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Stability for the 2D Anisotropic Magnetohydrodynamic Equations with Only Horizontal Magnetic Diffusion

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Abstract

This paper studies the stability and large-time behavior of perturbations around a large, constant magnetic field in a periodic, infinite channel under specific symmetry constraints. Mathematically, the perturbations are governed by the 2D incompressible magnetohydrodynamic equations with no velocity dissipation and only horizontal magnetic diffusion. This stability result is sharp in the sense that removing this horizontal magnetic diffusion leads to instability. The proof is nontrivial and involves delicate construction of a time-weighted energy functional. Our result rigorously confirms the stabilizing effect of a background magnetic field on electrically conducting fluids.

Keywords Background magnetic field · Magnetohydrodynamic equation · Horizontal magnetic diffusion · Stability

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1 Introduction

We consider the 2D incompressible magnetohydrodynamic (MHD) system with only horizontal magnetic diffusion

$$\begin{cases} \partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla)\widetilde{u} = -\nabla \widetilde{P} + (\widetilde{b} \cdot \nabla)\widetilde{b}, \\ \partial_t \widetilde{b} + (\widetilde{u} \cdot \nabla)\widetilde{b} = \eta\, \partial_1^2 \widetilde{b} + (\widetilde{b} \cdot \nabla)\widetilde{u}, \\ \nabla \cdot \widetilde{u} = \nabla \cdot \widetilde{b} = 0, \end{cases} \tag{1.1}$$

where $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)^\top$, $\widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2)^\top$ and $\widetilde{P}$ denote the velocity field of the fluid, the magnetic field and the scalar pressure, respectively. η is a positive constant and denotes the resistivity coefficient.The goal here is to understand the stability and large-time behavior of perturbations near a constant background magnetic field.

The spatial domain is taken to be $\Omega = \mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ represents a one-dimensional periodic domain. This configuration helps generate a spectral gap and eliminates the need for boundary conditions. The background magnetic field $\widetilde{b}^{(0)}$ is set as $\widetilde{b}^{(0)} \equiv e_1 := (1, 0)$. Together with the zero velocity field $\widetilde{u}^{(0)} \equiv 0$, they constitute a stationary solution $(\widetilde{u}^{(0)}, \widetilde{b}^{(0)})$. We consider perturbations (u, b) near this special steady state, namely

$$u := \widetilde{u} - \widetilde{u}^{(0)}, \qquad b := \widetilde{b} - \widetilde{b}^{(0)}.$$

It is easy to check that (u, b) satisfies

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla P + (b \cdot \nabla)b + \partial_1 b, & x \in \Omega,\ t > 0, \\ \partial_t b + (u \cdot \nabla)b = \eta\, \partial_1^2 b + (b \cdot \nabla)u + \partial_1 u, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x). \end{cases} \tag{1.2}$$

We seek the small-data global well-posedness and stability on (1.2) in the Sobolev setting H^3. Without loss of generality, we take $\mathbb{T}$ to be the interval $[-\frac{1}{2}, \frac{1}{2}]$. We restrict our consideration to the initial perturbation (u_0, b_0) with the following symmetries

$$\begin{aligned} &u_{01},\ b_{02} \text{ are odd periodic with respect to } x_1, \\ &u_{02},\ b_{01} \text{ are even periodic with respect to } x_1. \end{aligned} \tag{1.3}$$

Solutions of (1.2) in the Sobolev space H^3 are unique. The uniqueness allows us to verify that the corresponding solution shares the same property,

$$\begin{aligned} &u_1,\ b_2 \text{ are odd periodic with respect to } x_1, \\ &u_2,\ b_1,\ P \text{ are even periodic with respect to } x_1. \end{aligned} \tag{1.4}$$

One can verify (1.4) similarly as in the 3D anisotropic Boussinesq equations (Wu and Zhang 2021). These symmetry constraints help eliminate kernels associated with specific spectral components, thereby ensuring a spectral gap and facilitating decay.

There are several motivations for this study. The first is to gain understanding on the dynamics of the ideal MHD equations. When the magnetic induction process dominates over magnetic diffusion such as in strongly collisional plasma, the following ideal MHD system applies,

$$\begin{cases} \partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla)\widetilde{u} = -\nabla \widetilde{P} + (\widetilde{b} \cdot \nabla)\widetilde{b}, \\ \partial_t \widetilde{b} + (\widetilde{u} \cdot \nabla)\widetilde{b} = (\widetilde{b} \cdot \nabla)\widetilde{u}, \\ \nabla \cdot \widetilde{u} = \nabla \cdot \widetilde{b} = 0, \end{cases} \tag{1.5}$$

Mathematically, (1.5) is difficult to analyze due to the lack of dissipation and magnetic diffusion. In fact, many fundamental issues such as the global regularity and stability problems remain open even in the 2D case. A natural and important question is how much dissipation or magnetic diffusion one really needs to assess the stability and large-time behavior. This paper presents an important example of the 2D MHD system for which we can establish the stability and understand the precise large-time behavior when the system involves some minimal regularization.

The second motivation is to reveal the mechanism underlying the remarkable stabilizing phenomenon observed in many physical experiments (see, e.g., (Alemany et al. 1979; Alexakis 2011; Alfvén 1942; Bardos et al. 1988; Gallet and Doering 2015)). The result presented in this paper establishes this phenomenon as a mathematically rigorous fact for the MHD model.

This third is to solve an open problem in a special case. The 2D MHD equations with only magnetic diffusion

$$\begin{cases} \partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla)\widetilde{u} = -\nabla \widetilde{P} + (\widetilde{b} \cdot \nabla)\widetilde{b}, \\ \partial_t \widetilde{b} + (\widetilde{u} \cdot \nabla)\widetilde{b} = \eta\, \Delta \widetilde{b} + (\widetilde{b} \cdot \nabla)\widetilde{u}, \\ \nabla \cdot \widetilde{u} = \nabla \cdot \widetilde{b} = 0 \end{cases} \tag{1.6}$$

model magnetic reconnection and magnetic turbulence when the role of resistivity is important and the fluid viscosity can be ignored (see (Priest and Forbes 2000)). The global regularity and the stability problems on (1.6) have attracted considerable interests and there are many recent developments.

When the spatial domain is the whole space $\mathbb{R}^2$, establishing global well-posedness and stability near the trivial solution or around a background magnetic field for equation (1.6) remains a challenging open problem. The primary difficulty lies in proving that the vorticity ω is essentially bounded. The absence of velocity dissipation makes this seem impossible. As shown in Jiu et al. (2015), this is a critical problem; while we can bound the L^p-norm of ω for any $1 \le p < \infty$, a bound in the L^∞-norm remains elusive, and few results exist for the whole space case. Another closely related work for the whole space case is by Boardman et al. (2020), who examined a variant of (1.6) where the vorticity satisfies an Euler-like equation, including an additional term given by a singular integral operator. In this case, the vorticity is also not known to be essentially bounded. Nonetheless, Boardman et al. (2020) successfully addressed the stability problem and derived precise long-time behavior for the solutions near a constant background magnetic field.

Many more results are currently available for the periodic domain $\mathbb{T}^2$, which has a key advantage over the whole space due to a Poincaré-type inequality. In this setting, significant progress has been made on the small-data well-posedness problem. Zhou and Zhu (2018) established global classical solutions of (1.6) near a background magnetic field under symmetry and mean-zero conditions on the initial perturbation (u_0, b_0). Wei and Zhang (2020) proved global existence for small initial data near the trivial solution in the Sobolev space H^4, assuming a mean-zero condition only for b_0. A crucial ingredient in their proof is the fact that this mean-zero condition enables exponential decay of the magnetic field in H^1. However, the problem of global existence and stability near a background magnetic field remains open. Ye and Yin (2019) improved upon Wei and Zhang (2020) by lowering the regularity requirement on the initial data (u_0, b_0), allowing it to be in either critical Besov spaces or the Sobolev space $H^s \times H^{s-1}$ with $s > 2$. The mean-zero condition on b_0 is still required. It is worth noting that the Sobolev norms of solutions obtained in Wei and Zhang (2020) and Ye and Yin (2019) grow over time.

Currently, no regularity or stability results for equation (1.6) are available in the domain $\mathbb{T}\times\mathbb{R}$. The work of Ren and Zhao (2017) gives a rigorous proof of the damping of the velocity and magnetic field for the linearized inviscid MHD equations around strictly monotone positive magnetic field $B = (b(y), 0)$ in a finite channel $\mathbb{T} \times (0, 1)$. It is worth noting that there have been many other significant developments in this area (see, e.g., (Cao et al. 2014; Ji and Wu 2020; Jiu et al. 2015; Lai et al. 2022; Lei and Zhou 2009; Yamazaki 2014; Zhang 2022; Zhou and Zhu 2018)). This list is by no means exhaustive.

This paper is able to make progress on the stability problem even when the magnetic diffusion is only in the horizontal direction. As noted in the introduction, the spatial domain is chosen as $\Omega = \mathbb{T} \times \mathbb{R}$, and we further restrict our analysis to initial perturbations with specific symmetry. Our study is motivated by the stabilizing phenomena observed in physical experiments (Alemany et al. 1979; Alexakis 2011; Alfvén 1942; Bardos et al. 1988; Gallet and Doering 2015).

The velocity equation in (1.2) is the 2D Euler with Lorentz forcing. Solutions to the Euler equations can grow rather rapidly in time (see (Kiselev and Sverak 2014; Zlatos 2015, 66)). Therefore, without the magnetic field, the fluid itself is not stable. To solve the desired stability problem, we fully exploit the smoothing and stabilizing effect of the magnetic field on the fluid. Mathematically the coupling and interaction in (1.2) generates a wave structure that reveal the hidden stabilizing effect. It is not difficult to show that any sufficiently regular solution (u, b) of (1.2) satisfies

$$\begin{cases} \partial_{tt}u - \eta\partial_{11}\partial_t u - \partial_{11}u = (\partial_t - \eta\partial_{11})N_1 + \partial_1 N_2, \\ \partial_{tt}b - \eta\partial_{11}\partial_t b - \partial_{11}b = \partial_t N_2 + \partial_1 N_1, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.7}$$

where N_1 and N_2 are the nonlinear terms in (1.2), namely

$$N_1 = \mathbb{P}((b \cdot \nabla)b - (u \cdot \nabla)u), \quad N_2 = (b \cdot \nabla)u - (u \cdot \nabla)b$$

with $\mathbb{P}$ being the projection operator onto divergence-free vector fields. In comparison with (1.2), u and b in (1.7) actually satisfy the same linearized wave equation, which contains more regularizing terms. The two extra terms in the equation of u come from distinct sources: $-\eta\partial_{11}\partial_t u$ due to the horizontal magnetic diffusion and $-\partial_{11}u$ from the background magnetic field. In spite of these smoothing effects, there are complications. One challenge is that the dissipation from the background field is relatively weak. We will elaborate on this technical difficulty later.

With these preparation at our disposal, we are ready to state our main result.

Theorem 1.1 *Let $(u_0, b_0) \in H^3(\Omega)$ satisfy $\nabla \cdot u_0 = 0, \nabla \cdot b_0 = 0$ and (1.3). Then there exists sufficiently small $\delta_0 = \delta_0(\eta) > 0$ such that, for any $\delta \leq \delta_0$, if*

$$\|u_0\|_{H^3(\Omega)} + \|b_0\|_{H^3(\Omega)} \leq \delta, \tag{1.8}$$

then there exists a unique global solution $(u, b) \in C\big([0, \infty); H^3(\Omega)\big)$ of (1.2) satisfying

$$\|(u, b)(t)\|^2_{H^3(\Omega)} + \int_0^t \Big(\|\partial_1 u(\tau)\|^2_{H^2(\Omega)} + \eta\, \|\partial_1 b(\tau)\|^2_{H^3(\Omega)}\Big) d\tau \leq C\delta^2 \tag{1.9}$$

for any $t > 0$ and some uniform constant $C > 0$. In addition, we have the following time decay estimate:

$$\|u(t)\|_{H^1(\Omega)} + \|\nabla^2 u_2(t)\|_{L^2(\Omega)} + \|b_2(t)\|_{H^2(\Omega)} \leq C(1 + t)^{-1} \tag{1.10}$$

provided that δ is small enough.

The proof of Theorem 1.1 overcomes several major difficulties. The first is to construct a suitable energy functional. This is the key component of the bootstrapping argument. Philosophically the energy functional should involve the Sobolev norm of the solution and the time integral parts due to dissipation. In addition, it should have enough number of terms so that one can prove a closed energy inequality. Since we are seeking solutions in the Sobolev space H^3, the energy functional should naturally contain the H^3-norms of u and b as well as the time integral piece associated with the horizontal magnetic diffusion. Certainly the energy functional should also include the aforementioned enhanced dissipation revealed in the wave structure (1.7), especially the weak dissipation in the direction of the background magnetic field. Mathematically this dissipation provides one-derivative order lower smoothing than standard dissipation. These considerations prompt us to define the following functional

$$E_0(t) = \sup_{0\leq\tau\leq t} \left(\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\right) + \int_0^t (\|\partial_1 u(\tau)\|_{H^2}^2 + \eta\, \|\partial_1 b(\tau)\|_{H^3}^2)\, d\tau. \tag{1.11}$$

The time integral of $\|\partial_1 u(\tau)\|_{H^2}^2$ represents the enhanced dissipation and cannot be strengthened to the time integral of $\|\partial_1 u(\tau)\|_{H^3}^2$. This weaker dissipation makes it extremely difficult to control the Navier–Stokes nonlinearity term $u \cdot \nabla u$ in the H^3-estimate. When we apply the energy method to bound the H^3-norm of u or the H^2-norm of the vorticity $\omega = \nabla \times u$, we encounter the term

$$\begin{aligned}\sum_{i=1}^{2} \int \partial_i u \cdot \nabla(\partial_i \omega)\, \partial_i^2 \omega\, dx &= \int \partial_1 u \cdot \nabla \partial_1 \omega\, \partial_1^2 \omega\, dx + \int \partial_2 u_1\, \partial_1 \partial_2 \omega\, \partial_2^2 \omega\, dx \\ &\quad + \int \partial_2 u_2\, \partial_2^2 \omega\, \partial_2^2 \omega\, dx.\end{aligned}$$

It is clear that the most challenging term would be the one with vertical derivatives, namely

$$T_0^* = \int \partial_2 u_2\, \partial_2^2 \omega\, \partial_2^2 \omega\, dx = -\int \partial_1 u_1 \partial_2^2 \omega\, \partial_2^2 \omega\, dx.$$

It doesn't appear to be possible to bound it by the terms in E_0 defined in (1.11) due to the weak horizontal dissipation $\|\partial_1 u\|_{H^2}$. This motivated us to define a time-weighted energy functional E_1 by

$$\begin{aligned}E_1(t) = &\sup_{0\leq\tau\leq t} (1+\tau)^2 \left(\|u_2(\tau)\|_{H^2}^2 + \|b_2(\tau)\|_{H^2}^2\right) \\ &+ \int_0^t (1+\tau)^2 (\|\partial_1 \nabla u_2(\tau)\|_{L^2}^2 + \eta\, \|\partial_1 b_2(\tau)\|_{H^2}^2)\, d\tau.\end{aligned}$$

The definition of E_1 takes into account the decay properties of the solution (u, b). The total energy $E(t)$ is the sum of E_0 and E_1,

$$E(t) = E_0(t) + E_1(t).$$

As we shall see in Lemma 3.2, the difficult term can be suitably controlled in terms of $E_1(t)$, i.e.,

$$\int_0^t T_0^*(\tau)\, d\tau \leq C \int_0^t \|\partial_2 \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{4}} \|\partial_2^2 \partial_1 u_2\|_{L^2}^{\frac{1}{4}} \|\partial_2^2 \omega\|_{L^2}^2\, d\tau$$

$$
\begin{aligned}
&\le C \sup_{0\le\tau\le t} \|\partial_2^2\omega\|_{L^2}^2 (1+\tau)^{\frac14} \|\partial_2^2 u_2\|_{L^2}^{\frac14} \Big(\int_0^t (1+\tau)^2 \|\partial_2\partial_1 u_2(\tau)\|_{L^2}^2\, d\tau\Big)^{\frac14} \\
&\quad\times \Big(\int_0^t \|\partial_2^2\partial_1 u_2(\tau)\|_{L^2}^2\, d\tau\Big)^{\frac18} \Big(\int_0^t (1+\tau)^{-\frac65}\, d\tau\Big)^{\frac58} \\
&\le C E_0^{\frac98}(t) E_1^{\frac38}(t),
\end{aligned}
$$

which allows us to eventually establish the estimate

$$
E(t) \le C_1 E(0) + C_2 E^{\frac32}(t). \tag{1.12}
$$

Our main efforts are devoted to proving (1.12). A bootstrapping argument then leads to the desired global uniform bound on $E(t)$ for all time.

We briefly remark that the MHD models have been extensively investigated and important progress has been on various aspects of the MHD flow (see, e.g., (Cai and Lei 2018; Cao and Wu 2011; Cao et al. 2013; Deng and Zhang 2018; Dong et al. 2018, 2019; Du and Zhou 2015; Fan and Ozawa 2014; Fan et al. 2014; Fefferman et al. 2014, 2017; Feng et al. 2021; He et al. 2018; Hu and Lin 2014; Jiang and Jiang 2019, 2020, 2021; Li et al. 2017; Lai et al. 2021, 2022; Lin and Du 2013; Lin et al. 2015, 2020; Paicu and Zhu 2021; Pan et al. 2018; Ren et al. 2014, 2016; Schonbek et al. 1996; Suo and Jiu 2022; Tan and Wang 2018; Wan 2016; Wei and Zhang 2017; Wu 2018; Wu and Wu 2017; Wu et al. 2015, 55, Yamazaki 2014; Ye and Yin 2020; Yuan and Zhao 2018; Yang et al. 2019; Zhang 2014, 2016)).

The rest of this paper is organized as follows. Section 2 recalls several tools to be used in the proof of main estimate (1.12). In particular, we provide strong Poincaré-type inequalities and anisotropic Sobolev bounds for triple products. Sections 3 and 4 are devoted to the proofs of the estimates for $E_0(t)$ and $E_1(t)$, respectively. Section 5 completes the proof of Theorem 1.1.

2 Preliminary

This section recalls strong Poincaré-type inequalities and several anisotropic upper bounds for triple products. They will be used in the proof of (1.12).

To simplify the notation, we write

$$
\begin{aligned}
&\partial_i v = \partial_{x_i} v \ (i = 1, 2), \quad \|v\|_{H^s} = \|v\|_{H^s(\Omega)}, \\
&\int f(x)dx = \int_\Omega f(x)dx.
\end{aligned}
$$

We shall also use the norm notation: $\|(f, g)\|_{H^s}^2 = \|f\|_{H^s}^2 + \|g\|_{H^s}^2$. In additions, we use $\overline{f}$ for the average of f on $\mathbb{T}$, i.e.,

$$\overline{f} = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1.$$

This first lemma assesses the strong Poincaré-type inequalities involving only the x_1 partial derivative in homogeneous Sobolev space $\dot{H}^s(\Omega)$.

Lemma 2.1 *Let* $\overline{f} = 0$ *and* $f \in H^s(\Omega)$ *with* $s \geq 0$ *being an integer. Then the Poincaré-type inequality holds*

$$\|f\|_{\dot{H}^s(\Omega)} \leq C\|\partial_1 f\|_{\dot{H}^s(\Omega)}, \tag{2.1}$$

where $C > 0$ *is a pure constant. In particular, for any* (u, b) *with the symmetry properties in (1.4),*

$$\|u_i\|_{\dot{H}^s(\Omega)} \leq C\|\partial_1 u_i\|_{\dot{H}^s(\Omega)}, \ \ i = 1, 2, \tag{2.2}$$

$$\|b_2\|_{\dot{H}^s(\Omega)} \leq C\|\partial_1 b_2\|_{\dot{H}^s(\Omega)}. \tag{2.3}$$

Proof of Lemma 2.1 As $s = 0$, Dong et al. (2021) has shown the proof of (2.1) (see Lemma 4 for detailed). In fact, due to $\overline{f} = 0$, we can apply the 1-D Poincaré inequality to get

$$\|f\|_{L^2_{x_1}} \leq C\|\partial_1 f\|_{L^2_{x_1}}.$$

Taking the L^2-norm in x_2 yields

$$\|f\|_{L^2} \leq C\|\partial_1 f\|_{L^2}. \tag{2.4}$$

That means (2.1) holds as $s = 0$. For $s > 0$, it is easy to prove $\overline{\nabla^s f} = 0$. Then by (2.4) we can derive

$$\|f\|_{\dot{H}^s} \leq C\|\partial_1 f\|_{\dot{H}^s}$$

for any $s > 0$. For (2.2) and (2.3), it suffices to verify that u and b_2 satisfy the mean-zero condition. First, by (1.4) it is obvious that $\overline{u}_1 = \overline{b}_2 = 0$. For u_2, it is noted that by the incompressible condition for u, there exists a stream function ψ such that

$$u = \nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi), \tag{2.5}$$

which implies $\overline{u}_2 = 0$. This completes the proof of Lemma 2.1. □

We remark that a much sharp version of (2.1) has recently being obtained in Feng et al. (2023), but (2.1) is good enough for our purpose. The second lemma presents two anisotropic inequalities related to L^∞-norm and triple product. We refer to the proof in Dong et al. (2021) (see Lemma 3).

Lemma 2.2 *Assume* $\overline{f} = 0$ *and* $f, \partial_1 f \in H^1(\Omega)$, $g, \partial_2 g, h \in L^2(\Omega)$. *Then we have*

$$\|f\|_{L^\infty(\Omega)} \le C\|\partial_1 f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_2 f\|_{L^2(\Omega)}^{\frac{1}{4}} \|\partial_{12} f\|_{L^2(\Omega)}^{\frac{1}{4}} \le C\|\partial_1 f\|_{H^1(\Omega)}, \tag{2.6}$$

$$\iint_\Omega |fgh|dx_1dx_2 \le C\|\partial_1 f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_2 g\|_{L^2(\Omega)}^{\frac{1}{2}}\|h\|_{L^2(\Omega)} \tag{2.7}$$

for some pure constant $C > 0$.

In addition, we will use the simple fact that, for any (u, b) with the symmetry properties in (1.4),

$$\overline{\nabla^k u} = 0, \ \overline{\nabla^k \omega} = 0, \ \overline{\nabla^k b_2} = 0$$

for $k \ge 0$ an integer. This can be verified by the symmetry condition (1.4) together with (2.5).

3 Estimates of $E_0(t)$

This section proves the following estimate for $E_0(t)$:

Proposition 3.1 *Suppose that* (u_0, b_0) *satisfies the conditions in Theorem 1.1. Then we have*

$$E_0(t) \le \frac{1}{c_0}\Big(c_1 E(0) + Cc_2 E_0^{\frac{3}{2}}(t) + Cc_2 E_1^{\frac{3}{2}}(t)\Big), \tag{3.1}$$

where $c_0 = \min\{1 - \lambda_0(2 + \eta^2), \lambda_0\}$, $c_1 = 4 + 2\lambda_0$ *and* $c_2 = 1 + \frac{1}{\eta} + \lambda_0$ *with* $\lambda_0 < \frac{1}{2+\eta^2}$.

For the sake of clarity, we divide $E_0(t)$ into two parts,

$$E_0(t) = E_{0,0}(t) + E_{0,1}(t),$$

where

$$E_{0,0}(t) = \sup_{0\le\tau\le t}\Big(\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\Big) + \eta\int_0^t \|\partial_1 b(\tau)\|_{H^3}^2\, d\tau,$$

$$E_{0,1}(t) = \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2\, d\tau.$$

The first part $E_{0,0}$ includes essential terms required for estimating the H^3-norms of (u, b), namely the L^∞-in-time norm of $\|(u, b)\|_{H^3}$ and the time integral of terms related to the dissipative effects in the MHD system. This part alone does not suffice for our stability estimates. The second part $E_{0,1}$ exploits the enhanced dissipation on the velocity in the horizontal direction due to the background magnetic field in the same direction. The contributions from $E_{0,0}$ are relatively straightforward, as they are natural components of the energy. In contrast, the contribution $E_{0,1}$ comes from the hidden wave structure of the system and is relatively more challenging to estimate. By combining $E_{0,1}$ with $E_{0,0}$, we can effectively control all nonlinear terms involving horizontal derivatives.

Then the proof of the estimate for $E_0(t)$ is naturally split in two parts, which will be shown in two subsections. The first subsection bounds $\|(u,b)\|_{H^3}$ while the second subsection is to bound the velocity dissipation $\int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau$.

3.1 Bound for $E_{0,0}(t)$

We start with the estimates on $E_{0,0}(t)$. As explained in the introduction, the most difficult terms are the nonlinear integrals with vertical derivative for all terms. To overcome the difficulty, we exploit the stability effect and the time decay of the solution by introducing the time-weighted energy functional, i.e., $E_1(t)$. With the help of $E_1(t)$ and $E_{0,1}(t)$, we are able to establish the closed estimate for $E_{0,0}(t)$. It is worth noting that the term $E_0^{\frac{9}{8}} E_1^{\frac{3}{8}}$ with especial exponents appears in the upper bound, arising from the hard terms T_0^*, T_1^*, T_2^* and their associated estimates. While these exponents differ from those in other terms within $E_0(t)$ and $E_1(t)$, it actually can be bounded by $E_0^{\frac{3}{2}}(t) + E_1^{\frac{3}{2}}(t)$ by means of Young's inequality, which is fundamentally equivalent.

Lemma 3.2 *Suppose that the initial data* (u_0, b_0) *satisfies the conditions in Theorem 1.1. Then we have*

$$E_{0,0}(t) \leq 4E(0) + C\left(1 + \frac{1}{\eta}\right)\left(E_0^{\frac{3}{2}}(t) + E_1^{\frac{3}{2}}(t)\right) \tag{3.2}$$

for the constants $C > 0$ *independent of* η.

Proof of Lemma 3.2 We formally compute $\partial_t E_{0,0}(t)$, where the norm H^3 was chosen as $\|\cdot\|_{L^2} + \|\nabla^3 \cdot\|_{L^2}$ based on the fact of their equivalence. In the following proof, any terms involving $\|\partial_1 \nabla^3 b\|_{L^2}$ can be controlled by using the horizontal magnetic dissipation. However, in contrast to the setting of full dissipation, a major challenge concerns estimating higher derivatives of the velocity as well as estimating vertical derivatives of the magnetic field. To this end we crucially exploit the enhanced dissipation due to the background magnetic field and coupling and interactions.

Now, we will specifically present our estimation process. Firstly, it is clear that

$$\|(u,b)(t)\|_{L^2}^2 + 2\eta \int_0^t \|\partial_1 b(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{3.3}$$

Now we bound $\|(\nabla^3 u, \nabla^3 b)\|_{L^2}^2$. Before the proof, we show the following facts:

$$\|\omega\|_{\dot{H}^s} = \|\nabla u\|_{\dot{H}^s}, \ \ \|j\|_{\dot{H}^s} = \|\nabla b\|_{\dot{H}^s}$$

where $s \geq 0$ is an integer. In fact, by integration by parts and the incompressible condition, we have

$$\begin{aligned}\|\omega\|_{L^2}^2 &= \int (|\partial_1 u_2|^2 - 2\partial_1 u_2 \partial_2 u_1 + |\partial_2 u_1|^2)\, dx \\ &= \int (|\partial_1 u_2|^2 - 2\partial_1 u_1 \partial_2 u_2 + |\partial_2 u_1|^2)\, dx = \|\nabla u\|_{L^2}^2.\end{aligned}$$

Notice

$$\|(\nabla^3 u, \nabla^3 b)\|_{L^2}^2 = \|(\nabla^2 \omega, \nabla^2 j)\|_{L^2}^2 \leq 2\sum_{i=1}^{2} \|(\partial_i^2 \omega, \partial_i^2 j)\|_{L^2}^2. \tag{3.4}$$

Thus, it suffices to establish $\sum_{i=1}^{2} \|(\partial_i^2 \omega, \partial_i^2 j)\|_{L^2}$-estimates. Applying the operator $\nabla\times$ to (1.2), then (ω, j) satisfies

$$\begin{cases} \partial_t \omega + (u\cdot\nabla)\omega = (b\cdot\nabla) j + \partial_1 j, \\ \partial_t j + (u\cdot\nabla) j = \eta\, \partial_1^2 j + (b\cdot\nabla)\omega + \partial_1\omega + Q, \end{cases} \tag{3.5}$$

where

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

We apply ∂_i^2 to the system (3.5) and multiply the first and second equations of the resulting system by $\partial_i^2\omega$ and $\partial_i^2 j$, respectively. After integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\|(\partial_i^2\omega, \partial_i^2 j)\|_{L^2}^2 + \eta\sum_{i=1}^{2}\|\partial_1\partial_i^2 j\|_{L^2}^2 \\ &= -\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla)\omega\, \partial_i^2\omega dx + \sum_{i=1}^{2}\int \partial_i^2(b\cdot\nabla) j\, \partial_i^2\omega dx - \sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla) j\, \partial_i^2 j dx \\ &\quad + \sum_{i=1}^{2}\int \partial_i^2(b\cdot\nabla)\omega\, \partial_i^2 j dx + \sum_{i=1}^{2}\int \partial_i^2 Q\, \partial_i^2 j dx \\ &:= I_1 + I_2 + \cdots + I_5.\end{aligned} \tag{3.6}$$

(1) The bound for I_1.

By integration by parts, we first split I_1 in three parts,

$$\begin{aligned}I_1 &= -\sum_{i=1}^{2}\int (\partial_i^2 u\cdot\nabla\omega)\, \partial_i^2\omega dx - 2\sum_{i=1}^{2}\int \partial_i u\cdot\nabla(\partial_i\omega)\, \partial_i^2\omega dx \\ &= -\Big(\int \partial_1^2 u\cdot\nabla\omega\, \partial_1^2\omega dx + 2\int \partial_1 u\cdot\nabla(\partial_1\omega)\, \partial_1^2\omega dx + 2\int \partial_2 u_1\partial_1\partial_2\omega\, \partial_2^2\omega dx\Big) \\ &\quad - \Big(\int \partial_2^2 u_1\partial_1\omega\, \partial_2^2\omega dx + \int \partial_2^2 u_2\partial_2\omega\, \partial_2^2\omega dx\Big) - 2\int \partial_2 u_2\partial_2^2\omega\, \partial_2^2\omega dx\end{aligned}$$

$$= I_{11} + I_{12} - 2T_0^*.$$

Applying Hölder's inequality, Sobolev's inequality and the Poincaré inequality (2.6) with $f = \partial_2 u_1$ yield

$$\begin{aligned} I_{11} &\leq \|\partial_1^2 u\|_{L^4} \|\nabla\omega\|_{L^4} \|\partial_1^2\omega\|_{L^2} + 2\|\partial_1 u\|_{L^\infty} \|\partial_1\nabla\omega\|_{L^2}^2 + 2\|\partial_2 u_1\|_{L^\infty} \|\partial_1\partial_2\omega\|_{L^2} \|\partial_2^2\omega\|_{L^2} \\ &\leq C\|\partial_1^2 u\|_{H^1} \|\nabla\omega\|_{H^1} \|\partial_1^2\omega\|_{L^2} + C\|\partial_1 u\|_{H^2} \|\partial_1\nabla\omega\|_{L^2}^2 + C\|\partial_2\partial_1 u_1\|_{H^1} \|\partial_1\partial_2\omega\|_{L^2} \|\partial_2^2\omega\|_{L^2} \\ &\leq C\|u\|_{H^3} \|\partial_1 u\|_{H^2}^2. \end{aligned}$$

Due to $\overline{\partial_2^2 u_1} = \overline{\partial_2\omega} = 0$, we can use the anisotropic inequality (2.7) to bound I_{12} as

$$\begin{aligned} I_{12} \leq& C\|\partial_2^2\partial_1 u_1\|_{L^2} \|\partial_1\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2^2\omega\|_{L^2} \\ &+ C\|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\omega\|_{L^2} \|\partial_2^2\omega\|_{L^2} \\ \leq& C\|u\|_{H^3} \|\partial_1 u\|_{H^2}^2. \end{aligned}$$

T_0^* can not be closed directly by the energy functional $E_0(t)$ due to the lack of the vertical dissipation for u. We need to resort to the decay rates in $E_1(t)$. It will be handled at the end of this proof. Therefore, we obtain

$$I_1 \leq C\|u\|_{H^3} \|\partial_1 u\|_{H^2}^2 - 2T_0^*. \tag{3.7}$$

(2) The bound for I_2.

We proceed to bound I_2. Owing to $\overline{b}_1 \neq 0$, $\overline{j} \neq 0$, the proof will be more complicated than I_1. So are the bounds for I_3, I_4 and I_5. Similarly to I_1, we first decompose I_2 as

$$\begin{aligned} I_2 =& \sum_{k=1}^{2} C_2^k \int \partial_1^k b \cdot \nabla\partial_1^{2-k} j\, \partial_1^2\omega\, dx + \sum_{k=1}^{2} C_2^k \int \partial_2^k b_1\, \partial_2^{2-k}\partial_1 j\, \partial_2^2\omega\, dx \\ &+ \int \partial_2^2 b_2\, \partial_2 j\, \partial_2^2\omega\, dx + 2\int \partial_2 b_2\, \partial_2^2 j\, \partial_2^2\omega\, dx + \sum_{i=1}^{2} \int (b\cdot\nabla)\partial_i^2 j\, \partial_i^2\omega dx \\ =& I_{21} + I_{22} + I_{23} + 2T_1^* + \sum_{i=1}^{2} \int (b\cdot\nabla)\partial_i^2 j\, \partial_i^2\omega dx. \end{aligned}$$

It is easy to get the bound for I_{21}.

$$\begin{aligned} I_{21} &\leq \sum_{k=1}^{2} C_2^k \|\partial_1^k b\|_{L^\infty} \|\nabla\partial_1^{2-k} j\|_{L^2} \|\partial_1^2\omega\|_{L^2} \\ &\leq C\|b\|_{H^3} \|\partial_1 u\|_{H^2} \|\partial_1 b\|_{H^3}. \end{aligned}$$

For I_{22}, we first use integration by parts and then apply (2.7) with $f = \partial_2\omega$ to obtain

$$\begin{aligned} I_{22} =& -\sum_{k=1}^{2} \mathcal{C}_2^k \int (\partial_2^{k+1} b_1\, \partial_2^{2-k}\partial_1 j \ + \partial_2^k b_1\, \partial_2^{3-k}\partial_1 j)\, \partial_2\omega\, dx \\ \le& C\sum_{k=1}^{2} \mathcal{C}_2^k\, \|\partial_2^{k+1} b_1\|_{L^2} \|\partial_2^{2-k}\partial_1 j\|_{L^2}^{\frac12} \|\partial_2^{3-k}\partial_1 j\|_{L^2}^{\frac12} \|\partial_2\partial_1\omega\|_{L^2} \\ &+ C\sum_{k=1}^{2} \mathcal{C}_2^k\, \|\partial_2^{3-k}\partial_1 j\|_{L^2} \|\partial_2^k b_1\|_{L^2}^{\frac12} \|\partial_2^{k+1} b_1\|_{L^2}^{\frac12} \|\partial_2\partial_1\omega\|_{L^2} \\ \le& C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}. \end{aligned}$$

Similarly, by $\|\partial_2\omega\|_{L^2} \le C\|\partial_2\partial_1\omega\|_{L^2}$ we have

$$\begin{aligned} I_{23} &= -\int (\partial_2^3 b_2\, \partial_2 j + \partial_2^2 b_2\, \partial_2^2 j)\, \partial_2\omega\, dx \\ &\le \|\partial_2^3 b_2\|_{L^4}\|\partial_2 j\|_{L^4}\|\partial_2\omega\|_{L^2} + \|\partial_2^2 b_2\|_{L^\infty}\|\partial_2^2 j\|_{L^2}\|\partial_2\omega\|_{L^2} \\ &\le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}. \end{aligned}$$

The estimate of T_1^* possesses the same difficulty as T_0^*. We also bound it in the last step. Thus,

$$I_2 \le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3} + 2T_1^* + \sum_{i=1}^{2}\int (b\cdot\nabla)\partial_i^2 j\ \partial_i^2\omega dx. \tag{3.8}$$

(3) The bound for I_3.
I_3 can be bounded in a similar way. We first have

$$\begin{aligned} I_3 =& -\sum_{k=1}^{2} \mathcal{C}_2^k \int \partial_1^k u\cdot\nabla\partial_1^{2-k} j\ \partial_1^2 j\, dx - \sum_{k=1}^{2} \mathcal{C}_2^k \int \partial_2^k u_1\, \partial_2^{2-k}\partial_1 j\ \partial_2^2 j\, dx \\ &- \sum_{k=1}^{2} \mathcal{C}_2^k \int \partial_2^k u_2\, \partial_2^{3-k} j\ \partial_2^2 j\, dx \\ =& I_{31} + I_{32} + I_{33}. \end{aligned}$$

Applying Hölder's inequality and Sobolev's inequality to I_{31}, the anisotropic inequality (2.7) to I_{32}, we get

$$\begin{aligned} I_{31} + I_{32} \le& \sum_{k=1}^{2} \mathcal{C}_2^k\, \|\partial_1^k u\|_{L^4}\|\nabla\partial_1^{2-k} j\|_{L^2}\|\partial_1^2 j\|_{L^4} \\ &+ C\sum_{k=1}^{2} \mathcal{C}_2^k\, \|\partial_2^k\partial_1 u_1\|_{L^2}\|\partial_2^{2-k}\partial_1 j\|_{L^2}^{\frac12}\|\partial_2^{3-k}\partial_1 j\|_{L^2}^{\frac12}\|\partial_2^2 j\|_{L^2} \end{aligned}$$

$$\leq C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}.$$

By integration by parts, Sobolev's inequality, (2.6) and (2.2), I_{33} can be estimated as

$$\begin{aligned}
I_{33} =&4\int u_1\,\partial_1\partial_2^2 j\,\partial_2^2 j\,dx + \int \partial_2 u_1\,\partial_1\partial_2 j\,\partial_2^2 j\,dx + \int \partial_2 u_1\,\partial_2 j\,\partial_2^2\partial_1 j\,dx\\
\leq&4\|u_1\|_{L^\infty}\|\partial_1\partial_2^2 j\|_{L^2}\|\partial_2^2 j\|_{L^2} + \|\partial_2 u_1\|_{L^4}\|\partial_1\partial_2 j\|_{L^4}\|\partial_2^2 j\|_{L^2}\\
&+\|\partial_2 u_1\|_{L^4}\|\partial_2 j\|_{L^4}\|\partial_2^2\partial_1 j\|_{L^2}\\
\leq&C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}.
\end{aligned}$$

Consequently,

$$I_3 \leq C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}. \tag{3.9}$$

(4) The bound for I_4.
We now turn to I_4. We rewrite it as follows

$$\begin{aligned}
I_4 =&\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k\int \partial_i^k b_1\,\partial_1\partial_i^{2-k}\omega\,\partial_i^2 j\,dx + \sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k\int \partial_i^k b_2\,\partial_2\partial_i^{2-k}\omega\,\partial_i^2 j\,dx\\
&+\sum_{i=1}^{2}\int (b\cdot\nabla)\partial_i^2\omega\,\partial_i^2 j dx\\
=&I_{41} + I_{42} + \sum_{i=1}^{2}\int (b\cdot\nabla)\partial_i^2\omega\,\partial_i^2 j dx.
\end{aligned}$$

Applying integration by parts and the anisotropic inequality (2.7) with $f = \partial_i^{2-k}\omega$ yields

$$\begin{aligned}
I_{41} =& -\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k\int \partial_i^{2-k}\omega\Big(\partial_1\partial_i^k b_1\,\partial_i^2 j + \partial_i^k b_1\,\partial_i^2\partial_1 j\Big)\,dx\\
\leq& C\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k\,\|\partial_1\partial_i^{2-k}\omega\|_{L^2}\Big(\|\partial_1\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_2\partial_1\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_i^2 j\|_{L^2}\\
&+\|\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_2\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_1\partial_i^2 j\|_{L^2}\Big)\\
\leq& C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}.
\end{aligned}$$

Also, Sobolev's inequality together with $\|\Delta b_2\|_{L^\infty} \leq C\|\partial_1\Delta b_2\|_{H^1}, \|\partial_2\omega\|_{L^2} \leq C\|\partial_2\partial_1\omega\|_{L^2}$ leads to

$$I_{42} = \sum_{i=1}^{2}\int \partial_i^2 b_2\,\partial_2\omega\,\partial_i^2 j\,dx + 2\int \partial_1 b_2\,\partial_1\partial_2\,\omega\,\partial_1^2 j\,dx + 2\int \partial_2 b_2\partial_2^2\omega\,\partial_2^2 j\,dx$$

$$
\begin{aligned}
&\le \|\Delta b_2\|_{L^\infty}\|\partial_2\omega\|_{L^2}\|\Delta j\|_{L^2} + 2\|\partial_1 b_2\|_{L^\infty}\|\partial_2\partial_1\omega\|_{L^2}\|\partial_1^2 j\|_{L^2} + 2T_1^* \\
&\le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3} + 2T_1^*.
\end{aligned}
$$

Thus, we derive

$$
I_4 \le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3} + 2T_1^* + \sum_{i=1}^{2}\int (b\cdot\nabla)\partial_i^2\omega\, \partial_i^2 j dx. \tag{3.10}
$$

(5) The bound for I_5.

For the last term I_5, we need more subtle estimates.

$$
\begin{aligned}
I_5 &= 2\sum_{i=1}^{2}\int \partial_i^2\Big(\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2)\Big)\,\partial_i^2 j\,dx - 2\sum_{i=1}^{2}\int \partial_i^2\Big(\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)\Big)\,\partial_i^2 j\,dx \\
&:= I_{51} + I_{52}.
\end{aligned}
$$

It is simple to bound I_{51}. By (2.7) we infer

$$
\begin{aligned}
I_{51} &= 2\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int \partial_i^k\partial_1 b_1\,\partial_i^{2-k}(\partial_2 u_1 + \partial_1 u_2)\,\partial_i^2 j\,dx \\
&\quad + 2\sum_{i=1}^{2}\int \partial_1 b_1\,\partial_i^2(\partial_2 u_1 + \partial_1 u_2)\,\partial_i^2 j\,dx \\
&\le C\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k\|\partial_1\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_2\partial_1\partial_i^k b_1\|_{L^2}^{\frac12}\|\partial_i^{2-k}\partial_1\nabla u\|_{L^2}\|\partial_i^2 j\|_{L^2} \\
&\quad + 2\|\partial_1 b_1\|_{L^\infty}\|\Delta\partial_1 u_2\|_{L^2}\|\Delta j\|_{L^2} + 2T_2^* \\
&\le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3} + 2T_2^*,
\end{aligned}
$$

where

$$
T_2^* = \int \partial_1 b_1 \partial_2^3 u_1 \partial_2^2 j\,dx.
$$

To bound I_{52}, we first split it in four parts by integration by parts.

$$
\begin{aligned}
I_{52} &= -2\sum_{i=1}^{2}\sum_{k=0}^{2} C_2^k \int \partial_i^k\partial_1 u_1\,\partial_i^{2-k}(\partial_2 b_1 + \partial_1 b_2)\,\partial_i^2 j\,dx \\
&= 2\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int \partial_i^k u_1\,\partial_i^{2-k}(\partial_1\partial_2 b_1 + \partial_1^2 b_2)\,\partial_i^2 j\,dx \\
&\quad + 2\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int \partial_i^k u_1\,\partial_i^{2-k}(\partial_2 b_1 + \partial_1 b_2)\,\partial_i^2\partial_1 j\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i=1}^{2}\int u_1\,\partial_i^2(\partial_1\partial_2 b_1+\partial_1^2 b_2)\,\partial_i^2 j\,dx\\
&+2\sum_{i=1}^{2}\int u_1\,\partial_i^2(\partial_2 b_1+\partial_1 b_2)\,\partial_i^2\partial_1 j\,dx\\
:=&I_{52,1}+I_{52,2}+I_{52,3}+I_{52,4}.
\end{aligned}
$$

For $I_{52,1}$, $I_{52,2}$, by means of (2.7) we have

$$
\begin{aligned}
I_{52,1}+I_{52,2}\le{}& C\sum_{k=1}^{2}\mathcal{C}_2^k\,\|\nabla^k\partial_1 u_1\|_{L^2}\Big(\|\nabla^{3-k}\partial_1 b\|_{L^2}^{\frac12}\|\partial_2\nabla^{3-k}\partial_1 b\|_{L^2}^{\frac12}\|\nabla^2 j\|_{L^2}\\
&+\|\nabla^{3-k}b\|_{L^2}^{\frac12}\|\partial_2\nabla^{3-k}b\|_{L^2}^{\frac12}\|\partial_1\nabla^2 j\|_{L^2}\Big)\\
\le{}& C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}.
\end{aligned}
$$

For $I_{52,3}$, $I_{52,4}$, invoking Hölder's inequality, Sobolev's inequality and (2.6) yields

$$
\begin{aligned}
I_{52,3}+I_{52,4}&\le 2\|u_1\|_{L^\infty}(\|\nabla^3\partial_1 b\|_{L^2}\|\Delta j\|_{L^2}+\|\nabla^3 b\|_{L^2}\|\Delta\partial_1 j\|_{L^2})\\
&\le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3},
\end{aligned}
$$

which together with the estimates for $I_{52,1}$, $I_{52,2}$ and I_{51} derives

$$
I_5\le C\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3}+2T_2^*. \tag{3.11}
$$

Combining all estimates in (3.7)–(3.11) and integrating with respect to time, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{2}\|(\partial_i^2\omega,\partial_i^2 j)\|_{L^2}^2+2\eta\sum_{i=1}^{2}\int_0^t\|\partial_1\partial_i^2 j(\tau)\|_{L^2}^2\,d\tau\\
&\le C\int_0^t(\|u\|_{H^3}\|\partial_1 u\|_{H^2}^2+\|b\|_{H^3}\|\partial_1 u\|_{H^2}\|\partial_1 b\|_{H^3})\,d\tau\\
&+4\int_0^t(-T_0^*(\tau)+2T_1^*(\tau)+T_2^*(\tau))\,d\tau\\
&\le C\sup_{0\le\tau\le t}\|(u,b)(\tau)\|_{H^3}\int_0^t\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_1 b\|_{H^3}^2\Big)\,d\tau\\
&+4\int_0^t(-T_0^*(\tau)+2T_1^*(\tau)+T_2^*(\tau))\,d\tau
\end{aligned}
$$

$$\leq C\left(1+\frac{1}{\eta}\right) E_0^{\frac{3}{2}}(t) + E(0) + 4\int_0^t (-T_0^*(\tau) + 2T_1^*(\tau) + T_2^*(\tau))\,d\tau.$$

In what follows, our efforts focus on bounding T_0^*, T_1^* and T_2^*. Actually, T_0^*, T_1^* and T_2^* have the similar difficulties in essence. Hence, we only show the estimate of T_0^*. First, by the anisotropic inequality (2.6), we can get

$$T_0^* \leq \|\partial_2 u_2\|_{L^\infty} \|\partial_2^2 \omega\|_{L^2}^2 \leq C\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{4}} \|\partial_2^2\partial_1 u_2\|_{L^2}^{\frac{1}{4}} \|\partial_2^2\omega\|_{L^2}^2.$$

Then

$$\begin{aligned}
\int_0^t T_0^*(\tau)\,d\tau \leq & C \sup_{0\leq\tau\leq t} \|\partial_2^2\omega\|_{L^2}^2 (1+\tau)^{\frac{1}{4}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{4}} \\
& \times \int_0^t (1+\tau)^{\frac{1}{2}} \|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2\partial_1 u_2\|_{L^2}^{\frac{1}{4}} (1+\tau)^{-\frac{3}{4}}\,d\tau \\
\leq & C \sup_{0\leq\tau\leq t} \|\partial_2^2\omega\|_{L^2}^2 (1+\tau)^{\frac{1}{4}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{4}} \Big(\int_0^t (1+\tau)^2 \|\partial_2\partial_1 u_2(\tau)\|_{L^2}^2\,d\tau\Big)^{\frac{1}{4}} \\
& \times \Big(\int_0^t \|\partial_2^2\partial_1 u_2(\tau)\|_{L^2}^2\,d\tau\Big)^{\frac{1}{8}} \Big(\int_0^t (1+\tau)^{-\frac{6}{5}}\,d\tau\Big)^{\frac{5}{8}} \\
\leq & C E_0^{\frac{9}{8}}(t) E_1^{\frac{3}{8}}(t).
\end{aligned}$$

Similarly, we have for T_1^*, T_2^*

$$\int_0^t (T_1^*(\tau) + T_2^*(\tau))\,d\tau \leq C E_0^{\frac{9}{8}}(t) E_1^{\frac{3}{8}}(t).$$

Therefore, applying Young's inequality and (3.4), we can conclude

$$\begin{aligned}
&\|(\nabla^3 u, \nabla^3 b)(t)\|_{L^2}^2 + 2\eta \int_0^t \|\partial_1 \nabla^3 b(\tau)\|_{L^2}^2\,d\tau \\
&\quad \leq 2E(0) + C(1+\frac{1}{\eta}) E_0^{\frac{3}{2}}(t) + C E_0^{\frac{9}{8}}(t) E_1^{\frac{3}{8}}(t) \\
&\quad \leq 2E(0) + C(1+\frac{1}{\eta}) E_0^{\frac{3}{2}}(t) + C E_1^{\frac{3}{2}}(t).
\end{aligned} \tag{3.12}$$

Then using (3.12) and (3.3) yields

$$\frac{1}{2}\|(u,b)(t)\|_{H^3}^2 + \eta\int_0^t \|\partial_1 b(\tau)\|_{H^3}^2\,d\tau \le 2E(0) + C\left(1+\frac{1}{\eta}\right)\left(E_0^{\frac{3}{2}}(t) + E_1^{\frac{3}{2}}(t)\right). \tag{3.13}$$

This completes the proof of Lemma 3.2. □

3.2 Bound for $E_{0,1}(t)$

This subsection is devoted to constructing the horizontal dissipation for u in H^2 arising from the background magnetic field. That is to establish the bound for $E_{0,1}(t)$, which plays an important role in the establishment of closed bound for $E_{0,0}(t)$. However, the dissipation achieved in H^2 for u is weaker than that for b. The key reason for this lies in the dissipation term $\partial_1^2 b$ in the magnetic field equation. When we establish $\|\partial_1 u\|_{H^2}$, the linear term $\int \partial_1\nabla\omega\cdot\nabla\partial_1^2 j$ will emerge, which needs three-order dissipation $\|\partial_1 b\|_{H^3}^2$ to absorb it. In addition, the upper bound for $E_{0,1}(t)$ will generate $E_{0,0}$. While this is not a fundamental difficulty, we can address it by multiplying the bound in (3.14) by a sufficient small constant and combining it with (3.2) to eliminate the effect. The following lemma presents these results.

Lemma 3.3 *Assume that (u_0, b_0) satisfies the conditions in Theorem 1.1. Then we have, for a pure constant $C > 0$*

$$E_{0,1}(t) \le 2E(0) + (2+\eta^2)E_{0,0}(t) + C(1+\frac{1}{\eta})E_0^{\frac{3}{2}}(t). \tag{3.14}$$

Proof of Lemma 3.3 In order to establish the bound for $\int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau$, we shall make full advantage of the structure of (1.2) and (3.5). As in Lemma 3.2, it suffices to show the estimates for

$$\int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 d\tau \quad \text{and} \quad \int_0^t \|\partial_1\nabla\omega(\tau)\|_{L^2}^2 d\tau.$$

We first consider the L^2-inner product $(\partial_1 u, b)$. Then by virtues of the velocity equation and the magnetic equation in (1.2), we have

$$\begin{aligned}\frac{d}{dt}(\partial_1 u, b) &= (\partial_1 u_t, b) + (\partial_1 u, b_t)\\ &= \int \partial_1\Big(-(u\cdot\nabla)u + (b\cdot\nabla)b\Big)\cdot b\,dx - \|\partial_1 b\|_{L^2}^2\\ &\quad + \int \partial_1 u\cdot\Big(-(u\cdot\nabla)b + \eta\,\partial_1^2 b + (b\cdot\nabla)u\Big)dx + \|\partial_1 u\|_{L^2}^2,\end{aligned} \tag{3.15}$$

where $\int \partial_1(\nabla p)\cdot b dx=0$ by the incompressible condition. Similarly, it follows from the vorticity equations (3.5) that

$$\frac{d}{dt}(\partial_1\nabla\omega,\nabla j)=\int\partial_1\nabla\Big(-(u\cdot\nabla)\omega+(b\cdot\nabla)j\Big)\cdot\nabla j dx-\|\partial_1\nabla j\|_{L^2}^2 + \int\partial_1\nabla\omega\cdot\nabla\Big(-(u\cdot\nabla)j+\eta\,\partial_1^2 j+(b\cdot\nabla)\omega+Q\Big)dx+\|\partial_1\nabla\omega\|_{L^2}^2. \quad (3.16)$$

Adding (3.15) to (3.16), we obtain

$$\begin{aligned} &-\frac{d}{dt}\Big[(\partial_1 u,b)+(\partial_1\nabla\omega,\nabla j)\Big]+(\|\partial_1 u\|_{L^2}^2+\|\partial_1\nabla\omega\|_{L^2}^2)-(\|\partial_1 b\|_{L^2}^2+\|\partial_1\nabla j\|_{L^2}^2)\\ &=\int\Big(\partial_1(u\cdot\nabla)u-\partial_1(b\cdot\nabla)b\Big)\cdot b\,dx+\int\partial_1 u\cdot((u\cdot\nabla)b-(b\cdot\nabla)u)\,dx\\ &+\int\Big(\partial_1\nabla(u\cdot\nabla\omega)-\partial_1\nabla(b\cdot\nabla j)\Big)\cdot\nabla j\,dx\\ &+\int\partial_1\nabla\omega\cdot\Big(\nabla(u\cdot\nabla j)-\nabla(b\cdot\nabla\omega)\Big)\,dx\\ &-\int\partial_1\nabla\omega\cdot\nabla Q\,dx-\eta\int(\partial_1 u\cdot\partial_1^2 b+\partial_1\nabla\omega\cdot\nabla\partial_1^2 j)\,dx\\ &:=J_1+\cdots+J_6. \end{aligned} \quad (3.17)$$

Noticing $\overline{u}=\overline{b}_2=0$. We first use integration by parts and then apply Hölder's inequality, Sobolev's inequality, (2.6) and (2.2) to get

$$\begin{aligned} J_1+J_2&=-\int\Big((u\cdot\nabla)u-b_1\,\partial_1 b-b_2\,\partial_2 b\Big)\cdot\partial_1 b dx+\int\partial_1 u\cdot((u\cdot\nabla)b-(b\cdot\nabla)u)dx\\ &\le\Big(\|u\|_{L^\infty}\|\nabla u\|_{L^2}+\|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2}+\|b_2\|_{L^\infty}\|\partial_2 b\|_{L^2}\Big)\|\partial_1 b\|_{L^2}\\ &\quad+\Big(\|u\|_{L^\infty}\|\nabla b\|_{L^2}+\|b\|_{L^\infty}\|\nabla u\|_{L^2}\Big)\|\partial_1 u\|_{L^2}\\ &\le C(\|u\|_{H^1}+\|b\|_{H^2})(\|\partial_1 u\|_{H^1}^2+\|\partial_1 b\|_{H^1}^2). \end{aligned} \quad (3.18)$$

Similarly, J_3 can be bounded by

$$\begin{aligned} J_3&\le\Big(\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}+\|b_1\|_{L^\infty}\|\partial_1 j\|_{L^2}+\|b_2\|_{L^\infty}\|\partial_2 j\|_{L^2}\Big)\|\partial_1\Delta j\|_{L^2}\\ &\le C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_1 u\|_{H^1}^2+\|\partial_1 b\|_{H^3}^2). \end{aligned}$$

For J_4, we first divided it into several parts. Then a similar argument to J_1 reaches

$$\begin{aligned} J_4=&\int\partial_1\nabla\omega\cdot\Big(\nabla u\cdot\nabla j+u\cdot\nabla^2 j-\nabla b_1\partial_1\omega-\nabla b_2\partial_2\omega-b_1\partial_1\nabla\omega-b_2\partial_2\nabla\omega\Big)dx\\ \le&\|\partial_1\nabla\omega\|_{L^2}\Big(\|\nabla u\|_{L^\infty}\|\nabla j\|_{L^2}+\|u\|_{L^\infty}\|\nabla^2 j\|_{L^2}+\|\nabla b_1\|_{L^\infty}\|\partial_1\omega\|_{L^2}\\ &+\|\nabla b_2\|_{L^\infty}\|\partial_2\omega\|_{L^2}+\|b_1\|_{L^\infty}\|\partial_1\nabla\omega\|_{L^2}+\|b_2\|_{L^\infty}\|\partial_2\nabla\omega\|_{L^2}\Big) \end{aligned}$$

$$\leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|\partial_1 u\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2).$$

Also, J_5 can be bounded by

$$\begin{aligned} J_5 \leq & 2\int |\partial_1 \nabla \omega| \Big(|\partial_1 \nabla u_1|\,|\nabla b| + |\partial_1 u_1|\,|\nabla^2 b| + |\partial_1 \nabla b_1|\,|\nabla u| + |\partial_1 b_1|\,|\nabla^2 u| \Big)\,dx \\ \leq & \|\partial_1 \nabla \omega\|_{L^2} \Big(\|\nabla \partial_1 u_1\|_{L^2} \|\nabla b\|_{L^\infty} + \|\partial_1 u_1\|_{L^\infty} \|\nabla^2 b\|_{L^2} \\ & + \|\nabla \partial_1 b_1\|_{L^2} \|\nabla u\|_{L^\infty} + \|\partial_1 b_1\|_{L^\infty} \|\nabla^2 u\|_{L^2} \Big) \\ \leq & C(\|u\|_{H^3} + \|b\|_{H^3})(\|\partial_1 u\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2). \end{aligned}$$

Finally, it is obvious that

$$J_6 \leq \frac{1}{2}(\|\partial_1 u\|_{L^2}^2 + \|\partial_1 \nabla \omega\|_{L^2}^2) + \frac{\eta^2}{2}(\|\partial_1^2 b\|_{L^2}^2 + \|\partial_1 \nabla^2 j\|_{L^2}^2).$$

Inserting the estimates above in (3.17), we obtain

$$\begin{aligned} &(\|\partial_1 u\|_{L^2}^2 + \|\partial_1 \nabla \omega\|_{L^2}^2) - (2\|\partial_1 b\|_{L^2}^2 + 2\|\partial_1 \nabla j\|_{L^2}^2 + \eta^2 \|\partial_1^2 b\|_{L^2}^2 + \eta^2 \|\partial_1 \nabla^2 j\|_{L^2}^2) \\ &\quad \leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|\partial_1 u\|_{H^2}^2 + \|\partial_1 b\|_{H^3}^2) + 2\frac{d}{dt}\Big[(\partial_1 u, b) + (\partial_1 \nabla \omega, \nabla j)\Big]. \end{aligned} \tag{3.19}$$

Then integrating (3.19) over $[0, t]$ and applying Hölder's inequality yield

$$\begin{aligned} &\int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau - (2+\eta^2) \int_0^t \|\partial_1 b(\tau)\|_{H^3}^2 d\tau \\ \leq & 2(\|(\partial_1 u, b)\|_{L^2}^2 + \|(\partial_1 \nabla \omega, \nabla j)\|_{L^2}^2) + 2(\|(\partial_1 u_0, b_0)\|_{L^2}^2 + \|(\partial_1 \nabla \omega_0, \nabla j_0)\|_{L^2}^2) \\ & + C \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{H^3} + \|b(\tau)\|_{H^3}) \int_0^t (\|\partial_1 u(\tau)\|_{H^2}^2 + \|\partial_1 b(\tau)\|_{H^3}^2)\,d\tau, \end{aligned}$$

which implies the desired bound (3.14). This completes the proof of Lemma 3.3. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1 The inequality (3.1) is a direct consequence of Lemma 3.2 and Lemma 3.3. In fact, we can make the combination

$$(3.2) + \lambda_0 \times (3.14)$$

to get

$$(1 - \lambda_0(2+\eta^2)) E_{0,0}(t) + \lambda_0 E_{0,1}(t)$$

$$\leq (4+2\lambda_0)E(0) + C(1+\frac{1}{\eta}+\lambda_0)(E_0(t)^{\frac{3}{2}} + E_1(t)^{\frac{3}{2}}) \tag{3.20}$$

provided that $\lambda_0 < \frac{1}{2+\eta^2}$. Furthermore, we derive from (3.20) the desired bound (3.1). □

4 Estimates of $E_1(t)$

This section proves the *a priori* estimate on $E_1(t)$.

Proposition 4.1 *Assume (u_0, b_0) obeys the conditions stated in Theorem 1.1. Then the solution of the system (1.2) satisfies*

$$E_1(t) \leq \frac{1}{\widetilde{c}_0}\Big(C(1+\frac{1}{\eta})(\lambda+1)(E_0^{\frac{3}{2}}(t) + E_1(t)^{\frac{3}{2}}) + (\lambda+1)E(0) + C\widetilde{c}_1 E_0(t)\Big), \tag{4.1}$$

where

$$\widetilde{c}_0 = \min\left\{\lambda - \frac{1}{2},\ \lambda - c_0,\ \lambda(2\eta-\delta_1) - \frac{1}{2},\ \lambda(2\eta-\delta_1) - \frac{\eta^2}{2} - \frac{1}{2},\ \frac{1}{2} - \delta_2 - 2\delta_0\lambda\right\}$$

and

$$\widetilde{c}_1 = \lambda(\frac{1}{\delta_0} + \frac{1}{\delta_1\eta}) + \frac{1}{\delta_2\eta}$$

with $\delta_1 < 2\eta$, $\delta_2 < \frac{1}{2}$

$$\lambda > \max\left\{\frac{1}{2},\ c_0,\ \frac{1+\eta^2}{2(2\eta-\delta_1)}\right\} \text{ and } \delta_0 < \frac{\frac{1}{2}-\delta_2}{2\lambda}.$$

As aforementioned in the introduction, the time-weighted energy functional $E_1(t)$ serves to solve the most difficult terms with all vertical derivatives, i.e., T_0^*, T_1^* and T_2^* in $E_0(t)$. By making full of the decay rates in $E_1(t)$, we are able to control the growth of these hard items. Thereby, the closed bound (3.1) for E_0 can be established. This is the key part of the whole proof.

The proof is split into two subsections, which will be devoted to the estimates of $E_{1,0}(t)$ and $E_{1,1}(t)$, respectively, where

$$E_{1,0}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^2\Big(\|u_2(\tau)\|_{H^2}^2 + \|b_2(\tau)\|_{H^2}^2\Big) + \eta\int_0^t (1+\tau)^2\|\partial_1 b_2(\tau)\|_{H^2}^2\, d\tau,$$

$$E_{1,1}(t) = \int_0^t (1+\tau)^2\|\partial_1\nabla u_2(\tau)\|_{L^2}^2\, d\tau.$$

4.1 Bound for $E_{1,0}(t)$

Lemma 4.2 *Let (u,b) be the solution of the system (1.2). Then it holds*

$$
\begin{aligned}
&(1+t)^2\Big(\|(u_2,b_2)(t)\|_{L^2}^2+\|(\Delta u_2,\Delta b_2)(t)\|_{L^2}^2\Big)\\
&\quad+(2\eta-\delta_1)\int_0^t(1+\tau)^2\Big(\|\partial_1 b_2(\tau)\|_{L^2}^2+\|\partial_1\Delta b_2(\tau)\|_{L^2}^2\Big)\,d\tau\\
&\quad\le 2\delta_0\int_0^t(1+\tau)^2\|\partial_1\nabla u_2\|_{L^2}^2\,d\tau+\Big(1+\frac{1}{\eta}\Big)E_0^{\frac12}(t)E_1(t)+E(0)+C\left(\frac{1}{\delta_0}+\frac{1}{\delta_1\eta}\right)E_0(t),
\end{aligned}\tag{4.2}
$$

where δ_0,δ_1 are two positive constants.

Proof Due to the equivalence

$$
\|v\|_{H^2}\sim\|v\|_{L^2}+\|\Delta v\|_{L^2},
$$

it suffices to prove the time-weighted functional

$$
(1+t)^2\|(u_2,b_2)\|_{L^2}^2\ \text{ and }\ (1+t)^2\|(\Delta u_2,\Delta b_2)\|_{L^2}^2.
$$

Taking the L^2-inner product of the equations of u_2 and b_2 in (1.2) with (u_2,b_2), then multiplying the time weight$(1+t)^2$, we have

$$
\begin{aligned}
&\frac{d}{dt}(1+t)^2\|(u_2,b_2)\|_{L^2}^2+2\eta(1+t)^2\|\partial_1 b_2\|_{L^2}^2\\
&\quad=2(1+t)\|(u_2,b_2)\|_{L^2}^2-2(1+t)^2\int\partial_2 P\,u_2dx.
\end{aligned}\tag{4.3}
$$

By Poincaré-type inequality (2.2) and (2.3), the first term in (4.3) can be bounded as

$$
\begin{aligned}
2(1+t)\|(u_2,b_2)(t)\|_{L^2}^2&\le C(1+t)\|\partial_1^2u_2\|_{L^2}\|\partial_1u_2\|_{L^2}+C(1+t)\|\partial_1b_2\|_{L^2}^2\\
&\le(1+t)^2(\delta_0\|\partial_1^2u_2\|_{L^2}^2+\delta_1\|\partial_1b_2\|_{L^2}^2)\\
&\quad+C(\frac{1}{\delta_0}\|\partial_1u_2\|_{L^2}^2+\frac{1}{\delta_1}\|\partial_1b_2\|_{L^2}^2),
\end{aligned}\tag{4.4}
$$

where δ_0,δ_1 are two small positive constants to be determined later.

For the second term in (4.3), recalling $P=\Delta^{-1}\nabla\cdot(b\cdot\nabla b-u\cdot\nabla u)$, we have

$$
-2(1+t)^2\int\partial_2P\,u_2dx=2(1+t)^2\int\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u-b\cdot\nabla b)\,u_2\,dx.
$$

For the integral term involving u, we can apply Sobolev's inequality and (2.2) to get

$$
\int\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\,u_2\,dx
$$

$$
\begin{aligned}
&= \int \partial_2 \Delta^{-1} \partial_2 (u \cdot \nabla u_2)\, u_2 \, dx + \int \partial_2 \Delta^{-1} \partial_1 (u_1 \partial_1 u_1)\, u_2 \, dx \\
&\quad + \int \partial_2 \Delta^{-1} \partial_1 (u_2 \partial_2 u_1)\, u_2 \, dx \\
&\le C(\|u \cdot \nabla u_2\|_{L^2} + \|u_1 \partial_1 u_1\|_{L^2} + \|u_2 \partial_2 u_1\|_{L^2})\, \|u_2\|_{L^2} \\
&\le C(\|u\|_{L^\infty} \|\nabla u_2\|_{L^2} + \|u_1\|_{L^\infty} \|\partial_1 u_1\|_{L^2} + \|\partial_2 u_1\|_{L^\infty} \|u_2\|_{L^2})\, \|u_2\|_{L^2} \\
&\le C \|u\|_{H^3} \|\partial_1 \nabla u_2\|_{L^2}^2,
\end{aligned}
$$

where we have used the fact that the Riesz transform $\mathcal{R}_i = \partial_i (-\Delta)^{-\frac{1}{2}}$ is bounded in L^p for $1 \le p < \infty$. Similarly, by (2.3) we have

$$
\int \partial_2 \Delta^{-1} \nabla \cdot (b \cdot \nabla b) \cdot u_2 \, dx \le C \|b\|_{H^3} \|\partial_1 \nabla b_2\|_{L^2} \|\partial_1 \nabla u_2\|_{L^2}.
$$

Hence, we obtain

$$
-2(1+t)^2 \int \partial_2 P \, u_2 dx \le C(1+t)^2 \|(u,b)\|_{H^3} (\|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_1 \nabla b_2\|_{L^2}^2). \quad (4.5)
$$

Combining the estimates (4.4) with (4.5), we derive

$$
\begin{aligned}
&(1+t)^2 \|(u_2, b_2)\|_{L^2}^2 + (2\eta - \delta_1) \int_0^t (1+\tau)^2 \|\partial_1 b_2\|_{L^2}^2 \, d\tau \\
&\le \delta_0 \int_0^t (1+\tau)^2 \|\partial_1^2 u_2\|_{L^2}^2 \, d\tau + C \int_0^t (\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1 b_2\|_{L^2}^2) \, d\tau \\
&\quad + C \int_0^t (1+\tau)^2 \|(u,b)\|_{H^3} (\|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_1 \nabla b_2\|_{L^2}^2) \, d\tau + \|(u_{02}, b_{02})\|_{L^2}^2.
\end{aligned}
\quad (4.6)
$$

Now we focus on the second-order time-weighted energy estimate. We first take Δ to the equations of u_2, b_2, and multiply the resulted equations by $(1+t)^2(\Delta u_2, \Delta b_2)$ and then integrate in Ω,

$$
\begin{aligned}
&\frac{d}{dt}(1+t)^2 \|(\Delta u_2, \Delta b_2)\|_{L^2}^2 + 2\eta (1+t)^2 \|\partial_1 \Delta b_2\|_{L^2}^2 \\
&= 2(1+t) \|(\Delta u_2, \Delta b_2)(t)\|_{L^2}^2 - 2(1+t)^2 \int \partial_2 \Delta P \, \Delta u_2 dx \\
&\quad - 2(1+t)^2 \int \Delta (u \cdot \nabla u_2)\, \Delta u_2 \, dx + 2(1+t)^2 \int \Delta (b \cdot \nabla b_2)\, \Delta u_2 \, dx \\
&\quad - 2(1+t)^2 \int \Delta (u \cdot \nabla b_2)\, \Delta b_2 \, dx + 2(1+t)^2 \int \Delta (b \cdot \nabla u_2)\, \Delta b_2 \, dx
\end{aligned}
$$

$$:= K_1 + K_2 + \cdots + K_6. \tag{4.7}$$

The estimates for the right-side terms in (4.7) are complicated and subtle. We shall bound them one by one. By means of Sobolev's inequality $\|\nabla v\|_{L^2} \leq C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}$, Poincaré-type inequality (2.2) and (2.3), we obtain

$$\begin{aligned}
K_1 &\leq C(1+t)\|\nabla u_2\|_{L^2}\|\nabla^3 u_2\|_{L^2} + C(1+t)\|\partial_1 \Delta b_2\|_{L^2}^2 \\
&\leq C(1+t)\|\partial_1 \nabla u_2\|_{L^2}\|\nabla^3 u_2\|_{L^2} + C(1+t)\|\partial_1 \Delta b_2\|_{L^2}^2 \\
&\leq (1+t)^2(\delta_0\|\partial_1\nabla u_2\|_{L^2}^2 + \delta_1\|\partial_1\Delta b\|_{L^2}^2) + C(\frac{1}{\delta_0}\|\partial_1 u\|_{H^2}^2 + \frac{1}{\delta_1}\|\partial_1 b_2\|_{H^2}^2),
\end{aligned} \tag{4.8}$$

where we also use $\|\nabla^3 u_2\|_{L^2} = \|\partial_1 \nabla^2 u\|_{L^2}$.

Next we bound all the integral terms in (4.7). It is the most difficult to handle K_2 among all of them. Due to the weak dissipation of u_2, we need more elaborate argument. Firstly, by $\Delta P = \nabla \cdot (b \cdot \nabla b - u \cdot \nabla u)$, we have

$$\begin{aligned}
K_2 &= 2(1+t)^2 \int \partial_2 \nabla \cdot (u \cdot \nabla u) \cdot \Delta u_2 \, dx - 2(1+t)^2 \int \partial_2 \nabla \cdot (b \cdot \nabla b) \cdot \Delta u_2 \, dx \\
&:= 2(1+t)^2(K_{21} + K_{22})
\end{aligned}$$

By integration by parts, we decompose it into several parts as follows

$$\begin{aligned}
K_{21} &= -\int \partial_2 \nabla \nabla \cdot (u \cdot \nabla u) \cdot \nabla u_2 \, dx = -\sum_{i=1}^{2}\sum_{j=1}^{2} \int \partial_2 \nabla (\partial_j u_i \partial_i u_j) \cdot \nabla u_2 \, dx \\
&= -4 \int (\partial_2 \partial_1 u_1 \, \partial_1 \nabla u_1 + \partial_1 u_1 \, \partial_1 \partial_2 \nabla u_1) \cdot \nabla u_2 \, dx \\
&\quad - 2 \int (\partial_1 u_2 \, \partial_2^2 \nabla u_1 + \partial_1 \partial_2 u_2 \, \partial_2 \nabla u_1) \cdot \nabla u_2 \, dx - 2 \int \partial_1 \nabla u_2 \, \partial_2^2 u_1 \cdot \nabla u_2 \, dx \\
&\quad + 2 \int \partial_2 \nabla u_2 \cdot (\partial_2 \partial_1 u_1 \, \nabla u_2 + \partial_2 u_1 \, \partial_1 \nabla u_2) dx.
\end{aligned}$$

By the anisotropic inequalities (2.6) and (2.7) and Poincaré-type inequality (2.2), we derive

$$\begin{aligned}
K_{21} \leq\, & C\|\partial_2 \partial_1 u_1\|_{L^2}\|\partial_1 \nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \partial_2 \nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla u_2\|_{L^2} \\
&+ C\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \partial_2 \nabla u_1\|_{L^2}\|\partial_1 \nabla u_2\|_{L^2} \\
&+ C\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 \partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 \nabla u_1\|_{L^2}\|\partial_1 \nabla u_2\|_{L^2} \\
&+ C\|\partial_2 \partial_1 u_2\|_{L^2}\|\partial_2 \nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 \nabla u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla u_2\|_{L^2} \\
&+ C\|\partial_1 \nabla u_2\|_{L^2}\|\partial_2^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^3 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla u_2\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ C\|\partial_2\nabla u_2\|_{L^2}\|\partial_1\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 u_1\|_{L^2}^{\frac12}\|\partial_1\nabla u_2\|_{L^2}\\
&+ C\|\partial_2\nabla u_2\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_1\nabla u_2\|_{L^2}\\
\le{}& C\|\Delta u_2\|_{L^2}\|\partial_1 u\|_{H^2}\|\partial_1\nabla u_2\|_{L^2} + C\|u\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}^2, \qquad (4.9)
\end{aligned}
$$

where we use $\|\partial_1 u_1\|_{L^2} \le \|\partial_1^2 u_1\|_{L^2}$, $\|\partial_1 u_2\|_{L^2} \le \|\partial_1^2 u_2\|_{L^2}$.

Thanks to strong dissipation $\|\partial_1 b_2\|_{H^2}$, it is simpler to bound K_{22}. By integration by parts, K_{22} is first split into the following parts

$$
\begin{aligned}
K_{22} &= -\sum_{i=1}^{2}\sum_{j=1}^{2}\int \partial_2(\partial_j b_i\partial_i b_j)\,\Delta u_2\,dx\\
&= -4\int \partial_1 b_1\partial_2\partial_1 b_1\,\Delta u_2\,dx\\
&\quad + 2\int (\partial_1\partial_2\nabla b_2\,\partial_2 b_1 + \partial_1\partial_2 b_2\,\partial_2\nabla b_1 + \partial_1\nabla b_2\,\partial_2^2 b_1 + \partial_1 b_2\,\partial_2^2\nabla b_1)\cdot\nabla u_2\,dx.
\end{aligned}
$$

Then applying Sobolev's inequality and Poincaré-type inequality (2.2) and (2.3) yields

$$
\begin{aligned}
K_{22} &\le 4\|\partial_1 b_1\|_{L^2}\|\partial_1\partial_2 b_1\|_{L^\infty}\|\Delta u_2\|_{L^2} + 2\|\partial_2\partial_1\nabla b_2\|_{L^2}\|\partial_2 b_1\|_{L^\infty}\|\nabla u_2\|_{L^2}\\
&\quad + 4\|\partial_1\nabla b_2\|_{L^4}\|\partial_2\nabla b_1\|_{L^4}\|\nabla u_2\|_{L^2} + 2\|\partial_1 b_2\|_{L^\infty}\|\partial_2^2\nabla b_1\|_{L^2}\|\nabla u_2\|_{L^2}\\
&\le C\|\partial_1^2 b_1\|_{L^2}\|\partial_1 b\|_{H^3}\|\Delta u_2\|_{L^2} + C\|\partial_2\partial_1\nabla b_2\|_{L^2}\|\partial_2 b_1\|_{H^2}\|\nabla\partial_1 u_2\|_{L^2}\\
&\quad + C\|\partial_1\nabla b_2\|_{H^1}\|\partial_2\nabla b_1\|_{H^1}\|\partial_1\nabla u_2\|_{L^2} + C\|\partial_1 b_2\|_{H^2}\|\partial_2^2\nabla b_1\|_{L^2}\|\nabla\partial_1 u_2\|_{L^2}\\
&\le C\|\Delta u_2\|_{L^2}\|\partial_1 b_2\|_{H^2}\|\partial_1 b\|_{H^3} + C\|b\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}\|\partial_1 b_2\|_{H^2}. \qquad (4.10)
\end{aligned}
$$

Therefore, by (4.9) and (4.10),

$$
\begin{aligned}
K_2 &\le C(1+t)^2(\|\Delta u_2\|_{L^2}\|\partial_1 u\|_{H^2}\|\partial_1\nabla u_2\|_{L^2} + \|u\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}^2\\
&\quad + \|\Delta u_2\|_{L^2}\|\partial_1 b_2\|_{H^2}\|\partial_1 b\|_{H^3} + \|b\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}\|\partial_1 b_2\|_{H^2}). \qquad (4.11)
\end{aligned}
$$

We proceed to deal with K_3. First, we rewrite it as follows

$$
\begin{aligned}
K_3 &= -2(1+t)^2\Big(\int \Delta u\cdot\nabla u_2\,\Delta u_2\,dx + 2\int \nabla u\cdot\nabla(\nabla u_2)\,\Delta u_2\,dx\Big)\\
&:= K_{31} + K_{32}.
\end{aligned}
$$

For K_{31}, we further split it in four terms and then use Sobolev's inequality and (2.7), (2.2) to get

$$
\begin{aligned}
K_{31} &= -2(1+t)^2\Big(\int \Delta u\cdot\nabla u_2\,\partial_1^2 u_2\,dx + \int \partial_1^2 u\cdot\nabla u_2\,\partial_2^2 u_2\,dx\\
&\quad + \int \partial_2^2 u_1\partial_1 u_2\,\partial_2^2 u_2\,dx + \int \partial_2^2 u_2\partial_2 u_2\,\partial_2^2 u_2\,dx\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(1+t)^2\Big(\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Delta\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_2\|_{L^2}\|\partial_1^2 u_2\|_{L^2}\\
&\quad+\|\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_2\|_{L^2}\|\partial_2^2 u_2\|_{L^2}\\
&\quad+\|\partial_1\partial_2^2 u_1\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 u_2\|_{L^2}\\
&\quad+\|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^3 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_{L^2}\|\partial_2^2 u_2\|_{L^2}\Big)\\
&\leq C(1+t)^2(\|u\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}^2+\|\Delta u_2\|_{L^2}\|\partial_1 u\|_{H^2}\|\partial_1\nabla u_2\|_{L^2}).
\end{aligned}
$$

Similarly, by (2.6) and (2.7) we have

$$
\begin{aligned}
K_{32}&=-4(1+t)^2\Big(\int\partial_1 u\cdot\nabla\partial_1 u_2\Delta u_2\,dx+\int\partial_2 u_1\partial_1\partial_2 u_2\Delta u_2\,dx+\int\partial_2 u_2\partial_2^2 u_2\Delta u_2\,dx\Big)\\
&\leq C(1+t)^2\Big(\|\nabla u\|_{L^\infty}\|\partial_1\nabla u_2\|_{L^2}\|\Delta u_2\|_{L^2}+\|\partial_1\partial_2 u_2\|_{L^2}\|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^3 u_2\|_{L^2}^{\frac{1}{2}}\|\Delta u_2\|_{L^2}\Big)\\
&\leq C(1+t)^2\|\Delta u_2\|_{L^2}\|\partial_1 u\|_{H^2}\|\partial_1\nabla u_2\|_{L^2}.
\end{aligned}
$$

Thus,

$$
K_3\leq C(1+t)^2(\|u\|_{H^3}\|\partial_1\nabla u_2\|_{L^2}^2+\|\Delta u_2\|_{L^2}\|\partial_1 u\|_{H^2}\|\partial_1\nabla u_2\|_{L^2}). \tag{4.12}
$$

For K_4, we first divide K_4 into three parts.

$$
\begin{aligned}
K_4&=2(1+t)^2\Big(\int\Delta b\cdot\nabla b_2\,\Delta u_2\,dx+2\int\nabla b\cdot\nabla(\nabla b_2)\,\Delta u_2\,dx+\int b\cdot\nabla(\Delta b_2)\,\Delta u_2\,dx\Big)\\
&:=2(1+t)^2\Big(K_{41}+K_{42}+\int b\cdot\nabla(\Delta b_2)\,\Delta u_2\,dx\Big).
\end{aligned}
$$

To Bound K_{41}, we further split it and apply Sobolev's inequality and (2.2), (2.3) to get

$$
\begin{aligned}
K_{41}&=\int\partial_1^2 b\cdot\nabla b_2\,\Delta u_2\,dx+\int\partial_2^2 b_2\,\partial_2 b_2\,\Delta u_2\,dx\\
&\quad-\int(\partial_2^2\nabla b_1\,\partial_1 b_2+\partial_2^2 b_1\,\partial_1\nabla b_2)\cdot\nabla u_2\,dx\\
&\leq\|\partial_1^2 b\|_{L^4}\|\nabla b_2\|_{L^4}\|\Delta u_2\|_{L^2}+\|\partial_2^2 b_2\|_{L^4}\|\partial_2 b_2\|_{L^4}\|\Delta u_2\|_{L^2}\\
&\quad+\|\partial_2^2\nabla b_1\|_{L^2}\|\partial_1 b_2\|_{L^\infty}\|\nabla u_2\|_{L^2}+\|\partial_2^2 b_1\|_{L^4}\|\nabla\partial_1 b_2\|_{L^4}\|\nabla u_2\|_{L^2}\\
&\leq C\|\partial_1^2 b\|_{H^1}\|\nabla\partial_1 b_2\|_{H^1}\|\Delta u_2\|_{L^2}+C\|\partial_2^2 b_2\|_{H^1}\|\partial_2\partial_1 b_2\|_{H^1}\|\Delta u_2\|_{L^2}\\
&\quad+C\|\partial_2^2\nabla b_1\|_{L^2}\|\partial_1 b_2\|_{H^2}\|\nabla\partial_1 u_2\|_{L^2}+C\|\partial_2^2 b_1\|_{H^1}\|\nabla\partial_1 b_2\|_{H^1}\|\nabla\partial_1 u_2\|_{L^2}\\
&\leq C\|b\|_{H^3}\|\partial_1 b_2\|_{H^2}\|\partial_1\nabla u_2\|_{L^2}+C\|\partial_1 b\|_{H^2}\|\partial_1 b_2\|_{H^2}\|\Delta u_2\|_{L^2}.
\end{aligned}
$$

Similarly,

$$
K_{42}=2\int\partial_1 b\cdot\nabla\partial_1 b_2\,\Delta u_2\,dx+2\int\partial_2 b_2\,\partial_2^2 b_2\,\Delta u_2\,dx
$$

$$
\begin{aligned}
&- 2\int (\partial_2\nabla b_1\, \partial_1\partial_2 b_2 + \partial_2 b_1\, \partial_1\partial_2\nabla b_2)\cdot\nabla u_2\, dx \\
&\le 2\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1 b_2\|_{L^2}\|\Delta u_2\|_{L^2} + C\|\partial_2\partial_1 b_2\|_{L^2}\|\partial_2^2 b_2\|_{L^2}^{\frac12}\|\partial_2^3 b_2\|_{L^2}^{\frac12}\|\Delta u_2\|_{L^2} \\
&\quad + 2\|\partial_2\nabla b_1\|_{L^4}\|\partial_2\partial_1 b_2\|_{L^4}\|\nabla u_2\|_{L^2} + 2\|\partial_2 b_1\|_{L^\infty}\|\nabla\partial_2\partial_1 b_2\|_{L^2}\|\nabla u_2\|_{L^2} \\
&\le C\|b\|_{H^3}\|\partial_1 b_2\|_{H^2}\|\partial_1\nabla u_2\|_{L^2} + C\|\partial_1 b\|_{H^2}\|\partial_1 b_2\|_{H^2}\|\Delta u_2\|_{L^2}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
K_4 &\le C(1+t)^2\big(\|b\|_{H^3}\|\partial_1 b_2\|_{H^2}\|\partial_1\nabla u_2\|_{L^2} + \|\partial_1 b\|_{H^2}\|\partial_1 b_2\|_{H^2}\|\Delta u_2\|_{L^2}\big) \\
&\quad + 2(1+t)^2\int b\cdot\nabla(\Delta b_2)\,\Delta u_2\, dx.
\end{aligned} \tag{4.13}
$$

For I_5, it's easily to get

$$
\begin{aligned}
K_5 &= -2(1+t)^2\Big(\int \Delta u\cdot\nabla b_2\,\Delta b_2\, dx + 2\int \nabla u\cdot\nabla\nabla b_2\,\Delta b_2\, dx\Big) \\
&\le 2(1+t)^2\big(\|\Delta u\|_{L^4}\|\nabla b_2\|_{L^4}\|\Delta b_2\|_{L^2} + 2\|\nabla u\|_{L^\infty}\|\nabla^2 b_2\|_{L^2}\|\Delta b_2\|_{L^2}\big) \\
&\le C(1+t)^2\|u\|_{H^3}\|\partial_1 b_2\|_{H^2}^2.
\end{aligned} \tag{4.14}
$$

Also, the last term K_6 can be bounded as

$$
\begin{aligned}
K_6 &= 2(1+t)^2\Big(\int \Delta b\cdot\nabla u_2\,\Delta b_2\, dx + 2\int \nabla b_1\cdot\partial_1\nabla u_2\Delta b_2\, dx \\
&\quad + 2\int \nabla b_2\cdot\partial_2\nabla u_2\Delta b_2\, dx + \int b\cdot\nabla(\Delta u_2)\,\Delta b_2\, dx\Big) \\
&\le 2(1+t)^2\Big(\|\Delta b\|_{L^2}^{\frac12}\|\partial_2\Delta b\|_{L^2}^{\frac12}\|\partial_1\nabla u_2\|_{L^2}\|\Delta b_2\|_{L^2} + \|\nabla b_1\|_{L^\infty}\|\partial_1\nabla u_2\|_{L^2}\|\Delta b_2\|_{L^2} \\
&\quad + \|\nabla b_2\|_{L^4}\|\partial_2\nabla u_2\|_{L^4}\|\Delta b_2\|_{L^2} + \int b\cdot\nabla(\Delta u_2)\,\Delta b_2\, dx\Big) \\
&\le C(1+t)^2\|(u,b)\|_{H^3}(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2) \\
&\quad + 2(1+t)^2\int b\cdot\nabla(\Delta u_2)\,\Delta b_2\, dx.
\end{aligned} \tag{4.15}
$$

Inserting (4.8), (4.11), (4.12), (4.13), (4.14) and (4.15) in (4.7) and integrating in time, we obtain

$$
\begin{aligned}
&(1+t)^2\|(\Delta u_2, \Delta b_2)(t)\|_{L^2}^2 + (2\eta - \delta_1)\int_0^t (1+\tau)^2\|\partial_1\Delta b_2(\tau)\|_{L^2}^2\, d\tau \\
&\le \delta_0\int_0^t (1+\tau)^2\|\partial_1\nabla u_2\|_{L^2}^2\, d\tau + C\int_0^t \Big(\frac{1}{\delta_0}\|\partial_1 u\|_{H^2}^2 + \frac{1}{\delta_1}\|\partial_1 b\|_{H^2}^2\Big)\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_0^t (1+\tau)^2 \|(u,b)\|_{H^3} (\|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2)\, d\tau \\
&+ C\int_0^t (1+\tau)^2 (\|\partial_1 u\|_{H^2} + \|\partial_1 b\|_{H^3})\, (\|\partial_1 \nabla u_2\|_{L^2} + \|\partial_1 b_2\|_{H^2})\, \|\Delta u_2\|_{L^2}\, d\tau \\
&+ \|(\Delta u_{02}, \Delta b_{02})\|_{L^2}^2.
\end{aligned} \tag{4.16}
$$

It is noted that

$$
\begin{aligned}
&\int_0^t (1+\tau)^2 \|(u,b)\|_{H^3} \left(\|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2\right) d\tau \\
&\quad \le C \sup_{0\le\tau\le t} \|(u,b)(\tau)\|_{H^3} \int_0^t (1+\tau)^2 (\|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2)\, d\tau \\
&\quad \le C(1+\frac{1}{\eta}) E_0^{\frac{1}{2}}(t) E_1(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t (1+\tau)^2 (\|\partial_1 u\|_{H^2} + \|\partial_1 b\|_{H^3})(\|\partial_1 \nabla u_2\|_{L^2} + \|\partial_1 b_2\|_{H^2}) \|\Delta u_2\|_{L^2}\, d\tau \\
&\quad \le \sup_{0\le\tau\le t} (1+\tau) \|\Delta u_2\|_{L^2} \int_0^t (\|\partial_1 u\|_{H^2} + \|\partial_1 b\|_{H^3})(1+\tau)(\|\partial_1 \nabla u_2\|_{L^2} + \|\partial_1 b_2\|_{H^2})\, d\tau \\
&\le C(1+\frac{1}{\eta}) E_0^{\frac{1}{2}}(t) E_1(t).
\end{aligned}
$$

Then by (4.6) and (4.16) we conclude

$$
\begin{aligned}
&(1+t)^2 \left(\|(u_2, b_2)(t)\|_{L^2}^2 + \|(\Delta u_2, \Delta b_2)(t)\|_{L^2}^2\right) \\
&\quad + (2\eta - \delta_1) \int_0^t (1+\tau)^2 \left(\|\partial_1 b_2(\tau)\|_{L^2}^2 + \|\partial_1 \Delta b_2(\tau)\|_{L^2}^2\right) d\tau \\
&\le 2\delta_0 \int_0^t (1+\tau)^2 \|\partial_1 \nabla u_2\|_{L^2}^2\, d\tau + C\left(1+\frac{1}{\eta}\right) E_0^{\frac{1}{2}}(t) E_1(t) + E(0) + C\left(\frac{1}{\delta_0} + \frac{1}{\delta_1 \eta}\right) E_0(t).
\end{aligned}
$$

This completes the proof of Lemma 4.2.

□

4.2 Bound for $E_{1,1}(t)$

To establish the bound for $E_1(t)$, it remains to bound $E_{1,1}(t)$. This can be done by making use of the linear term in (1.2). We have the following result.

Lemma 4.3 *Let (u,b) be the solution of the system (1.2). Then it holds*

$$\begin{aligned}
&\int_0^t (1+\tau)^2\Big(\frac{1}{2}-\delta_2\Big)\|\partial_1\nabla u_2\|_{L^2}^2\,d\tau \\
&\quad-\int_0^t (1+\tau)^2\Big(\frac{1}{2}\|\partial_1 b_2\|_{L^2}^2+\frac{1+\eta^2}{2}\|\partial_1\Delta b_2\|_{L^2}^2\Big)\,d\tau \\
&\quad\le \frac{1}{2}(1+t)^2(\|\partial_1\nabla u_2\|_{L^2}^2+\|\nabla b_2\|_{L^2}^2)+C(1+\frac{1}{\eta})E_0^{\frac{1}{2}}(t)E_1(t) \\
&\quad+E(0)+C\frac{1}{\delta_2\eta}E_0(t),
\end{aligned}\tag{4.17}$$

where δ_2 is a positive constant.

Proof of Lemma 4.3 Similarly to (3.17), we introduce the time-weighted inner product $(1+t)^2(\partial_1\nabla u_2, \nabla b_2)$ to get

$$\begin{aligned}
&(1+t)^2\|\partial_1\nabla u_2\|_{L^2}^2-(1+t)^2\|\partial_1\nabla b_2\|_{L^2}^2-\frac{d}{dt}(1+t)^2(\partial_1\nabla u_2,\nabla b_2) \\
&\quad=-2(1+t)(\partial_1\nabla u_2,\nabla b_2) \\
&\qquad+(1+t)^2(\partial_1\nabla u_2,\nabla(u\cdot\nabla b_2))-\eta(1+t)^2(\partial_1\nabla u_2,\partial_1^2\nabla b_2))-(1+t)^2(\partial_1\nabla u_2,\nabla(b\cdot\nabla u_2)) \\
&\qquad+(1+t)^2(\partial_1\nabla(u\cdot\nabla u_2),\nabla b_2)+(1+t)^2(\partial_1\nabla\partial_2 P,\nabla b_2)-(1+t)^2(\partial_1\nabla(b\cdot\nabla b_2),\nabla b_2) \\
&\quad:=H_1+H_2+\cdots+H_7.
\end{aligned}$$

Firstly, by Hölder's inequality and (2.3), we have

$$H_1+H_3\le\Big(\frac{1}{2}+\delta_2\Big)(1+t)^2\|\partial_1\nabla u_2\|_{L^2}^2+\frac{\eta^2}{2}(1+t)^2\|\partial_1^2\nabla b_2\|_{L^2}^2+C\frac{1}{\delta_2}\|\partial_1\nabla b_2\|_{L^2}^2.$$

where $\delta_2>0$ is a small pure constant.

For H_2, a simple application of Hölder's inequality, Sobolev's inequality as well as (2.3) leads to

$$\begin{aligned}
H_2&=(1+t)^2\int(\nabla u\cdot\nabla b_2\cdot\partial_1\nabla u_2+u\cdot\nabla(\nabla b_2)\cdot\partial_1\nabla u_2)dx \\
&\le(1+t)^2(\|\nabla u\|_{L^\infty}\|\nabla b_2\|_{L^2}\|\partial_1\nabla u_2\|_{L^2}+\|u\|_{L^\infty}\|\nabla^2 b_2\|_{L^2}\|\partial_1\nabla u_2\|_{L^2}) \\
&\le C(1+t)^2\|u\|_{H^3}\|\partial_1 b_2\|_{H^2}\|\partial_1\nabla u_2\|_{L^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}H_7 &= (1+t)^2 \int (\nabla b \cdot \nabla b_2 \cdot \partial_1 \nabla b_2 + b \cdot \nabla(\nabla b_2) \cdot \partial_1 \nabla b_2) dx \\ &\le C(1+t)^2 \|b\|_{H^3} \|\partial_1 b_2\|_{H^2}^2.\end{aligned}$$

For H_4, we first divide it in three parts and then use Hölder's inequality, Sobolev's inequality, (2.2) and (2.3) to get

$$\begin{aligned}H_4 &= (1+t)^2 \int (\nabla b \cdot \nabla u_2 + b_1 \partial_1 \nabla u_2 + b_2 \partial_2 \nabla u_2) \cdot \partial_1 \nabla u_2 dx \\ &\le (1+t)^2 (\|\nabla b\|_{L^\infty} \|\nabla u_2\|_{L^2} + \|b_1\|_{L^\infty} \|\nabla \partial_1 u_2\|_{L^2} \\ &\quad + \|b_2\|_{L^4} \|\nabla \partial_2 u_2\|_{L^4}) \|\partial_1 \nabla u_2\|_{L^2} \\ &\le C(1+t)^2 \|(u,b)\|_{H^3} (\|\partial_1 b_2\|_{H^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2).\end{aligned}$$

H_5 can be bounded with a similar argument.

$$\begin{aligned}H_5 &= -(1+t)^2 \int (\nabla u \cdot \nabla u_2 + u_1 \partial_1 \nabla u_2 + u_2 \partial_2 \nabla u_2) \cdot \partial_1 \nabla b_2 dx \\ &\le (1+t)^2 (\|\nabla u\|_{L^\infty} \|\nabla u_2\|_{L^2} \|\partial_1 \nabla b_2\|_{L^2} + \|u_1\|_{L^\infty} \|\nabla \partial_1 u_2\|_{L^2} \|\partial_1 \nabla b_2\|_{L^2} \\ &\quad + \|\partial_1 u_2\|_{L^2} \|\nabla \partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_2^2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_2\|_{L^2}) \\ &\le C(1+t)^2 \|u\|_{H^3} (\|\partial_1 b_2\|_{H^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2),\end{aligned}$$

where we have used $\|\partial_1 u_2\|_{L^2} \le C\|\partial_1^2 u_2\|_{L^2}$. Now we handle H_6. Invoking $\Delta P = \nabla \cdot (b \cdot \nabla b - u \cdot \nabla u)$, we have

$$\begin{aligned}H_6 &= -(1+t)^2 \int \nabla \partial_2 \Delta^{-1} \nabla \cdot (b \cdot \nabla b - u \cdot \nabla u) \cdot \nabla \partial_1 b_2 \, dx \\ &= -(1+t)^2 \sum_{i=1}^{2} \sum_{j=1}^{2} \int \nabla \partial_2 \Delta^{-1} (\partial_j b_i \partial_i b_j - \partial_j u_i \partial_i u_j) \cdot \nabla \partial_1 b_2 \, dx \\ &\le (1+t)^2 \sum_{i=1}^{2} \sum_{j=1}^{2} \|\partial_j b_i \partial_i b_j - \partial_j u_i \partial_i u_j\|_{L^2} \|\nabla \partial_1 b_2\|_{L^2} \\ &\le C(1+t)^2 \|\nabla \partial_1 b_2\|_{L^2} (\|\partial_1 u_1\|_{L^\infty} \|\partial_1 u_1\|_{L^2} + \|\partial_2 u_1\|_{L^\infty} \|\partial_1 u_2\|_{L^2} \\ &\quad + \|\partial_1 b_1\|_{L^\infty} \|\partial_1 b_1\|_{L^2} + \|\partial_2 b_1\|_{L^\infty} \|\partial_1 b_2\|_{L^2}) \\ &\le C(1+t)^2 \|(u,b)\|_{H^3} (\|\partial_1 \nabla b_2\|_{L^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2),\end{aligned}$$

where we have used

$$\|\partial_1 u_1\|_{L^2} \le C\|\partial_1^2 u_1\|_{L^2}, \ \|\partial_1 u_2\|_{L^2} \le C\|\partial_1^2 u_2\|_{L^2},$$

$$\|\partial_1 b_1\|_{L^2} \le C\|\partial_1^2 b_1\|_{L^2},\ \|\partial_1 b_2\|_{L^2} \le C\|\partial_1^2 b_2\|_{L^2}.$$

Collecting all estimates for H_1 through H_7, integrating in time and using Hölder's inequality yield

$$\int_0^t (1+\tau)^2\Big(\frac{1}{2}-\delta_2\Big)\|\partial_1\nabla u_2\|_{L^2}^2\,d\tau - \int_0^t (1+\tau)^2\Big(\frac{1}{2}\|\partial_1 b_2\|_{L^2}^2 + \frac{1+\eta^2}{2}\|\partial_1\Delta b_2\|_{L^2}^2\Big)\,d\tau$$
$$\le \frac{1}{2}(1+t)^2(\|\partial_1\nabla u_2\|_{L^2}^2 + \|\nabla b_2\|_{L^2}^2) + C(1+\frac{1}{\eta})E_0^{\frac{1}{2}}(t)E_1(t) + E(0) + C\frac{1}{\delta_2\eta}E_0(t).$$

Here we have used

$$\|\partial_1\nabla b_2\|_{L^2}^2 = -\int \partial_1 b_2\,\partial_1\Delta b_2 dx \le \frac{1}{2}(\|\partial_1 b_2\|_{L^2}^2 + \|\partial_1\Delta b_2\|_{L^2}^2).$$

This completes the proof of Lemma 4.3. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition (4.1) According to Lemma 4.2 and Lemma 4.3, we make the calculation as follows

$$\lambda \times (4.2) + (4.17).$$

That is

$$\begin{aligned}
&(1+t)^2\Big(\lambda\|(u_2,b_2)(t)\|_{L^2}^2 + (\lambda-\frac{1}{2})\|\Delta u_2\|_{L^2}^2 + (\lambda-c_0)\|\Delta b_2\|_{L^2}^2\Big)\\
&\quad+\Big(\lambda(2\eta-\delta_1)-\frac{1}{2}\Big)\int_0^t (1+\tau)^2\,\|\partial_1 b_2(\tau)\|_{L^2}^2\,d\tau\\
&\quad+\Big(\lambda(2\eta-\delta_1)-\frac{\eta^2}{2}-\frac{1}{2}\Big)\int_0^t (1+\tau)^2\|\partial_1\Delta b_2(\tau)\|_{L^2}^2\,d\tau\\
&\quad+\Big(\frac{1}{2}-\delta_2-2\delta_0\lambda\Big)\int_0^t (1+\tau)^2\|\partial_1\nabla u_2\|_{L^2}^2\,d\tau\\
&\quad\le C(1+\frac{1}{\eta})(\lambda+1)E_0^{\frac{1}{2}}(t)E_1(t) + (\lambda+1)E(0) + C\Big(\lambda(\frac{1}{\delta_0}+\frac{1}{\delta_1\eta}) + \frac{1}{\delta_2\eta}\Big)E_0(t),
\end{aligned}\tag{4.18}$$

where we have used

$$\frac{1}{2}\|\nabla b_2\|_{L^2}^2 \le c_0\|\partial_1\nabla b_2\|_{L^2}^2$$

for some pure constant $c_0 > 0$. Now for some given sufficiently small $\delta_1 < 2\eta$ and $\delta_2 < \frac{1}{2}$, we can select λ and δ_0 to satisfy

$$\lambda > \max\left\{\frac{1}{2},\ c_0,\ \frac{1+\eta^2}{2(2\eta-\delta_1)}\right\} \text{ and } \delta_0 < \frac{\frac{1}{2}-\delta_2}{2\lambda}.$$

Then from (4.18) we derive

$$\begin{aligned}
&(1+t)^2\Big(\|(u_2,b_2)(t)\|_{L^2}^2 + \|(\Delta u_2,\Delta b_2)\|_{L^2}^2\Big)\\
&\quad + \int_0^t (1+\tau)^2\Big(\eta\|\partial_1 b_2(\tau)\|_{L^2}^2 + \eta\|\partial_1\Delta b_2(\tau)\|_{L^2}^2 + \|\partial_1\nabla u_2\|_{L^2}^2\Big)\\
&\quad \le \frac{1}{\widetilde{c}_0}\Big(C(1+\frac{1}{\eta})(\lambda+1)(E_0^{\frac{3}{2}}(t) + E_1(t)^{\frac{3}{2}}) + (\lambda+1)E(0) + C\widetilde{c}_1 E_0(t)\Big),
\end{aligned}$$

where

$$\widetilde{c}_0 = \min\left\{\lambda-\frac{1}{2},\ \lambda-c_0,\ \frac{\lambda(2\eta-\delta_1)}{\eta}-\frac{1}{2\eta},\ \frac{\lambda(2\eta-\delta_1)}{\eta}-\frac{\eta}{2}-\frac{1}{2\eta},\ \frac{1}{2}-\delta_2-2\delta_0\lambda\right\}$$

and

$$\widetilde{c}_1 = \lambda(\frac{1}{\delta\,0}+\frac{1}{\delta_1\eta}) + \frac{1}{\delta_2\eta}.$$

This completes the proof of (4.1).

□

5 Proof of Theorem 1.1

In this section, we will apply the bootstrapping argument (see (Tao 2006), p.21) to prove Theorem 1.1. Before the proof, we first show the *a priori estimate* for $E(t)$.

Proposition 5.1 *Suppose that the initial data* (u_0, b_0) *satisfies the conditions in Theorem 1.1. Then for two positive constants* C_1, C_2*, it holds*

$$E(t) \le C_1 E(0) + C_2 E^{\frac{3}{2}}(t). \tag{5.1}$$

Proof of Proposition 5.1 Making a simple calculation (3.1) $+\,\epsilon_1 \times$(4.1) with ϵ_1 be a positive constant and applying Young's inequality yields

$$\begin{aligned}
(1-\epsilon_1\frac{C\widetilde{c}_1}{\widetilde{c}_0})E_0(t) + \epsilon_1 E_1(t) &\le \Big(\frac{c_1}{c_0} + \frac{\epsilon_1(\lambda+1)}{\widetilde{c}_0}\Big)E(0)\\
&+C\Big(\frac{c_2}{c_0} + \frac{\epsilon_1(1+\frac{1}{\eta})(1+\lambda)}{\widetilde{c}_0}\Big)E(t)^{\frac{3}{2}}.
\end{aligned}$$

Now we take $\epsilon_1 > 0$ sufficiently small such that $1 - \epsilon_1 \frac{C\widetilde{c}_1}{\widetilde{c}_0} > 0$. Then we derive, for two positive constants $C_1(\eta)$, $C_2(\eta)$

$$E(t) \le C_1 E(0) + C_2 E^{\frac{3}{2}}(t),$$

where

$$C_1(\eta) = \frac{1}{C_3}\Big(\frac{c_1}{c_0} + \frac{\epsilon_1(\lambda+1)}{\widetilde{c}_0}\Big), \quad C_2(\eta) = \frac{1}{C_3}\Big(\frac{c_2}{c_0} + \frac{\epsilon_1(1+\frac{1}{\eta})(1+\lambda)}{\widetilde{c}_0}\Big)$$

with $C_3 = \min\{(1 - \epsilon_1 \frac{C\widetilde{c}_1}{\widetilde{c}_0}), \epsilon_1\}$. This completes the proof of Proposition 5.1. □

With (5.1) at our proposal, we are able to prove Theorem 1.1.

Proof of Theorem 1.1 We will utilize the bootstrapping argument to prove the global existence of smooth solutions. To initiate the bootstrapping argument, we start with the ansatz

$$E(t) \le \frac{1}{4C_2^2}. \tag{5.2}$$

Then it suffices to prove that $E(t)$ actually admits a smaller bound. This can be achieved via (5.1). Invoking (5.2) and the initial data assumption (1.8), we infer

$$E(t) \le C_1\delta^2 + \frac{1}{2}E(t)$$

or

$$E(t) \le 2C_1\delta^2.$$

Then if we select δ sufficiently small to obey

$$\delta \le \frac{1}{4\sqrt{C_1}C_2},$$

we can derive

$$E(t) \le \frac{1}{8C_2^2}.$$

The bootstrapping argument then asserts the desired global bound

$$E(t) \le C\delta^2. \tag{5.3}$$

As a result, the uniform upper bound (1.9) and the decay rates (1.10) follow from (5.3) immediately. The proof of Theorem 1.1 is therefore complete. □

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