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Stability for the 3D magneto-micropolar fluids with only velocity dissipation near a background magnetic field

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Abstract

Many biological fluids such as blood contain ions that can interact with an applied magnetic field, and can thus be treated as electrically conducting fluids. In many recent studies related to bioengineering such as the development of magnetic tracers and medical applications, the magneto-micropolar systems have been used to model the dynamics of these biofluids. A natural question related to these studies is the influence of a static background magnetic field on the stability and long-time behavior of the biofluids. This paper intends to present a rigorous theory in terms of the magneto-micropolar system with only velocity dissipation but without magnetic dissipation and angular dissipation. The spatial domain is taken to be a periodic box and the background magnetic field is assumed to satisfy a Diophantine condition. This Diophantine condition is satisfied by almost every vector field. We establish the asymptotic stability of any perturbation and its precise long-time behavior. This result reflects the stabilizing effect of the background magnetic field. Without it, it is almost impossible to analyze this 3D nonlinear magneto-micropolar system. The mathematical objects underlying the smoothing and stabilizing effect are the wave structures hidden in this magneto-micropolar system.

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1. Introduction and main results

The compressible viscous magneto-micropolar equations model the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids in which they are immersed (see, e.g., [2]). The governing equations can be written as

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-(\mu+\zeta)\Delta\mathbf{u}-(\lambda+\mu-\zeta)\nabla\operatorname{div}\mathbf{u}+\nabla P=2\zeta\nabla\times\mathbf{w}+(\nabla\times\mathbf{B})\times\mathbf{B},\\ \rho\partial_t\mathbf{w}+\rho\mathbf{u}\cdot\nabla\mathbf{w}-\mu'\Delta\mathbf{w}-(\lambda'+\mu')\nabla\operatorname{div}\mathbf{w}+4\zeta\mathbf{w}=2\zeta\nabla\times\mathbf{u},\\ \partial_t\mathbf{B}+\mathbf{u}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{u}+\mathbf{B}\operatorname{div}\mathbf{u}-\sigma\Delta\mathbf{B}=0,\\ \operatorname{div}\mathbf{B}=0,\end{cases}\tag{1.1}$$

where the unknown functions ρ, $\mathbf{u}$, $\mathbf{w}$, $P=P(\rho)$ and $\mathbf{B}$ denote the fluid density, velocity, microrotational velocity, pressure, and the magnetic field, respectively. The shear viscosity μ, the bulk viscosity λ and the microrotation viscosity ζ satisfy the physical restrictions

$$\mu>0,\quad 2\mu+3\lambda-4\zeta\geqslant 0.\tag{1.2}$$

μ' and λ' are the angular viscosities satisfying

$$\mu'>0,\quad 2\mu'+3\lambda'\geqslant 0\tag{1.3}$$

while the constant $\sigma\geq 0$ is the magnetic diffusivity.

The magneto-micropolar equations have recently being used to model the micropolar biomagnetic flows such as blood flow in a magnetic field. There have been numerous research studies related to bioengineering such as the development of magnetic devices for cell separation, development of magnetic tracers and medical applications (see, e.e., [9,33]). The majority of biological fluids are considered as biomagnetic, mainly because they contain ions which interact with the applied magnetic field. Blood in particular, has erythrocytes that have the tendency to orient with their disk plane parallel to the magnetic field direction [12]. Therefore, blood can be modeled as an electrically conducting fluid which exhibits magnetization, such that magnetohydrodynamics (MHD) could also be incorporated into the mathematical model [25].

A natural question proposed and studied in [9,12,25] is the influence of a background magnetic field on the stability of magneto-micropolar fluids. [9] focuses on the numerical algorithm and computation. [25] contains some simple linear analysis. The goal of this paper is to present a rigorous theory on the nonlinear stability and long-time behavior. We focus on the following 3D compressible magneto-micropolar fluids with only velocity dissipation

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\quad t>0,\ x\in\mathbb{T}^3,\\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-(\mu+\zeta)\Delta\mathbf{u}-(\lambda+\mu-\zeta)\nabla\operatorname{div}\mathbf{u}+\nabla P=2\zeta\nabla\times\mathbf{w}+(\nabla\times\mathbf{B})\times\mathbf{B},\\ \rho\partial_t\mathbf{w}+\rho\mathbf{u}\cdot\nabla\mathbf{w}+4\zeta\mathbf{w}=2\zeta\nabla\times\mathbf{u},\\ \partial_t\mathbf{B}+\mathbf{u}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{u}+\mathbf{B}\operatorname{div}\mathbf{u}=0,\\ \operatorname{div}\mathbf{B}=0,\\ (\rho,\mathbf{u},\mathbf{w},\mathbf{B})|_{t=0}=(\rho_0,\mathbf{u}_0,\mathbf{w}_0,\mathbf{B}_0),\end{cases}\tag{1.4}$$

where $\mathbb{T}^3$ denotes a periodic domain. Suitable for biofluids, this compressible model involves only velocity dissipation and no magnetic diffusion or microrotation velocity dissipation.

The mathematical setup is to study the stability of perturbations near the steady-state solution induced by a background magnetic field

$$(\rho^{(0)}, \mathbf{u}^{(0)}, \mathbf{w}^{(0)}, \mathbf{B}^{(0)}) = (\bar{\rho}, 0, 0, \mathbf{n}),$$

where $\bar{\rho} > 0$ is constant and $\mathbf{n}$ is a constant vector. Without loss of generality, we assume

$$\left|\mathbb{T}^3\right| = 1, \qquad \bar{\rho} = \int_{\mathbb{T}^3} \rho_0 \, dx = 1.$$

In addition, we further assume the following initial averages are zero,

$$\int_{\mathbb{T}^3} \rho_0 \mathbf{u}_0 \, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \mathbf{B}_0 \, dx = 0. \tag{1.5}$$

It is easy to check that, for sufficiently regular solutions, these averages are conserved in time,

$$\int_{\mathbb{T}^3} \rho \, dx = \bar{\rho}, \quad \int_{\mathbb{T}^3} \rho \mathbf{u} \, dx = 0, \quad \text{and} \quad \int_{\mathbb{T}^3} \mathbf{B} \, dx = 0. \tag{1.6}$$

To write the equations for the perturbations, we set

$$a \stackrel{\text{def}}{=} \rho - 1, \;\; \bar{\mu}(\rho) \stackrel{\text{def}}{=} \frac{\mu + \zeta}{\rho}, \;\; \bar{\lambda}(\rho) \stackrel{\text{def}}{=} \frac{\lambda + \mu - \zeta}{\rho}, \;\; I(a) \stackrel{\text{def}}{=} \frac{a}{1+a} \quad \text{and} \quad k(a) \stackrel{\text{def}}{=} \frac{P'(1+a)}{1+a} - 1.$$

Still writing $\mathbf{B}$ for the perturbation $\mathbf{B} - \mathbf{n}$, we find that the perturbation $(a, \mathbf{u}, \mathbf{w}, \mathbf{B})$ satisfies the following system

$$\begin{cases} \partial_t a + \operatorname{div} \mathbf{u} = f_1, \\ \partial_t \mathbf{u} - \operatorname{div}(\bar{\mu}(\rho)\nabla \mathbf{u}) - \nabla(\bar{\lambda}(\rho)\operatorname{div} \mathbf{u}) + \nabla a = 2\zeta \nabla \times \mathbf{w} + \mathbf{n} \cdot \nabla \mathbf{B} - \nabla(\mathbf{n} \cdot \mathbf{B}) + f_2, \\ \partial_t \mathbf{w} + 4\zeta \mathbf{w} = 2\zeta \nabla \times \mathbf{u} + f_3, \\ \partial_t \mathbf{B} = \mathbf{n} \cdot \nabla \mathbf{u} - \mathbf{n} \operatorname{div} \mathbf{u} + f_4, \\ \operatorname{div} \mathbf{B} = 0, \\ (a, \mathbf{u}, \mathbf{w}, \mathbf{B})|_{t=0} = (a_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{B}_0), \end{cases} \tag{1.7}$$

where

$$\begin{aligned} f_1 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla a - a \operatorname{div} \mathbf{u}, \\ f_2 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \mathbf{B} + k(a)\nabla a + 2\zeta I(a) \nabla \times \mathbf{w} + (\mu + \zeta)(\nabla I(a))\nabla \mathbf{u} \\ &\qquad + (\lambda + \mu - \zeta)(\nabla I(a))\operatorname{div} \mathbf{u} - I(a)(\mathbf{n} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{B} - \mathbf{n} \nabla \mathbf{B} + \mathbf{B} \nabla \mathbf{B}), \\ f_3 &\stackrel{\text{def}}{=} -\mathbf{u} \cdot \nabla \mathbf{w} - 4\zeta I(a)\mathbf{w} + 2\zeta I(a) \nabla \times \mathbf{u}, \end{aligned}$$

$$f_4 \stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\operatorname{div}\mathbf{u}. \tag{1.8}$$

To state our results, we recall the Diophantine condition. A vector field $\mathbf{n}\in\mathbb{R}^3$ is said to satisfy the Diophantine condition if for any $\mathbf{k}\in\mathbb{Z}^3\setminus\{0\}$,

$$|\mathbf{n}\cdot\mathbf{k}|\ge \frac{c}{|\mathbf{k}|^r}\quad \text{for some } c>0 \text{ and } r>2. \tag{1.9}$$

As demonstrated in [5], almost every vector field in $\mathbb{R}^3$ actually satisfies (1.9). Certainly there are vector fields that do not obey this condition such as $\mathbf{e}_1=(1,0,0)$.

The first main result can be stated as follows.

Theorem 1.1. *Assume* $\mathbf{n}$ *satisfies the Diophantine condition (1.9). Let* $N\ge 4r+7$ *with* $r>2$*. Consider the system (1.7) with the initial data* $(a_0,\mathbf{u}_0,\mathbf{w}_0,\mathbf{B}_0)$ *satisfying*

$$a_0\in H^N(\mathbb{T}^3),\quad c_0\le a_0\le c_0^{-1},\quad \mathbf{u}_0\in H^N(\mathbb{T}^3),\quad \mathbf{w}_0\in H^N(\mathbb{T}^3),\quad \mathbf{B}_0\in H^N(\mathbb{T}^3)$$

for some constant $c_0>0$ *and* (1.5)*. Then there exists a small constant* ε *such that, if*

$$\|a_0\|_{H^N}+\|\mathbf{u}_0\|_{H^N}+\|\mathbf{w}_0\|_{H^N}+\|\mathbf{B}_0\|_{H^N}\le\varepsilon,$$

then the system (1.7) *admits a unique global solution* $(a,\mathbf{u},\mathbf{w},\mathbf{B})\in C([0,\infty);H^N)$*. Moreover, for* $r+4\le\beta\le N$*, there holds*

$$\|a(t)\|_{H^\beta}+\|\mathbf{u}(t)\|_{H^\beta}+\|\mathbf{w}(t)\|_{H^\beta}+\|\mathbf{B}(t)\|_{H^\beta}\le C\varepsilon(1+t)^{-\frac{3(N-\beta)}{2(N-r-4)}} \tag{1.10}$$

for any $t\ge 0$.

Remark 1.1. *Inspired by recent work in [39], we shall consider the global well-posedness of* (1.7) *with some large initial data in the further work.*

When the density is a constant, the system (1.4) reduces to the following 3D incompressible magneto-micropolar equations

$$\begin{cases}\partial_t\mathbf{u}-(\mu+\zeta)\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{B}\cdot\nabla\mathbf{B}+2\zeta\nabla\times\mathbf{w},\\ \partial_t\mathbf{w}+\mathbf{u}\cdot\nabla\mathbf{w}+4\zeta\mathbf{w}=2\zeta\nabla\times\mathbf{u},\\ \partial_t\mathbf{B}+\mathbf{u}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{u},\\ \operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{B}=0,\\ (\mathbf{u},\mathbf{w},\mathbf{B})|_{t=0}=(\mathbf{u}_0,\mathbf{w}_0,\mathbf{B}_0).\end{cases} \tag{1.11}$$

Correspondingly the perturbation for the density a becomes zero and the system (1.7) for the perturbations is reduced to

$$\begin{cases} \partial_t \mathbf{u} - (\mu+\zeta)\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{n}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{B} + 2\zeta\nabla\times\mathbf{w}, \\ \partial_t \mathbf{w} + \mathbf{u}\cdot\nabla\mathbf{w} + 4\zeta\mathbf{w} = 2\zeta\nabla\times\mathbf{u}, \\ \partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{n}\cdot\nabla\mathbf{u} + \mathbf{B}\cdot\nabla\mathbf{u}, \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{B} = 0, \\ (\mathbf{u},\mathbf{B})|_{t=0} = (\mathbf{u}_0,\mathbf{B}_0). \end{cases} \tag{1.12}$$

As a consequence of Theorem 1.1 or its derivation, we obtain the following result for the incompressible magneto-micropolar system (1.12).

Corollary 1.2. *Let $N \geq 4r+7$ with $r > 2$. Assume $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{B}_0) \in H^N(\mathbb{T}^3)$ and*

$$\int_{\mathbb{T}^3} \mathbf{u}_0\,dx = \int_{\mathbb{T}^3} \mathbf{w}_0\,dx = \int_{\mathbb{T}^3} \mathbf{B}_0\,dx = 0.$$

Then there exists a small constant ε such that, if

$$\|\mathbf{u}_0\|_{H^N} + \|\mathbf{w}_0\|_{H^N} + \|\mathbf{B}_0\|_{H^N} \leq \varepsilon, \tag{1.13}$$

then the system (1.12) admits a unique global solution $(\mathbf{u},\mathbf{w},\mathbf{B}) \in C([0,\infty); H^N)$. Moreover, the solution enjoys the following decay estimate, for any $r+4 \leq \beta \leq N$,

$$\|\mathbf{u}(t)\|_{H^\beta} + \|\mathbf{w}(t)\|_{H^\beta} + \|\mathbf{B}(t)\|_{H^\beta} \leq C(1+t)^{-\frac{3(N-\beta)}{2(N-r-4)}} \tag{1.14}$$

for any $t > 0$.

We explain the main difficulties and strategies involved in the proof. The major difficulty is the lack of dissipation in the equations of a, $\mathbf{B}$ and $\mathbf{w}$. In general solutions to transport equations without damping or dissipation would grow in time and thus destroy the stability. This paper explores the enhanced dissipation from two different sources, the background magnetic field and the interaction due to the coupling in the magneto-micropolar system. We provide some details to give a more precise account of these enhanced dissipations. First, we explain how the background magnetic field and the interaction between $\mathbf{u}$ and $\mathbf{B}$ generate smoothing and stabilizing effect in the direction of the magnetic field. Applying $\mathbb{P}$ to $(1.7)_2$ gives rise to

$$\partial_t \mathbb{P}\mathbf{u} - (\mu+\zeta)\Delta\mathbb{P}\mathbf{u} = 2\zeta\mathbb{P}\nabla\times\mathbf{w} + \mathbf{n}\cdot\nabla\mathbf{B} + \mathbb{P}\widetilde{f_2}. \tag{1.15}$$

Combining (1.15) with the equation of $\mathbf{B}$ in (1.7)

$$\partial_t \mathbf{B} - \mathbf{n}\cdot\nabla\mathbf{u} + \mathbf{n}\operatorname{div}\mathbf{u} = f_4,$$

we find that $\mathbb{P}\mathbf{u}$ and $\mathbf{B}$ both satisfy the damped degenerate wave equations

$$\partial_{tt}\mathbb{P}\mathbf{u} - (\mu+\zeta)\Delta\partial_t\mathbb{P}\mathbf{u} - (\mathbf{n}\cdot\nabla)^2\mathbb{P}\mathbf{u} = M_1,$$
$$\partial_{tt}\mathbf{B} - (\mu+\zeta)\Delta\partial_t\mathbf{B} - (\mathbf{n}\cdot\nabla)^2\mathbf{B} = M_2,$$

where M_1 and M_2 are other terms. The term $(\mathbf{n}\cdot\nabla)^2\mathbf{B}$ would allow us to extract the dissipation in the direction of $\mathbf{n}$. In fact, we can show the time integrability of $\|\mathbf{n}\cdot\nabla\mathbf{B}\|^2_{H^{r+3}}$. More details can be found in Proposition 3.3. Moreover, combining with the Poincaré type inequality induced by the Diophantine condition on $\mathbf{n}$ (Lemma 2.1), we are then able to control the time integrability of $\|\mathbf{B}\|_{L^\infty}$ and $\|\nabla\mathbf{B}\|_{L^\infty}$.

Next, we explain the dissipation in the density perturbation a. Although the equation for a appears to be a typical transport equation, a careful examination of the combined quantity

$$d \stackrel{\text{def}}{=} a + \mathbf{n}\cdot\mathbf{B}$$

allows us to extract the dissipative properties of a. Let $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denote the standard Leray projector operator and $\mathbb{Q} = I - \mathbb{P}$. Applying the operator $\mathbb{Q}$ to $(1.7)_2$ yields

$$\partial_t\mathbb{Q}\mathbf{u} - \nu\Delta\mathbb{Q}\mathbf{u} + \nabla a + \nabla(\mathbf{n}\cdot\mathbf{B}) = \mathbb{Q}\widetilde{f_2},$$

where $\nu \stackrel{\text{def}}{=} \lambda + 2\mu$, and we have used $\mathbb{Q}(\mathbf{n}\cdot\nabla\mathbf{B}) = 0$, $\mathbb{Q}(\nabla(\mathbf{n}\cdot\mathbf{B})) = \nabla(\mathbf{n}\cdot\mathbf{B})$. Now we consider the following system including a, $\mathbb{Q}\mathbf{u}$ and $\mathbf{B}$,

$$\begin{cases} \partial_t a + \operatorname{div}\mathbf{u} = f_1, \\ \partial_t\mathbb{Q}\mathbf{u} - \nu\Delta\mathbb{Q}\mathbf{u} + \nabla a + \nabla(\mathbf{n}\cdot\mathbf{B}) = \mathbb{Q}\widetilde{f_2}, \\ \partial_t\mathbf{B} = \mathbf{n}\cdot\nabla\mathbf{u} - \mathbf{n}\operatorname{div}\mathbf{u} + f_4. \end{cases}$$

Introducing a new auxiliary function $\mathbf{G} \stackrel{\text{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{\nu}\Delta^{-1}\nabla(a + \mathbf{n}\cdot\mathbf{B})$, we deduce that $(d, \mathbf{G})$ satisfy the following system

$$\begin{cases} \partial_t d + \dfrac{1}{\nu}(|\mathbf{n}|^2+1)d + (|\mathbf{n}|^2+1)\operatorname{div}\mathbf{G} = \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} + f_1 + f_4\cdot\mathbf{n}, \\ \partial_t\mathbf{G} - \nu\Delta\mathbf{G} = \dfrac{1}{\nu}(|\mathbf{n}|^2+1)\mathbb{Q}\mathbf{u} - \dfrac{1}{\nu}\Delta^{-1}\nabla(\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}) + \mathbb{Q}\widetilde{f_2} - \dfrac{1}{\nu}\Delta^{-1}\nabla(f_1 + f_4\cdot\mathbf{n}). \end{cases}$$

Obviously, the above system clearly reveals the damping in $d = a + \mathbf{n}\cdot\mathbf{B}$ and the dissipation in G. These are the essential ingredients in the proof of our main result.

Before ending this section, we give some extended review on results related to the magneto-micropolar systems. Due to its physical applications and mathematical complexity, the compressible viscous magneto-micropolar system has been extensively studied (see, e.g., [11,23,24]). If the effect of angular velocity field of the particle's rotation is omitted, i.e., $\mathbf{w} = 0$, then the system (1.1) reduces to compressible magnetohydrodynamic (MHD) equations. The presence of magnetic field and its interaction with the hydrodynamic motion in MHD flows generally makes the analysis more complicated, and many fundamental problems for the compressible MHD system remain open even for the one-dimensional case. There have been substantial developments (see, e.g., [3,4,16] for the one-dimensional case and [15,18,17,37] for the multi-dimensional case). In [14], Kawashima obtained the global existence of smooth solutions to the compressible MHD system in two dimensions, provided that the initial data are closed to some constant state. Furthermore, the global existence and time decay rate of smooth solutions to the linearized two-dimensional MHD equations have been investigated in [32]. [10,26] studied the well-posedness in critical framework for MHD equations. When the magnetic field is neglected (that is, $\mathbf{B} = 0$), system (1.1) reduces to the compressible micropolar fluid equations. The theory of micropolar fluids introduced by Eringen in the 1960s (see [6,7]) is a significant step toward the generalization of the classical compressible Navier-Stokes model. Due to the profound physical background and important mathematical significance, the compressible micropolar fluid equations have been extensively studied, such as long-time behavior of solutions [20,21], low Mach number limit of solutions [28,29] and pointwise estimates of the smooth solutions [35], among many other results.

The compressible magneto-micropolar equations have been studied by a number of authors. For multi-dimensional compressible magneto-micropolar equations, Amirat and Hamdache [1] proved the global existence of weak solutions with finite energy which generalized Lions' pioneering work [19] and results by Feireisl etal. [8]. Recently, Wei et al. [34] constructed the global existence and optimal time decay rates of solutions to the system (1.1). Xu and Zhong [38] proved the local existence and uniqueness of strong solutions in bounded domains or the whole space $\mathbb{R}^3$. In 2021, Song [27] obtained the global well-posedness for the 3-D compressible micropolar system in the critical Besov space when the system has a special viscosity coefficients. This result can be fully extended to magneto-micropolar fluids (1.1).

The rest of this paper is organized as follows. The next section recalls several important calculus inequalities. Section 3 is devoted to the proof of Theorem 1.1. Several propositions are stated and used. Section 4 presents the proofs of the propositions used in the proof of Theorem 1.1.

2. Preliminaries

This section reviews analysis tools including several calculus inequalities to be used in the subsequent sections.

Lemma 2.1. ([36]) *Assume $\mathbf{n} \in \mathbb{R}^3$ satisfies the Diophantine condition (1.9). Let $s \in \mathbb{R}$. Then, for any function f satisfies $\nabla f \in H^{s+r}(\mathbb{T}^3)$ and $\int_{\mathbb{T}^3} f\,dx = 0$,*

$$\|f\|_{H^s(\mathbb{T}^3)} \leq C\|\mathbf{n}\cdot\nabla f\|_{H^{s+r}(\mathbb{T}^3)}.$$

Lemma 2.2. *Let $\mathbf{n} \in \mathbb{R}^3$ satisfy (1.9) and $\rho - 1 \in L^2(\mathbb{T}^3)$ satisfy*

$$\|\rho - 1\|_{L^2} \leq \frac{1}{2}, \quad \text{and} \int_{\mathbb{T}^3} \rho\mathbf{u}\,dx = 0. \tag{2.3}$$

Then for any $s \geq 0$,

$$\|\mathbf{u}\|_{H^s} \leq C\|\mathbf{n}\cdot\nabla\mathbf{u}\|_{H^{s+r}}. \tag{2.4}$$

Proof. For any $s > 0$, there holds

$$\|\mathbf{u}\|_{H^s} \approx \|\mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{\dot{H}^s}.$$

Hence, in view of Lemma 2.1, we have

$$\begin{aligned}\|\mathbf{u}\|_{H^s} \approx& \|\mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{\dot{H}^s} \\ \lesssim& \|\mathbf{u}\|_{L^2} + \|\mathbf{n}\cdot\nabla\mathbf{u}\|_{H^{s+r}}.\end{aligned} \tag{2.5}$$

Next, we only need to verify (2.4) holds for $s = 0$. Denote $\bar{\mathbf{u}}$ the mean of $\mathbf{u}$, there holds

$$\|\mathbf{u}\|_{L^2} \le \|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2} + \|\bar{\mathbf{u}}\|_{L^2} \le C\|\mathbf{n}\cdot\nabla\mathbf{u}\|_{H^r} + |\bar{\mathbf{u}}|.$$

But we note from (2.3) that

$$|\bar{\mathbf{u}}| = \left|\int_{\mathbb{T}^3} (\rho - 1)\mathbf{u}\,dx\right| \le \|\rho - 1\|_{L^2}\|\mathbf{u}\|_{L^2} \le \frac{1}{2}\|\mathbf{u}\|_{L^2}.$$

Putting two estimates together implies that,

$$\|\mathbf{u}\|_{L^2} \le C\|\mathbf{n}\cdot\nabla\mathbf{u}\|_{H^r}, \tag{2.6}$$

from which and (2.5), we arrive at (2.4). This finishes the proof of the lemma. □

Lemma 2.7. ([13]) *Let $s \ge 0$. Then there exists a constant C such that, for any $f, g \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, we have*

$$\|fg\|_{H^s} \le C(\|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}).$$

Lemma 2.8. ([13]) *Let $s > 0$. Then there exists a constant C such that, for any $f \in H^s(\mathbb{T}^3) \cap W^{1,\infty}(\mathbb{T}^3)$, $g \in H^{s-1}(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, there holds*

$$\left\|[\Lambda^s, f\cdot\nabla]g\right\|_{L^2} \le C(\|\nabla f\|_{L^\infty}\left\|\Lambda^s g\right\|_{L^2} + \left\|\Lambda^s f\right\|_{L^2}\|\nabla g\|_{L^\infty}).$$

Lemma 2.9. ([31]) *Let $s > 0$ and $f \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$. Assume that F is a smooth function on $\mathbb{R}$ with $F(0) = 0$. Then we have*

$$\|F(f)\|_{H^s} \le C(1 + \|f\|_{L^\infty})^{[s]+1}\|f\|_{H^s},$$

where the constant C depends on $\sup_{k\le[s]+2, t\le\|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$.

3. Proof of Theorem 1.1

Based on the linearization, construction of approximating solutions and application of compactness argument, given the initial data $(a_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{B}_0) \in H^N(\mathbb{T}^3)$, the local-in-time well-posedness of the system (1.4) in the Sobolev setting $H^N(\mathbb{T}^3)$ with sufficiently large N can be shown following standard approaches (see, e.g., [14,22]). Therefore, we may assume that there exists $T > 0$ such that the system (1.4) has a unique solution $(a, \mathbf{u}, \mathbf{w}, \mathbf{B}) \in C([0, T]; H^N)$. Moreover, it holds that

$$\frac{1}{2}c_0 \le a(t,x) \le 2c_0^{-1}, \quad \text{for any } t \in [0,T]. \tag{3.1}$$

Therefore, by the standard bootstrapping argument (see, e.g., [30]), to prove Theorem 1.1, it suffices to derive the *a priori* estimates. To do this, we may assume that

$$\sup_{t\in[0,T]} \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N} \le \delta, \tag{3.2}$$

for some $0<\delta<1$ to be determined later.

In what follows, we divide the proof of Theorem 1.1 into five subsections, which we shall admit for the time being. Moreover, in order to show our proof ideas more clearly, we shall postpone the proof of Propositions 3.1-3.5 in the next section.

3.1. Basic energy estimate

Proposition 3.1. *Let $(a,\mathbf{u},\mathbf{w},\mathbf{B}) \in C([0,T];H^N)$ be a solution to the system (1.7). There holds the following basic energy inequality.*

$$\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{L^2}^2 + \frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + (\lambda+\mu-\zeta)\|\mathrm{div}\,\mathbf{u}\|_{L^2}^2 + \frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{L^2}^2 \le 0. \tag{3.3}$$

3.2. High-order energy estimate

Proposition 3.2. *Let $(a,\mathbf{u},\mathbf{w},\mathbf{B}) \in C([0,T];H^N)$ be a solution to the system (1.7). For any $0\le \ell \le N$, there holds*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^\ell}^2 &+ \frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^\ell}^2 + (\lambda+\mu-\zeta)\|\mathrm{div}\,\mathbf{u}\|_{H^\ell}^2 + \frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{H^\ell}^2\\ &\le C\Upsilon_\infty(t)\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^\ell}^2\end{aligned} \tag{3.4}$$

with

$$\begin{aligned}\Upsilon_\infty(t) \overset{\text{def}}{=} &(1+\|a\|_{L^\infty}^2)\|(a,\mathbf{w},\mathbf{B})\|_{L^\infty}^2 + (1+\|a\|_{L^\infty})\|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty}\\ &+ (1+\|(a,\nabla\mathbf{u},\mathbf{B})\|_{L^\infty}^2)\|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty}^2.\end{aligned} \tag{3.5}$$

3.3. The dissipation of the magnetic field B

Proposition 3.3. *Assume that*

$$\sup_{t\in[0,T]} \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N} \le \delta, \tag{3.6}$$

for some $0<\delta<1$. Then there holds that

$$\begin{aligned}&\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2-\sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx\\&\quad\le C\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\,\|\mathbf{w}\|_{H^{r+4}}^2+C\delta^2\,\|a+\mathbf{n}\cdot\mathbf{B}\|_{H^{r+4}}^2. \qquad (3.7)\end{aligned}$$

3.4. The dissipation of the density a

Let

$$d\stackrel{\text{def}}{=}a+\mathbf{n}\cdot\mathbf{B},\qquad\text{and}\quad\mathbf{G}\stackrel{\text{def}}{=}\mathbb{Q}\mathbf{u}-\frac{1}{\nu}\Delta^{-1}\nabla d,\quad\nu\stackrel{\text{def}}{=}\lambda+2\mu.$$

Proposition 3.4. *Let $(a,\mathbf{u},\mathbf{w},\mathbf{B})\in C([0,T];H^N)$ be a solution to the system (1.7). Assume that*

$$\sup_{t\in[0,T]}\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N}\le\delta, \qquad (3.8)$$

for some $0<\delta<1$. Then there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(d,\mathbf{G})\|_{H^{r+4}}^2+\frac{1}{\nu}\|d\|_{H^{r+4}}^2+\nu\,\|\nabla\mathbf{G}\|_{H^{r+4}}^2\\&\quad\le C\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2+C\delta^2\,\|(d,\mathbf{w})\|_{H^{r+4}}^2. \qquad (3.9)\end{aligned}$$

3.5. The derivation of the differential inequality for the energy

Let $\widetilde{c}>1$ and set

$$\mathcal{E}(t)=\widetilde{c}\Big(\|(a,\mathbf{u},\mathbf{w},\mathbf{B},d,\mathbf{G})\|_{H^{r+4}}^2\Big)-\sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx,$$

and

$$\begin{aligned}\mathcal{D}(t)=\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2&+\widetilde{c}\Big(\frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^{r+4}}^2+(\lambda+\mu-\zeta)\,\|\mathrm{div}\,\mathbf{u}\|_{H^{r+4}}^2\\&+\frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{H^{r+4}}^2+\frac{1}{\nu}\|d\|_{H^{r+4}}^2+\nu\,\|\nabla\mathbf{G}\|_{H^{r+4}}^2\Big).\end{aligned}$$

Proposition 3.5. *Let $(a,\mathbf{u},\mathbf{w},\mathbf{B})\in C([0,T];H^N)$ be a solution to the system (1.4) and $\widetilde{c}$ be a suitable large constant determined later. Assume that*

$$\sup_{t\in[0,T]}\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N}\le\delta, \qquad (3.10)$$

for some $0<\delta<1$. Then there holds

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)+\mathcal{D}(t)\le\,&C\delta(1+\delta^3)\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\\&+C\delta(1+\delta^3)\,\|(d,\mathbf{w})\|_{H^{r+4}}^2+C\delta^2(\delta^2+1)\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2. \qquad (3.11)\end{aligned}$$

With the above five propositions in hand, we now begin to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Thanks to $a = d - \mathbf{n}\cdot\mathbf{B}$ and

$$\Big|\sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx\Big| \le C\,\|\mathbf{u}\|_{H^{r+3}}\,\|\mathbf{B}\|_{H^{r+4}}\,, \tag{3.12}$$

we can take $\widetilde{c} > 1$ such that

$$\mathcal{E}(t) \ge \|(\mathbf{u},\mathbf{w},\mathbf{B},d,\mathbf{G})\|^2_{H^{r+4}}\,.$$

Hence, by choosing $\delta > 0$ small enough, we can get from (3.11) that

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{2}\mathcal{D}(t) \le 0. \tag{3.13}$$

For any $N \ge 4r+7$, we invoke the interpolation inequality and Lemmas 2.1, 2.2

$$\|\mathbf{B}\|^2_{H^{r+4}} \le \|\mathbf{B}\|^{\frac32}_{H^3}\,\|\mathbf{B}\|^{\frac12}_{H^N} \le C\delta^{\frac12}\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|^{\frac32}_{H^{r+3}}$$

to obtain

$$\begin{aligned}
\mathcal{E}(t) \le& C(\|(d,\mathbf{w})\|^2_{H^{r+4}} + \|(\mathbf{u},\mathbf{G})\|^2_{H^{r+4}} + \|\mathbf{B}\|^2_{H^{r+4}})\\
\le& C\,\|(d,\mathbf{w})\|^{\frac32}_{H^{r+4}}\,\|d\|^{\frac12}_{H^{r+4}} + C\,\|(\mathbf{u},\mathbf{G})\|^{\frac32}_{H^3}\,\|(\mathbf{u},\mathbf{G})\|^{\frac12}_{H^N} + C\,\|\mathbf{B}\|^{\frac32}_{H^3}\,\|\mathbf{B}\|^{\frac12}_{H^N}\\
\le& C\,\|(d,\mathbf{w})\|^{\frac32}_{H^{r+4}}\,\|(d,\mathbf{w})\|^{\frac12}_{H^N} + C\,\|(\mathbf{u},\mathbf{G})\|^{\frac32}_{H^3}\,\|(\mathbf{u},\mathbf{G})\|^{\frac12}_{H^N} + C\,\|\mathbf{B}\|^{\frac32}_{H^3}\,\|\mathbf{B}\|^{\frac12}_{H^N}\\
\le& C\delta^{\frac12}\,\|(d,\mathbf{w})\|^{\frac32}_{H^{r+4}} + C\delta^{\frac12}\,\|\nabla(\mathbf{u},\mathbf{G})\|^{\frac32}_{H^{r+4}} + C\delta^{\frac12}\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|^{\frac32}_{H^{r+3}}\\
\le& C(\mathcal{D}(t))^{\frac34}.
\end{aligned}$$

Inserting this inequality into (3.13) gives

$$\frac{d}{dt}\mathcal{E}(t) + c(\mathcal{E}(t))^{\frac43} \le 0.$$

It then follows easily that

$$\mathcal{E}(t) \le C(1+t)^{-3}. \tag{3.14}$$

Taking $\ell = N$ in (3.4) and using Sobolev's inequalities, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\,\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|^2_{H^N} + \frac{\mu}{2}\,\|\nabla\mathbf{u}\|^2_{H^N} + (\lambda+\mu-\zeta)\,\|\mathrm{div}\,\mathbf{u}\|^2_{H^N} + \frac{4\mu\zeta}{\mu+2\zeta}\,\|\mathbf{w}\|^2_{H^N}\\
&\le C Z(t)\,\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|^2_{H^N}
\end{aligned} \tag{3.15}$$

with

$$Z(t) \stackrel{\text{def}}{=} \|(d, \mathbf{u}, \mathbf{B})\|_{H^3} + \|(d, \mathbf{u}, \mathbf{B})\|_{H^3}^2 + \|d\|_{H^3}^2 \|\mathbf{B}\|_{H^3}^2 + \|\mathbf{B}\|_{H^3}^4 .$$

Clearly, the decay upper bound in (3.14) implies

$$\int_0^t Z(\tau)\, d\tau \le C.$$

Applying Gronwall's inequality to (3.15) yields

$$\|(a, \mathbf{u}, \mathbf{w}, \mathbf{B})\|_{H^N}^2 \le C \|(a_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{B}_0)\|_{H^N}^2 \le C\varepsilon^2.$$

By taking ε to be sufficiently small, say $\sqrt{C}\varepsilon \le \delta/2$, we obtain

$$\|(a, \mathbf{u}, \mathbf{w}, \mathbf{B})\|_{H^N} \le \frac{\delta}{2}.$$

The standard bootstrapping argument then implies that the local solution can be extended as a global one in time. Finally we prove the decay rate in (1.10). It follows from (3.14), that

$$\|(a, \mathbf{u}, \mathbf{w}, \mathbf{B})(t)\|_{H^{r+4}} \le C(1+t)^{-\frac{3}{2}}. \tag{3.16}$$

(1.10) is a consequence of (3.16) and the following interpolation inequality, for any $r+4 \le \beta < N$,

$$\|f(t)\|_{H^\beta} \le \|f(t)\|_{H^{r+4}}^{\frac{N-\beta}{N-r-4}} \|f(t)\|_{H^N}^{\frac{\beta-r-4}{N-r-4}} .$$

This completes the proof of Theorem 1.1. □

4. Proof of the propositions

The remaining part of the work is to prove Propositions 3.1–3.5. Without loss of generality, we here make the following assumption

$$\sup_{t\in\mathbb{R}_+,\, x\in\mathbb{T}^3} |a(t,x)| \le \frac{1}{2}. \tag{4.1}$$

Because of $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$, (4.1) is ensured by the fact that the solution constructed here has small norm in $H^2(\mathbb{T}^3)$. It then follows from Lemma 2.9 that the following composition estimate holds,

$$\|I(a)\|_{H^s} \le C\|a\|_{H^s}, \quad \text{for any } s > 0. \tag{4.2}$$

4.1. Proof of Proposition 3.1

Denote by $g(\rho)$ the potential energy density, namely

$$g(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{P(\tau) - P(\bar{\rho})}{\tau^2} \, d\tau.$$

For any fixed positive constant c_0, if $c_0 \le \rho \le c_0^{-1}$, then

$$g(\rho) \sim (\rho - \bar{\rho})^2.$$

The standard basic energy estimate gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} \left(2g(\rho) + \rho|\mathbf{u}|^2 + \rho|\mathbf{w}|^2 + |\mathbf{B}|^2\right) dx \\
&\quad + (\mu + \zeta)\, \|\nabla \mathbf{u}\|_{L^2}^2 + (\lambda + \mu - \zeta)\, \|\mathrm{div}\, \mathbf{u}\|_{L^2}^2 + 4\zeta\, \|\mathbf{w}\|_{L^2}^2 \\
&= \int_{\mathbb{T}^3} 2\zeta \nabla \times \mathbf{w} \cdot \mathbf{u} \, dx + \int_{\mathbb{T}^3} 2\zeta \nabla \times \mathbf{u} \cdot \mathbf{w} \, dx \\
&= \int_{\mathbb{T}^3} 4\zeta \nabla \times \mathbf{u} \cdot \mathbf{w} \, dx \\
&\lesssim \frac{\mu + 2\zeta}{2}\, \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{8\zeta^2}{\mu + 2\zeta}\, \|\mathbf{w}\|_{L^2}^2
\end{aligned} \tag{4.3}$$

where we have used the following cancellations

$$\int_{\mathbb{T}^3} (\mathbf{n} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\mathbb{T}^3} (\mathbf{n} \cdot \nabla \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{u}) \cdot \mathbf{B} \, dx = 0,$$

$$\int_{\mathbb{T}^3} (\mathbf{B}\nabla \mathbf{B} + \mathbf{n}\nabla \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\mathbb{T}^3} (\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{n}\mathrm{div}\, \mathbf{u} + \mathbf{B}\mathrm{div}\, \mathbf{u}) \cdot \mathbf{B} \, dx = 0.$$

Therefore, (4.3) further implies that

$$\frac{1}{2}\frac{d}{dt}\, \|(a, \mathbf{u}, \mathbf{w}, \mathbf{B})\|_{L^2}^2 + \frac{\mu}{2}\, \|\nabla \mathbf{u}\|_{L^2}^2 + (\lambda + \mu - \zeta)\, \|\mathrm{div}\, \mathbf{u}\|_{L^2}^2 + \frac{4\mu\zeta}{\mu + 2\zeta}\, \|\mathbf{w}\|_{L^2}^2 \le 0. \tag{4.4}$$

4.2. Proof of Proposition 3.2

(3.4) with $\ell = 0$ is the basic energy inequality in (4.4). Now we consider the case when $\ell \ge 1$. Writing $\Lambda = \sqrt{-\Delta}$ and applying Λ^s with $1 \le s \le \ell$ to (1.7) and then taking L^2 inner product with $(\Lambda^s a, \Lambda^s \mathbf{u}, \Lambda^s \mathbf{w}, \Lambda^s \mathbf{B})$ yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(\Lambda^s a, \Lambda^s \mathbf{u}, \Lambda^s \mathbf{w}, \Lambda^s \mathbf{B})\right\|_{L^2}^2 + 4\zeta \int\limits_{\mathbb{T}^3} \Lambda^s \mathbf{w}\cdot\Lambda^s \mathbf{w}\, dx \\
&\quad - \int\limits_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\bar{\mu}(\rho)\nabla \mathbf{u})\cdot \Lambda^s \mathbf{u}\, dx - \int\limits_{\mathbb{T}^3} \Lambda^s \nabla(\bar{\lambda}(\rho)\mathrm{div}\,\mathbf{u})\cdot\Lambda^s \mathbf{u}\, dx \\
&= \int\limits_{\mathbb{T}^3} \Lambda^s f_1\cdot\Lambda^s a\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s f_2\cdot\Lambda^s \mathbf{u}\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s f_3\cdot\Lambda^s \mathbf{w}\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s f_4\cdot\Lambda^s \mathbf{B}\, dx \\
&\quad + \int\limits_{\mathbb{T}^3} 2\zeta\Lambda^s\nabla\times\mathbf{w}\cdot\Lambda^s\mathbf{u}\, dx + \int\limits_{\mathbb{T}^3} 2\zeta\Lambda^s\nabla\times\mathbf{u}\cdot\Lambda^s\mathbf{w}\, dx, \qquad (4.5)
\end{aligned}$$

where we have used the following cancellations

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,\mathbf{u}\cdot\Lambda^s a\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s\nabla a\cdot\Lambda^s\mathbf{u}\, dx = 0, \\
&\int\limits_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\cdot\Lambda^s\mathbf{u}\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\cdot\Lambda^s\mathbf{B}\, dx = 0, \\
&\int\limits_{\mathbb{T}^3} \Lambda^s\nabla(\mathbf{n}\cdot\mathbf{B})\cdot\Lambda^s\mathbf{u}\, dx + \int\limits_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\mathrm{div}\,\mathbf{u})\cdot\Lambda^s\mathbf{B}\, dx = 0.
\end{aligned}$$

The second term on the left-hand side of (4.5) can be written as

$$\begin{aligned}
&- \int\limits_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\bar{\mu}(\rho)\nabla \mathbf{u})\cdot \Lambda^s \mathbf{u}\, dx \\
&= \int\limits_{\mathbb{T}^3} \Lambda^s(\bar{\mu}(\rho)\nabla\mathbf{u})\cdot\nabla\Lambda^s\mathbf{u}\, dx \\
&= \int\limits_{\mathbb{T}^3} \bar{\mu}(\rho)\nabla\Lambda^s\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\, dx + \int\limits_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho)]\nabla\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\, dx. \qquad (4.6)
\end{aligned}$$

Due to (3.1), we have for any $t \in [0, T]$ that

$$\int\limits_{\mathbb{T}^3} \bar{\mu}(\rho)\nabla\Lambda^s\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\, dx \geq c_0^{-1}\mu\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2. \qquad (4.7)$$

For the last term in (4.6), we first rewrite this term into

$$\int\limits_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho)]\nabla\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\, dx = \int\limits_{\mathbb{T}^3} [\Lambda^s, \bar{\mu}(\rho) - \mu + \mu]\nabla\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\, dx$$

$$= -\int_{\mathbb{T}^3} [\Lambda^s, \mu I(a)]\nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u}\, dx.$$

Then, with the aid of (4.2), Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{T}^3} [\Lambda^s, \mu I(a)]\nabla \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u}\, dx\Big| \\
&\quad \le C \left\|\nabla \Lambda^s \mathbf{u}\right\|_{L^2} (\|\nabla I(a)\|_{L^\infty} \left\|\Lambda^s \mathbf{u}\right\|_{L^2} + \|\nabla \mathbf{u}\|_{L^\infty} \left\|\Lambda^s I(a)\right\|_{L^2}) \\
&\quad \le \frac{c_0^{-1}}{2}\mu \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|\nabla a\|_{L^\infty}^2 \left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty}^2 \left\|\Lambda^s a\right\|_{L^2}^2\big).
\end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6) leads to

$$\begin{aligned}
-\int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\bar{\mu}(\rho)\nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u}\, dx \ge& \frac{c_0^{-1}}{2}\mu \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 \\
&- C\big(\|\nabla a\|_{L^\infty}^2 \left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty}^2 \left\|\Lambda^s a\right\|_{L^2}^2\big).
\end{aligned}$$

The third term on the left-hand side of (4.5) can be dealt with similarly. Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(\Lambda^s a, \Lambda^s \mathbf{u}, \Lambda^s \mathbf{B})\right\|_{L^2}^2 + c_0^{-1}(\mu+\zeta)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 \\
&\quad + c_0^{-1}(\lambda+\mu-\zeta)\left\|\Lambda^s \mathrm{div}\mathbf{u}\right\|_{L^2}^2 + 4\zeta\|\Lambda^s \mathbf{w}\|_{L^2}^2 \\
&\le C\big(\|\nabla a\|_{L^\infty}^2 \left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty}^2 \left\|\Lambda^s a\right\|_{L^2}^2\big) \\
&\quad + \int_{\mathbb{T}^3} \Lambda^s f_1 \cdot \Lambda^s a\, dx + \int_{\mathbb{T}^3} \Lambda^s f_2 \cdot \Lambda^s \mathbf{u}\, dx + \int_{\mathbb{T}^3} \Lambda^s f_3 \cdot \Lambda^s \mathbf{w}\, dx \\
&\quad + \int_{\mathbb{T}^3} \Lambda^s f_4 \cdot \Lambda^s \mathbf{B}\, dx + \int_{\mathbb{T}^3} 2\zeta \Lambda^s \nabla\times\mathbf{w} \cdot \Lambda^s \mathbf{u}\, dx + \int_{\mathbb{T}^3} 2\zeta \Lambda^s \nabla\times\mathbf{u} \cdot \Lambda^s \mathbf{w}\, dx.
\end{aligned} \tag{4.9}$$

We now estimate successively terms on the right hand side of (4.9). To bound the first term in f_1, we rewrite it into

$$\begin{aligned}
\int_{\mathbb{T}^3} \Lambda^s(\mathbf{u}\cdot\nabla a)\cdot\Lambda^s a\, dx =& \int_{\mathbb{T}^3} (\Lambda^s(\mathbf{u}\cdot\nabla a) - \mathbf{u}\cdot\nabla\Lambda^s a)\cdot\Lambda^s a\, dx + \int_{\mathbb{T}^3} \mathbf{u}\cdot\nabla\Lambda^s a\cdot\Lambda^s a\, dx \\
&\overset{\mathrm{def}}{=} A_1 + A_2.
\end{aligned} \tag{4.10}$$

By Lemma 2.8,

$$\begin{aligned} A_1 \leq & C \left\| [\Lambda^s, \mathbf{u} \cdot \nabla] a \right\|_{L^2} \left\| \Lambda^s a \right\|_{L^2} \\ \leq & C (\| \nabla \mathbf{u} \|_{L^\infty} \left\| \Lambda^s a \right\|_{L^2} + \left\| \Lambda^s \mathbf{u} \right\|_{L^2} \| \nabla a \|_{L^\infty}) \left\| \Lambda^s a \right\|_{L^2} \\ \leq & C (\| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla a \|_{L^\infty}) (\left\| \Lambda^s a \right\|_{L^2}^2 + \left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2). \end{aligned} \tag{4.11}$$

By integration by parts,

$$A_2 \leq C \| \nabla \mathbf{u} \|_{L^\infty} \left\| \Lambda^s a \right\|_{L^2}^2 . \tag{4.12}$$

For the second term in f_1, it follows from Lemma 2.7 that

$$\begin{aligned} \int_{\mathbb{T}^3} \Lambda^s (a \operatorname{div} \mathbf{u}) \cdot \Lambda^s a \, dx \leq & C\big(\| \operatorname{div} \mathbf{u} \|_{L^\infty} \| a \|_{H^s} + \| \operatorname{div} \mathbf{u} \|_{H^s} \| a \|_{L^\infty} \big) \left\| \Lambda^s a \right\|_{L^2} \\ \leq & \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 + C (\| \nabla \mathbf{u} \|_{L^\infty} + \| a \|_{L^\infty}^2) \left\| \Lambda^s a \right\|_{L^2}^2 , \end{aligned} \tag{4.13}$$

where we have used the inequalities

$$\| a \|_{H^s} \leq C \left\| \Lambda^s a \right\|_{L^2} \quad \text{and} \quad \| \operatorname{div} \mathbf{u} \|_{H^s} \leq C \| \Lambda^{s+1} \mathbf{u} \|_{L^2}.$$

Collecting (4.11), (4.12) and (4.13), we can get

$$\begin{aligned} \int_{\mathbb{T}^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \leq & \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 \\ & + C (\| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla a \|_{L^\infty} + \| a \|_{L^\infty}^2) (\left\| \Lambda^s a \right\|_{L^2}^2 + \left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2). \end{aligned} \tag{4.14}$$

For the first term in f_4, a similar process as in (4.11) and (4.12) yields

$$\int_{\mathbb{T}^3} \Lambda^s (\mathbf{u} \cdot \nabla \mathbf{B}) \cdot \Lambda^s \mathbf{B} \, dx \leq C (\| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla \mathbf{B} \|_{L^\infty}) (\left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2 + \left\| \Lambda^s \mathbf{B} \right\|_{L^2}^2).$$

For the last two terms in f_4, a derivation similar to (4.13) gives

$$\int_{\mathbb{T}^3} \Lambda^s (\mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \operatorname{div} \mathbf{u}) \cdot \Lambda^s \mathbf{B} \, dx \leq \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 + C (\| \nabla \mathbf{u} \|_{L^\infty} + \| \mathbf{B} \|_{L^\infty}^2) \left\| \Lambda^s \mathbf{B} \right\|_{L^2}^2 .$$

Therefore,

$$\begin{aligned} \int_{\mathbb{T}^3} \Lambda^s f_4 \cdot \Lambda^s \mathbf{B} \, dx \leq & \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 \\ & + C (\| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla \mathbf{B} \|_{L^\infty} + \| \mathbf{B} \|_{L^\infty}^2) (\left\| \Lambda^s \mathbf{u} \right\|_{L^2}^2 + \left\| \Lambda^s \mathbf{B} \right\|_{L^2}^2). \end{aligned} \tag{4.15}$$

In the following, we bound the terms in f_2. To do so, we write

$$\int_{\mathbb{T}^3} \Lambda^s f_2 \cdot \Lambda^s \mathbf{u}\, dx = \sum_{i=3}^{10} A_i \tag{4.16}$$

with

$$\begin{aligned}
A_3 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Lambda^s\mathbf{u}\, dx,\\
A_4 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(\mathbf{B}\cdot\nabla\mathbf{B})\cdot\Lambda^s\mathbf{u}\, dx,\\
A_5 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(k(a)\nabla a)\cdot\Lambda^s\mathbf{u}\, dx,\\
A_6 &\stackrel{\text{def}}{=} 2\int_{\mathbb{T}^3} \Lambda^s(\zeta I(a)\nabla\times\mathbf{w})\cdot\Lambda^s\mathbf{u}\, dx,\\
A_7 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s((\mu+\zeta)(\nabla I(a))\nabla\mathbf{u})\cdot\Lambda^s\mathbf{u}\, dx,\\
A_8 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s((\lambda+\mu-\zeta)(\nabla I(a))\mathrm{div}\,\mathbf{u})\cdot\Lambda^s\mathbf{u}\, dx,\\
A_9 &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(I(a)(\mathbf{n}\cdot\nabla\mathbf{B}-\mathbf{n}\nabla\mathbf{B}))\cdot\Lambda^s\mathbf{u}\, dx,\\
A_{10} &\stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \Lambda^s(I(a)(\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\nabla\mathbf{B}))\cdot\Lambda^s\mathbf{u}\, dx.
\end{aligned}$$

The term A_3 can be bounded as in (4.10) to get

$$A_3 \le C\,\|\nabla\mathbf{u}\|_{L^\infty}\,\left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2.$$

We next deal with the term A_4. In view of $\mathrm{div}\,\mathbf{B}=0$, one can write

$$\begin{aligned}
A_4 &= \int_{\mathbb{T}^3} \Lambda^s \mathrm{div}\,(\mathbf{B}\otimes\mathbf{B})\cdot\Lambda^s\mathbf{u}\, dx\\
&\le C\|\mathbf{B}\|_{L^\infty}\|\mathbf{B}\|_{H^s}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}\\
&\le \frac{c_0^{-1}\mu}{16}\|\Lambda^{s+1}\mathbf{u}\|_{L^2}^2 + C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^s}^2.
\end{aligned}$$

It follows from Lemma 2.7 and (4.2), that

$$
\begin{aligned}
A_5 \leq & C\big(\|\nabla a\|_{L^\infty}\|k(a)\|_{H^{s-1}} + \|\nabla a\|_{H^{s-1}}\|k(a)\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} \\
\leq & \frac{c_0^{-1}\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C(\|\nabla a\|_{L^\infty}^2 + \|a\|_{L^\infty}^2)\|a\|_{H^s}^2,
\end{aligned}
$$

where we have used the fact that $\|k(a)\|_{L^\infty} \leq \|a\|_{L^\infty}$. Similarly,

$$
\begin{aligned}
A_6 = & 2\zeta \int_{\mathbb{T}^3} \Lambda^s(I(a)\nabla\times\mathbf{w})\cdot\Lambda^s\mathbf{u}\,dx \\
\leq & C\big(\|I(a)\|_{L^\infty}\|\nabla\times\mathbf{w}\|_{H^{s-1}} + \|I(a)\|_{H^{s-1}}\|\nabla\times\mathbf{w}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} \\
\leq & C\big(\|a\|_{L^\infty}\|\nabla\mathbf{w}\|_{H^{s-1}} + \|a\|_{H^{s-1}}\|\nabla\times\mathbf{w}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} \\
\leq & C\|a\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^{s+1}}\|\mathbf{w}\|_{H^s} + C\|\nabla\mathbf{w}\|_{L^\infty}\|(a,\mathbf{w})\|_{H^s}^2 \\
\leq & \frac{c_0^{-1}\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|a\|_{L^\infty}^2\|\mathbf{w}\|_{H^s}^2 + \|\nabla\mathbf{w}\|_{L^\infty}^2\|a\|_{H^s}^2\big), \\
A_7 + A_8 \leq & C\big(\|\nabla I(a)\|_{L^\infty}\left\|\Lambda^s\mathbf{u}\right\|_{L^2} + \|\nabla I(a)\|_{H^{s-1}}\|\nabla\mathbf{u}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} \\
\leq & \frac{c_0^{-1}\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|\nabla a\|_{L^\infty}^2\left\|\Lambda^s\mathbf{u}\right\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2\|a\|_{H^s}^2\big),
\end{aligned}
$$

and

$$
\begin{aligned}
A_9 \leq & C\big(\|I(a)\|_{L^\infty}\|\nabla\mathbf{B}\|_{H^{s-1}} + \|I(a)\|_{H^{s-1}}\|\nabla\mathbf{B}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2} \\
\leq & \frac{c_0^{-1}\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|a\|_{L^\infty}^2\|\mathbf{B}\|_{H^s}^2 + \|\nabla\mathbf{B}\|_{L^\infty}^2\|a\|_{H^s}^2\big).
\end{aligned} \tag{4.17}
$$

For the last term A_{10}, we use Lemmas 2.7 and 2.9 again to get

$$
A_{10} \leq C\big(\|I(a)\|_{L^\infty}\|\mathbf{B}\nabla\mathbf{B}\|_{H^{s-1}} + \|I(a)\|_{H^{s-1}}\|\mathbf{B}\nabla\mathbf{B}\|_{L^\infty}\big)\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}. \tag{4.18}
$$

Due to

$$
\|\mathbf{B}\nabla\mathbf{B}\|_{H^{s-1}} \leq C\|\mathbf{B}\|_{L^\infty}\|\mathbf{B}\|_{H^s},
$$

which, together with (4.18), leads to

$$
A_{10} \leq \frac{c_0^{-1}\mu}{16}\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|a\|_{L^\infty}^2\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^s}^2 + \|\mathbf{B}\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{L^\infty}^2\|a\|_{H^s}^2\big).
$$

Inserting the bounds from A_3 to A_{10} in (4.16), we get

$$\int_{\mathbb{T}^3} \Lambda^s f_2 \cdot \Lambda^s \mathbf{u}\, dx \leq \frac{5c_0^{-1}\mu}{16} \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2 + C\big(\|\nabla \mathbf{u}\|_{L^\infty} + \|(\nabla a, \nabla \mathbf{u}, \nabla \mathbf{B})\|_{L^\infty}^2 + \|(a,\mathbf{B})\|_{L^\infty}^2 + \|a\|_{L^\infty}^2 \|\mathbf{B}\|_{L^\infty}^2 + \|\mathbf{B}\|_{L^\infty}^2 \|\nabla \mathbf{B}\|_{L^\infty}^2 \big) \|(a,\mathbf{u},\mathbf{B})\|_{H^s}^2. \tag{4.19}$$

For the first term in f_3, a similar process as in (4.11) and (4.12) yields

$$\int_{\mathbb{T}^3} \Lambda^s (\mathbf{u}\cdot\nabla\mathbf{w}) \cdot \Lambda^s \mathbf{w}\, dx \leq C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{w}\|_{L^\infty})(\left\|\Lambda^s \mathbf{u}\right\|_{L^2}^2 + \left\|\Lambda^s \mathbf{w}\right\|_{L^2}^2). \tag{4.20}$$

For the second term in f_3, we have

$$\begin{aligned}\int_{\mathbb{T}^3} \Lambda^s (4\zeta I(a)\mathbf{w}) \cdot \Lambda^s \mathbf{w}\, dx &\leq C\big(\|I(a)\|_{L^\infty}\|\mathbf{w}\|_{H^s} + \|I(a)\|_{H^s}\|\mathbf{w}\|_{L^\infty}\big) \left\|\Lambda^s \mathbf{w}\right\|_{L^2} \\ &\leq C\big(\|a\|_{L^\infty}\|\mathbf{w}\|_{H^s} + \|I(a)\|_{H^s}\|\mathbf{w}\|_{L^\infty}\big) \left\|\Lambda^s \mathbf{w}\right\|_{L^2} \\ &\leq C\|a\|_{L^\infty}\|\mathbf{w}\|_{H^s}^2 + C\|\mathbf{w}\|_{L^\infty}\|(a,\mathbf{w})\|_{H^s}^2.\end{aligned} \tag{4.21}$$

Similarly

$$\begin{aligned}\int_{\mathbb{T}^3} \Lambda^s (I(a)\nabla\times\mathbf{u}) \cdot \Lambda^s \mathbf{w}\, dx &\leq C\big(\|I(a)\|_{L^\infty}\|\nabla\times\mathbf{u}\|_{H^s} + \|I(a)\|_{H^s}\|\nabla\times\mathbf{u}\|_{L^\infty}\big) \left\|\Lambda^s \mathbf{w}\right\|_{L^2} \\ &\leq C\big(\|a\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^s} + \|a\|_{H^s}\|\nabla\times\mathbf{u}\|_{L^\infty}\big) \left\|\Lambda^s \mathbf{w}\right\|_{L^2} \\ &\leq C\|a\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^s}\|\mathbf{w}\|_{H^s} + C\|\nabla\mathbf{u}\|_{L^\infty}\|(a,\mathbf{w})\|_{H^s}^2 \\ &\leq \varepsilon\|\nabla\mathbf{u}\|_{H^s}^2 + C\|a\|_{L^\infty}^2\|\mathbf{w}\|_{H^s}^2 + C\|\nabla\mathbf{u}\|_{L^\infty}\|(a,\mathbf{w})\|_{H^s}^2.\end{aligned} \tag{4.22}$$

For the last two terms on the right-hand side of (4.5), according to the integrating by parts, Hölder's and Young's inequalities, we have

$$\int_{\mathbb{T}^3} 2\zeta\Lambda^s \nabla\times\mathbf{w}\cdot\Lambda^s\mathbf{u}\, dx + \int_{\mathbb{T}^3} 2\zeta\Lambda^s\nabla\times\mathbf{u}\cdot\Lambda^s\mathbf{w}\, dx \lesssim \frac{\mu+2\zeta}{2}\left\|\nabla\Lambda^s\mathbf{u}\right\|_{L^2}^2 + \frac{8\zeta^2}{\mu+2\zeta}\left\|\Lambda^s\mathbf{w}\right\|_{L^2}^2. \tag{4.23}$$

Plugging (4.14), (4.15), (4.19), (4.20), (4.21), (4.22) and (4.23) into (4.9) and combining with basic energy inequality (4.4), we can arrive at the desired estimate (3.4) by summing up for any $1 \leq s \leq \ell$.

4.3. Proof of Proposition 3.3

First, we rewrite $(1.7)_2$ as follows

$$\partial_t \mathbf{u} - (\mu+\zeta)\Delta\mathbf{u} - (\lambda+\mu-\zeta)\nabla \mathrm{div}\mathbf{u} + \nabla a = 2\zeta\nabla\times\mathbf{w} + \mathbf{n}\cdot\nabla\mathbf{B} - \nabla(\mathbf{n}\cdot\mathbf{B}) + \widetilde{f_2}, \tag{4.24}$$

where $\widetilde{f_2}$ is slightly different from f_2, namely

$$\widetilde{f_2} \stackrel{\text{def}}{=} -\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{B}\cdot\nabla\mathbf{B}+\mathbf{B}\nabla\mathbf{B}+k(a)\nabla a+2\zeta I(a)\nabla\times\mathbf{w}$$
$$-I(a)((\mu+\zeta)\Delta\mathbf{u}+(\lambda+\mu-\zeta)\nabla\operatorname{div}\mathbf{u})-I(a)(\mathbf{n}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{n}\nabla\mathbf{B}+\mathbf{B}\nabla\mathbf{B}). \tag{4.25}$$

Applying the standard Leray projector operator $\mathbb{P}$ to (4.24) gives

$$\partial_t\mathbb{P}\mathbf{u}-(\mu+\zeta)\Delta\mathbb{P}\mathbf{u}=2\zeta\mathbb{P}\nabla\times\mathbf{w}+\mathbf{n}\cdot\nabla\mathbf{B}+\mathbb{P}\widetilde{f_2}, \tag{4.26}$$

where we have used $\mathbb{P}(\mathbf{n}\cdot\nabla\mathbf{B})=\mathbf{n}\cdot\nabla\mathbf{B}$, $\mathbb{P}(\nabla(\mathbf{n}\cdot\mathbf{B}))=0$.

Applying $\Lambda^s(0\le s\le r+3)$ to (4.26), and taking the L^2-inner product with $\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})$, yield that

$$\begin{aligned}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\right\|_{L^2}^2=&\int_{\mathbb{T}^3}\Lambda^s\partial_t\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx+2\zeta\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\nabla\times\mathbf{w}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx\\&-(\mu+\zeta)\int_{\mathbb{T}^3}\Lambda^s\Delta\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx-\int_{\mathbb{T}^3}\Lambda^s(\mathbb{P}\widetilde{f_2})\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx.\end{aligned} \tag{4.27}$$

By Hölder's inequality,

$$\begin{aligned}2\zeta\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\nabla\times\mathbf{w}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx&\le\frac{1}{8}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\right\|_{L^2}^2+C\left\|\Lambda^{s+1}\mathbf{w}\right\|_{L^2}^2,\\(\mu+\zeta)\int_{\mathbb{T}^3}\Lambda^s\Delta\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx&\le\frac{1}{8}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\right\|_{L^2}^2+C\left\|\Lambda^{s+2}\mathbf{u}\right\|_{L^2}^2,\\\int_{\mathbb{T}^3}\Lambda^s(\mathbb{P}\widetilde{f_2})\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx&\le\frac{1}{8}\left\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\right\|_{L^2}^2+C\left\|\Lambda^s\widetilde{f_2}\right\|_{L^2}^2.\end{aligned}$$

In what follows, we shift the time derivative in the first term on the right-hand side of (4.27) and use the fourth equation in (1.7) to get

$$\begin{aligned}&\int_{\mathbb{T}^3}\Lambda^s\partial_t\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx\\&=\frac{d}{dt}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx-\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\partial_t\mathbf{B})\,dx\\&=\frac{d}{dt}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx+\int_{\mathbb{T}^3}\Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s\partial_t\mathbf{B}\,dx\\&=\frac{d}{dt}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx+\int_{\mathbb{T}^3}\Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\,dx\end{aligned}$$

$$-\int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s(\mathbf{n}\mathrm{div}\,\mathbf{u})\,dx + \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s f_4\,dx. \tag{4.28}$$

The last three terms in (4.28) can be bounded by

$$\int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\,dx - \int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s(\mathbf{n}\mathrm{div}\,\mathbf{u})\,dx \le C\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2,$$

and

$$\int_{\mathbb{T}^3} \Lambda^s(\mathbf{n}\cdot\nabla\mathbb{P}\mathbf{u})\cdot\Lambda^s f_4\,dx \le C(\|\Lambda^s f_4\|_{L^2}^2 + \left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2).$$

Collecting the estimates above, we can infer from (4.27) that

$$\begin{aligned} &\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\|_{L^2}^2 - \frac{d}{dt}\sum_{0\le s\le r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx \\ &\le C(\left\|\Lambda^{s+2}\mathbf{u}\right\|_{L^2}^2 + \left\|\Lambda^{s+1}\mathbf{w}\right\|_{L^2}^2 + \|\Lambda^s\widetilde{f_2}\|_{L^2}^2 + \|\Lambda^s f_4\|_{L^2}^2). \end{aligned} \tag{4.29}$$

It follows from Lemma 2.7 and (3.6), that

$$\begin{aligned} \|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{B})\|_{L^2}^2 \le& C\big(\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{H^s}^2 + \|\mathbf{u}\|_{H^s}^2\|\nabla\mathbf{B}\|_{L^\infty}^2\big) \\ \le& C\big(\|\mathbf{u}\|_{H^3}^2\|\mathbf{B}\|_{H^N}^2 + \|\mathbf{u}\|_{H^N}^2\|\mathbf{B}\|_{H^3}^2\big) \\ \le& C\delta^2(\|\mathbf{u}\|_{H^3}^2 + \|\mathbf{B}\|_{H^3}^2). \end{aligned} \tag{4.30}$$

Similarly,

$$\|\Lambda^s(\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{B}\mathrm{div}\,\mathbf{u})\|_{L^2}^2 \le C\delta^2(\|\mathbf{u}\|_{H^3}^2 + \|\mathbf{B}\|_{H^3}^2).$$

Moreover, we conclude, by Lemma 2.1, that

$$\|\mathbf{B}\|_{H^3}^2 \le C\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2,$$

from which we get

$$\|\Lambda^s f_4\|_{L^2}^2 \le C\delta^2\,\|\mathbf{u}\|_{H^3}^2 + C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2. \tag{4.31}$$

We now deal with the terms in $\widetilde{f_2}$. The term $\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}^2$ can be bounded as in (4.30),

$$\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}^2 \le C\delta^2\left\|\Lambda^{s+1}\mathbf{u}\right\|_{L^2}^2. \tag{4.32}$$

Thanks to Lemma 2.7 again, we have

$$\begin{aligned}\left\|\Lambda^s(\mathbf{B}\cdot\nabla\mathbf{B}+\mathbf{B}\nabla\mathbf{B})\right\|_{L^2}^2 \leq & C(\|\mathbf{B}\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{H^s}^2+\|\nabla\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^s}^2)\\ \leq & C(\|\mathbf{B}\|_{H^2}^2\|\mathbf{B}\|_{H^{s+1}}^2+\|\nabla\mathbf{B}\|_{H^2}^2\|\mathbf{B}\|_{H^s}^2)\\ \leq & C\|\mathbf{B}\|_{H^{s+1}}^2\|\mathbf{B}\|_{H^3}^2\\ \leq & C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2.\end{aligned}\tag{4.33}$$

The term $\mathbb{P}(k(a)\nabla a)=0$ since $k(a)\nabla a$ can be written as a gradient. It then follows from Lemma 2.7, that

$$\begin{aligned}\left\|\Lambda^s(I(a)\Delta\mathbf{u})\right\|_{L^2}^2 \leq & C(\|I(a)\|_{L^\infty}^2\|\Delta\mathbf{u}\|_{H^s}^2+\|\Delta\mathbf{u}\|_{L^\infty}^2\|I(a)\|_{H^s}^2)\\ \leq & C(\|a\|_{H^2}^2\|\mathbf{u}\|_{H^{s+2}}^2+\|\mathbf{u}\|_{H^4}^2\|a\|_{H^s}^2)\\ \leq & C\delta^2\|\mathbf{u}\|_{H^{s+2}}^2+C\delta^2\|\mathbf{u}\|_{H^4}^2,\end{aligned}\tag{4.34}$$

and

$$\begin{aligned}\left\|\Lambda^s(I(a)\nabla\times\mathbf{w})\right\|_{L^2}^2 \leq & C(\|I(a)\|_{L^\infty}^2\|\nabla\times\mathbf{w}\|_{H^s}^2+\|\nabla\times\mathbf{w}\|_{L^\infty}^2\|I(a)\|_{H^s}^2)\\ \leq & C(\|a\|_{H^2}^2\|\mathbf{w}\|_{H^{s+1}}^2+\|\mathbf{w}\|_{H^4}^2\|a\|_{H^s}^2)\\ \leq & C\delta^2\|\mathbf{w}\|_{H^{r+4}}^2.\end{aligned}\tag{4.35}$$

The term $I(a)\nabla\mathrm{div}\,\mathbf{u}$ can be dealt with similarly. The last term in $\widetilde{f_2}$ can be bounded as in (4.17), (4.18) and (4.33) to get

$$\begin{aligned}\left\|\Lambda^s(I(a)\mathbf{n}\nabla\mathbf{B})\right\|_{L^2}^2 \leq & C(\|I(a)\|_{L^\infty}^2\|\mathbf{n}\nabla\mathbf{B}\|_{H^s}^2+\|\mathbf{n}\nabla\mathbf{B}\|_{L^\infty}^2\|I(a)\|_{H^s}^2)\\ \leq & C(\|a\|_{H^3}^2\|\mathbf{n}\nabla\mathbf{B}\|_{H^s}^2+\|\mathbf{B}\|_{H^3}^2\|a\|_{H^s}^2)\\ \leq & C(\|a+\mathbf{n}\cdot\mathbf{B}-\mathbf{n}\cdot\mathbf{B}\|_{H^3}^2\|\mathbf{B}\|_{H^N}^2+\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\|a\|_{H^N}^2)\\ \leq & C((\|a+\mathbf{n}\cdot\mathbf{B}\|_{H^3}^2+\|\mathbf{B}\|_{H^3}^2)\|\mathbf{B}\|_{H^N}^2+\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\|a\|_{H^N}^2)\\ \leq & C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2,\end{aligned}\tag{4.36}$$

and

$$\left\|\Lambda^s(I(a)(\mathbf{n}\cdot\nabla\mathbf{B})\right\|_{L^2}^2+\left\|\Lambda^s(I(a)(\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\nabla\mathbf{B})\right\|_{L^2}^2\leq C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2.\tag{4.37}$$

Combining with (4.32), (4.33), (4.34), (4.35) and (4.37) gives rise to

$$\left\|\Lambda^s\widetilde{f_2}\right\|_{L^2}^2\leq C\delta^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\|\mathbf{w}\|_{H^{r+4}}^2+C\delta^2\|a+\mathbf{n}\cdot\mathbf{B}\|_{H^{r+3}}^2.\tag{4.38}$$

Finally, inserting (4.31) and (4.38) in (4.29) and taking δ small enough, we deduce that (3.7) holds.

4.4. Proof of Proposition 3.4

The equations of a and $\mathbf{B}$ in (1.7) do not contain any dissipative or damping terms. But we do need these stabilizing effects in order to prove the desired stability results. This subsection explores the structure of (1.7) and discovers that the equation of the combined quantity

$$a + \mathbf{n} \cdot \mathbf{B}$$

and the equation of the gradient part $\mathbb{Q}\mathbf{u}$ of $\mathbf{u}$ form a system with smoothing and stabilizing effects. Combining the bound for $a + \mathbf{n} \cdot \mathbf{B}$ and $\mathbf{n} \cdot \nabla\mathbf{B}$ allows us to control a. For that, applying the operator $\mathbb{Q}$ to (4.24) yields

$$\partial_t \mathbb{Q}\mathbf{u} - \nu\Delta\mathbb{Q}\mathbf{u} + \nabla a + \nabla(\mathbf{n} \cdot \mathbf{B}) = \mathbb{Q}\widetilde{f_2}, \tag{4.39}$$

where $\mathbb{Q} = I - \mathbb{Q}$, $\nu \stackrel{\text{def}}{=} \lambda + 2\mu$, and we have used $\mathbb{Q}(\mathbf{n} \cdot \nabla\mathbf{B}) = 0$, $\mathbb{Q}(\nabla(\mathbf{n} \cdot \mathbf{B})) = \nabla(\mathbf{n} \cdot \mathbf{B})$. Now we consider the following system including a, $\mathbb{Q}\mathbf{u}$ and $\mathbf{B}$,

$$\begin{cases} \partial_t a + \operatorname{div}\mathbf{u} = f_1, \\ \partial_t \mathbb{Q}\mathbf{u} - \nu\Delta\mathbb{Q}\mathbf{u} + \nabla a + \nabla(\mathbf{n} \cdot \mathbf{B}) = \mathbb{Q}\widetilde{f_2}, \\ \partial_t \mathbf{B} = \mathbf{n} \cdot \nabla\mathbf{u} - \mathbf{n}\operatorname{div}\mathbf{u} + f_4, \end{cases} \tag{4.40}$$

and introduce a new auxiliary function $\mathbf{G} \stackrel{\text{def}}{=} \mathbb{Q}\mathbf{u} - \frac{1}{\nu}\Delta^{-1}\nabla(a + \mathbf{n} \cdot \mathbf{B})$. In order to get the equation of $\mathbf{G}$, it follows from the third equation in (4.40) that

$$\partial_t(\mathbf{n} \cdot \mathbf{B}) = \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{n}\operatorname{div}\mathbf{u} + f_4 \cdot \mathbf{n},$$

which together with the equation of a in (4.40) implies that

$$\partial_t(a + \mathbf{n} \cdot \mathbf{B}) = \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n} - (|\mathbf{n}|^2 + 1)\operatorname{div}\mathbf{u} + f_1 + f_4 \cdot \mathbf{n}. \tag{4.41}$$

Let $d \stackrel{\text{def}}{=} a + \mathbf{n} \cdot \mathbf{B}$, by the definition of $\mathbb{Q} = \nabla\Delta^{-1}\operatorname{div}$, we note that $\operatorname{div}\mathbf{u} = \operatorname{div}\mathbb{Q}\mathbf{u} = \operatorname{div}\mathbf{G} + \frac{1}{\nu}d$. Hence, we conclude that $(d, \mathbf{G})$ satisfy the following system

$$\begin{cases} \partial_t d + \dfrac{1}{\nu}(|\mathbf{n}|^2 + 1)d + (|\mathbf{n}|^2 + 1)\operatorname{div}\mathbf{G} = \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n} + f_1 + f_4 \cdot \mathbf{n}, \\ \partial_t \mathbf{G} - \nu\Delta\mathbf{G} = \dfrac{1}{\nu}(|\mathbf{n}|^2 + 1)\mathbb{Q}\mathbf{u} - \dfrac{1}{\nu}\Delta^{-1}\nabla(\mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n}) + \mathbb{Q}\widetilde{f_2} - \dfrac{1}{\nu}\Delta^{-1}\nabla(f_1 + f_4 \cdot \mathbf{n}). \end{cases} \tag{4.42}$$

Applying Λ^m $(m \geq 0)$ to the first equation in (4.42), and multiplying it by $\Lambda^m d$ lead to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\Lambda^m d\right\|_{L^2}^2 + \frac{1}{\nu}(|\mathbf{n}|^2 + 1)\left\|\Lambda^m d\right\|_{L^2}^2 = \int_{\mathbb{T}^3} \Lambda^m(\mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n}) \cdot \Lambda^m d\, dx \\ - (|\mathbf{n}|^2 + 1)\int_{\mathbb{T}^3} \Lambda^m\operatorname{div}\mathbf{G} \cdot \Lambda^m d\, dx + \int_{\mathbb{T}^3} \Lambda^m(f_1 + f_4 \cdot \mathbf{n}) \cdot \Lambda^m d\, dx \end{aligned}$$

$$\le C(\left\|\Lambda^m \nabla \mathbf{u}\right\|_{L^2}\left\|\Lambda^m d\right\|_{L^2}+\left\|\Lambda^m \operatorname{div} \mathbf{G}\right\|_{L^2}\left\|\Lambda^m d\right\|_{L^2}+\int_{\mathbb{T}^3} \Lambda^m(f_1+f_4\cdot \mathbf{n})\cdot \Lambda^m d\,dx)$$

$$\le \frac{1}{8\nu}\left\|\Lambda^m d\right\|_{L^2}^2+C(\left\|\Lambda^{m+1}\mathbf{u}\right\|_{L^2}^2+\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2+\left\|\Lambda^m f_1\right\|_{L^2}^2+\left\|\Lambda^m f_4\right\|_{L^2}^2)$$

from which we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^m d\right\|_{L^2}^2+\frac{1}{2\nu}(|\mathbf{n}|^2+1)\left\|\Lambda^m d\right\|_{L^2}^2\\&\quad\le C(\left\|\Lambda^{m+1}\mathbf{u}\right\|_{L^2}^2+\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2+\left\|\Lambda^m f_1\right\|_{L^2}^2+\left\|\Lambda^m f_4\right\|_{L^2}^2).\end{aligned}\tag{4.43}$$

By the equation of $\mathbf{G}$ in (4.42) there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^m \mathbf{G}\right\|_{L^2}^2+\nu\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2\\&\quad=\frac{1}{\nu}(|\mathbf{n}|^2+1)\int_{\mathbb{T}^3}\Lambda^m \mathbb{Q}\mathbf{u}\cdot\Lambda^m\mathbf{G}\,dx-\frac{1}{\nu}\int_{\mathbb{T}^3}\Lambda^m(\Delta^{-1}\nabla(\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}))\cdot\Lambda^m\mathbf{G}\,dx\\&\qquad+\int_{\mathbb{T}^3}\Lambda^m\mathbb{Q}\widetilde{f_2}\cdot\Lambda^m\mathbf{G}\,dx+\frac{1}{\nu}\int_{\mathbb{T}^3}\Lambda^m\Delta^{-1}\nabla(f_1+f_4\cdot\mathbf{n})\cdot\Lambda^m G\,dx.\end{aligned}\tag{4.44}$$

For $m=0$, we get by the Young inequality and the Poincaré inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{G}\|_{L^2}^2+\nu\|\nabla\mathbf{G}\|_{L^2}^2\\&\quad=\frac{1}{\nu}(|\mathbf{n}|^2+1)\int_{\mathbb{T}^3}\mathbb{Q}\mathbf{u}\cdot\mathbf{G}\,dx-\frac{1}{\nu}\int_{\mathbb{T}^3}(\Delta^{-1}\nabla(\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}))\cdot\mathbf{G}\,dx\\&\qquad+\int_{\mathbb{T}^3}\mathbb{Q}\widetilde{f_2}\cdot\mathbf{G}\,dx+\frac{1}{\nu}\int_{\mathbb{T}^3}\Delta^{-1}\nabla(f_1+f_4\cdot\mathbf{n})\cdot G\,dx\\&\quad\le C(\|\mathbf{u}\|_{L^2}+\|\widetilde{f_2}\|_{L^2}^2+\left\|\Delta^{-1}\nabla(f_1+f_4\cdot\mathbf{n})\right\|_{L^2})\|\mathbf{G}\|_{L^2}\\&\quad\le\frac{\nu}{2}\|\nabla\mathbf{G}\|_{L^2}^2+C\big(\|\mathbf{u}\|_{H^1}^2+\|(f_1,f_4)\|_{H^{-1}}^2+\|\widetilde{f_2}\|_{L^2}^2\big).\end{aligned}\tag{4.45}$$

For $1\le m\le N$, we get by the integration by parts and the Young inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^m\mathbf{G}\right\|_{L^2}^2+\nu\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2\\&\quad\le C\left\|\Lambda^{m-1}\mathbf{u}\right\|_{L^2}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}+C\left\|\Lambda^{m-1}\widetilde{f_2}\right\|_{L^2}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}\\&\qquad+C\left\|\Lambda^{m-2}(f_1+f_4\cdot\mathbf{n})\right\|_{L^2}\left\|\Lambda^{m+1}G\right\|_{L^2}\end{aligned}$$

$$\leq \frac{\nu}{4}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2 + C\left\|\Lambda^{m-1}\mathbf{u}\right\|_{L^2}^2$$
$$+ C\left\|\Lambda^{m-2}f_1\right\|_{L^2}^2 + C\left\|\Lambda^{m-2}f_4\right\|_{L^2}^2 + C\left\|\Lambda^{m-1}\widetilde{f_2}\right\|_{L^2}^2$$
$$\leq \frac{\nu}{4}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2 + C\left\|\Lambda^{m+1}\mathbf{u}\right\|_{L^2}^2$$
$$+ C\left\|\Lambda^{m-2}f_1\right\|_{L^2}^2 + C\left\|\Lambda^{m-2}f_4\right\|_{L^2}^2 + C\left\|\Lambda^{m-1}\widetilde{f_2}\right\|_{L^2}^2$$

from which and (4.45), we have for any $0 \leq m \leq N$ that

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^m\mathbf{G}\right\|_{L^2}^2 + \frac{\nu}{2}\left\|\Lambda^{m+1}\mathbf{G}\right\|_{L^2}^2 \leq C(\|\nabla\mathbf{u}\|_{H^m}^2 + \|f_1\|_{H^m}^2 + \|\widetilde{f_2}\|_{H^{m-1}}^2 + \|f_4\|_{H^m}^2). \tag{4.46}$$

Multiplying (4.46) by a suitable large constant and then adding to (4.43), we get

$$\frac{d}{dt}\|(d,\mathbf{G})\|_{H^m}^2 + \frac{1}{\nu}\|d\|_{H^m}^2 + \nu\|\nabla\mathbf{G}\|_{H^m}^2 \leq C(\|\nabla\mathbf{u}\|_{H^m}^2 + \|f_1\|_{H^m}^2 + \|\widetilde{f_2}\|_{H^{m-1}}^2 + \|f_4\|_{H^m}^2). \tag{4.47}$$

In particular, taking $m = r+4$ in (4.47) gives

$$\frac{d}{dt}\|(d,\mathbf{G})\|_{H^{r+4}}^2 + \frac{1}{\nu}\|d\|_{H^{r+4}}^2 + \nu\|\nabla\mathbf{G}\|_{H^{r+4}}^2 \leq C(\|\nabla\mathbf{u}\|_{H^{r+4}}^2 + \|f_1\|_{H^{r+4}}^2 + \|\widetilde{f_2}\|_{H^{r+3}}^2 + \|f_4\|_{H^{r+4}}^2). \tag{4.48}$$

We bound term by term above in what follows. Employing Lemma 2.7, we have

$$\begin{aligned}\|f_1\|_{H^{r+4}}^2 &\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\nabla a\|_{H^{r+4}}^2 + \|a\|_{H^{r+4}}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\ &\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}^2\|a\|_{H^N}^2\\ &\leq C\delta^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2.\end{aligned} \tag{4.49}$$

Similarly,

$$\begin{aligned}\|f_4\|_{H^{r+4}}^2 &\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\nabla\mathbf{B}\|_{H^{r+4}}^2 + \|\mathbf{B}\|_{H^{r+4}}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\ &\leq C(\|\mathbf{u}\|_{H^{r+4}}^2\|\mathbf{B}\|_{H^N}^2 + \|\mathbf{B}\|_{H^N}^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2)\\ &\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}^2\|\mathbf{B}\|_{H^N}^2\\ &\leq C\delta^2\|\nabla\mathbf{u}\|_{H^{r+4}}^2.\end{aligned} \tag{4.50}$$

Finally we estimate $\|\widetilde{f_2}\|_{H^{r+3}}^2$ and start with the first term $\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^{r+3}}^2$. By Lemma 2.7, we have

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^{r+3}}^2 \le& C\,\|\mathbf{u}\|_{H^{r+4}}^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\\ \le& C\,\|\mathbf{u}\|_{H^{N}}^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\\ \le& C\delta^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\,.\end{aligned}$$

It follows from Lemma 2.1, that

$$\begin{aligned}\|\mathbf{B}\cdot\nabla\mathbf{B}+\mathbf{B}\nabla\mathbf{B}\|_{H^{r+3}}^2 \le& C\,\|\mathbf{B}\|_{H^3}^2\,\|\nabla\mathbf{B}\|_{H^{r+3}}^2\\ \le& C\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,\|\mathbf{B}\|_{H^N}^2\\ \le& C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,.\end{aligned}$$

With the aid of Lemma 2.7 again, we can deduce that

$$\begin{aligned}\|k(a)\nabla a\|_{H^{r+3}}^2 \le& C\left(\|\nabla a\|_{H^{r+3}}^2\,\|k(a)\|_{L^\infty}^2+\|k(a)\|_{H^{r+3}}^2\,\|\nabla a\|_{L^\infty}^2\right)\\ \le& C\,\|a\|_{H^3}^2\,\|a\|_{H^N}^2\\ \le& C\,\|d-\mathbf{n}\cdot\mathbf{B}\|_{H^3}^2\,\|a\|_{H^N}^2\\ \le& C(\|d\|_{H^3}^2+\|\mathbf{B}\|_{H^3}^2)\,\|a\|_{H^N}^2\\ \le& C\delta^2\,\|d\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,,\\ \|I(a)\nabla\times\mathbf{w}\|_{H^{r+3}}^2 \le& C\,\|a\|_{H^N}^2\,\|\nabla\times\mathbf{w}\|_{H^{r+3}}^2\le C\delta^2\,\|\mathbf{w}\|_{H^{r+4}}^2\,,\end{aligned}$$

and

$$\begin{aligned}\|I(a)(\mu\Delta\mathbf{u}+(\lambda+\mu-\zeta)\nabla\mathrm{div}\,\mathbf{u})\|_{H^{r+3}}^2 \le& C\,\|a\|_{H^N}^2\,\|\Delta\mathbf{u}\|_{H^{r+3}}^2\\ \le& C\delta^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2\,.\end{aligned}$$

By Lemmas 2.7 and 2.9, and (4.2), we infer that

$$\begin{aligned}\|I(a)(\mathbf{n}\cdot\nabla\mathbf{B}-\mathbf{n}\nabla\mathbf{B})\|_{H^{r+3}}^2 \le& C(\|I(a)\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{H^{r+3}}^2+\|\nabla\mathbf{B}\|_{L^\infty}^2\|I(a)\|_{H^{r+3}}^2)\\ \le& C(\|\mathbf{B}\|_{H^N}^2\,\|a\|_{H^3}^2+\|\mathbf{B}\|_{H^3}^2\,\|a\|_{H^{r+4}}^2)\\ \le& C\delta^2\,\|d-\mathbf{n}\cdot\mathbf{B}\|_{H^3}^2+C\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,\|a\|_{H^N}^2\\ \le& C\delta^2(\|d\|_{H^3}^2+\|\mathbf{B}\|_{H^3}^2)+C\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,\|a\|_{H^N}^2\\ \le& C\delta^2\,\|d\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\,.\end{aligned}\tag{4.51}$$

The last term in $\|\widetilde{f_2}\|_{H^{r+3}}^2$ can be dealt with similarly as (4.51). Collecting the estimates above yields

$$\|\widetilde{f_2}\|_{H^{r+3}}^2\le C\delta^2\,\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2+C\delta^2\,\|d\|_{H^{r+4}}^2+C\delta^2\,\|\mathbf{w}\|_{H^{r+4}}^2\,.\tag{4.52}$$

Inserting (4.49), (4.50) and (4.52) in (4.48), we conclude that (3.9) holds.

4.5. Proof of Proposition 3.5

Taking $\ell = r+4$ in (3.4) leads to

$$\frac{1}{2}\frac{d}{dt}\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2+\frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^{r+4}}^2+(\lambda+\mu-\zeta)\|\mathrm{div}\,\mathbf{u}\|_{H^{r+4}}^2+\frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{H^{r+4}}^2 \\ \leq C\Upsilon_\infty(t)\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2. \tag{4.53}$$

Multiplying (4.53) by a suitable large constant and then adding to (3.9) lead to

$$\frac{d}{dt}\|(a,\mathbf{u},\mathbf{w},\mathbf{B},d,\mathbf{G})\|_{H^{r+4}}^2+\frac{1}{\nu}\|d\|_{H^{r+4}}^2+\zeta\|\mathbf{w}\|_{H^{r+4}}^2 \\ +\frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^{r+4}}^2+(\lambda+\mu-\zeta)\|\mathrm{div}\,\mathbf{u}\|_{H^{r+4}}^2+\frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{H^{r+4}}^2+\nu\|\nabla\mathbf{G}\|_{H^{r+4}}^2 \\ \leq C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2+C\delta^2\|d\|_{H^{r+4}}^2+C\delta^2\|\mathbf{w}\|_{H^{r+4}}^2 \\ +C\Upsilon_\infty(t)\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2. \tag{4.54}$$

In what follows, we have to bound the last term in (4.54). At first, we use (4.1) and the embedding relation to get

$$\begin{aligned}\Upsilon_\infty(t)\leq& C\|(a,\mathbf{w},\mathbf{B})\|_{H^2}^2+C\|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{H^2}\\ &+C(1+\|(a,\nabla\mathbf{u},\mathbf{B})\|_{H^2}^2)\|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{H^2}^2\\ \leq& C\|(a,\mathbf{w},\mathbf{B})\|_{H^N}^2+C\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N}\\ &+C(1+\|(a,\mathbf{u},\mathbf{B})\|_{H^N}^2)\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^N}^2\\ \leq& C\delta(1+\delta^3),\end{aligned} \tag{4.55}$$

from which and the definition $d = a+\mathbf{n}\cdot\mathbf{B}$ we can get that,

$$\begin{aligned}\Upsilon_\infty(t)\|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2\leq& C\Upsilon_\infty(t)\|(d-\mathbf{n}\cdot\mathbf{B},\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2\\ \leq& C\Upsilon_\infty(t)\|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2+C\Upsilon_\infty(t)\|\mathbf{B}\|_{H^{r+4}}^2\\ \leq& C\delta(1+\delta^3)\|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2+C\Upsilon_\infty(t)\|\mathbf{B}\|_{H^{r+4}}^2.\end{aligned} \tag{4.56}$$

Next, we only need to deal with the term $\Upsilon_\infty(t)\|\mathbf{B}\|_{H^{r+4}}^2$. By using the embedding relation again and Lemma 2.1, we have

$$\begin{aligned}(1+\|a\|_{L^\infty}^2)\|(a,\mathbf{w},\mathbf{B})\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2\leq& C\|(a,\mathbf{w})\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2+C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2\\ \leq& C\|(d-\mathbf{n}\cdot\mathbf{B},\mathbf{w})\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2+C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2\\ \leq& C\|(d,\mathbf{w})\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2+C\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2\\ \leq& C\|(d,\mathbf{w})\|_{L^\infty}^2\|\mathbf{B}\|_{H^{r+4}}^2+C\|\mathbf{B}\|_{H^3}^2\|\mathbf{B}\|_{H^{r+4}}^2\\ \leq& C\|\mathbf{B}\|_{H^N}^2\|(d,\mathbf{w})\|_{H^{r+4}}^2+C\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2\|\mathbf{B}\|_{H^{r+4}}^2\end{aligned}$$

$$
\begin{aligned}
&\le C \|\mathbf{B}\|_{H^N}^2 \|(d,\mathbf{w})\|_{H^{r+4}}^2 + C \|\mathbf{B}\|_{H^N}^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 \\
&\le C\delta^2 \|(d,\mathbf{w})\|_{H^{r+4}}^2 + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 . \qquad (4.57)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty}^2 \|\mathbf{B}\|_{H^{r+4}}^2 &\le C \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^3}^2 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(d,\mathbf{u},\mathbf{w})\|_{H^3}^2 \|\mathbf{B}\|_{H^{r+4}}^2 + C \|\mathbf{B}\|_{H^3}^2 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C\delta^2 \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 , \qquad (4.58)
\end{aligned}
$$

$$
\begin{aligned}
&\|(a,\nabla\mathbf{u},\mathbf{B})\|_{L^\infty}^2 \|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty}^2 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(a,\mathbf{B})\|_{H^3}^2 \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^3}^2 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^3}^4 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(d,\mathbf{u},\mathbf{w})\|_{H^3}^4 \|\mathbf{B}\|_{H^{r+4}}^2 + C \|\mathbf{B}\|_{H^3}^4 \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C\delta^2 \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C\delta^4 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 , \qquad (4.59)
\end{aligned}
$$

$$
\begin{aligned}
&(1+\|a\|_{L^\infty}) \|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty} \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^3} \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(d-\mathbf{n}\cdot\mathbf{B},\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^3} \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C \|(d,\mathbf{u},\mathbf{w})\|_{H^3} \|\mathbf{B}\|_{H^{r+4}}^2 + C \|\mathbf{B}\|_{H^3} \|\mathbf{B}\|_{H^{r+4}}^2 \\
&\le C\delta^2 \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C \|\mathbf{B}\|_{H^{r+4}}^4 + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 . \qquad (4.60)
\end{aligned}
$$

For any $N \ge 4r+7$, we use the interpolation inequality and Lemma 2.1 that

$$
\|\mathbf{B}\|_{H^{r+4}}^4 \le \|\mathbf{B}\|_{H^3}^3 \|\mathbf{B}\|_{H^N} \le C\delta \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^3 \le C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2
$$

from which we have

$$
(1+\|a\|_{L^\infty}) \|(\nabla a,\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{B})\|_{L^\infty} \|\mathbf{B}\|_{H^{r+4}}^2 \le C\delta^2 \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C\delta^2 \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 . \qquad (4.61)
$$

Collecting (4.57)–(4.61), we get

$$
\mathrm{Y}_\infty(t) \|\mathbf{B}\|_{H^{r+4}}^2 \le C\delta^2 \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C\delta^2(\delta^2+1) \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 , \qquad (4.62)
$$

from which, we can infer from (4.56) that

$$
\mathrm{Y}_\infty(t) \|(a,\mathbf{u},\mathbf{w},\mathbf{B})\|_{H^{r+4}}^2 \le C\delta(1+\delta^3) \|(d,\mathbf{u},\mathbf{w})\|_{H^{r+4}}^2 + C\delta^2(\delta^2+1) \|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2 . \qquad (4.63)
$$

Inserting (4.63) into (4.54) and noticing the Lemma 2.2 imply that

$$\begin{aligned}
&\frac{d}{dt}\|(a,\mathbf{u},\mathbf{w},\mathbf{B},d,\mathbf{G})\|_{H^{r+4}}^2+\frac{1}{\nu}\|d\|_{H^{r+4}}^2+\zeta\|\mathbf{w}\|_{H^{r+4}}^2\\
&\quad+\frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^{r+4}}^2+(\lambda+\mu-\zeta)\|\operatorname{div}\mathbf{u}\|_{H^{r+4}}^2+\frac{4\mu\zeta}{\mu+2\zeta}\|\mathbf{w}\|_{H^{r+4}}^2+\nu\|\nabla\mathbf{G}\|_{H^{r+4}}^2\\
&\leq C\delta(1+\delta^3)\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\delta(1+\delta^3)\|(d,\mathbf{w})\|_{H^{r+4}}^2+C\delta^2(\delta^2+1)\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2.
\end{aligned} \tag{4.64}$$

From Proposition 3.3 and the Poincaré inequality, we deduce that

$$\|\mathbf{n}\cdot\nabla\mathbf{B}\|_{H^{r+3}}^2-\sum_{0\leq s\leq r+3}\int_{\mathbb{T}^3}\Lambda^s\mathbb{P}\mathbf{u}\cdot\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{B})\,dx\leq C\|\nabla\mathbf{u}\|_{H^{r+4}}^2+C\|(d,\mathbf{w})\|_{H^{r+4}}^2. \tag{4.65}$$

Let $\widetilde{c}>1$, multiplying (4.64) by $\widetilde{c}$ and then adding to (4.65) lead to (3.11).

CRediT authorship contribution statement

Xiaoping Zhai, Jiahong Wu and Fuyi Xu contributed equally to this work.

Ethical approval

We certify that this manuscript is original and has not been published and will not be submitted elsewhere for publication while being considered by Journal of Differential Equations.

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Data availability

Data and materials sharing not applicable to this article as no data and materials were generated or analyzed during the current study.

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