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Calculus of Variations

Global large solutions to a multi-dimensional compressible magnetohydrodynamic flows with a nonlinear initial constraint

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Abstract
The paper examines the Cauchy problem of the multi-dimensional compressible magnetohydrodynamic (MHD) flows. Under a nonlinear constraint on the initial data, we are able to construct global large solutions of this model in critical Besov spaces. Moreover, we provide an example of initial data satisfying this nonlinear constraint, but the norms of each component of the initial velocity field u_0 and magnetic field B_0 are large.

Mathematics Subject Classification 35Q35 · 35A01 · 35A02.

1 Introduction

1.1 The setting and motivation

This paper intends to construct a class of large global solutions to the compressible magnetohydrodynamic (MHD) flows with initial data satisfying a nonlinear constraint. The compressible MHD equations considered here model the dynamics of electrically conducting compressible fluids. Mathematically, the compressible MHD system consists of the compressible Navier-Stokes equations of fluid dynamics and the Maxwell equations of electromagnetism (see, e.g., [3, 19, 28, 29, 32, 34]),

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$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = J \times B, \\ \varepsilon_0\partial_t E + J = \dfrac{1}{\mu_0}(\nabla \times B), \\ \partial_t B + \nabla \times E = 0, \\ \operatorname{div}B = 0. \end{cases} \tag{1.1}$$

Here ρ, u and $P = P(\rho)$ represent the density, velocity and the pressure of the fluid motion, while E, B and J denote the electric field, the magnetic field and the electric current density, respectively. The coefficients $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$ denote the bulk and shear viscosities, respectively. They are assumed to be smooth functions of ρ and satisfy in the neighborhood of some reference constant density $\bar{\rho} > 0$ the conditions

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0. \tag{1.2}$$

$D(u) \overset{\text{def}}{=} \frac{1}{2}(\nabla u + {}^T\nabla u)$ is the deformation tensor. ε_0 and μ_0 are the permittivity and the permeability, respectively.

In magnetohydrodynamics [28, 29], the displacement current can be neglected and the term $\varepsilon_0\partial_t E$ disappear. Then system (1.1), together with Ohm's law $J = \sigma_0(E + u \times B)$, allows us to convert (1.1) to the following system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = B \cdot \nabla B - \dfrac{1}{2}\nabla(|B|^2), \\ \partial_t B + (\operatorname{div}u)B + u \cdot \nabla B - B \cdot \nabla u = \nu\Delta B, \\ \operatorname{div}B = 0, \end{cases} \tag{1.3}$$

where the positive constant ν is the magnetic diffusivity coefficient. The state equation $P = P(\rho)$ is assumed to satisfy $P'(\rho) > 0$ for all $\rho > 0$. Our attention will be focused on (1.3) in $\mathbb{R}^n$ subject to the initial data

$$(\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0). \tag{1.4}$$

Due to its wide applicability in physics and mathematical importance, the compressible MHD system has attracted considerable interests. Fundamental issues such as the global existence, regularity and stability problems have been extensively investigated and significant progress has been made. Kawashima proved in [25] the global (in time) existence and asymptotic stability of smooth solutions to a general class of quasilinear symmretric hyperbolic-parabolic composite systems, under the smallness assumptions on the initial data and the dissipation condition on the linearized systems. These results applied to the 3D compressible MHD equations yields a unique small global H^3-solution. Global smooth solutions close to a constant steady state in the 2D case were established in [26] while the time decay rates for the 2D linearized MHD equations obtained in [38]. Small smooth global solutions to the 1D compressible MHD equations can be found in [27]. Chen and Wang [7, 8] showed the existence and continuous dependence of large H^1-solutions to the 1D compressible MHD equations. Global existence of large energy weak solutions to the 3D compressible MHD equations were obtained by Hu and Wang [23, 24]. In addition, Suen established the global existence and uniqueness of intermediate weak solutions to the 3D compressible MHD system with small data in $H^s(\mathbb{R}^3)$ for $1/2 < s < 1$ [35]. We also mention that Li, Xu and

Zhang proved the global well-posedness of classical solutions for regular initial data with small energy but possibly large oscillations, with the flow density allowed to contain vacuum states [30].

Now let us recall that some well-posedness results seek solutions in critical spaces. The compressible MHD system has a scaling invariance property. If (ρ, u, B) is a solution of (1.3), then $(\rho_\lambda, u_\lambda, B_\lambda)$ with the modified pressure $\lambda^2 P$ is also a solution of (1.3), where

$$\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \quad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \quad \text{and} \quad B_\lambda(x,t) = \lambda B(\lambda x, \lambda^2 t).$$

Critical spaces are those that preserve the norms under the aforementioned scaling. Solutions of the compressible Navier–Stokes equations in critical spaces have been obtained by Danchin [14–16] and Zhang [44]. Global strong solutions to the 3D isentropic compressible MHD equations were obtained in Chen and Tan [10] and Hao [20] when the initial data are close to a stable equilibrium state in critical L^2-Besov spaces. Xu, Zhang, Wu and Caccetta were able to extend it to the 3D non-isentropic full compressible MHD equations case [42]. More recently, Shi and Xu proved the global existence of strong solutions to the multi-dimensional compressible MHD equations when the initial data are close to a stable equilibrium state in the L^p-based critical Bsov spaces [36]. There are certainly many other important results available on the compressible MHD equations and our brief summary simply cannot cover them all (see, e.g., [9, 28, 29]).

Motivated by the stabilizing phenomenon observed in physical experiments on electrically conducting fluids, some recent studies on the compressible MHD equations focus on the global existence and stability problem when the MHD system has only partial dissipation. The work of Wu and Wu [39] developed a systematic approach to solve the stability problem on the 2D compressible MHD equations without magnetic diffusion in the whole space $\mathbb{R}^2$. Hu studied this problem in the Besov setting [22]. Recently, Wu and Zhu [40] established the global existence of smooth solutions, stability and large-time decay rates for the 2D compressible MHD equations without magnetic diffusion in a periodic domain when the initial data is close to a background magnetic field. The stability problem on the 3D compressible viscous non-resistive MHD system near a background magnetic field remains an outstanding open problem. A very recent work of Dong, Wu and Zhai [17] was able to solve the global well-posedness and stability problem on a special $2\frac{1}{2}$-D compressible viscous non-resistive MHD system near a steady-state solution.

The goal of this paper is to establish the global existence of solutions in critical Besov spaces under a nonlinear smallness constraint on the initial data. This constraint actually allows initial velocity and magnetic field to be large in every direction (see Remarks 1.2–1.3). This result is partially motivated by well-posedness theory on the inhomogeneous incompressible fluids such as Navier–Stokes and MHD equations (see [33, 41, 43]).

1.2 Main results

We state our main result. We introduce a few notations. Let $\mathcal{P}$ and $\mathcal{Q}$ be the orthogonal projectors onto divergence-free and potential vector fields, respectively, namely

$$\mathcal{P} = I - \nabla\Delta^{-1}\nabla\cdot \quad \text{and} \quad \mathcal{Q} = \nabla\Delta^{-1}\nabla.$$

Let $\dot{B}^{s,t}_{2,p}$ denote the hybrid Besov space given by the Definition 2.2. In addition, the space-time spaces $\widetilde{\mathcal{C}}\Big([0,\infty); \dot{B}^{s,t}_{2,p}\Big)$ and $\widetilde{L}^\infty(\mathbb{R}^+; \dot{B}^{s,t}_{2,p})$ are defined in Sect. 2.

Theorem 1.1 *Let $n \geq 2$ and p satisfy $2 \leq p \leq \min(4, \frac{2n}{n-2})$ with $p \neq 4$ if $n = 2$. There exist two positive constants c_0 and C_0 depending only on p, n, λ, μ, such that, for any*

$$a_0 \in \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}, \ \mathcal{P}u_0 \in \dot{B}_{p,1}^{\frac{n}{p}-1}, \mathcal{Q}u_0 \in \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}, \ B_0 \in \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}$$

with $\operatorname{div} B_0 = 0$ *satisfying*

$$\begin{aligned} A_0 \triangleq \Big(\|a_0\|_{\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}} + \|\mathcal{Q}u_0\|_{\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}} + C_L\Big) \\ \times \exp\Big\{C_0\big(\|\mathcal{P}u_0\|^2_{\dot{B}_{p,1}^{\frac{n}{p}-1}} + \|B_0\|^2_{\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}}\big)\Big\} \leq \frac{c_0}{2C_0} \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} C_L \triangleq \Big\| \Big(& e^{t\mu\Delta}\mathcal{P}u_0 \cdot \nabla e^{t\mu\Delta}\mathcal{P}u_0, e^{t\mu\Delta}\mathcal{P}u_0 \cdot \nabla e^{t\nu\Delta}B_0, \\ & e^{t\nu\Delta}B_0 \cdot \nabla e^{t\mu\Delta}\mathcal{P}u_0, e^{t\nu\Delta}B_0 \cdot \nabla e^{t\nu\Delta}B_0\Big)\Big\|_{L^1_t(\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1})}, \end{aligned}$$

then the Cauchy problem (1.3)–(1.4) admits a global solution (a, u, B) with

$$\begin{aligned} a &\in \widetilde{\mathcal{C}}\Big([0,\infty); \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}\Big) \cap L^1\Big(\mathbb{R}^+; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}}\Big), \\ \mathcal{Q}u &\in \widetilde{\mathcal{C}}\Big([0,\infty); \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\Big) \cap L^1\Big(\mathbb{R}^+; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\Big), \\ \mathcal{P}u &\in \widetilde{\mathcal{C}}\Big([0,\infty); \dot{B}_{p,1}^{\frac{n}{p}-1}\Big) \cap L^1\Big(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{n}{p}+1}\Big), \\ B &\in \widetilde{\mathcal{C}}\Big([0,\infty); \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\Big) \cap L^1\Big(\mathbb{R}^+; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\Big). \end{aligned}$$

Furthermore, the following estimate holds

$$\begin{aligned} &\|a\|_{\widetilde{L}^\infty\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}\big)\cap L^1\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}}\big)} + \|\mathcal{Q}u\|_{\widetilde{L}^\infty\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\big)\cap L^1\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\big)} \\ &+ \|\mathcal{P}u - e^{t\mu\Delta}\mathcal{P}u_0\|_{\widetilde{L}^\infty\big(\mathbb{R}^+;\dot{B}_{p,1}^{\frac{n}{p}-1}\big)\cap L^1\big(\mathbb{R}^+;\dot{B}_{p,1}^{\frac{n}{p}+1}\big)} + \|B\|_{\widetilde{L}^\infty\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\big)\cap L^1\big(\mathbb{R}^+;\dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\big)} \\ &\lesssim A_0 + \|B_0\|_{\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}}. \end{aligned} \tag{1.6}$$

We remark that there exist indeed initial datas satisfying the nonlinear constraint condition (1.5).

Remark 1.2 Let $n \geq 2$, $p \in (\frac{n}{n-1}, 2n)$, $\alpha, \varepsilon \in (0,1)$ and f, g be in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. We define the divergence-free vector fields by

$$\omega_{0,\varepsilon,i}(x) \stackrel{\text{def}}{=} \frac{(-\log\varepsilon)^{\frac{1}{6}}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(0,\ldots,0,\partial_{i+1}\varphi^i_\varepsilon, -\partial_i\varphi^i_\varepsilon, 0,\ldots,0), \quad i = 1,\ldots,n-1,$$

$$\varpi_{0,\varepsilon,i}(x) \stackrel{\text{def}}{=} \frac{(-\log\varepsilon)^{\frac{1}{6}}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(0,\ldots,0,\partial_{i+1}\phi^i_\varepsilon, -\partial_i\phi^i_\varepsilon, 0,\ldots,0), \quad i = 1,\ldots,n-1,$$

$$\omega_{0,\varepsilon,n}(x) \stackrel{\text{def}}{=} \frac{(-\log\varepsilon)^{\frac{1}{6}}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(-\partial_n\varphi_\varepsilon^n, 0, \dots, 0, \partial_1\varphi_\varepsilon^n),$$

$$\varpi_{0,\varepsilon,n}(x) \stackrel{\text{def}}{=} \frac{(-\log\varepsilon)^{\frac{1}{6}}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(-\partial_n\phi_\varepsilon^n, 0, \dots, 0, \partial_1\phi_\varepsilon^n),$$

where

$$\varphi_\varepsilon^1(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_n}{\varepsilon}\Big) f\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3, \dots, x_n\Big),$$

$$\phi_\varepsilon^1(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_n}{\varepsilon}\Big) g\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3, \dots, x_n\Big),$$

$$\varphi_\varepsilon^i(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_{i-1}}{\varepsilon}\Big) f\Big(x_1, \dots, x_i, \frac{x_{i+1}}{\varepsilon^\alpha}, x_{i+2}, \dots, x_n\Big), \quad i = 2, \dots, n-1,$$

$$\phi_\varepsilon^i(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_{i-1}}{\varepsilon}\Big) g\Big(x_1, \dots, x_i, \frac{x_{i+1}}{\varepsilon^\alpha}, x_{i+2}, \dots, x_n\Big), \quad i = 2, \dots, n-1,$$

$$\varphi_\varepsilon^n(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_{n-1}}{\varepsilon}\Big) f\Big(\frac{x_1}{\varepsilon^\alpha}, x_2, \dots, x_n\Big),$$

$$\phi_\varepsilon^n(x) \stackrel{\text{def}}{=} \sin\Big(\frac{x_{n-1}}{\varepsilon}\Big) g\Big(\frac{x_1}{\varepsilon^\alpha}, x_2, \dots, x_n\Big).$$

If α and ε are sufficiently small, then the divergence-free vector and magnetic field

$$\mathcal{P}u_{0,\varepsilon} \stackrel{\text{def}}{=} \sum_{1\le i\le n} \omega_{0,\varepsilon,i}, \qquad B_{0,\varepsilon} \stackrel{\text{def}}{=} \sum_{1\le i\le n} \varpi_{0,\varepsilon,i}$$

satisfy

$$\|(\mathcal{P}u_{0,\varepsilon}^i, B_{0,\varepsilon}^i)\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \approx (-\log\varepsilon)^{\frac{1}{6}}, \quad i = 1, \dots, n, \tag{1.7}$$

and

$$\begin{aligned}\|\Big(&e^{t\mu\Delta}\mathcal{P}u_{0,\varepsilon}\cdot\nabla e^{t\mu\Delta}\mathcal{P}u_{0,\varepsilon}, e^{t\mu\Delta}\mathcal{P}u_{0,\varepsilon}\cdot\nabla e^{t\nu\Delta}B_{0,\varepsilon},\\ &e^{t\nu\Delta}B_{0,\varepsilon}\cdot\nabla e^{t\mu\Delta}\mathcal{P}u_{0,\varepsilon}, e^{t\nu\Delta}B_{0,\varepsilon}\cdot\nabla e^{t\nu\Delta}B_{0,\varepsilon}\Big)\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{\frac{n}{p}-1})} \lesssim (-\log\varepsilon)^{\frac{1}{3}}\varepsilon^\eta,\end{aligned} \tag{1.8}$$

for some constant $\eta > 0$.

Next we construct another example which satisfies (1.5) but allows each component of the initial velocity u_0 and magnetic field B_0 to be arbitrarily large in $\dot{B}_{p,1}^{\frac{n}{p}-1}$.

Remark 1.3 Let $\xi \in \mathcal{S}(\mathbb{R}^n)$. Choosing an initial data $(a_{0,\varepsilon}, u_{0,\varepsilon}, B_{0,\varepsilon}) \stackrel{\text{def}}{=} (\varepsilon\xi, \varepsilon\nabla\xi + \mathcal{P}u_{0,\varepsilon}, B_{0,\varepsilon})$ with $\mathcal{P}u_{0,\varepsilon}$ defined in the above Remark. On one hand, due to $\|\cdot\|_{\dot{B}_{2,p}^{s,t}} \le \|\cdot\|_{\dot{B}_{2,1}^{s}} + \|\cdot\|_{\dot{B}_{p,1}^{t}}$, the left-hand of (1.5) for $(a_{0,\varepsilon}, u_{0,\varepsilon}, B_{0,\varepsilon})$ is bounded by

$$C\big(\varepsilon + (-\log\varepsilon)^{\frac{1}{3}}\varepsilon^\eta\big)e^{(-\log\varepsilon)^{\frac{1}{3}}} \to 0, \qquad \varepsilon \to 0.$$

Thus, when ε is small enough, the initial data $(a_{0,\varepsilon}, u_{0,\varepsilon}, B_{0,\varepsilon})$ generates a unique global solution to (3.1). On the other hand, combining with (1.7) and the following relations

$$\|u_{0,\varepsilon}^i\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \ge \|\mathcal{P}u_{0,\varepsilon}^i\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} - \varepsilon\|\partial_i\xi\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \approx (-\log\varepsilon)^{\frac{1}{6}},$$

$$\|B_{0,\varepsilon}^i\|_{\dot{B}_{p,1}^{\frac{n}{p}-1}} \approx (-\log\varepsilon)^{\frac{1}{6}},$$

for $i = 1, \ldots, n$, we find that $\|(u_{0,\varepsilon}, B_{0,\varepsilon})\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$ can be arbitrarily large in every direction if ε is small enough.

1.3 Strategy on the proof of Theorem 1.1

We briefly explain the main difficulties and strategies involved in the proof.

First, due to the mixed hyperbolic-parabolic property of system (1.3), we need to handle the low and high frequencies differently. For low frequencies, system (1.3) has to be treated by means of hyperbolic energy methods. That is, we must treat the low frequencies regime only in spaces based on L^2, as it is classical that hyperbolic systems are ill-posed in general L^p spaces. In contrast, for the high frequencies, a L^p approach may be used. In order to cover more general values of the integration parameter p in the high frequencies, we need to exploit the damping and smoothing effect of the density field. However, the density equation in (1.3) involves no damping or dissipation. To tackle this problem, we introduce a suitable effective velocity, following the ideas of Hoff [18] and Haspot [21] on the compressible Navier–Stokes equations. To be more precise, we set

$$a = \rho - 1, \quad u_L \stackrel{\text{def}}{=} e^{\mu t\Delta}\mathcal{P}u_0, \quad B_L \stackrel{\text{def}}{=} e^{\nu t\Delta}B_0, \quad \bar{u} = u - u_L, \quad \bar{B} = B - B_L.$$

The effective velocity field ω is then defined to be

$$\omega = \mathcal{Q}\bar{u} + (-\Delta)^{-1}\nabla a.$$

We can then verify that ω satisfies the heat equation involving some harmless lower-order terms while a governs a transport equation with an effective damping term. Taking advantage of the smoothing effect from the wave structure, we then suitably bound the high frequencies of a.

Second, due to the lack of the smallness condition on $\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$ and $\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}$, the definitions $u_L \stackrel{\text{def}}{=} e^{\mu t\Delta}\mathcal{P}u_0$, $B_L \stackrel{\text{def}}{=} e^{\nu t\Delta}B_0$ and Lemma 2.14 imply that

$$\|(u_L, B_L)\|_{\tilde{L}^\infty\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)\cap L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}$$

is no longer small. Therefore, we need to try to avoid the appearance of them. Our proof of Theorem 1.1 is very different from the usual construction of global small solutions in [4, 13, 14, 21, 36] and is much more challenging. In addition, due to the lack of divergence-free condition on the velocity field, the handling of the nonlinear terms such as $(\mathrm{div}u)B$, $B \cdot \nabla B$, $u \cdot \nabla B$ and $B \cdot \nabla u$ are much more difficult in comparison with the incompressible MHD equations. A natural idea is to further decompose $\bar{u}$ into the divergence-free part and the gradient part, namely

$$\bar{u} = \mathcal{P}\bar{u} + \mathcal{Q}\bar{u}.$$

We explain our approach on how to bound the two nonlinear terms $\mathcal{Q}(u \cdot \nabla u)$ and $B \cdot \nabla u$. By the decomposition $u = \bar{u} + u_L$, one has

$$\mathcal{Q}(u \cdot \nabla u) = \mathcal{Q}\big(\bar{u} \cdot \nabla\bar{u} + \bar{u} \cdot \nabla u_L + u_L \cdot \nabla\bar{u} + u_L \cdot \nabla u_L\big).$$

It's relatively easy to control these terms in the high frequencies based on the general L^p-framework. However, the most troublesome term in the above decomposition is $\mathcal{Q}(u_L \cdot$

$\nabla\bar{u}$) in the low frequencies. Indeed, using Lemma 2.9, interpolation inequality and Young's inequality, we may get

$$\begin{aligned}\int_0^t \|\mathcal{Q}(u_L\cdot\nabla\bar{u})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\nabla\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau+\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\ &\lesssim \|u_L\|_{\tilde{L}^\infty(\dot{B}^{\frac{n}{p}-1}_{p,1})}\Big(\|\mathcal{P}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}+\|\mathcal{Q}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}\right)}\Big)\\ &\quad+\varepsilon\Big(\|\mathcal{P}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}+\|\mathcal{Q}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}\right)}\Big)\\ &\quad+\|u_L\|_{\tilde{L}^\infty(\dot{B}^{\frac{n}{p}-1}_{p,1})}\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau.\end{aligned}$$

Obviously, the first term on the right-hand side of the above inequality is difficult since $\|u_L\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1})}$ has no smallness condition. To overcome it, $\mathrm{div}u_L=0$ and some non-so-classical commutators estimates (see Lemma 2.11) play a vital role. To deal with the term $B\cdot\nabla u$, we use $B=\bar{B}+B_L$ and $\bar{u}=\mathcal{P}\bar{u}+\mathcal{Q}\bar{u}$ to decompose it into

$$\begin{aligned}B\cdot\nabla u&=\bar{B}\cdot\nabla\bar{u}+\bar{B}\cdot\nabla u_L+B_L\cdot\nabla\bar{u}+B_L\cdot\nabla u_L\\ &=\bar{B}\cdot\nabla\big(\mathcal{P}\bar{u}+\mathcal{Q}\bar{u}\big)+\bar{B}\cdot\nabla u_L+B_L\cdot\nabla\big(\mathcal{P}\bar{u}+\mathcal{Q}\bar{u}\big)+B_L\cdot\nabla u_L.\end{aligned}$$

To bound $B_L\cdot\nabla\mathcal{P}\bar{u}$ above, we need to estimate the low frequency part

$$\int_0^t\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|B_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^2}d\tau.$$

This is difficult. In fact, if we bound it directly by Hölder's inequality, we would get

$$\int_0^t\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|B_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^2}d\tau\le\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{P}\bar{u}\|^\ell_{\dot{B}^{\frac{n}{2}}_{2,1}}d\tau.$$

Since $\mathcal{P}\bar{u}$ lies in a L^p-type space, we are unable to control the term. On the other hand, if we apply the usual product laws Corollary 2.7, we would obtain $\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau+\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau$. Then, the term $\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau$ will also creates serious difficulty because $\|B_L\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1})}$ has no smallness condition. To tackle this problem, we employ various analysis tools such as the Littlewood-Paley decomposition and Fourier frequency localization technique.

The rest of this paper is divided into two sections. Section 2 introduces various analysis tools such as the Littlewood-Paley decomposition, Besov spaces and related tools. In addition, we prove a non-standard commutator estimate. Section 1 is devoted to the proof of Theorem 1.1.

2 Preliminaries

This section reviews several analysis tools including the Littlewood-Paley decomposition, the Besov and related spaces, and bounds on products and commutators in Besov space settings. In particular, the hybrid Besov spaces will be defined and their properties revealed. In addition, a special commutator estimate is proven here, which will be used in the subsequent section.

We start with the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ valued in the interval [0, 1] and supported in $\mathcal{C} = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

The homogeneous frequency localization operators $\dot{\Delta}_j$ and $\dot{S}_j$ are defined by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u \quad \text{for} \quad \dot{S}_j u = \sum_{j'\leq j-1} \dot{\Delta}_{j'} u.$$

With our choice of φ, one can easily verify that

$$\dot{\Delta}_q \dot{\Delta}_k u = 0 \quad \text{for } |q-k| \geq 2 \quad \text{and} \quad \dot{\Delta}_q(\dot{S}_{k-1} u \dot{\Delta}_k u) = 0 \quad \text{for } |q-k| \geq 5.$$

We denote by $\mathcal{S}'_h(\mathbb{R}^n)$ the space of tempered distribution such that, for any $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\lim_{j\to-\infty} \dot{S}_j u = 0$$

in the distributional sense. For any $u \in \mathcal{S}'(\mathbb{R}^n)$, the following decomposition holds

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u, \quad u \in \mathcal{S}'_h(\mathbb{R}^n).$$

We first recall the definition of homogeneous Besov spaces (see, e.g., [2, 14, 31]).

Definition 2.1 Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} = \Big\{ u \in \mathcal{S}'_h(\mathbb{R}^n) : \|u\|_{\dot{B}^s_{p,r}} < \infty \Big\},$$

where

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \big\| 2^{js} \|\dot{\Delta}_j u\|_{L^p} \big\|_{l^r}.$$

Hybrid Besov spaces are Besov spaces with different regularity indices for high and low frequencies (see, e.g., [2, 13, 14]).

Definition 2.2 Let $s, t \in \mathbb{R}$ and $1 \leq p \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of all the distributions u in $\mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^{s,t}_{2,p}} \stackrel{\text{def}}{=} \sum_{j\leq j_0} 2^{js} \|\dot{\Delta}_j u\|_{L^2} + \sum_{j\geq j_0+1} 2^{jt} \|\dot{\Delta}_j u\|_{L^p} < \infty.$$

Lemma 2.3 *The hybrid Besov spaces have the following properties*
(i) Embedding relations:

$$\dot{B}^{s_2,t}_{2,p} \hookrightarrow \dot{B}^{s_1,t}_{2,p} \quad s_1 \geq s_2,$$
$$\dot{B}^{s,t_2}_{2,p} \hookrightarrow \dot{B}^{s,t_1}_{2,p} \quad t_1 \leq t_2,$$
$$\dot{B}^{s}_{2,1} \hookrightarrow \dot{B}^{s,s-\frac{n}{2}+\frac{n}{p}}_{2,p} \hookrightarrow \dot{B}^{s-\frac{n}{2}+\frac{n}{p}}_{p,1} \quad p \geq 2.$$

(ii) Interpolation: for $s_1, s_2, t_1, t_2 \in \mathbb{R}, \theta \in (0,1)$ and $p \in [1,\infty]$, we have

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2, \theta t_1+(1-\theta)t_2}_{2,p}} \leq \|u\|^{\theta}_{\dot{B}^{s_1,t_1}_{2,p}} \|u\|^{1-\theta}_{\dot{B}^{s_2,t_2}_{2,p}}.$$

We shall also need the following homogeneous space-time Besov spaces.

Definition 2.4 Let $s \in \mathbb{R}$, $(r, \rho, p) \in [1, +\infty]^3$ and $T \in (0, +\infty]$. We say that $u \in L^\rho_T(\dot{B}^s_{p,r})$, if

$$\|u\|_{L^\rho_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big\| \big\| 2^{qs} \|\dot{\Delta}_q u\|_{L^p} \big\|_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

The Besov-Chemin-Lerner spaces $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})$ are defined in [6].

Definition 2.5 Let $(r, \rho, p) \in [1, +\infty]^3$, $s \le \frac{n}{p}$, and $T \in (0, +\infty]$. We define $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})$ as the completion of $C([0, T]; \mathcal{S}'_h)$ by the norm

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big\| 2^{js} \|\dot{\Delta}_j u(t)\|_{L^\rho(0,T;L^p)} \Big\|_{\ell^r}.$$

Obviously, $\widetilde{L}^1_T(\dot{B}^s_{p,1}) = L^1_T(\dot{B}^s_{p,1})$. By Minkowski's inequality, we have the following relations

$$L^\rho_T(\dot{B}^s_{p,r}) \hookrightarrow \widetilde{L}^\rho_T(\dot{B}^s_{p,r}) \text{ if } \ r \ge \rho \quad \text{and} \quad \widetilde{L}^\rho_T(\dot{B}^s_{p,r}) \hookrightarrow L^\rho_T(\dot{B}^s_{p,r}) \text{ if } \ \rho \ge r.$$

The norm for the space-time hybrid Besov space is given by

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^{s,t}_{2,p})} \stackrel{\text{def}}{=} \sum_{j \le j_0} 2^{js} \|\dot{\Delta}_j u\|_{L^q_T(L^2)} + \sum_{j \ge j_0+1} 2^{jt} \|\dot{\Delta}_j u\|_{L^q_T(L^p)}.$$

The following notations will be used frequently, for any $u \in \mathcal{S}'_h(\mathbb{R}^n)$,

$$\|u\|^l_{\dot{B}^s_{p,1}} \stackrel{\text{def}}{=} \sum_{j \le j_0} 2^{js} \|\dot{\Delta}_j u\|_{L^p}, \quad \|u\|^h_{\dot{B}^s_{p,1}} \stackrel{\text{def}}{=} \sum_{j \ge j_0+1} 2^{js} \|\dot{\Delta}_j u\|_{L^p},$$

$$\|u\|^l_{\widetilde{L}^q_T(\dot{B}^s_{p,1})} \stackrel{\text{def}}{=} \sum_{j \le j_0} 2^{js} \|\dot{\Delta}_j u\|_{L^q_T(L^p)}, \quad \|u\|^h_{\widetilde{L}^q_T(\dot{B}^s_{p,1})} \stackrel{\text{def}}{=} \sum_{j \ge j_0+1} 2^{js} \|\dot{\Delta}_j u\|_{L^q_T(L^p)},$$

$$\widetilde{\mathcal{C}}_T(\dot{B}^s_{p,r}) \stackrel{\text{def}}{=} \mathcal{C}_T(\dot{B}^s_{p,r}) \cap \widetilde{L}^\infty_T(\dot{B}^s_{p,r}), \quad \widetilde{\mathcal{C}}_T(\dot{B}^{s,t}_{2,p}) \stackrel{\text{def}}{=} \mathcal{C}_T(\dot{B}^{s,t}_{2,p}) \cap \widetilde{L}^\infty_T(\dot{B}^{s,t}_{2,p}).$$

The basic tool of the paradifferential calculus is Bony's decomposition [1]. Formally, the product of two tempered distributions u and v may be decomposed into

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v)$$

with

$$\dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \quad \widetilde{\dot{\Delta}}_j v = \sum_{|j-j'| \le 1} \dot{\Delta}_{j'} v.$$

The paraproduct and remainder operators satisfy the estimates stated in the following lemma. Throughout the rest of this paper, C denotes a positive generic constant that may vary from line to line and $A \lesssim B$ means $A \le CB$, and $[A, B] = AB - BA$ denotes the standard commutator operator.

Lemma 2.6 *[2] Let $s, s_1, s_2 \in \mathbb{R}$, $t < 0$, $1 \le p, p_1, p_2, r, r_1, r_2 \le \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1$. Then it holds that*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\dot{B}^s_{p_2,r}}, \quad \|\dot{T}_u v\|_{\dot{B}^{s+t}_{p,r}} \lesssim \|u\|_{\dot{B}^t_{p_1,\infty}} \|v\|_{\dot{B}^s_{p_2,r}}, \tag{2.1}$$

$$\|\dot{R}(u,v)\|_{\dot{B}^{s}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}, \qquad for \quad s_1+s_2>0, \tag{2.2}$$

$$\|\dot{R}(u,v)\|_{\dot{B}^{0}_{p,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}, \qquad for \quad s_1+s_2=0. \tag{2.3}$$

Corollary 2.7 *(i) Let $s, s_1, s_2 \in \mathbb{R}$ with $s = s_1 + s_2$. Assume p satisfies*

$$\begin{cases} 2 \le p \le 4, & if \quad n < 2s_1, \\ 2 \le p \le \min\left(4, \dfrac{2n}{n-2s_1}\right), & if \quad n \ge 2s_1, \end{cases}$$

then we have

$$\|\dot{T}_u v\|_{\dot{B}^{\frac{n}{2}-s}_{2,1}} \lesssim \|u\|_{\dot{B}^{\frac{n}{p}-s_1}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}-s_2}_{p,1}}.$$

(ii) Let $s \in [0,+\infty)$, $s_1, s_2 \in \mathbb{R}$ and $s = s_1 + s_2$. Then for all $n > s$ and $p \in [2,4] \cap [2, \frac{2n}{s})$,

$$\|\dot{R}(u,v)\|_{\dot{B}^{\frac{n}{2}-s}_{2,1}} \lesssim \|u\|_{\dot{B}^{\frac{n}{p}-s_1}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}-s_2}_{p,1}}.$$

We shall also use the following notations.

$$\text{For some } k_0 \in \mathbb{Z}, \quad z^\ell \overset{\text{def}}{=} \sum_{j \le k_0} \dot{\Delta}_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} z - z^\ell,$$

$$\|z\|^{\ell}_{\dot{B}^{s}_{2,1}} \overset{\text{def}}{=} \sum_{j \le k_0} 2^{js} \|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|^{h}_{\dot{B}^{s}_{2,1}} \overset{\text{def}}{=} \sum_{j \ge k_0} 2^{js} \|\dot{\Delta}_j z\|_{L^2}.$$

It is easy to see that

$$\|z^\ell\|_{\dot{B}^{s}_{2,1}} \lesssim \|z\|^{\ell}_{\dot{B}^{s}_{2,1}} \quad \text{and} \quad \|z^h\|_{\dot{B}^{s}_{2,1}} \lesssim \|z\|^{h}_{\dot{B}^{s}_{2,1}}.$$

Corollary 2.8 *[15] Let $n \ge 2$ and $2 \le p \le (4, \frac{2n}{n-2})$. Then, for all $s \in \mathbb{R}$, $j_0 \in \mathbb{Z}$,*

$$\|\dot{T}_u v^\ell\|^{\ell}_{\dot{B}^{\frac{n}{2}}_{2,1}} \lesssim \|u\|_{\dot{B}^{-1}_{\infty,\infty}} \|v^\ell\|_{\dot{B}^{\frac{n}{2}+1}_{2,1}},$$

$$\|\dot{T}_u v^h\|^{\ell}_{\dot{B}^{\frac{n}{2}}_{2,1}} \lesssim 2^{j_0} \|u\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|v^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.$$

Lemma 2.9 *[16] Let $1 \le p \le \infty$, $s_1, s_2 \le \frac{n}{p}$ and $s_1 + s_2 > n \max(0, \frac{2}{p} - 1)$, then it holds that*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{p}}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,1}} \|v\|_{\dot{B}^{s_2}_{p,1}}.$$

Let the real numbers s_1, s_2, p_1 and p_2 be such that

$$s_1 + s_2 > 0,\, s_1 \le \frac{n}{p_1},\, s_2 \le \frac{n}{p_2},\, s_1 \ge s_2,\, \frac{1}{p_1} + \frac{1}{p_2} \le 1.$$

Then, we have

$$\|uv\|_{\dot{B}^{s_2}_{q,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,1}} \|v\|_{\dot{B}^{s_2}_{p_2,1}} \qquad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s_1}{n}.$$

Lemma 2.10 *[2] Assume that F is a smooth function on $\mathbb{R}^n \setminus \{0\}$ and is homogeneous of degree k. Let $\sigma \in (0,1]$, $s \in \mathbb{R}$ and $1 \le p, p_1, p_2, r \le \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,*

$$\|[T_u, F(D)]v\|_{\dot{B}^{s-k+\sigma}_{p,r}} \lesssim \|\nabla u\|_{\dot{B}^{\sigma-1}_{p_1,1}} \|v\|_{\dot{B}^{s}_{p_2,r}}.$$

We shall now state and prove an important commutator estimate to be used in the proof of our main result.

Lemma 2.11 *Let $n \geq 2$ and p satisfy $2 \leq p \leq \min(4, \frac{2n}{n-2})$ with $p \neq 4$ if $n = 2$. Assume that σ is a smooth function on $\mathbb{R}^n$ and is homogeneous of degree 1. Then*

$$\sum_{j\in\mathbb{Z}} 2^{j(\frac{n}{2}-1)} \big\|[\dot{\Delta}_j, u]\sigma(D)v\big\|_{L^2} \lesssim \|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{2.4}$$

Proof By Bony's decomposition,

$$\begin{aligned}\sigma(D)v = &- [\dot{T}_u, \dot{\Delta}_j]\sigma(D)v - \dot{T}_{\sigma(D)\dot{\Delta}_j v} u - \dot{R}(u, \sigma(D)\dot{\Delta}_j v)\\ &+ \dot{\Delta}_j(\dot{T}_{\sigma(D)v} u) + \dot{\Delta}_j \dot{R}(\sigma(D)v, u).\end{aligned}$$

Using the definition of paraproduct, we get

$$\begin{aligned}
&[\dot{T}_u, \dot{\Delta}_j]\sigma(D)v\\
&= \sum_{|j'-j|\leq 4} [\dot{S}_{j'-1}u, \dot{\Delta}_j]\sigma(D)\dot{\Delta}_{j'}v\\
&= \sum_{|j'-j|\leq 4} \dot{S}_{j'-1}u(\dot{\Delta}_j\sigma(D)\dot{\Delta}_{j'}v) - \dot{\Delta}_j(\dot{S}_{j'-1}u\sigma(D)\dot{\Delta}_{j'}v)\\
&= \sum_{|j'-j|\leq 4} \dot{S}_{j'-1}u 2^{jn} \int \varphi(2^j y)\sigma(D)\dot{\Delta}_{j'}v(x-y)dy\\
&\quad - \sum_{|j'-j|\leq 4} 2^{jn} \int \varphi(2^j y)\Big(\dot{S}_{j'-1}u(x-y)\sigma(D)\dot{\Delta}_{j'}v(x-y)\Big)dy\\
&= \sum_{|j'-j|\leq 4} \dot{S}_{j'-1}u \int \varphi(z)\sigma(D)\dot{\Delta}_{j'}v(x-2^{-j}z)dz\\
&\quad - \sum_{|j'-j|\leq 4} \int \varphi(z)\Big(\dot{S}_{j'-1}u(x-2^{-j}z)\sigma(D)\dot{\Delta}_{j'}v(x-2^{-j}z)\Big)dz\\
&= \sum_{|j'-j|\leq 4} \int \varphi(z)\Big(\dot{S}_{j'-1}u(x) - \dot{S}_{j'-1}u(x-2^{-j}z)\Big)\sigma(D)\dot{\Delta}_{j'}v(x-2^{-j}z)dz\\
&= \sum_{|j'-j|\leq 4} \int \varphi(z)\Big(\dot{S}_{j'-1}u(x) - \dot{S}_{j'-1}u(x-2^{-j}z)\Big)\sigma(D)\tilde{\dot{\Delta}}_{j'}\dot{\Delta}_{j'}v(x-2^{-j}z)dz\\
&\lesssim \sum_{|j'-j|\leq 4} \int_0^1 \int \varphi(z)|\nabla\dot{S}_{j'-1}u(x-\tau 2^{-j}z)|(2^{-j}z)2^{j'}(\sigma\tilde{\varphi})(2^{-j}D)\dot{\Delta}_{j'}v(x-2^{-j}z)dzd\tau,
\end{aligned}$$

where $\tilde{\varphi}$ is a smooth function supported in an annulus. Applying Hölder's inequality, Bernstein's inequality of Fourier multipliers, $\frac{1}{2} = \frac{1}{p} + \frac{1}{p^*}$, $\frac{n}{p^*} - 1 \leq 0$ and $p \leq p^*$, we get

$$
\begin{aligned}
&2^{j\left(\frac{n}{2}-1\right)}\|[\dot{T}_u,\dot{\Delta}_j]\sigma(D)v\|_{L^2}\\
&\lesssim \sum_{|j'-j|\le 4} 2^{j\left(\frac{n}{2}-1\right)}\|\nabla\dot{S}_{j'-1}u\|_{L^{p^*}}\|\dot{\Delta}_{j'}v\|_{L^p}\\
&\lesssim \sum_{|j'-j|\le 4} 2^{j\left(\frac{n}{2}-1\right)}\sum_{k\le j'-2}2^{-k\left(\frac{n}{p^*}-1\right)}2^{k\left(\frac{n}{p^*}-1\right)}\|\nabla\dot{\Delta}_k u\|_{L^{p^*}}2^{-\frac{n}{p}j'}2^{\frac{n}{p}j'}\|\dot{\Delta}_{j'}v\|_{L^p}\\
&\lesssim \sum_{|j'-j|\le 4} 2^{j(\frac{n}{2}-1)}2^{-\frac{n}{p}j'}\sum_{k\le j'-2}2^{-k(\frac{n}{p^*}-1)}c_j\|\nabla u\|_{\dot{B}^{\frac{n}{p^*}-1}_{p^*,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,\infty}}\\
&\lesssim \sum_{|j'-j|\le 4} 2^{(j-j')(\frac{n}{2}-1)}c_j\|\nabla u\|_{\dot{B}^{\frac{n}{p^*}-1}_{p^*,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,\infty}}\\
&\lesssim c_j\|\nabla u\|_{\dot{B}^{\frac{n}{p^*}-1}_{p^*,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\\
&\lesssim c_j\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}
\end{aligned}
\tag{2.5}
$$

with $\|c_j\|_{l^1}\le 1$. Then summing up over j in (2.5) immediately yields that

$$
\sum_{j\in\mathbb{Z}}2^{j(\frac{n}{2}-1)}\|[\dot{T}_u,\dot{\Delta}_j]\sigma(D)v\|_{L^2}\lesssim\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{2.6}
$$

For the term $\dot{T}_{\sigma(D)\dot{\Delta}_j v}u$, we have

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}2^{j(\frac{n}{2}-1)}\|\dot{T}_{\sigma(D)\dot{\Delta}_j v}u\|_{L^2}&\lesssim\sum_{j\in\mathbb{Z}}2^{j(\frac{n}{2}-1)}\sum_{j'\ge j-3}\|\dot{S}_{j'-1}\sigma(D)\dot{\Delta}_j v\|_{L^p}\|\dot{\Delta}_{j'}u\|_{L^{p^*}}\\
&\lesssim\sum_{j\in\mathbb{Z}}2^{j\frac{n}{2}}\sum_{j'\ge j-3}2^{-\frac{n}{p}j}2^{\frac{n}{p}j}\|\dot{\Delta}_j v\|_{L^p}2^{-\frac{n}{p^*}j'}2^{\frac{n}{p^*}j'}\|\dot{\Delta}_{j'}u\|_{L^{p^*}}\\
&\lesssim\sum_{j\in\mathbb{Z}}\sum_{j'\ge j-3}2^{(j-j')\frac{n}{p^*}}2^{\frac{n}{p}j}\|\dot{\Delta}_j v\|_{L^p}2^{\frac{n}{p^*}j'}\|\dot{\Delta}_{j'}u\|_{L^{p^*}}\\
&\lesssim\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.
\end{aligned}
\tag{2.7}
$$

Similarly, we get

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}2^{j\left(\frac{n}{2}-1\right)}\|\dot{R}(\sigma(D)\dot{\Delta}_j v,u)\|_{L^2}&\lesssim\sum_{j\in\mathbb{Z}}2^{j(\frac{n}{2}-1)}\sum_{|j-j'|\le 2}\|\dot{\Delta}_{j'}u\Delta_{j'}(\sigma(D)\dot{\Delta}_j v)\|_{L^2}\\
&\lesssim\sum_{j\in\mathbb{Z}}2^{j\frac{n}{2}}\sum_{|j-j'|\le 2}2^{-\frac{n}{p}j'}2^{\frac{n}{p}j'}\|\dot{\Delta}_{j'}u\|_{L^p}2^{-\frac{n}{p^*}j}2^{\frac{n}{p^*}j}\|\dot{\Delta}_j v\|_{L^{p^*}}\\
&\lesssim\sum_{j\in\mathbb{Z}}\sum_{|j-j'|\le 2}2^{(j-j')\frac{n}{p}}2^{\frac{n}{p}j'}\|\dot{\Delta}_{j'}u\|_{L^p}2^{\frac{n}{p^*}j}\|\dot{\Delta}_j v\|_{L^{p^*}}\\
&\lesssim\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.
\end{aligned}
\tag{2.8}
$$

Finally, it follows from Corollary 2.7 that

$$\begin{aligned}
\|\dot{T}_{\sigma(D)v}u\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &\lesssim \|\sigma(D)v\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}},\\
\|\dot{R}(\sigma(D)v,u)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} &\lesssim \|\sigma(D)v\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.
\end{aligned} \tag{2.9}$$

Putting (2.6)–(2.9) all together, we finally conclude that (2.4) holds. This completes the proof of Lemma 2.11.

To bound the compositions of functions involved in system (3.1), we need the following lemma. □

Lemma 2.12 *[11, 12] Let $1 \le p, r \le \infty$. Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function with $F(0) = 0$. Then for any $s > 0$ and any $a \in \dot{B}^s_{p,r} \cap L^\infty$, we have $F(a) \in \dot{B}^s_{p,r} \cap L^\infty$ and*

$$\|F(a)\|_{\dot{B}^s_{p,r}} \le C(\|a\|_{L^\infty}, F')\|a\|_{\dot{B}^s_{p,r}}.$$

In the case $s > -n\min(\frac{1}{p}, 1-\frac{1}{p})$ and $a \in \dot{B}^s_{p,1} \cap \dot{B}^{\frac{n}{p}}_{p,1}$, we have $F(a) \in \dot{B}^s_{p,r} \cap \dot{B}^{\frac{n}{p}}_{p,1}$ and

$$\|F(a)\|_{\dot{B}^s_{p,r}} \le C\Big(1 + \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\|a\|_{\dot{B}^s_{p,r}}.$$

The following Bernstein's inequalities will be frequently used.

Lemma 2.13 *[5] Let $1 \le p_1 \le p_2 \le +\infty$. Assume that $u \in L^{p_1}(\mathbb{R}^n)$, then for any $\gamma \in (\mathbb{N} \cup \{0\})^n$, there exist constants C_1 and C_2 independent of u and q such that*

$$\begin{aligned}
&\mathrm{supp}\hat{u} \subseteq \{|\xi| \le A_0 2^q\} \Rightarrow \|\partial^\gamma u\|_{p_2} \le C_1 2^{q|\gamma| + qn\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|u\|_{p_1},\\
&\mathrm{supp}\hat{u} \subseteq \{A_1 2^q \le |\xi| \le A_2 2^q\} \Rightarrow \|u\|_{p_1} \le C_2 2^{-q|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta u\|_{p_1}.
\end{aligned}$$

Finally, we recall the parabolic regularity estimate for the heat equation.

Lemma 2.14 *[2] Let $s \in \mathbb{R}$, $1 \le \lambda, p, r \le \infty$, $\mu > 0$. Let u be the solution of*

$$\begin{cases} \partial_t u - \mu\Delta u = f,\\ u|_{t=0} = u_0. \end{cases}$$

Then

$$\mu^{\frac{1}{\lambda}}\|u\|_{\widetilde{L}^\lambda_T\left(\dot{B}^{s+\frac{2}{\lambda}}_{p,r}\right)} \lesssim \|u_0\|_{\dot{B}^s_{p,r}} + \|f\|_{\widetilde{L}^1_T\left(\dot{B}^s_{p,r}\right)}.$$

3 The proof of Theorem 1.1

This section proves Theorem 1.1. For the sake of clarity, we divide the proof into five main parts.

3.1 Reformulation of the original system (1.3)-(1.4)

We reformulate the original system (1.3)–(1.4) into a different form that is more convenient for our purpose. Without loss of generality, we assume the initial density ρ_0 is a perturbation near the constant density 1 and set

$$a \stackrel{\text{def}}{=} \rho - 1$$

and assume $P'(1) = 1$. We remark that the coefficients $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$ are allowed to be constants, but we treat them as general smooth functions of ρ. Then, in term of the new variables (a, u, B), system (1.3)–(1.4) can be written as

$$\begin{cases} \partial_t a + \mathrm{div} u = F, \\ \partial_t u - \mathcal{A}u + \nabla a = G, \\ \partial_t B - \nu \Delta B = H, \\ (a, u, B)|_{t=0} = (a_0, u_0, B_0), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} F &\stackrel{\text{def}}{=} -\mathrm{div}(au), \\ G &\stackrel{\text{def}}{=} -u \cdot \nabla u - L_1(a)\mathcal{A}u - L_2(a)\nabla a \\ &\quad + L_3(a)\Big(\mathrm{div}\big(2\widetilde{\mu}(a)D(u)\big) + \nabla\big(\widetilde{\lambda}(a)\mathrm{div} u\big) + B \cdot \nabla B - \frac{1}{2}\nabla(|B|^2)\Big), \\ H &\stackrel{\text{def}}{=} -(\mathrm{div} u)B - u \cdot \nabla B + B \cdot \nabla u \end{aligned}$$

with

$$\begin{aligned} &\mathcal{A}u \stackrel{\text{def}}{=} \mu \Delta u + (\lambda + \mu)\nabla \mathrm{div} u, \quad L_1(a) \stackrel{\text{def}}{=} \frac{a}{1+a}, \qquad L_2(a) \stackrel{\text{def}}{=} \frac{P'(1+a)}{1+a} - P'(1), \\ &L_3(a) \stackrel{\text{def}}{=} \frac{1}{1+a}, \qquad \lambda \stackrel{\text{def}}{=} \lambda(1), \qquad \mu \stackrel{\text{def}}{=} \mu(1), \\ &\widetilde{\mu}(a) \stackrel{\text{def}}{=} \mu(1+a) - \mu(1), \qquad \widetilde{\lambda}(a) \stackrel{\text{def}}{=} \lambda(1+a) - \lambda(1). \end{aligned}$$

Let $u_L \stackrel{\text{def}}{=} e^{\mu t \Delta}\mathcal{P}u_0$ and $B_L \stackrel{\text{def}}{=} e^{\nu t \Delta}B_0$ be the solutions to the heat equations

$$\begin{cases} \partial_t u_L - \mu \Delta u_L = 0, \\ u_L|_{t=0} = \mathcal{P}u_0, \end{cases}$$

and

$$\begin{cases} \partial_t B_L - \nu \Delta B_L = 0, \\ B_L|_{t=0} = B_0, \end{cases}$$

respectively. Clearly, by Lemma 2.14, we have

$$\begin{aligned} &\|u_L\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^s_{p,1})} + \mu\|u_L\|_{L^1(\mathbb{R}^+;\dot{B}^{s+2}_{p,1})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^s_{p,1}}, \\ &\|B_L\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^s_{p,1})} + \nu\|B_L\|_{L^1(\mathbb{R}^+;\dot{B}^{s+2}_{p,1})} \lesssim \|B_0\|_{\dot{B}^s_{p,1}}, \end{aligned} \tag{3.2}$$

whenever $\mathcal{P}u_0, B_0 \in \dot{B}^s_{p,1}$ for $s \in \mathbb{R}$ and $1 \le p \le \infty$.

In what follows, we will seek a solution (a, u, B) of system (3.1) with $u = \bar{u} + u_L$, $B = \bar{B} + B_L$. Obviously, $\mathrm{div} u_L = 0$, $\mathrm{div} B_L = \mathrm{div} \bar{B} = 0$. It then follows from system (3.1) that

$$\begin{cases} \partial_t a + \mathrm{div}\bar{u} = f_1, \\ \partial_t \bar{u} - \mathcal{A}\bar{u} + \nabla a = g_1, \\ \partial_t \bar{B} - \nu\Delta\bar{B} = h_1, \\ (a, \bar{u}, \bar{B})|_{t=0} = (a_0, \mathcal{Q}u_0, 0), \end{cases} \tag{3.3}$$

where

$$\begin{aligned} f_1 &\stackrel{\text{def}}{=} -\bar{u}\cdot\nabla a - u_L\cdot\nabla a - a\mathrm{div}\bar{u}, \\ g_1 &\stackrel{\text{def}}{=} -\bar{u}\cdot\nabla\bar{u} - \bar{u}\cdot\nabla u_L - u_L\cdot\nabla\bar{u} - u_L\cdot\nabla u_L - L_1(a)\mathcal{A}(\bar{u}+u_L) - L_2(a)\nabla a \\ &\quad + L_3(a)\Big(\mathrm{div}\big(2\widetilde{\mu}(a)D(\bar{u}+u_L)\big) + \nabla\big(\widetilde{\lambda}(a)\mathrm{div}\bar{u}\big) \\ &\quad + \bar{B}\cdot\nabla\bar{B} + \bar{B}\cdot\nabla B_L + B_L\cdot\nabla\bar{B} + B_L\cdot\nabla B_L\Big), \\ h_1 &= -(\mathrm{div}\bar{u})\bar{B} - (\mathrm{div}\bar{u})B_L - \bar{u}\cdot\nabla\bar{B} - \bar{u}\cdot\nabla B_L - u_L\cdot\nabla\bar{B} - u_L\cdot\nabla B_L \\ &\quad + \bar{B}\cdot\nabla\bar{u} + \bar{B}\cdot\nabla u_L + B_L\cdot\nabla\bar{u} + B_L\cdot\nabla u_L. \end{aligned} \tag{3.4}$$

Applying operators $\mathcal{P}$ and $\mathcal{Q}$ to the equation of $\bar{u}$ in system (3.3), respectively, we can decompose it into two new systems. One is the nonlinear heat equation for $\mathcal{P}\bar{u}$,

$$\partial_t\mathcal{P}\bar{u} - \mu\Delta\mathcal{P}\bar{u} = \mathcal{P}g_1, \tag{3.5}$$

and the other is a coupled hyperbolic-parabolic system of $(a, \mathcal{Q}\bar{u})$,

$$\begin{cases} \partial_t a + \mathrm{div}\mathcal{Q}\bar{u} = f_1, \\ \partial_t\mathcal{Q}\bar{u} - \Delta\mathcal{Q}\bar{u} + \nabla a = \mathcal{Q}g_1, \end{cases} \tag{3.6}$$

where we have set $\lambda + 2\mu = 1$ for simplicity. In addition, we have also used the following properties of $\mathcal{P}$ and $\mathcal{Q}$,

$$\mathcal{P}u = \mathcal{P}\bar{u} + u_L, \quad \mathcal{Q}u = \mathcal{Q}\bar{u}, \quad \mathcal{P}\bar{u}_0 = 0.$$

According to the local existence and uniqueness result on strong solutions to system (3.3) in [36], there exists a positive time T such that system (3.3) has a unique solution $(a, \bar{u}, \bar{B}) \in X(t)$ for all $t \le T$, where

$$\begin{aligned} X(t) \stackrel{\text{def}}{=} \Big\{ & a \in \widetilde{\mathcal{C}}\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}\Big) \cap L^1\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}}\Big), \\ & \mathcal{P}\bar{u} \in \widetilde{\mathcal{C}}\Big([0,t]; \dot{B}_{p,1}^{\frac{n}{p}-1}\Big) \cap L^1\Big([0,t]; \dot{B}_{p,1}^{\frac{n}{p}+1}\Big), \\ & \mathcal{Q}\bar{u} \in \widetilde{\mathcal{C}}\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\Big) \cap L^1\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\Big), \\ & \bar{B} \in \widetilde{\mathcal{C}}\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\Big) \cap L^1\Big([0,t]; \dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}\Big) \Big\} \end{aligned}$$

endowed with the norm

$$\begin{aligned} \|(a,\bar{u},\bar{B})\|_{X(t)} \stackrel{\text{def}}{=} & \|a\|_{\widetilde{L}_t^\infty\left(\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}}\right)} + \|(\mathcal{Q}\bar{u},\bar{B})\|_{\widetilde{L}_t^\infty\left(\dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}\right)} + \|\mathcal{P}\bar{u}\|_{\widetilde{L}_t^\infty\left(\dot{B}_{p,1}^{\frac{n}{p}-1}\right)} \\ & + \int_0^t \Big(\|a\|_{\dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}}} + \|(\mathcal{Q}\bar{u},\bar{B})\|_{\dot{B}_{2,p}^{\frac{n}{2}+1,\frac{n}{p}+1}} + \|\mathcal{P}\bar{u}\|_{\dot{B}_{p,1}^{\frac{n}{p}+1}}\Big) d\tau. \end{aligned}$$

Let T^* be the life-span of the solution. In what follows, we shall prove that $T^* = \infty$ and (1.6) holds. We shall employ a bootstrapping argument to prove the desired result. An abstract statement on the bootstrapping argument can be found in Tao [37].

3.2 Estimates on the high frequencies

We shall construct *a priori* estimate based on the general L^p-framework on the high frequencies for system (3.3). By Lemma 2.14, we have

$$\begin{aligned} &\|\mathcal{P}\bar{u}\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu\|\mathcal{P}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \lesssim \||\mathcal{P}g_1\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}, \\ &\|\bar{B}\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \nu\|\bar{B}\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \lesssim \|h_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}. \end{aligned} \tag{3.7}$$

To handle the high frequencies of the coupled system (3.6), we rewrite it as

$$\begin{cases} \partial_t a + \mathrm{div}\mathcal{Q}\bar{u} = f_1, \\ \partial_t(\mathcal{Q}\bar{u} + (-\Delta)^{-1}\nabla a) - \Delta(\mathcal{Q}\bar{u} + (-\Delta)^{-1}\nabla a) = \mathcal{Q}g_1 + (-\Delta)^{-1}\nabla\partial_t a. \end{cases} \tag{3.8}$$

We introduce a viscous effective flux

$$\omega \stackrel{\text{def}}{=} \mathcal{Q}\bar{u} + (-\Delta)^{-1}\nabla a = \Delta^{-1}\nabla\mathrm{div}\bar{u} + (-\Delta)^{-1}\nabla a = \nabla(-\Delta)^{-1}(a - \mathrm{div}\bar{u}).$$

Clearly,

$$\mathrm{div}\omega = -a + \mathrm{div}\mathcal{Q}\bar{u}.$$

It follows from (3.8) that ω satisfies

$$\partial_t\omega - \Delta\omega = \nabla(-\Delta)^{-1}(f_1 - \mathrm{div}g_1) + \omega - (-\Delta)^{-1}\nabla a. \tag{3.9}$$

By Lemma 2.14, we have

$$\begin{aligned} &\|\omega\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \\ &\lesssim \|\omega_0\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|f_1 - \mathrm{div}g_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-2}_{p,1}\right)} + \|\omega - \nabla(-\Delta)^{-1}a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}. \end{aligned} \tag{3.10}$$

Thanks to the high-frequency cutoff, we get

$$\begin{aligned} \|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} &\lesssim 2^{-2j_0}\|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}, \\ \|\nabla(-\Delta)^{-1}a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} &\lesssim 2^{-2j_0}\|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)}, \end{aligned}$$

which, together with (3.10), yields that

$$\begin{aligned} &\|\omega\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \\ &\lesssim \|\omega_0\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|f_1 - \mathrm{div}g_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-2}_{p,1}\right)} + 2^{-2j_0}\|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + 2^{-2j_0}\|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)}. \end{aligned} \tag{3.11}$$

On the other hand, we may rewrite the first equation in (3.3) as the following transport system with damping term,

$$\partial_t a + a + (\bar{u} + u_L)\cdot\nabla a + a\mathrm{div}\bar{u} = -\mathrm{div}\omega.$$

Applying the operator $\dot{\Delta}_j$ to the equation above gives

$$\partial_t \dot{\Delta}_j a + \dot{\Delta}_j a + (\bar{u} + u_L)\cdot\nabla\dot{\Delta}_j a = -\dot{\Delta}_j(a\mathrm{div}\bar{u}) - \dot{\Delta}_j\mathrm{div}\omega + \dot{R}_j \tag{3.12}$$

with $\dot{R}_j = [(\bar{u} + u_L)\cdot\nabla, \dot{\Delta}_j]a$. By the commutator estimate in Lemma 2.11,

$$\|\dot{R}_j\|_{L^p} \lesssim c_k 2^{-k\frac{n}{p}} \|\nabla(\bar{u} + u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}, \qquad \sum_{k\in\mathbb{Z}} c_k = 1.$$

Applying Lemma 2.9 yields

$$\|a\mathrm{div}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\mathrm{div}\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.$$

Therefore, evaluating the L^p norm of $\dot{\Delta}_j a$ via (3.12) and integrating in time, then multiplying by $2^{j\frac{n}{p}}$ and summing up over $j \ge j_0$, we obtain

$$\|a\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} + \|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} \lesssim \|a_0\|^h_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|\mathrm{div}\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} + \int_0^t \|\nabla(\bar{u} + u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau. \tag{3.13}$$

Combining (3.11) with (3.13), and then choosing j_0 large enough yield that

$$\begin{aligned} &\|\omega\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|a\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} + \|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} \\ &\lesssim \|\omega_0\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|a_0\|^h_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|f_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-2}_{p,1}\right)} + \|g_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \int_0^t \|\nabla(\bar{u} + u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau. \end{aligned} \tag{3.14}$$

Moreover, according to the definition of ω, we have

$$\begin{aligned} &\|\mathcal{Q}\bar{u}\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\mathcal{Q}\bar{u}\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \\ &\lesssim \|\omega\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\omega\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|a\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} + \|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)}, \end{aligned}$$

which together with (3.14) ensures that

$$\begin{aligned} &\|\mathcal{Q}\bar{u}\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|\mathcal{Q}\bar{u}\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|a\|^h_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} + \|a\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}}_{p,1}\right)} \\ &\lesssim \|\mathcal{Q}u_0\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|a_0\|^h_{\dot{B}^{\frac{n}{p}}_{p,1}} + \|f_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-2}_{p,1}\right)} + \|g_1\|^h_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \\ &\quad + \int_0^t \|\nabla(\bar{u} + u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau. \end{aligned} \tag{3.15}$$

3.3 Estimates in the low frequencies

We now turn to the *a priori* estimate of system (3.6) in the low frequencies based on the L^2-framework. Applying $\dot\Delta_j$ to (3.6), and then using the standard energy estimates, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\dot\Delta_j a\|_{L^2}^2+(\mathrm{div}\dot\Delta_j\mathcal{Q}\bar u,\dot\Delta_j a)=(\dot\Delta_j f_1,\dot\Delta_j a),\\
&\frac{1}{2}\frac{d}{dt}\|\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}^2+\|\nabla\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}^2+(\nabla\dot\Delta_j a,\dot\Delta_j\mathcal{Q}\bar u)=(\dot\Delta_j\mathcal{Q}g_1,\dot\Delta_j\mathcal{Q}\bar u),\\
&\frac{1}{2}\frac{d}{dt}\|\nabla\dot\Delta_j a\|_{L^2}^2+(\nabla\mathrm{div}\dot\Delta_j\mathcal{Q}\bar u,\nabla\dot\Delta_j a)=(\nabla\dot\Delta_j f_1,\nabla\dot\Delta_j a),\\
&\frac{1}{2}\frac{d}{dt}\|\dot\Delta_j\bar B\|_{L^2}^2+\nu\|\nabla\dot\Delta_j\bar B\|_{L^2}^2=(\dot\Delta_j h_1,\dot\Delta_j\bar B),\\
&\frac{d}{dt}(\dot\Delta_j\mathcal{Q}\bar u,\nabla\dot\Delta_j a)+\|\nabla\dot\Delta_j a\|_{L^2}^2-\|\nabla\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}^2-(\Delta\dot\Delta_j\mathcal{Q}\bar u,\nabla\dot\Delta_j a)\\
&\qquad\qquad=(\dot\Delta_j\mathcal{Q}g_1,\nabla\dot\Delta_j a)+(\nabla\dot\Delta_j f_1,\dot\Delta_j\mathcal{Q}\bar u).
\end{aligned}
\tag{3.16}
$$

Summing up the equations above yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+\frac{1}{2}\|\nabla\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}^2+\frac{1}{2}\|\nabla\dot\Delta_j a\|_{L^2}^2+\nu\|\nabla\dot\Delta_j\bar B\|_{L^2}^2\\
&\quad=(\dot\Delta_j f_1,\dot\Delta_j a)+(\dot\Delta_j\mathcal{Q}g_1,\dot\Delta_j\mathcal{Q}\bar u)+\frac{1}{2}(\nabla\dot\Delta_j f_1,\nabla\dot\Delta_j a)+(\dot\Delta_j h_1,\dot\Delta_j\bar B)\\
&\quad\ +\frac{1}{2}(\dot\Delta_j\mathcal{Q}g_1,\nabla\dot\Delta_j a)+\frac{1}{2}(\nabla\dot\Delta_j f_1,\dot\Delta_j\mathcal{Q}\bar u),
\end{aligned}
\tag{3.17}
$$

where

$$
\mathcal{L}_j^2\overset{\mathrm{def}}{=}\|\dot\Delta_j a\|_{L^2}^2+\frac{1}{2}\|\nabla\dot\Delta_j a\|_{L^2}^2+\|\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}^2+\|\dot\Delta_j\bar B\|_{L^2}^2+(\dot\Delta_j\mathcal{Q}\bar u,\nabla\dot\Delta_j a).
$$

To overcome the difficulties due to the lack of smallness in $u_L=e^{\mu t\Delta}\mathcal{P}u_0$ and $B_L=e^{\nu t\Delta}B_0$, we make use of various analysis tools such as commutator estimate and the important fact that $\mathrm{div}u_L=\mathrm{div}B_L=\mathrm{div}\bar B=0$. We rewrite the terms on the right-hand side of (3.17) as follows.

$$
(\dot\Delta_j f_1,\dot\Delta_j a)=\big(\dot\Delta_j(f_1+u_L\cdot\nabla a),\dot\Delta_j a\big)-\big([\dot\Delta_j,u_L\cdot\nabla]a,\dot\Delta_j a\big),
\tag{3.18}
$$

$$
\begin{aligned}
(\nabla\dot\Delta_j f_1,\nabla\dot\Delta_j a)&=\big(\nabla\dot\Delta_j(f_1+u_L\cdot\nabla a),\nabla\dot\Delta_j a\big)-\big(\nabla[\dot\Delta_j,u_L\cdot\nabla]a,\nabla\dot\Delta_j a\big)\\
&\quad+\big(\nabla u_L\cdot\nabla\dot\Delta_j a,\nabla\dot\Delta_j a\big),
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
(\nabla\dot\Delta_j f_1,\dot\Delta_j\mathcal{Q}\bar u)&=\big(\nabla\dot\Delta_j(f_1+u_L\cdot\nabla a),\dot\Delta_j\mathcal{Q}\bar u\big)-\big(\nabla[\dot\Delta_j,u_L\cdot\nabla]a,\dot\Delta_j\mathcal{Q}\bar u\big)\\
&\quad-\big(\nabla u_L\cdot\nabla\dot\Delta_j a,\dot\Delta_j\mathcal{Q}\bar u\big),
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
(\dot\Delta_j\mathcal{Q}g_1,\dot\Delta_j\mathcal{Q}\bar u)&=\big(\dot\Delta_j\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B),\dot\Delta_j\mathcal{Q}\bar u\big)\\
&\quad-\big([\dot\Delta_j,u_L\cdot\nabla]\mathcal{Q}\bar u,\dot\Delta_j\mathcal{Q}\bar u\big)+\big([\dot\Delta_j,L_3(a)]\mathrm{div}(B_L\otimes\bar B),\dot\Delta_j\mathcal{Q}\bar u\big)\\
&\quad+\big(L_3(a)[\dot\Delta_j,B_L\cdot\nabla]\bar B,\dot\Delta_j\mathcal{Q}\bar u\big)+\big(L_3(a)B_L\cdot\nabla\dot\Delta_j\bar B,\dot\Delta_j\mathcal{Q}\bar u\big)\\
&\quad+\big([\dot\Delta_j,\mathcal{Q}L_3(a)]\mathrm{div}(B_L\otimes\bar B),\dot\Delta_j\mathcal{Q}\bar u\big)\\
&\quad+\big(\mathcal{Q}L_3(a)[\dot\Delta_j,B_L\cdot\nabla]\bar B,\dot\Delta_j\mathcal{Q}\bar u\big)+\big(\mathcal{Q}(L_3(a))B_L\cdot\nabla\dot\Delta_j\bar B,\dot\Delta_j\mathcal{Q}\bar u\big),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
(\dot\Delta_j\mathcal{Q}g_1,\nabla\dot\Delta_j a)&=\big(\dot\Delta_j\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B),\nabla\dot\Delta_j a\big)\\
&\quad-\big([\dot\Delta_j,u_L\cdot\nabla]\mathcal{Q}\bar u,\nabla\dot\Delta_j a\big)+\big([\dot\Delta_j,L_3(a)]\mathrm{div}(B_L\otimes\bar B),\nabla\dot\Delta_j a\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\big(L_3(a)[\dot\Delta_j, B_L]\nabla\bar B, \nabla\dot\Delta_j a\big)+\big(L_3(a)B_L\cdot\nabla\dot\Delta_j\bar B, \nabla\dot\Delta_j a\big)\\
&+\big([\dot\Delta_j, \mathcal{Q}L_3(a)]\mathrm{div}(B_L\otimes\bar B), \nabla\dot\Delta_j a\big)+\big(\mathcal{Q}L_3(a)[\dot\Delta_j, B_L]\nabla\bar B, \nabla\dot\Delta_j a\big)\\
&+\big(\mathcal{Q}L_3(a)B_L\cdot\nabla\dot\Delta_j\bar B, \nabla\dot\Delta_j a\big), \qquad (3.22)\\
&(\dot\Delta_j h_1, \dot\Delta_j\bar B)=\big(\dot\Delta_j(h_1+u_L\cdot\nabla\bar B+B_L\cdot\mathrm{div}\bar u-B_L\cdot\nabla\bar u), \Delta_j\bar B\big)\\
&-\big([\dot\Delta_j, u_L\cdot\nabla]\bar B, \Delta_j\bar B\big)-\big([\dot\Delta_j, B_L]\mathrm{div}\bar u, \Delta_j\bar B\big)\\
&-\big(B_L\cdot\mathrm{div}\dot\Delta_j\mathcal{Q}\bar u, \dot\Delta_j\bar B\big)+\big([\dot\Delta_j, B_L\cdot\nabla]\bar u, \Delta_j\bar B\big)\\
&+\big(B_L\cdot\nabla\dot\Delta_j\mathcal{Q}\bar u, \Delta_j\bar B\big)+\big(B_L\cdot\nabla\dot\Delta_j\mathcal{P}\bar u, \Delta_j\bar B\big). \qquad (3.23)
\end{aligned}
$$

Inserting (3.18)–(3.23) in (3.17), integration by parts and using the fact

$$\mathcal{L}_j^2\approx\|(\dot\Delta_j a, \dot\Delta_j\mathcal{Q}\bar u, \dot\Delta_j\bar B)\|_{L^2}^2,$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_j^2+2^{2j}\mathcal{L}_j^2\\
&\lesssim\Big((1+2^j)\|\dot\Delta_j(f_1+u_L\cdot\nabla a)\|_{L^2}+\|\dot\Delta_j\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B)\|_{L^2}\\
&\quad+\|h_1+u_L\cdot\nabla\bar B+B_L\cdot\mathrm{div}\bar u-B_L\cdot\nabla\bar u\|_{L^2}+\|[\dot\Delta_j, L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}\\
&\quad+\|L_3(a)\|_{L^\infty}\|[\dot\Delta_j, B_L\cdot\nabla]\bar B\|_{L^2}+\|L_3(a)B_L\|_{L^\infty}\|\nabla\dot\Delta_j\bar B\|_{L^2}\\
&\quad+\|[\dot\Delta_j, \mathcal{Q}L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}+\|\mathcal{Q}L_3(a)\|_{L^\infty}\|[\dot\Delta_j, B_L\cdot\nabla]\bar B\|_{L^2}\\
&\quad+\|\mathcal{Q}(L_3(a))B_L\|_{L^\infty}\|\nabla\dot\Delta_j\bar B\|_{L^2}+\|[\dot\Delta_j, u_L\cdot\nabla]\bar B\|_{L^2}+\|[\dot\Delta_j, B_L]\mathrm{div}\mathcal{Q}\bar u\|_{L^2}\\
&\quad+\|B_L\|_{L^\infty}\|\mathrm{div}\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}+\|B_L\|_{L^\infty}\|\nabla\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}+(1+2^j)\|[\dot\Delta_j, u_L\cdot\nabla]a\|_{L^2}\\
&\quad+\|[\dot\Delta_j, u_L\cdot\nabla]\mathcal{Q}\bar u\|_{L^2}+2^j\|\nabla u_L\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}+\|[\dot\Delta_j, B_L\cdot\nabla]\bar u\|_{L^2}\\
&\quad+\|B_L\cdot\nabla\dot\Delta_j\mathcal{P}\bar u\|_{L^2}\Big)\mathcal{L}_j,
\end{aligned}
\tag{3.24}
$$

which leads to, for $j\le j_0$,

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_j+2^{2j}\mathcal{L}_j\\
&\lesssim\|\dot\Delta_j(f_1+u_L\cdot\nabla a)\|_{L^2}+\|\dot\Delta_j\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B)\|_{L^2}\\
&\quad+\|h_1+u_L\cdot\nabla\bar B+B_L\cdot\mathrm{div}\bar u-B_L\cdot\nabla\bar u\|_{L^2}+\|[\dot\Delta_j, L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}\\
&\quad+\|L_3(a)\|_{L^\infty}\|[\dot\Delta_j, B_L\cdot\nabla]\bar B\|_{L^2}+\|L_3(a)B_L\|_{L^\infty}\|\nabla\dot\Delta_j\bar B\|_{L^2}\\
&\quad+\|[\dot\Delta_j, \mathcal{Q}L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}+\|\mathcal{Q}L_3(a)\|_{L^\infty}\|[\dot\Delta_j, B_L\cdot\nabla]\bar B\|_{L^2}\\
&\quad+\|\mathcal{Q}(L_3(a))B_L\|_{L^\infty}\|\nabla\dot\Delta_j\bar B\|_{L^2}+\|[\dot\Delta_j, u_L\cdot\nabla]\bar B\|_{L^2}+\|[\dot\Delta_j, B_L]\mathrm{div}\mathcal{Q}\bar u\|_{L^2}\\
&\quad+\|B_L\|_{L^\infty}\|\mathrm{div}\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}+\|B_L\|_{L^\infty}\|\nabla\dot\Delta_j\mathcal{Q}\bar u\|_{L^2}+\|[\dot\Delta_j, u_L\cdot\nabla]a\|_{L^2}\\
&\quad+\|[\dot\Delta_j, u_L\cdot\nabla]\mathcal{Q}\bar u\|_{L^2}+\|\nabla u_L\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}+\|[\dot\Delta_j, B_L\cdot\nabla]\bar u\|_{L^2}+\|B_L\cdot\nabla\dot\Delta_j\mathcal{P}\bar u\|_{L^2}.
\end{aligned}
\tag{3.25}
$$

By Lemmas 2.11–2.12, the commutators in (3.25) are bounded by

$$
\begin{aligned}
&\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j, L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}\\
&\qquad\lesssim\|L_3(a)\|_{\dot B^{\frac{n}{p}}_{p,1}}\|B_L\otimes\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}}\lesssim\|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big(1+\|a\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big),
\end{aligned}
$$

$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,\mathcal{Q}L_3(a)]\mathrm{div}(B_L\otimes\bar B)\|_{L^2}$$
$$\lesssim \|L_3(a)\|_{\dot B^{\frac{n}{p}}_{p,1}}\|B_L\otimes\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}}\lesssim \|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big(1+\|a\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big),$$
$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,B_L\cdot\nabla]\bar B\|_{L^2}\lesssim \|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}},$$
$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,u_L\cdot\nabla]\bar B\|_{L^2}\lesssim \|u_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}},$$
$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,B_L\cdot\nabla]\bar u\|_{L^2}\lesssim \|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar u\|_{\dot B^{\frac{n}{p}}_{p,1}},$$

$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,B_L]\mathrm{div}\mathcal{Q}\bar u\|_{L^2}\lesssim \|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar u\|_{\dot B^{\frac{n}{p}}_{p,1}},$$
$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,u_L\cdot\nabla]a\|_{L^2}\lesssim \|u_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|a\|_{\dot B^{\frac{n}{p}}_{p,1}},$$
$$\sum_{j\le j_0} 2^{j(\frac{n}{2}-1)}\|[\dot\Delta_j,u_L\cdot\nabla]\mathcal{Q}\bar u\|_{L^2}\lesssim \|u_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar u\|_{\dot B^{\frac{n}{p}}_{p,1}}.$$

Inserting these estimates in (3.25), integrating over $[0,t]$, multiplying by $2^{j(\frac{n}{2}-1)}$ and then summing up on $j\le j_0$, we obtain

$$\begin{aligned}
&\|(a,\mathcal{Q}\bar u,\bar B)\|^l_{\widetilde L^\infty_t(\dot B^{\frac{n}{2}-1}_{2,1})}+\int_0^t\|(a,\mathcal{Q}\bar u,\bar B)\|^l_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau\\
&\lesssim \|a_0\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}+\|\mathcal{Q}u_0\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}+\int_0^t\|f_1+u_L\cdot\nabla a\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\quad+\int_0^t\|\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B)\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\quad+\int_0^t\|h_1+u_L\cdot\nabla\bar B+B_L\cdot\mathrm{div}\bar u-B_L\cdot\nabla\bar u\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\quad+\int_0^t\|(u_L,B_L)\|_{\dot B^{\frac{n}{p}}_{p,1}}\|(a,\bar u,\bar B)\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big(1+\|a\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big)d\tau\\
&\quad+\int_0^t\|L_3(a)B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|^l_{\dot B^{\frac{n}{2}}_{2,1}}d\tau+\int_0^t\|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\mathcal{Q}\bar u\|^l_{\dot B^{\frac{n}{2}}_{2,1}}d\tau\\
&\quad+\int_0^t\|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\|a\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau+\int_0^t\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|B_L\cdot\nabla\dot\Delta_j\mathcal{P}\bar u\|_{L^2}d\tau,
\end{aligned}\tag{3.26}$$

where we used the fact $\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ is continuously embedded into $L^\infty(\mathbb{R}^n)$. We bound the last term in (3.26), namely

$$\int_0^t\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|B_L\cdot\nabla\dot\Delta_j\mathcal{P}\bar u\|_{L^2}d\tau.$$

Applying the Littlewood-Paley decomposition to B_L and the quasi-orthogonality property of $\dot{\Delta}_j$, we have, for any fixed j,

$$B_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}=\sum_k\dot{\Delta}_kB_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}=\sum_{|k-j|\le 2}\dot{\Delta}_kB_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}.$$

By Hölder's inequality and Bernstein's inequality, for $\frac{1}{2}=\frac{1}{p}+\frac{1}{p^*}$ and $p\le p^*$, we have

$$\begin{aligned}\|B_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^2}&\le\sum_{|k-j|\le 2}\|\dot{\Delta}_kB_L\|_{L^p}\|\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^{p^*}}\\&\le\sum_{|k-j|\le 2}2^{j(\frac{n}{p}-\frac{n}{p^*})}\|\dot{\Delta}_kB_L\|_{L^p}\|\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^p}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{j\le j_0}2^{j(\frac{n}{2}-1)}\|B_L\cdot\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^2}\\&\quad\lesssim\sum_{j\le j_0}\sum_{|k-j|\le 2}2^{j(\frac{n}{2}-1)}2^{j(\frac{n}{p}-\frac{n}{p^*})}2^{-k\frac{n}{p}}2^{-j(\frac{n}{p}-1)}2^{k\frac{n}{p}}\|\dot{\Delta}_kB_L\|_{L^p}2^{j(\frac{n}{p}-1)}\|\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^p}\\&\quad\lesssim\sum_{j\le j_0}\sum_{|k-j|\le 2}2^{(j-k)\frac{n}{p}}2^{k\frac{n}{p}}\|\dot{\Delta}_kB_L\|_{L^p}2^{j(\frac{n}{p}-1)}\|\nabla\dot{\Delta}_j\mathcal{P}\bar{u}\|_{L^p}\\&\quad\lesssim\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.\end{aligned}$$

Inserting this bound in (3.26) leads to

$$\begin{aligned}&\|(a,\mathcal{Q}\bar{u},\bar{B})\|^l_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1}_{2,1})}+\int_0^t\|(a,\mathcal{Q}\bar{u},\bar{B})\|^l_{\dot{B}^{\frac{n}{2}+1}_{2,1}}d\tau\\&\quad\lesssim\|a_0\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\mathcal{Q}u_0\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\int_0^t\|f_1+u_L\cdot\nabla a\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\qquad+\int_0^t\|\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar{u}-L_3(a)B_L\cdot\nabla\bar{B})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\qquad+\int_0^t\|h_1+u_L\cdot\nabla\bar{B}+B_L\cdot\mathrm{div}\bar{u}-B_L\cdot\nabla\bar{u}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\qquad+\int_0^t\|(u_L,B_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|(a,\bar{u},\bar{B})\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)d\tau\\&\qquad+\int_0^t\|L_3(a)B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{B}\|^l_{\dot{B}^{\frac{n}{2}}_{2,1}}d\tau+\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{Q}\bar{u}\|^l_{\dot{B}^{\frac{n}{2}}_{2,1}}d\tau\\&\qquad+\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau+\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau.\end{aligned}\tag{3.27}$$

3.4 Nonlinear estimates

This subsection is devoted to dealing with the nonlinear estimates. Putting (3.7), (3.15) and (3.27) together, and then using the definitions of hybrid Besov spaces, we conclude that

$$
\begin{aligned}
\|(a,\bar u,\bar B)\|_{X(t)} \lesssim{}& \|a_0\|_{\dot B^{\frac n2-1,\frac np}_{2,p}} + \|\mathcal{Q}u_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}} + \int_0^t \|f_1\|^h_{\dot B^{\frac np-2}_{p,1}}d\tau + \int_0^t \|g_1\|_{\dot B^{\frac np-1}_{p,1}}d\tau\\
&+\int_0^t \|h_1\|^h_{\dot B^{\frac np-1}_{p,1}}d\tau + \int_0^t \|\mathrm{div}(a\bar u)\|^l_{\dot B^{\frac n2-1}_{2,1}}d\tau\\
&+\int_0^t \|\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u - L_3(a)B_L\cdot\nabla\bar B)\|^l_{\dot B^{\frac n2-1}_{2,1}}d\tau\\
&+\int_0^t \|h_1+u_L\cdot\nabla\bar B + B_L\cdot \mathrm{div}\bar u - B_L\cdot\nabla\mathcal{Q}\bar u\|^l_{\dot B^{\frac n2-1}_{2,1}}d\tau\\
&+\int_0^t \|B_L\|_{\dot B^{\frac np}_{p,1}}\|\bar B\|^l_{\dot B^{\frac n2}_{2,1}}(1+\|a\|_{\dot B^{\frac np}_{p,1}})d\tau + \int_0^t \|\nabla(\bar u+u_L)\|_{\dot B^{\frac np}_{p,1}}\|a\|_{\dot B^{\frac np}_{p,1}}d\tau\\
&+\int_0^t \|(u_L,B_L)\|_{\dot B^{\frac np}_{p,1}}\|(a,\bar u,\bar B)\|_{\dot B^{\frac np}_{p,1}}\Big(1+\|a\|_{\dot B^{\frac np}_{p,1}}\Big)d\tau\\
&+\int_0^t \|B_L\|_{\dot B^{\frac np}_{p,1}}\|\mathcal{Q}\bar u\|^l_{\dot B^{\frac n2}_{2,1}}d\tau + \int_0^t \|B_L\|_{\dot B^{\frac np}_{p,1}}\|\mathcal{P}\bar u\|_{\dot B^{\frac np}_{p,1}}d\tau\\
&+\int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}}\|a\|^l_{\dot B^{\frac n2-1}_{2,1}}d\tau.
\end{aligned}
\tag{3.28}
$$

We bound the nonlinear terms on right-hand side of (3.28). We start with the terms in f_1. Thanks to Lemma 2.9, interpolation inequalities and Young's inequality,

$$
\begin{aligned}
\int_0^t \|\mathrm{div}(a\bar u)\|^h_{\dot B^{\frac np-2}_{p,1}}d\tau \lesssim{}& \int_0^t \|a\bar u\|_{\dot B^{\frac np}_{p,1}}d\tau\\
\lesssim{}& \int_0^t \|\bar u\|_{\dot B^{\frac np}_{p,1}}\|a\|_{\dot B^{\frac np}_{p,1}}d\tau\\
\lesssim{}& \int_0^t \Big(\|\bar u\|^{\frac12}_{\dot B^{\frac np-1}_{p,1}}\|\bar u\|^{\frac12}_{\dot B^{\frac np+1}_{p,1}}\Big)\Big(\|a^\ell\|^{\frac12}_{\dot B^{\frac n2-1}_{2,1}}\|a^\ell\|^{\frac12}_{\dot B^{\frac n2+1}_{2,1}} + \|a^h\|^{\frac12}_{\dot B^{\frac np}_{p,1}}\|a^h\|^{\frac12}_{\dot B^{\frac np}_{p,1}}\Big)d\tau\\
\lesssim{}& \int_0^t \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac np-1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}\Big)\|a\|_{\dot B^{\frac n2+1,\frac np}_{2,p}}d\tau\\
&+\int_0^t \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac np+1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac n2+1,\frac np+1}_{2,p}}\Big)\|a\|_{\dot B^{\frac n2-1,\frac np}_{2,p}}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t \|u_L\cdot\nabla a\|^h_{\dot B^{\frac np-2}_{p,1}}d\tau \lesssim{}& \int_0^t \|u_L\cdot\nabla a\|^h_{\dot B^{\frac np-1}_{p,1}}d\tau\\
\lesssim{}& \int_0^t \|u_L\|_{\dot B^{\frac np}_{p,1}}\|\nabla a\|_{\dot B^{\frac np-1}_{p,1}}d\tau\\
\lesssim{}& \int_0^t \|u_L\|_{\dot B^{\frac np}_{p,1}}\Big(\|a^\ell\|_{\dot B^{\frac np}_{p,1}} + \|a^h\|_{\dot B^{\frac np}_{p,1}}\Big)d\tau
\end{aligned}
$$

$$\lesssim \int_0^t \|u_L\|^{\frac12}_{\dot B^{\frac np-1}_{p,1}} \|u_L\|^{\frac12}_{\dot B^{\frac np+1}_{p,1}} \Big(\|a^\ell\|^{\frac12}_{\dot B^{\frac n2-1}_{2,1}} \|a^\ell\|^{\frac12}_{\dot B^{\frac n2+1}_{2,1}} + \|a^h\|^{\frac12}_{\dot B^{\frac np}_{p,1}} \|a^h\|^{\frac12}_{\dot B^{\frac np}_{p,1}} \Big) d\tau$$

$$\lesssim \varepsilon \|a\|_{L^1_t(\dot B^{\frac n2+1,\frac np}_{2,p})} + \frac{1}{\varepsilon} \|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}} \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|a\|_{\dot B^{\frac n2-1,\frac np}_{2,p}} d\tau.$$

We turn to the terms in g_1. By Lemma 2.9, we first infer that

$$\begin{aligned}
\int_0^t \|\bar u \cdot \nabla \bar u\|_{\dot B^{\frac np-1}_{p,1}} d\tau &\lesssim \int_0^t \|\bar u\|_{\dot B^{\frac np-1}_{p,1}} \|\bar u\|_{\dot B^{\frac np+1}_{p,1}} d\tau \\
&\lesssim \int_0^t \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac np-1}_{p,1}} \Big) \\
&\quad \times \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac np+1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac n2+1,\frac np+1}_{2,p}} \Big) d\tau, \\
\int_0^t \|\bar u \cdot \nabla u_L + u_L \cdot \nabla u_L\|_{\dot B^{\frac np-1}_{p,1}} d\tau &\lesssim \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|\bar u\|_{\dot B^{\frac np-1}_{p,1}} d\tau + \|u_L \cdot \nabla u_L\|_{L^1_t(\dot B^{\frac np-1}_{p,1})} \\
&\lesssim \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac np-1}_{p,1}} \Big) d\tau + C_L.
\end{aligned}$$

It follows from Lemma 2.9, interpolation inequalities and Young's inequality that

$$\begin{aligned}
\int_0^t \|u_L \cdot \nabla \bar u\|_{\dot B^{\frac np-1}_{p,1}} d\tau &\lesssim \int_0^t \|u_L\|_{\dot B^{\frac np}_{p,1}} \|\bar u\|_{\dot B^{\frac np}_{p,1}} d\tau \\
&\lesssim \varepsilon \|\bar u\|_{L^1_t(\dot B^{\frac np+1}_{p,1})} + \frac{1}{\varepsilon} \|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}} \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|\bar u\|_{\dot B^{\frac np-1}_{p,1}} d\tau \\
&\lesssim \varepsilon \Big(\|\mathcal{P}\bar u\|_{L^1_t(\dot B^{\frac np+1}_{p,1})} + \|\mathcal{Q}\bar u\|_{L^1_t(\dot B^{\frac n2+1,\frac np+1}_{2,p})} \Big) \\
&\quad + \frac{1}{\varepsilon} \|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}} \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac np-1}_{p,1}} \Big) d\tau.
\end{aligned}$$

By Lemmas 2.9 and 2.12, we have

$$\begin{aligned}
&\int_0^t \|L_1(a)\mathcal{A}(\bar u + u_L)\|_{\dot B^{\frac np-1}_{p,1}} d\tau \\
&\lesssim \int_0^t \|L_1(a)\|_{\dot B^{\frac np}_{p,1}} \|\bar u\|_{\dot B^{\frac np+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|L_1(a)\|_{\dot B^{\frac np}_{p,1}} d\tau \\
&\lesssim \int_0^t \|a\|_{\dot B^{\frac np}_{p,1}} \|\bar u\|_{\dot B^{\frac np+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|a\|_{\dot B^{\frac np}_{p,1}} d\tau \\
&\lesssim \int_0^t \|a\|_{\dot B^{\frac n2-1,\frac np}_{2,p}} \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac np+1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac n2+1,\frac np+1}_{2,p}} \Big) d\tau + \int_0^t \|u_L\|_{\dot B^{\frac np+1}_{p,1}} \|a\|_{\dot B^{\frac n2-1,\frac np}_{2,p}} d\tau
\end{aligned}$$

and

$$\begin{aligned}\int_0^t \|L_2(a)\nabla a\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|L_2(a)\|_{\dot B^{\frac{n}{p}}_{p,1}} \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\lesssim \int_0^t \Big(\|a^\ell\|_{\dot B^{\frac{n}{p}}_{p,1}} + \|a^h\|_{\dot B^{\frac{n}{p}}_{p,1}}\Big)^2 d\tau \\ &\lesssim \int_0^t \Big(\|a^\ell\|^{\frac12}_{\dot B^{\frac{n}{2}-1}_{2,1}} \|a^\ell\|^{\frac12}_{\dot B^{\frac{n}{2}+1}_{2,1}} + \|a^h\|^{\frac12}_{\dot B^{\frac{n}{p}}_{p,1}} \|a^h\|^{\frac12}_{\dot B^{\frac{n}{p}}_{p,1}}\Big)^2 d\tau \\ &\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \|a\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}} d\tau.\end{aligned}$$

To handle the term $L_3(a)\Big(\mathrm{div}\big(2\widetilde{\mu}(a)D(\bar u + u_L)\big) + \nabla\big(\widetilde{\lambda}(a))\mathrm{div}\bar u\big)\Big)$, we first rewrite it as follows

$$\begin{aligned}&L_3(a)\Big(\mathrm{div}\big(2\widetilde{\mu}(a)D(\bar u + u_L)\big) + \nabla\big(\widetilde{\lambda}(a))\mathrm{div}\bar u\big)\Big) \\ &= 2\frac{\tilde\mu(a)}{1+a}\mathrm{div}\big(D(\bar u + u_L)\big) + 2\frac{\widetilde{\mu}'(a)}{1+a}D(\bar u + u_L)\cdot\nabla a + \frac{\tilde\lambda(a)}{1+a}\nabla\mathrm{div}\bar u + \frac{\widetilde{\lambda}'(a)}{1+a}\mathrm{div}\bar u\cdot\nabla a \\ &\triangleq 2\frac{\tilde\mu(a)}{1+a}\mathrm{div}\big(D(\bar u + u_L)\big) + 2\nabla\big(L_4(a)\big)D(\bar u + u_L) + \frac{\tilde\lambda(a)}{1+a}\nabla\mathrm{div}\bar u + \nabla\big(L_5(a)\big)\mathrm{div}\bar u,\end{aligned}$$

where we have used the facts $\frac{\widetilde{\mu}'(a)}{1+a}\nabla a \overset{\text{def}}{=} \nabla\big(L_4(a)\big)$ and $\frac{\widetilde{\lambda}'(a)}{1+a}\nabla a \overset{\text{def}}{=} \nabla\big(L_5(a)\big)$ for some smooth functions $L_4(a)$ and $L_5(a)$ vanishing at 0. Since $\frac{\tilde\mu(a)}{1+a}$, $\frac{\tilde\lambda(a)}{1+a}$, $L_4(a)$ and $L_5(a)$ are smooth functions vanishing at 0, we obtain, by Lemmas 2.9 and 2.12,

$$\begin{aligned}&\int_0^t \Big\|\frac{\tilde\mu(a)}{1+a}\mathrm{div}\big(D(\bar u + u_L)\big)\Big\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \Big\|\frac{\tilde\mu(a)}{1+a}\Big\|_{\dot B^{\frac{n}{p}}_{p,1}} \|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \Big\|\frac{\tilde\mu(a)}{1+a}\Big\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} \|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} d\tau,\end{aligned}$$

$$\begin{aligned}&\int_0^t \|\nabla\big(L_4(a)\big)D(\bar u + u_L)\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \|L_4(a)\|_{\dot B^{\frac{n}{p}}_{p,1}} \|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \|L_4(a)\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} \|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau \\ &\quad\lesssim \int_0^t \|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) d\tau + \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} d\tau,\end{aligned}$$

$$
\begin{aligned}
&\int_0^t \|\frac{\tilde{\lambda}(a)}{1+a}\nabla\mathrm{div}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \lesssim \int_0^t \|\frac{\tilde{\lambda}(a)}{1+a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\nabla\mathrm{div}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \\
&\quad \lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} d\tau \\
&\quad \lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) d\tau, \\
&\int_0^t \|\nabla\big(L_5(a)\big)\mathrm{div}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \lesssim \int_0^t \|L_5(a)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} d\tau \\
&\quad \lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} d\tau \\
&\quad \lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) d\tau.
\end{aligned}
$$

By Lemmas 2.9 and 2.12, we deduce that

$$
\begin{aligned}
&\int_0^t \|L_3(a)\bar{B}\cdot\nabla\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau \\
&\quad \lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau, \\
&\int_0^t \|L_3(a)\bar{B}\cdot\nabla B_L + L_3(a)B_L\cdot\nabla B_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \\
&\quad \lesssim \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau + \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}\Big) \|B_L\cdot\nabla B_L\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \\
&\quad \lesssim \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau + \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big) C_L.
\end{aligned}
$$

Using Lemmas 2.9 and 2.12, and an interpolation inequality leads to

$$
\begin{aligned}
\int_0^t \|L_3(a)B_L\cdot\nabla\bar{B}\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) \|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau \\
&\lesssim \varepsilon\Big(1+\|a\|_{\tilde{L}^\infty_t\big(\dot{B}^{\frac{n}{p}}_{p,1}\big)}\Big)\|\bar{B}\|_{L^1_t\big(\dot{B}^{\frac{n}{p}+1}_{p,1}\big)} \\
&\quad + \frac{1}{\varepsilon}\|B_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau \\
&\lesssim \varepsilon\Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)\|\bar{B}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})} \\
&\quad + \frac{1}{\varepsilon}\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau.
\end{aligned}
$$

To bound the term $\int_0^t \|h_1\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau$, we employ Lemma 2.9 to obtain

$$\begin{aligned}
\int_0^t \|\mathrm{div}\bar u\cdot\bar B\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau\\
&\lesssim \int_0^t \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\Big)\|\bar B\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} d\tau,\\
\int_0^t \|\bar u\cdot\nabla\bar B\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau\\
&\lesssim \int_0^t \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\Big)\|\bar B\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} d\tau,\\
\int_0^t \|\bar u\cdot\nabla B_L\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|B_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau\\
&\lesssim \int_0^t \Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\Big)\|B_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} d\tau.
\end{aligned}$$

By Lemma 2.9, and an interpolation inequality and Young's inequality, we get

$$\begin{aligned}
\int_0^t \|\mathrm{div}\bar u\cdot B_L\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|\bar u\|_{\dot B^{\frac{n}{p}}_{p,1}}\|B_L\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau\\
&\lesssim \varepsilon\|\bar u\|_{L^1_t(\dot B^{\frac{n}{p}+1}_{p,1})} + \frac{1}{\varepsilon}\|B_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\int_0^t \|B_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\|\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau\\
&\lesssim \varepsilon\Big(\|\mathcal{P}\bar u\|_{L^1_t(\dot B^{\frac{n}{p}+1}_{p,1})} + \|\mathcal{Q}\bar u\|_{L^1_t(\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}\Big)\\
&\quad + \frac{1}{\varepsilon}\|B_0\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\int_0^t \|B_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\Big) d\tau,\\
\int_0^t \|u_L\cdot\nabla\bar B\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau &\lesssim \int_0^t \|u_L\|_{\dot B^{\frac{n}{p}}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{p}}_{p,1}} d\tau\\
&\lesssim \varepsilon\|\bar B\|_{L^1_t(\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})} + \frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\int_0^t \|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\|\bar B\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} d\tau.
\end{aligned}$$

It follows from the definition of $\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}$ that

$$\int_0^t \|u_L\cdot\nabla B_L\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau \lesssim \|u_L\cdot\nabla B_L\|_{L^1_t(\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p})} \lesssim C_L.$$

Similarly, we also obtain the corresponding estimates of other terms such as $\bar B\cdot\nabla\bar u$, $\bar B\cdot\nabla u_L$, $B_L\cdot\nabla\bar u$, $B_L\cdot\nabla u_L$. We omit the details. Thus,

$$\begin{aligned}
&\int_0^t \|f_1\|^h_{\dot B^{\frac{n}{p}-2}_{p,1}} d\tau + \int_0^t \|g_1\|_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau + \int_0^t \|h_1\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}} d\tau\\
&\quad\lesssim \varepsilon\Big(\|(a,\bar u,\bar B)\|_{X(t)} + \|(a,\bar u,\bar B)\|^2_{X(t)}\Big) + \|(a,\bar u,\bar B)\|^2_{X(t)} + \|(a,\bar u,\bar B)\|^3_{X(t)}\\
&\qquad + \int_0^t \Big((\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}} + 1)\|u_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}} + (\frac{1}{\varepsilon}\|B_0\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + 1)\|B_L\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\Big)
\end{aligned}$$

$$
\times\Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}+\big(\|(\mathcal{Q}\bar u,\bar B)\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+\|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\big)\big(1+\|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\big)\Big)d\tau
+\Big(1+\|a\|_{\tilde L^\infty_t(\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)C_L. \tag{3.29}
$$

To handle the term $\int_0^t\|\mathrm{div}(a\bar u)\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau$, we use Bony's decomposition to obtain

$$
a\bar u=\dot T_a\bar u+\dot R(a,\bar u)+\dot T_{\bar u}a^\ell+\dot T_{\bar u}a^h.
$$

It follows from Corollaries 2.7 and 2.8 that

$$
\begin{aligned}
\int_0^t(\|\dot T_a\bar u\|^l_{\dot B^{\frac{n}{2}}_{2,1}}+\|\dot R(a,\bar u)\|^l_{\dot B^{\frac{n}{2}}_{2,1}})d\tau&\lesssim\int_0^t\|a\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}}d\tau\\
&\lesssim\int_0^t\|a\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}+1}_{p,1}}+\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big)d\tau,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
\int_0^t\|\dot T_{\bar u}a^\ell\|^l_{\dot B^{\frac{n}{2}}_{2,1}}d\tau&\lesssim\int_0^t\|\bar u\|_{\dot B^{-1}_{\infty,\infty}}\|a^\ell\|_{\dot B^{\frac{n}{2}+1}_{2,1}}d\tau\\
&\lesssim\int_0^t\Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}+\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big)\|a\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}}d\tau,
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
\int_0^t\|\dot T_{\bar u}a^h\|^l_{\dot B^{\frac{n}{2}}_{2,1}}d\tau&\lesssim2^{j_0}\int_0^t\|\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|a^h\|_{\dot B^{\frac{n}{p}}_{p,1}}d\tau\\
&\lesssim2^{j_0}\int_0^t\Big(\|\mathcal{P}\bar u\|_{\dot B^{\frac{n}{p}-1}_{p,1}}+\|\mathcal{Q}\bar u\|_{\dot B^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big)\|a\|_{\dot B^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}}d\tau.
\end{aligned} \tag{3.32}
$$

To bound the term $\int_0^t\|\mathcal{Q}(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B)\|^l_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau$, we observe that

$$
\begin{aligned}
&\mathcal{Q}\Big(g_1+u_L\cdot\nabla\mathcal{Q}\bar u-L_3(a)B_L\cdot\nabla\bar B\Big)\\
&=-\mathcal{Q}\Big(u_L\cdot\nabla\mathcal{P}\bar u+\bar u\cdot\nabla\bar u+\bar u\cdot\nabla u_L+u_L\cdot\nabla u_L+L_1(a)\mathcal{A}(\bar u+u_L)+L_2(a)\nabla a\\
&\quad-L_3(a)\Big(\mathrm{div}\big(2\widetilde\mu(a)D(\bar u+u_L)\big)+\nabla\big(\widetilde\lambda(a)\mathrm{div}\bar u\big)+B_L\cdot\nabla B_L+\bar B\cdot\nabla B_L+\bar B\cdot\nabla\bar B\Big)\Big)\\
&=-\mathcal{Q}\Big(u_L\cdot\nabla\mathcal{P}\bar u+\bar u\cdot\nabla\bar u+\bar u\cdot\nabla u_L+u_L\cdot\nabla u_L+L_1(a)\mathcal{A}(\bar u+u_L)+L_2(a)\nabla a\\
&\quad-2\frac{\tilde\mu(a)}{1+a}\mathrm{div}\big(D(\bar u+u_L)\big)-2\nabla\big(L_4(a)\big)D(\bar u+u_L)-\frac{\tilde\lambda(a)}{1+a}\nabla\mathrm{div}\bar u-\nabla\big(L_5(a)\big)\mathrm{div}\bar u\\
&\quad+L_3(a)B_L\cdot\nabla B_L+L_3(a)\bar B\cdot\nabla B_L+L_3(a)\bar B\cdot\nabla\bar B\Big).
\end{aligned}
$$

We bound term by term above in what follows. To deal with the term $\mathcal{Q}(u_L\cdot\nabla\mathcal{P}\bar{u})$, we use Bony's decomposition to obtain

$$\mathcal{Q}(u_L\cdot\nabla\mathcal{P}\bar{u})=[\mathcal{Q},\dot{T}_{u_{L,k}}]\partial_k\mathcal{P}\bar{u}+\mathcal{Q}(\dot{T}_{\partial_k\mathcal{P}\bar{u}}u_{L,k}+\dot{R}(u_{L,k},\partial_k\mathcal{P}\bar{u}).$$

Applying Lemma 2.10, we get

$$\begin{aligned}\int_0^t\|[\mathcal{Q},\dot{T}_{u_{L,k}}]\partial_k\mathcal{P}\bar{u}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau&\lesssim\int_0^t\|\nabla u_L\|_{\dot{B}^{\frac{n}{p^*}-1}_{p^*,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\&\lesssim\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau,\end{aligned}$$

where we used the facts $\frac{n}{p^*}-1\le 0$ and $p\le p^*$ with $\frac{1}{p}+\frac{1}{p^*}=\frac{1}{2}$. By Corollary 2.7, we obtain

$$\int_0^t\Big(\|\dot{T}_{\partial_k\mathcal{P}\bar{u}}u_{L,k}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\dot{R}(u_{L,k},\partial_k\mathcal{P}\bar{u})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\Big)d\tau\lesssim\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau.$$

By an interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}&\int_0^t\|\mathcal{Q}(u_L\cdot\nabla\mathcal{P}\bar{u})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\lesssim\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\&\lesssim\int_0^t\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau\\&\le\varepsilon\Big(\|\mathcal{P}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}+\|\mathcal{Q}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}\right)}\Big)\\&\quad+\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau.\end{aligned}$$

Applying Corollary 2.7, we deduce that

$$\begin{aligned}&\int_0^t\|\bar{u}\cdot\nabla u_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\lesssim\int_0^t\|\dot{T}_{\bar{u}}\nabla u_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau+\int_0^t\|\dot{T}_{\nabla u_L}\bar{u}+\dot{R}(\bar{u},\nabla u_L)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\&\lesssim\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau+\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\&\lesssim\varepsilon\Big(\|\mathcal{P}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}+\|\mathcal{Q}\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}\right)}\Big)\\&\quad+\Big(1+\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^t \|\bar{u}\cdot\nabla\bar{u}+u_L\cdot\nabla u_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\lesssim \int_0^t \|\dot{T}_{\bar{u}}\nabla\bar{u}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau+\int_0^t \|\dot{T}_{\nabla\bar{u}}\bar{u}+\dot{R}(\bar{u},\nabla\bar{u})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau+\|u_L\cdot\nabla u_L\|^l_{L^1_t\left(\dot{B}^{\frac{n}{2}-1}_{2,1}\right)}\\
&\lesssim \int_0^t \|\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau+\int_0^t \|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau+C_L\\
&\lesssim \int_0^t \|\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau+C_L\\
&\lesssim \int_0^t \left(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\right)\left(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\right)d\tau+C_L.
\end{aligned}
$$

To bound the term $L_2(a)\nabla a$, it follows from Bony's decomposition that

$$
L_2(a)\nabla a=\dot{T}_{\nabla a}L_2(a)+\dot{R}(L_2(a),\nabla a)+\dot{T}_{L_2(a)}\nabla a^\ell+\dot{T}_{L_2(a)}\nabla a^h.
$$

By Corollary 2.7, Lemma 2.12, and an interpolation inequality, we get

$$
\begin{aligned}
\int_0^t \|\dot{T}_{\nabla a}L_2(a)+\dot{R}(L_2(a),\nabla a)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|L_2(a)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla a\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\
&\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
&\lesssim \int_0^t \left(\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|a^h\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a^h\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)^2d\tau\\
&\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\|a\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t \|\dot{T}_{L_2(a)}\nabla a^\ell\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|L_2(a)\|_{L^\infty}\|\nabla a^\ell\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a^\ell\|_{\dot{B}^{\frac{n}{2}}_{2,1}}d\tau\\
&\lesssim \int_0^t \left(\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|a^h\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a^h\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\\
&\quad\times\left(\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|a^\ell\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{2}+1}_{2,1}}\right)d\tau\\
&\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\|a\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}}d\tau.
\end{aligned}
$$

By Corollary 2.7 and Lemma 2.12, we have

$$\begin{aligned}\int_0^t \|\dot{T}_{L_2(a)}\nabla a^h\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|L_2(a)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\nabla a^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\ &\lesssim \int_0^t \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\|a\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|a^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\ &\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\|a\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau.\end{aligned}$$

To handle the term $L_1(a)\mathcal{A}(\bar{u}+u_L)$, we decompose it into

$$L_1(a)\mathcal{A}(\bar{u}+u_L)=\dot{T}_{L_1(a)}\mathcal{A}(\bar{u}+u_L)+\dot{T}_{\mathcal{A}(\bar{u}+u_L)}L_1(a)+\dot{R}\big(L_1(a),\mathcal{A}(\bar{u}+u_L)\big).$$

which, together with the low-frequency cutoff, Corollary 2.7 and Lemma 2.12, yields

$$\begin{aligned}&\int_0^t \|\dot{T}_{L_1(a)}\mathcal{A}(\bar{u}+u_L)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\ &\lesssim \int_0^t \|\dot{T}_{L_1(a)}\mathcal{A}(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{2}-2}_{2,1}}d\tau\\ &\lesssim \int_0^t \|\bar{u}+u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|L_1(a)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\ &\lesssim \int_0^t \|\bar{u}+u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)\|a\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\ &\lesssim \int_0^t \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big)\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau\\ &\quad+\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \Big(\|\dot{T}_{\mathcal{A}(\bar{u}+u_L)}L_1(a)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|\dot{R}(L_1(a),\mathcal{A}(\bar{u}+u_L))\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\Big)d\tau\\ &\lesssim \int_0^t \|\bar{u}+u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|L_1(a)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\ &\lesssim \int_0^t \|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau+\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\ &\lesssim \int_0^t \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big)\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau+\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau.\end{aligned}$$

Similarly, we can bound the terms $2\frac{\tilde{\mu}(a)}{1+a}\mathrm{div}\big(D(\bar{u}+u_L)\big)$ and $\frac{\tilde{\lambda}(a)}{1+a}\nabla\mathrm{div}\bar{u}$, omit them here. To bound the term $\nabla\big(L_4(a)\big)D(\bar{u}+u_L)$, use Bony's decomposition yields that

$$\begin{aligned}\nabla\big(L_4(a)\big)D(\bar{u}+u_L)&=\dot{T}_{\nabla(L_4(a))}D(\bar{u}+u_L)+\dot{T}_{D(\bar{u}+u_L)}\nabla\big(L_4(a)\big)\\ &\quad+\dot{R}\big(\nabla\big(L_4(a)\big),D(\bar{u}+u_L)\big).\end{aligned}$$

Thanks to Corollary 2.7 and Lemma 2.12, we have

$$
\begin{aligned}
&\int_0^t \|\dot{T}_{\nabla\left(L_4(a)\right)} D(\bar{u}+u_L) + \dot{R}\big(\nabla\big(L_4(a)\big), D(\bar{u}+u_L)\big)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \int_0^t \|D(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\nabla\big(L_4(a)\big)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \\
&\lesssim \int_0^t \|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau + \int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau \\
&\lesssim \int_0^t \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} d\tau + \int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} d\tau.
\end{aligned}
$$

It follows from Corollary 2.7, Lemma 2.12, an interpolation inequality and Young's inequality that

$$
\begin{aligned}
&\int_0^t \|\dot{T}_{D(\bar{u}+u_L)} \nabla\big(L_4(a)\big)\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \int_0^t \|\dot{T}_{D(\bar{u}+u_L)} \nabla\big(L_4(a)\big)\|_{\dot{B}^{\frac{n}{2}-2}_{2,1}} d\tau \\
&\lesssim \int_0^t \|\nabla(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\nabla\big(L_4(a)\big)\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} d\tau \\
&\lesssim \int_0^t \|(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}} d\tau \\
&\lesssim \int_0^t \Big(\|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big) \\
&\qquad \times \Big(\|a^\ell\|^{\frac12}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \|a^\ell\|^{\frac12}_{\dot{B}^{\frac{n}{2}+1}_{2,1}} + \|a^h\|^{\frac12}_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a^h\|^{\frac12}_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau \\
&\lesssim \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big) d\tau \\
&\quad + \int_0^t \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big) d\tau \\
&\quad + \varepsilon \|a\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p})} + \frac{1}{\varepsilon} \|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}} d\tau.
\end{aligned}
$$

Employing Corollary 2.7, Lemmas 2.9 and 2.12 yields

$$
\begin{aligned}
&\int_0^t \|L_3(a)\bar{B}\cdot\nabla\bar{B}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \int_0^t \|\bar{B}\cdot\nabla\bar{B}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \Big(1 + \|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau \\
&\lesssim \int_0^t \Big(\|\dot{T}_{\bar{B}}\nabla\bar{B}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\dot{T}_{\nabla\bar{B}}\bar{B} + \dot{R}(\nabla\bar{B},\bar{B})\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\Big)\Big(1 + \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau \\
&\lesssim \int_0^t \Big(\|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\Big(1 + \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau \\
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau.
\end{aligned}
$$

By Corollary 2.7, Lemmas 2.9 and 2.12, an interpolation inequality and Young's inequality, we deduce that

$$
\begin{aligned}
&\int_0^t \|L_3(a)\bar{B}\cdot\nabla B_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \int_0^t \|\bar{B}\cdot\nabla B_L\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big) d\tau \\
&\lesssim \int_0^t \Big(\|\dot{T}_{\bar{B}}\nabla B_L\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\dot{T}_{\nabla B_L}\bar{B} + \dot{R}(\bar{B},\nabla B_L)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau\Big)\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau \\
&\lesssim \int_0^t \Big(\|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau \\
&\lesssim \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau + \varepsilon\|\bar{B}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})} \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big) \\
&\quad + \frac{1}{\varepsilon}\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} \Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big) d\tau.
\end{aligned}
$$

Employing Corollary 2.7, Lemma 2.12 and the definition of $\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}$, we have

$$
\begin{aligned}
\int_0^t \|L_3(a)B_L\cdot\nabla B_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau &\lesssim \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}\Big)\int_0^t \|B_L\cdot\nabla B_L\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big) C_L.
\end{aligned}
$$

Putting the above estimates together gives

$$
\begin{aligned}
&\int_0^t \|\mathcal{Q}(g_1 + u_L\cdot\nabla\mathcal{Q}\bar{u} - L_3(a)B_L\cdot\nabla\bar{B})\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}} d\tau \\
&\lesssim \varepsilon\Big(\|(a,\bar{u},\bar{B})\|_{X(t)} + \|(a,\bar{u},\bar{B})\|^2_{X(t)}\Big) + \|(a,\bar{u},\bar{B})\|^2_{X(t)} + \|(a,\bar{u},\bar{B})\|^3_{X(t)} \\
&\quad + \int_0^t \Big((\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + 1)\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} + (\frac{1}{\varepsilon}\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + 1)\|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big) \\
&\quad \times \Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \big(\|(\mathcal{Q}\bar{u},\bar{B})\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\big)\big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\big)\Big) d\tau \\
&\quad + \Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big) C_L.
\end{aligned} \tag{3.33}
$$

To bound the term $\int_0^t \|h_1+u_L\cdot\nabla\bar{B}+B_L\cdot\mathrm{div}\bar{u}-B_L\cdot\nabla\bar{u}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau$, use Bony's decomposition, Corollary 2.7, an interpolation inequality and Young's inequality, we have

$$\begin{aligned}
\int_0^t \|\mathrm{div}\bar{u}\cdot\bar{B}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|\dot{T}_{\bar{B}}\mathrm{div}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau + \int_0^t \|\dot{T}_{\mathrm{div}\bar{u}}\bar{B} + \dot{R}(\bar{B},\mathrm{div}\bar{u})\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau \\
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau + \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau \\
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big)d\tau \\
&\quad + \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau, \\
\int_0^t \|\bar{u}\cdot\nabla\bar{B}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|\dot{T}_{\bar{u}}\nabla\bar{B}\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau + \int_0^t \|\dot{T}_{\nabla\bar{B}}\bar{u} + \dot{R}(\bar{u},\nabla\bar{B})\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau \\
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau + \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau \\
&\lesssim \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau \\
&\quad + \int_0^t \|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big)d\tau.
\end{aligned}$$

According to Corollary 2.7, an interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}
\int_0^t \|\bar{u}\cdot\nabla B_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau &\lesssim \int_0^t \|\dot{T}_{\bar{u}}\nabla B_L\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau + \int_0^t \|\dot{T}_{\nabla B_L}\bar{u} + \dot{R}(\bar{u},\nabla B_L)\|_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau \\
&\lesssim \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau + \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau \\
&\lesssim \int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau \\
&\quad + \varepsilon\Big(\|\mathcal{P}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})} + \|\mathcal{Q}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}\Big) \\
&\quad + \|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}} + \|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big)d\tau.
\end{aligned}$$

It follows from the definition of $\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}$ that

$$\int_0^t \|u_L\cdot\nabla B_L\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau \lesssim \|u_L\cdot\nabla B_L\|_{L^1_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p})} \lesssim C_L.$$

Similarly, we can bound the terms $\bar{B}\cdot\nabla\bar{u}$, $\bar{B}\cdot\nabla u_L$, $B_L\cdot\nabla u_L$. Hence,

$$\begin{aligned}
&\int_0^t \|h_1+u_L\cdot\nabla\bar{B}+B_L\cdot\mathrm{div}\bar{u}-B_L\cdot\nabla\bar{u}\|^l_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau\\
&\lesssim \varepsilon\|(a,\bar{u},\bar{B})\|_{X(t)}+\|(a,\bar{u},\bar{B})\|^2_{X(t)}+C_L+\int_0^t\Big(\big(\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+1\big)\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\\
&\quad+\big(\frac{1}{\varepsilon}\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}+1\big)\|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big)\Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(\mathcal{Q}\bar{u},\bar{B})\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big)d\tau.
\end{aligned}\tag{3.34}$$

To handle the term $\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{B}\|^\ell_{\dot{B}^{\frac{n}{2}}_{2,1}}\big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\big)d\tau$, apply an interpolation inequality and Young's inequality yields

$$\begin{aligned}
&\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{B}\|^\ell_{\dot{B}^{\frac{n}{2}}_{2,1}}\big(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\big)d\tau\\
&\lesssim\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{B}\|^\ell_{\dot{B}^{\frac{n}{2}}_{2,1}}\big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\big)d\tau\\
&\lesssim\varepsilon\Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)\|\bar{B}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}\\
&\quad+\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\int_0^t\|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau.
\end{aligned}\tag{3.35}$$

For the term $\int_0^t\|\nabla(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau$, we have

$$\begin{aligned}
&\int_0^t\|\nabla(\bar{u}+u_L)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
&\lesssim\int_0^t\Big(\|\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big)\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
&\lesssim\int_0^t\Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,P}}+\|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big)\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau.
\end{aligned}\tag{3.36}$$

To bound the terms $\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|(a,\bar{u},\bar{B})\|_{\dot{B}^{\frac{n}{p}}_{p,1}}(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}})d\tau$, thanks to an interpolation inequality and Young's inequality implies that

$$\begin{aligned}
&\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}})d\tau\\
&\lesssim\int_0^t\|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big(\|a^\ell\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|a^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau\\
&\lesssim\int_0^t\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|a^\ell\|^{\frac12}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}\|a^\ell\|^{\frac12}_{\dot{B}^{\frac{n}{2}+1}_{2,1}}+\|a^h\|^{\frac12}_{\dot{B}^{\frac{n}{p}}_{p,1}}\|a^h\|^{\frac12}_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\\
&\quad\times\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau\\
&\lesssim\varepsilon\|a\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}}_{2,p})}\Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)
\end{aligned}$$

$$+\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau, \tag{3.37}$$

$$\begin{aligned}
&\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}})d\tau\\
&\lesssim \int_0^t \|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau\\
&\lesssim \varepsilon\Big(\|\mathcal{P}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})}+\|\mathcal{Q}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}\Big)\Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)\\
&\quad+\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\Big(\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big)\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
&\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{B}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}(1+\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}})d\tau\\
&\lesssim \int_0^t \|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u_L\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{B}\|^{\frac12}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{B}\|^{\frac12}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau\\
&\lesssim \varepsilon\|\bar{B}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}\Big(1+\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p})}\Big)\\
&\quad+\frac{1}{\varepsilon}\|\mathcal{P}u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{B}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\Big(1+\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}\Big)d\tau.
\end{aligned} \tag{3.39}$$

Similarly,

$$\begin{aligned}
&\int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{Q}\bar{u}\|^\ell_{\dot{B}^{\frac{n}{2}}_{2,1}}d\tau\\
&\lesssim \varepsilon\|\mathcal{Q}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{2}+1,\frac{n}{p}+1}_{2,p})}+\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\mathcal{Q}\bar{u}\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}d\tau
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&\int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
&\lesssim \varepsilon\|\mathcal{P}\bar{u}\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})}+\|B_0\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}-1}_{2,p}}\int_0^t \|B_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\mathcal{P}\bar{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau.
\end{aligned} \tag{3.41}$$

For the last term, we conclude, from the definition of $\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}$, that

$$\int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|^\ell_{\dot{B}^{\frac{n}{2}-1}_{2,1}}d\tau \lesssim \int_0^t \|u_L\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot{B}^{\frac{n}{2}-1,\frac{n}{p}}_{2,p}}d\tau. \tag{3.42}$$

3.5 Bootstrapping argument

The goal of this subsection is to complete the proof of our main theorem by using the bootstrapping argument. Plugging (3.29)–(3.42) into (3.28), we finally arrive at that, for any

$t < T^*$,

$$
\begin{aligned}
&\|(a,\bar u,\bar B)\|_{X(t)}\\
&\lesssim \|a_0\|_{\dot B^{\frac n2-1,\frac np}_{2,p}}+\|\mathcal{Q}u_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}+\varepsilon\|(a,\bar u,\bar B)\|_{X(t)}+\|(a,\bar u,\bar B)\|^2_{X(t)}\\
&\quad+\|(a,\bar u,\bar B)\|^3_{X(t)}+C_L\Big(1+\|(a,\bar u,\bar B)\|_{X(t)}\Big)\\
&\quad+\Big(\frac1\varepsilon\big(\|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}}+\|B_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}\big)+1\Big)\int_0^t\|(B_L,u_L)(\tau)\|_{\dot B^{\frac np+1}_{p,1}}\|(a,\bar u,\bar B)\|_{X(\tau)}d\tau\\
&\quad+\Big(\frac1\varepsilon\big(\|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}}+\|B_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}\big)+1\Big)\int_0^t\|(B_L,u_L)(\tau)\|_{\dot B^{\frac np+1}_{p,1}}d\tau\|(a,\bar u,\bar B)\|^2_{X(t)}.
\end{aligned}
\tag{3.43}
$$

Let $c_0 \ll 1$ and C_0 in (1.5) be large enough. Define

$$
T^{**}\overset{\text{def}}{=}\sup\Big\{t\in[0,T^*):\|(a,\bar u,\bar B)\|_{X(t)}\le c_0\Big\}. \tag{3.44}
$$

In what follows, we shall prove that $T^{**}=\infty$ under the assumption of (1.5). For any $t < T^{**}$, using (3.44) and (1.5), then (3.43) can be recast by

$$
\begin{aligned}
&\|(a,\bar u,\bar B)\|_{X(t)}\\
&\lesssim \|a_0\|_{\dot B^{\frac n2-1,\frac np}_{2,p}}+\|\mathcal{Q}u_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}+C_L\\
&\quad+\Big(\frac1\varepsilon\big(\|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}}+\|B_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}\big)+1\Big)\int_0^t\|(B_L,u_L)(\tau)\|_{\dot B^{\frac np+1}_{p,1}}\|(a,\bar u,\bar B)\|_{X(\tau)}d\tau.
\end{aligned}
$$

Then, for any $t < T^{\star\star}$ and a sufficiently large positive constant C_0, it follows from Gronwall's inequality and (1.5), that

$$
\begin{aligned}
&\|(a,\bar u,\bar B)\|_{X(t)}\\
&\le C\Big(\|a_0\|_{\dot B^{\frac n2-1,\frac np}_{2,p}}+\|\mathcal{Q}u_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}+C_L\Big)\\
&\quad\times\exp\Big\{\Big(\frac1\varepsilon\big(\|\mathcal{P}u_0\|_{\dot B^{\frac np-1}_{p,1}}+\|B_0\|_{\dot B^{\frac n2-1\frac np-1}_{2,p}}\big)+1\Big)\int_0^t\|(B_L,u_L)(\tau)\|_{\dot B^{\frac np+1}_{p,1}}d\tau\Big\}\\
&\le C_0\Big(\|a_0\|_{\dot B^{\frac n2-1,\frac np}_{2,p}}+\|\mathcal{Q}u_0\|_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}+C_L\Big)\exp\Big\{C_0\big(\|\mathcal{P}u_0\|^2_{\dot B^{\frac np-1}_{p,1}}+\|B_0\|^2_{\dot B^{\frac n2-1,\frac np-1}_{2,p}}\big)\Big\}\\
&\le\frac{c_0}{2},
\end{aligned}
\tag{3.45}
$$

which implies that $T^{\star\star}=\infty$ by the standard bootstrapping argument. Then combining with $\|(a,\bar u,\bar B)\|_{X(t)}\le c_0$ and (3.2) with $s=\frac np-1$ for all $t\ge 0$, we conclude that the Cauchy problem (1.3)–(1.4) admits a global solution (a,u,B). The proof of Theorem 1.1 is completed.

Author Contributions This work was carried out in collaboration of three authors. Wu and Xu proposed the question and presented some ideas of the proof. Gao carried out the study of the existence and drafted the manuscript. All authors read and approved the final manuscript.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no potential Conflict of interest with respect to the research of this article.

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