)}ds =: M_{51}+M_{52}.
$$

Recalling the estimates (4.24) and (4.25) for $K_{10,1}$ and $K_{10,2}$, we get

$$
M_{51} \leqslant C\delta^2(1+t)^{-\frac{5}{4}}.
$$

The estimate for M_{52} is subtle. We need to bound it in different subdomains of A_2. It holds that

$$
\begin{aligned}
M_{52} \leqslant{}& C\int_0^t \mathrm{e}^{-c_3(t-s)}\|\mathrm{e}^{-c_2|\xi_h|^2(t-s)}\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_{21}\cup A_{23})}ds\\
&+ C\int_0^t \mathrm{e}^{-c_3(t-s)}\|\mathrm{e}^{-c_2\xi_3^2(t-s)}\widehat{\mathbb{P}\nabla\cdot Q}\|_{L^2(A_{22})}ds\\
=:{}& M_{52,1}+M_{52,2}.
\end{aligned}
$$

By (4.7),

$$
M_{52,1} \leqslant C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}\|\nabla\cdot Q\|_{L^2_{x_3}L^1_{x_1x_2}}ds.
$$

By Hölder's inequality, Minkowski's inequality and Sobolev's inequality,

$$
\|fg\|_{L^2_{x_3}L^1_{x_1x_2}} \leqslant \|\|f\|_{L^2_{x_1x_2}}\|g\|_{L^2_{x_1x_2}}\|_{L^2_{x_3}} \leqslant C\|\|f\|_{L^\infty_{x_3}}\|_{L^2_{x_1x_2}}\|g\|_{L^2} \leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_3 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}.
$$

Therefore,

$$
\begin{aligned}
M_{52,1} &\leqslant C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}(\|\nabla^2 u\|_{L^2}\|\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\tau\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}\|\nabla\tau\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla\tau\|_{L^2}^{\frac{1}{2}})ds\\
&\leqslant C\delta^2\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}((1+s)^{-1}+(1+s)^{-\frac{5}{4}})ds\\
&\leqslant C\delta^2(1+t)^{-1}.
\end{aligned}
$$

For $M_{52,2}$, we further decompose it as

$$
\begin{aligned}
M_{52,2} \leqslant{}& C\sum_{i=1}^{3}\int_0^t \mathrm{e}^{-c_3(t-s)}\|\mathrm{e}^{-c_2\xi_3^2(t-s)}\widehat{\partial_3 Q_{i3}}\|_{L^2}ds\\
&+ C\sum_{i=1}^{3}\sum_{j=1}^{2}\int_0^t \mathrm{e}^{-c_3(t-s)}\|\mathrm{e}^{-c_2\xi_3^2(t-s)}\widehat{\partial_j Q_{ij}}\|_{L^2}ds
\end{aligned}
$$

$$\leqslant C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}\|Q\|_{L^2}ds + C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{4}}\|\nabla_h Q\|_{L^2_{x_1x_2}L^1_{x_3}}ds. \tag{4.37}$$

The first integral can be bounded by

$$\begin{aligned}\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}\|Q\|_{L^2}ds &\leqslant C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}\|\nabla u\|_{L^2}\|\tau\|_{H^2}ds \\ &\leqslant C\delta^2\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{5}{4}}ds \leqslant C\delta^2(1+t)^{-\frac{5}{4}}.\end{aligned}$$

We state two inequalities, i.e.,

$$\|fg\|_{L^2_{x_1x_2}L^1_{x_3}} \leqslant \|\|f\|_{L^2_{x_3}}\|g\|_{L^2_{x_3}}\|_{L^2_{x_1x_2}} \leqslant C\|f\|_{L^2}\|g\|_{L^2_{x_3}L^\infty_{x_1x_2}} \leqslant C\|f\|_{L^2}\|g\|_{H^2}, \tag{4.38}$$

$$\begin{aligned}\|fg\|_{L^2_{x_1x_2}L^1_{x_3}} &\leqslant \|\|f\|_{L^2_{x_3}}\|g\|_{L^2_{x_3}}\|_{L^2_{x_1x_2}} \leqslant C\|f\|_{L^2_{x_3}L^\infty_{x_1}L^2_{x_2}}\|g\|_{L^2_{x_3}L^2_{x_1}L^\infty_{x_2}} \\ &\leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}.\end{aligned} \tag{4.39}$$

Applying (4.38) and (4.39) to the second integral in (4.37) yields

$$\begin{aligned}&\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{4}}\|\nabla_h Q\|_{L^2_{x_1x_2}L^1_{x_3}}ds \\ &\leqslant C\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{4}}(\|\nabla\nabla_h u\|_{L^2}\|\tau\|_{H^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\tau\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h\tau\|_{L^2}^{\frac{1}{2}})ds \\ &\leqslant C\delta^2\int_0^t \mathrm{e}^{-c_3(t-s)}(t-s)^{-\frac{1}{4}}(1+s)^{-\frac{5}{4}}ds \leqslant C\delta^2(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Hence, $M_{52,2} \leqslant C\delta^2(1+t)^{-\frac{5}{4}}$. Therefore, $M_{52} \leqslant C\delta^2(1+t)^{-1}$. Consequently, $M_5 \leqslant C\delta^2(1+t)^{-1}$. Collecting the estimates for M_1 through M_5, we conclude $\|\mathbb{P}\nabla\cdot\tau(t)\|_{L^2} \leqslant C\delta(1+t)^{-1}$. This completes the proof of the decay rate for $\|\mathbb{P}\nabla\cdot\tau\|_{L^2}$. □

5 Conclusions and discussions

In this paper, we investigate the stability of the 3D Oldroyd-B model with partial mixed dissipation. By discovering that the coupling and interaction of the fluid velocity u and τ generates extra smoothing and stabilization, we overcome the difficulties of the lack of the horizontal dissipation or damping in the equation of τ and solve the global well-posedness and the stability problem of (1.1). In addition, we also present the implicit decay rates of the solution that reveals the precise large-time behavior of the solution. In the future, we are interested in the 2D Oldroyd-B model with partial dissipation, i.e., the system with horizontal viscosity in the equation of u and vertical dissipation in the equation of τ. Compared with the three-dimensional system, the Sobolev inequalities in two dimensions are critical. This makes the stability problem of the 2D Oldroyd-B model extremely challenging and we have to seek the new strategy and idea.

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