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Stability of 3D perturbations to 2D Navier–Stokes flows with vertical dissipation

Chongsheng Cao [a], Jiahong Wu [b],*

[a] Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, United States of America
[b] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, United States of America

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ABSTRACT

The stability of 3D perturbations to 2D Navier–Stokes solutions is a fundamental problem with significant mathematical and physical implications. Stability has been established for the 3D Navier–Stokes equations with full dissipation or anisotropic dissipation in two spatial directions. This paper intends to understand the stability problem under the situation when dissipation is present only in one direction. We establish stability results in the case where dissipation acts only in the vertical direction and on the vertical average of the solution.

1. Introduction

Many real-world flows, like in the atmosphere and oceans, are effectively two-dimensional (2D) due to strong stratification or rotation(see, e.g. [1,2]). Understanding the stability of these 2D flows under 3D perturbations is crucial for explaining the emergence and persistence of 2D behavior in such systems. Mathematically the 3D incompressible Navier–Stokes equations are extremely difficult to analyze. If 2D flows can be shown to be stable under 3D perturbations, this justifies using simplified 2D models in place of full 3D dynamics in certain regimes.

This type of stability problem has been investigated by several authors (see, e.g. [3–5]). The work of Ponce, Racke, Sideris, and Titi showed that any 2D solution in the class $L^1 \cap H^1$ of the 2D Navier–Stokes is stable under 3D H^1 perturbations [5]. All these previous results rely on full dissipation in the 3D Navier–Stokes equations. Our main objective is to relax this requirement and still establish stability. Prior work has succeeded in proving stability when full dissipation is reduced to anisotropic dissipation in two directions [4,6–8]. This paper focuses on the case with dissipation only in one direction, partly motivated by applications in physical fluid systems.

Consider the 2D Navier–Stokes system in the 2D periodic box $\mathbb{T}^2 = [0,1]^2$,

$$\begin{cases} U_t + (U\cdot\nabla)U + \nabla q = \nu \Delta U, \\ \nabla\cdot U = 0, \\ U(t=0) = U_0 \,. \end{cases} \tag{1.1}$$

It is well-known that, for any $U_0 \in H^k(\mathbb{T}^2)$ with $k \geq 0$, (1.1) has a unique global solution U satisfying

$$U \in C([0,\infty); H^k(\mathbb{T}^2)) \cap L^2(0,\infty; H^{k+1}(\mathbb{T}^2)).$$

* Corresponding author.
E-mail addresses: caoc@fiu.edu (C. Cao), jwu29@nd.edu (J. Wu).

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In addition, if U_0 has mean zero,

$$\iint U_0\, dxdy = 0,$$

then there exists a constant $a = a(\nu) > 0$ such that

$$\|U(t)\|_{H^k} \le Ce^{-at}\|U_0\|_{H^k} \le Me^{-at},\ k \le 4. \tag{1.2}$$

Our objective is to study the stability of 3D perturbations near this 2D solution. More precisely, we analyze the 3D Navier–Stokes equations that govern perturbations around U,

$$\begin{cases} \partial_t u + (u+U)\cdot\nabla u + u\cdot\nabla U = -\nabla p + \nu\partial_3^2 u + \nu\Delta\bar{u}, & x\in\mathbb{T}^3,\ t>0,\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where the spatial domain is taken to be the 3D periodic box $\mathbb{T}^3 = [0,1]^3$, u denotes the velocity filed, p the pressure and ν the viscosity. Here $\nu\partial_3^2 u$ is dissipation acting only in the vertical direction, where $\bar{u}$ is defined as

$$\bar{u}(x_1,x_2) = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u(x_1,x_2,x_3)\,dx_3.$$

This study is motivated by turbulence models involving anisotropic viscosity, where vertical viscosity is enhanced due to strong stratification, while horizontal mixing occurs on a larger scale (see, e.g., [9]).

We establish the following theorem.

Theorem 1.1. *Consider (1.3) with $\nu > 0$. Assume $u_0 \in H^3(\mathbb{T}^3)$ with $\nabla\cdot u_0 = 0$. There exists a suitable constant $C_0(\nu, \|U_0\|_{H^3}) > 0$ such that, if*

$$\|u_0\|_{H^3} \le C_0\,\nu,$$

then (1.3) has a unique global solution $u \in L^\infty([0,\infty); H^3(\mathbb{T}^3))$. Furthermore, u remains uniformly bounded. That is, there is $C_1(\nu, \|U_0\|_{H^3}, C_0)$, such that for any $t > 0$,

$$\|u(\cdot,t)\|_{H^3(\mathbb{T}^3)} \le C_1(\nu, \|U_0\|_{H^3}, C_0). \tag{1.4}$$

2. Proof of Theorem 1.1

We prove Theorem 1.1 in this section. As a preparatory step, we first state two tool lemmas. We express

$$f = \bar{f} + \widetilde{f},$$

where $\bar{f}$ denotes the vertical average of f and $\widetilde{f}$ denotes the oscillation part,

$$\bar{f}(x_1,x_2) = \int_{\mathbb{T}} f(x_1,x_2,x_3)\,dx_3 \quad \text{and} \quad \widetilde{f} = f - \bar{f}. \tag{2.1}$$

The following lemma outlines some basic properties of $\bar{f}$ and $\widetilde{f}$.

Lemma 2.1. *Let $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1). The following properties hold:*

(1) The average and oscillation commute with any derivatives, namely

$$\overline{\partial_i f} = \partial_i\bar{f}, \quad \widetilde{\partial_i f} = \partial_i\widetilde{f}.$$

As a special consequence, if u is divergence-free, $\nabla\cdot u = 0$, then $\bar{u}$ and $\widetilde{u}$ are also divergence-free, $\nabla\cdot\bar{u} = 0$ and $\nabla\cdot\widetilde{u} = 0$.

(2) $\bar{f}$ and $\widetilde{f}$ are orthogonal. More precisely, for $f \in H^k(\mathbb{T}^3)$ with any non-negative integer k, the inner product of $\bar{f}$ and $\widetilde{f}$ in H^k is zero,

$$\int_{\mathbb{T}^3} \partial^\alpha\bar{f}(x)\,\partial^\alpha\widetilde{f}(x)\,dx = 0$$

for any multi-index α with $|\alpha| \le k$. As a special consequence,

$$\|f\|_{\dot{H}^k}^2 = \|\bar{f}\|_{\dot{H}^k}^2 + \|\tilde{f}\|_{\dot{H}^k}^2$$

and

$$\|\bar{f}\|_{H^k} \le \|f\|_{H^k} \quad \text{and} \quad \|\tilde{f}\|_{H^k} \le \|f\|_{H^k}.$$

(3) $\widetilde{f}$ satisfies the strong Poincaré type inequality

$$\|\widetilde{f}\|_{L^2} \le C\,\|\partial_3\widetilde{f}\|_{L^2} \tag{2.2}$$

Several versions of Lemma 2.1 for different types of spatial domains can be found in [10–13].

Throughout this paper, we adopt the following notation for anisotropic Lebesgue spaces

$$\|f\|_{L^p_{x_1}L^q_{x_2}L^r_{x_3}} := \|\|\|f\|_{L^p_{x_1}(\mathbb{T})}\|_{L^q_{x_2}(\mathbb{T})}\|_{L^r_{x_3}(\mathbb{T})}.$$

The subscripts x_1, x_2 and x_3 indicate in which direction the norm is taken. The notation for anisotropic Lebesgue and Sobolev norms should be understood similarly.

The following lemma provides an anisotropic upper bound on the integral of triple products, which is particularly useful when estimating the nonlinear terms of PDEs with anisotropic dissipation.

Lemma 2.2. *Assume that $f, \partial_1 f, g, \partial_2 g, h, \partial_3 h$ are all in $L^2(\mathbb{T}^3)$. Then, for a constant C independent of f, g and h,*

$$\begin{aligned}\left|\int_{\mathbb{T}^3} f(x)\,g(x)\,h(x)\,dx\right| \le C\,\|f\|_{L^2}^{\frac12}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac12}\,\|g\|_{L^2}^{\frac12}(\|g\|_{L^2}+\|\partial_2 g\|_{L^2})^{\frac12}\\ \times\,\|h\|_{L^2}^{\frac12}(\|h\|_{L^2}+\|\partial_3 h\|_{L^2})^{\frac12}.\end{aligned}$$

As a special consequence, if h just has the vertical oscillation part, then

$$\begin{aligned}\left|\int_{\mathbb{T}^3} f(x)\,g(x)\,\widetilde{h}(x)\,dx\right| \le C\,\|f\|_{L^2}^{\frac12}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac12}\,\|g\|_{L^2}^{\frac12}(\|g\|_{L^2}+\|\partial_2 g\|_{L^2})^{\frac12}\\ \times\,\|\widetilde{h}\|_{L^2}^{\frac12}\|\partial_3\widetilde{h}\|_{L^2}^{\frac12}.\end{aligned}$$

We will also use the following 2D version of the anisotropic upper bound.

Lemma 2.3. *Assume that $f, \partial_1 f, g, \partial_2 g, h$ are all in $L^2(\mathbb{T}^2)$. Then, for a constant C independent of f, g and h,*

$$\left|\int_{\mathbb{T}^2} f(x)\,g(x)\,h(x)\,dx\right| \le C\,\|f\|_{L^2}^{\frac12}(\|f\|_{L^2}+\|\partial_1 f\|_{L^2})^{\frac12}\,\|g\|_{L^2}^{\frac12}(\|g\|_{L^2}+\|\partial_2 g\|_{L^2})^{\frac12}\,\|h\|_{L^2}.$$

Proof of Theorem 1.1. We first take the L^2-inner product of (1.3) with u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \nu\|\partial_3 u(t)\|_{L^2}^2 + \nu\|\nabla\bar{u}\|_{L^2}^2 \le \|\nabla U\|_{L^\infty}\|u\|_{L^2}^2. \tag{2.3}$$

Let $\omega = \nabla\times u$ be the vorticity and $\Omega = \nabla\times U$

$$\partial_t\omega + (u+U)\cdot\nabla\omega + u\cdot\nabla\Omega - (\Omega+\omega)\cdot\nabla u - \omega\cdot\nabla U - \nu\partial_3^2\omega - \nu\Delta\bar{\omega} = 0. \tag{2.4}$$

Due to the equivalence of the two norms $\|u\|_{H^3}$ and $\|u\|_{L^2}+\|D^3u\|_{L^2}$, it suffices to estimate $\|D^3u\|_{L^2}$. We recall that $\|D^3u\|_{L^2}$ is comparable to $\|\Delta\omega\|_{L^2}$. Applying Δ to (2.4) and taking the inner product with $\Delta\omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_{L^2}^2 + \nu\|\partial_3\Delta\omega\|_{L^2}^2 + \nu\|\nabla\Delta\bar{\omega}\|_{L^2}^2 = I_1 + I_2 + I_3, \tag{2.5}$$

where

$$\begin{aligned}I_1 &= -\int \Delta(u\cdot\nabla\omega)\cdot\Delta\omega\,dx, \quad I_2 = \int \Delta(\omega\cdot\nabla u)\cdot\Delta\omega\,dx,\\ I_3 &= -\int \Delta(U\cdot\nabla\omega + u\cdot\nabla\Omega - \Omega\cdot\nabla u - \omega\cdot\nabla U)\cdot\Delta\omega\,dx.\end{aligned}$$

Due to $\nabla\cdot u = 0$,

$$I_1 = -\int \Delta u\cdot\nabla\omega\cdot\Delta\omega\,dx - 2\int \nabla u\cdot\nabla\nabla\omega\cdot\Delta\omega\,dx \tag{2.6}$$

and

$$I_2 = \int \Delta\omega\cdot\nabla u\cdot\Delta\omega\,dx + 2\int \nabla\omega\cdot\nabla\nabla u\cdot\Delta\omega\,dx + \int \omega\cdot\nabla\Delta u\cdot\Delta\omega\,dx. \tag{2.7}$$

Moreover,

$$\begin{aligned}I_3 = &-\int \Delta U\cdot\nabla\omega\cdot\Delta\omega\,dx - 2\int \nabla U\cdot\nabla\nabla\omega\cdot\Delta\omega\,dx\\ &-\int \Delta(u\cdot\nabla\Omega - \Omega\cdot\nabla u - \omega\cdot\nabla U)\cdot\Delta\omega\,dx.\end{aligned} \tag{2.8}$$

We first bound the terms in I_3. By Hölder's and Sobolev's inequalities,

$$\begin{aligned}&-\int \Delta U\cdot\nabla\omega\cdot\Delta\omega\,dx \le \|\Delta U\|_{L^3}\|\nabla\omega\|_{L^6}\,\|\Delta\omega\|_{L^2} \le C\,\|U\|_{H^3}\,\|\Delta\omega\|_{L^2}^2,\\ &-2\int \nabla U\cdot\nabla\nabla\omega\cdot\Delta\omega\,dx \le \|\nabla U\|_{L^\infty}\,\|\Delta\omega\|_{L^2}^2 \le C\,\|U\|_{H^3}\,\|\Delta\omega\|_{L^2}^2.\end{aligned}$$

After applying Leibniz's rule for differentiation, along with Hölder's and Sobolev inequalities, we obtain

$$
\begin{aligned}
&-\int \Delta(u\cdot\nabla\Omega-\Omega\cdot\nabla u-\omega\cdot\nabla U)\cdot\Delta\omega\,dx\\
&\le \|\Delta u\|_{L^6}\,\|\nabla\Omega\|_{L^3}\,\|\Delta\omega\|_{L^2}+2\|\nabla u\|_{L^\infty}\,\|\Delta\Omega\|_{L^2}\,\|\Delta\omega\|_{L^2}\\
&\quad+\|u\|_{L^\infty}\,\|U\|_{H^4}\,\|\Delta\omega\|_{L^2}+\|\Omega\|_{L^\infty}\,\|\Delta\omega\|_{L^2}^2\\
&\quad+\|\nabla U\|_{L^\infty}\,\|\Delta\omega\|_{L^2}^2+\|\nabla\omega\|_{L^6}\,\|\Delta U\|_{L^3}\,\|\Delta\omega\|_{L^2}\\
&\le C\,\|U\|_{H^4}\,\|\Delta\omega\|_{L^2}^2.
\end{aligned}
$$

That is,

$$
|I_3|\le C\|U\|_{H^4}\|\Delta\omega\|_{L^2}^2. \tag{2.9}
$$

For I_1 and I_2, we will use the following estimates to bound all these terms.

Proposition 2.4. *Assume that f,g,h are all elements of matrix $(\partial_j u^k)_{3\times3}$. Then, for a constant C independent of f, g and h,*

$$
\begin{aligned}
&\Big|\int_{\mathbb{T}^3} f(x)\,\partial_{lj}g(x)\,\partial_{ik}h(x)\,dx\Big|+\Big|\int_{\mathbb{T}^3}\partial_l f(x)\,\partial_j g(x)\,\partial_{ik}h(x)\,dx\Big|\\
&\le C\,\|\omega\|_{H^2}(\|\partial_3\Delta\omega\|_{L^2}^2+\|\nabla\Delta\bar\omega\|_{L^2}^2).
\end{aligned}
$$

Proof. We first write

$$
f=\bar f+\widetilde f,\quad g=\bar g+\widetilde g\quad\text{and}\quad h=\bar h+\widetilde h.
$$

The integral can be written

$$
\begin{aligned}
&\Big|\int_{\mathbb{T}^3} f(x)\,\partial_{lj}g(x)\,\partial_{ik}h(x)\,dx\Big|+\Big|\int_{\mathbb{T}^3}\partial_l f(x)\,\partial_j g(x)\,\partial_{ik}h(x)\,dx\Big|\\
&\le\int_{\mathbb{T}^3}\big[|f(x)\,\partial_{lj}g(x)|+|\partial_l f(x)\,\partial_j g(x)|\big]\;|\partial_{ik}h(x)|\,dx\\
&\le E_1+E_2+E_3+E_4+E_5,
\end{aligned}
$$

where

$$
\begin{aligned}
&E_1=\int(|\partial_l\widetilde f\partial_j\widetilde g|+|\widetilde f\partial_{lj}\widetilde g|)\,|\partial_{ki}\widetilde h|\,dx, &&E_2=\int(|\partial_l\bar f\partial_j\widetilde g|+|\bar f\partial_{lj}\widetilde g|)\,|\partial_{ki}\widetilde h|\,dx,\\
&E_3=\int(|\partial_l\widetilde f\partial_j\bar g|+|\widetilde f\partial_{lj}\bar g|)\,|\partial_{ki}\widetilde h|\,dx, &&E_4=\int(|\partial_l\widetilde f\partial_j\widetilde g|+|\widetilde f\partial_{lj}\widetilde g|)\,|\partial_{ki}\bar h|\,dx,\\
&E_5=\int(|\partial_l\bar f\partial_j\bar g|+|\bar f\partial_{lj}\bar g|)\,|\partial_{ki}\bar h|\,dx.
\end{aligned}
$$

We will use Lemma 2.2 and then Lemma 2.1 to obtain bounds for these terms.

$$
\begin{aligned}
|E_1|&\le C\,\|\partial_l\widetilde f\|_{L^2}^{\frac12}\|\partial_l\widetilde f\|_{H^1}^{\frac12}\,\|\partial_j\widetilde g\|_{L^2}^{\frac12}\|\partial_j\widetilde g\|_{H^1}^{\frac12}\,\|\partial_{ki}\widetilde h\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde h\|_{L^2}^{\frac12}\\
&\quad+\|\widetilde f\|_{L^\infty}\|\partial_{lj}\widetilde g\|_{L^2}\|\partial_{ki}\widetilde h\|_{L^2}\\
&\le C\,\|\partial_l\widetilde f\|_{H^1}\|\partial_{3j}\widetilde g\|_{H^1}\|\partial_{3ki}\widetilde h\|_{L^2}+C\|\widetilde f\|_{H^2}\|\partial_{3lj}\widetilde g\|_{L^2}\|\partial_{3ki}\widetilde h\|_{L^2}\\
&\le C\,\|\omega\|_{H^2}(\|\partial_3\Delta\omega\|_{L^2}^2+\|\nabla\Delta\bar\omega\|_{L^2}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
|E_2|&\le C\,\|\partial_l\bar f\|_{L^2}^{\frac12}\|\partial_l\bar f\|_{H^1}^{\frac12}\,\|\partial_j\widetilde g\|_{L^2}^{\frac12}\|\partial_j\widetilde g\|_{H^1}^{\frac12}\,\|\partial_{ki}\widetilde h\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde h\|_{L^2}^{\frac12}\\
&\quad+\|\bar f\|_{L^\infty}\|\partial_{lj}\widetilde g\|_{L^2}\|\partial_{ki}\widetilde h\|_{L^2}\\
&\le C\,\|\partial_l\bar f\|_{H^1}\|\partial_{3j}\widetilde g\|_{H^1}\|\partial_{3ki}\widetilde h\|_{L^2}+C\,\|\bar f\|_{L^\infty}\|\partial_{lj}\widetilde g\|_{L^2}\|\partial_{ki}\widetilde h\|_{L^2}\\
&\le C\,\|\omega\|_{H^2}(\|\partial_3\Delta\omega\|_{L^2}^2+\|\nabla\Delta\bar\omega\|_{L^2}^2).
\end{aligned}
$$

And also

$$
\begin{aligned}
|E_3|&\le C\,\|\partial_l\widetilde f\|_{L^2}^{\frac12}\|\partial_l\widetilde f\|_{H^1}^{\frac12}\,\|\partial_j\bar g\|_{L^2}^{\frac12}\|\partial_j\bar g\|_{H^1}^{\frac12}\,\|\partial_{ki}\widetilde h\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde h\|_{L^2}^{\frac12}\\
&\quad+\|\widetilde f\|_{L^\infty}\|\partial_{lj}\bar g\|_{L^2}\|\partial_{ki}\widetilde h\|_{L^2}\\
&\le C\,\|\partial_{3l}\widetilde f\|_{H^1}\|\partial_j\bar g\|_{H^1}\|\partial_{3ki}\widetilde h\|_{L^2}+C\|\partial_3\widetilde f\|_{H^2}\|\partial_{lj}\bar g\|_{L^2}\|\partial_{3ki}\widetilde h\|_{L^2}\\
&\le C\,\|\omega\|_{H^2}(\|\partial_3\Delta\omega\|_{L^2}^2+\|\nabla\Delta\bar\omega\|_{L^2}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
|E_4| &\leq C\, \|\partial_l \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_l \widetilde{f}\|_{H^1}^{\frac{3}{4}} \|\partial_j \widetilde{g}\|_{L^2}^{\frac{1}{4}} \|\partial_j \widetilde{g}\|_{H^1}^{\frac{3}{4}} \|\partial_{ki} \bar{h}\|_{L^2} + \|\widetilde{f}\|_{L^\infty} \|\partial_{lj} \widetilde{g}\|_{L^2} \|\partial_{ki} \bar{h}\|_{L^2} \\
&\leq C\, \|\partial_{3l} \widetilde{f}\|_{H^1} \|\partial_{3j} \widetilde{g}\|_{H^1} \|\partial_{ki} \bar{h}\|_{L^2} + C \|\partial_3 \widetilde{f}\|_{H^2} \|\partial_{3lj} \widetilde{g}\|_{L^2} \|\partial_{ki} \bar{h}\|_{L^2} \\
&\leq C\, \|\omega\|_{H^2} (\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\nabla \Delta\bar{\omega}\|_{L^2}^2).
\end{aligned}
$$

By 2.3, we have

$$
\begin{aligned}
|E_5| &\leq C\, \|\partial_l \bar{f}\|_{L^2}^{\frac{1}{2}} \|\partial_l \bar{f}\|_{H^1}^{\frac{1}{2}} \|\partial_j \bar{g}\|_{L^2}^{\frac{1}{2}} \|\partial_j \bar{g}\|_{H^1}^{\frac{1}{2}} \|\partial_{ki} \bar{h}\|_{L^2} + \|\bar{f}\|_{L^\infty} \|\partial_{lj} \bar{g}\|_{L^2} \|\partial_{ki} \bar{h}\|_{L^2} \\
&\leq C\, \|\bar{\omega}\|_{H^2} \|\nabla \Delta\bar{\omega}\|_{L^2}^2.
\end{aligned}
$$

This completes the proof of Proposition 2.4. □

We continue the proof of Theorem 1.1. By Proposition 2.4, I_1 and I_2 can be bounded by

$$
|I_1| + |I_2| \leq C\, \|\omega\|_{H^2} (\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\nabla \Delta\bar{\omega}\|_{L^2}^2). \tag{2.10}
$$

Inserting the bounds (2.9) and (2.10) in (2.5) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|\Delta\omega\|_{L^2}^2 + \nu \|\partial_3 \Delta\omega\|_{L^2}^2 + \nu \|\nabla \Delta\bar{\omega}\|_{L^2}^2 \\
&\leq C\, \|\omega\|_{H^2} (\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\nabla \Delta\bar{\omega}\|_{L^2}^2) + C \|U\|_{H^4} \|\Delta\omega\|_{L^2}^2.
\end{aligned} \tag{2.11}
$$

Adding (2.3) and (2.11) yields

$$
\frac{d}{dt} \|u\|_{H^3}^2 + (2\nu - C_2 \|u\|_{H^3}) \left(\|\partial_3 u\|_{H^3}^2 + \nu \|\nabla \bar{u}\|_{H^3}^2 \right) \leq C_3\, \|U\|_{H^4} \|u\|_{H^3}^2. \tag{2.12}
$$

We use the bootstrapping argument to show that, if the initial data u_0 satisfies

$$
\|u_0\|_{H^3} \leq C_2^{-1}\, e^{-C_3 M/(2a)}\, \nu,
$$

then (1.3) has a unique global solution u satisfying (1.4). In fact, if we make the ansatz that

$$
2\nu - C_2\, \|u(t)\|_{H^3} \geq 0 \quad \text{or} \quad \|u(t)\|_{H^3} \leq 2 C_2^{-1} \nu.
$$

Then, by applying Gronwall's inequality and the bound in (1.2), we obtain from (2.12) that

$$
\|u(t)\|_{H^3}^2 \leq \|u_0\|_{H^3}^2 e^{C_3 \int_0^t \|U(\tau)\|_{H^4} d\tau} \leq \|u_0\|_{H^3}^2 e^{C_3 M/a} \leq (C_2^{-1} \nu)^2
$$

or

$$
\|u(t)\|_{H^3} \leq C_2^{-1} \nu. \tag{2.13}
$$

The bootstrapping argument then implies that (2.13) holds for all $t > 0$. This completes the proof of Theorem 1.1. □

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Data availability

No data was used for the research described in the article.

References

[1] J. Pedlosky, Geophysical Fluid Dynamics, Springer, 1987.
[2] R. Salmon, Lectures on Geophysical Fluid Dynamics, Oxford University Press, 1998.
[3] C. Bardos, M. Lopes Filho, H.N. Lopes, D. Niu, E.S. Titi, Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking, SIAM J. Math. Anal. 45 (2013) 1871–1885.
[4] L. Ma, J. Wu, Q. Zhang, Stability of 3D perturbations near a special 2D solution to the rotating Boussinesq equations, Stud. Appl. Math. 148 (2022) 1624–1655.
[5] G. Ponce, R. Racke, T.C. Sideris, E.S. Titi, Global stability of large solutions to the 3D Navier–Stokes equations, Comm. Math. Phys. 159 (1994) 329–341.
[6] R. Ji, W. Luo, L. Jiang, Stability of the 3D incompressible Navier–Stokes equations with fractional horizontal dissipation, Appl. Math. Comput. 448 (2023) 127934.
[7] R. Ji, J. Wu, W. Yang, Stability and optimal decay for the 3D Navier–Stokes equations with horizontal dissipation, J. Differential Equations 290 (2021) 57–77.
[8] H. Shang, Y. Zhai, Stability and large time decay for the three-dimensional anisotropic magnetohydrodynamic equations, Z. Angew. Math. Phys. 73 (2022) 71.
[9] A.J. Majda, P.R. Kramer, Simplified models for turbulent diffusion: Theory, numerical modeling, and applications, Phys. Rep. 314 (1999) 237–574.

[10] C. Cao, J. Wu, Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation, Arch. Ration. Mech. Anal. 208 (2013) 985–1004.
[11] B. Dong, J. Wu, X. Xu, N. Zhu, Stability and exponential decay for the 2D anisotropic Navier–Stokes equations with horizontal dissipation, J. Math. Fluid Mech. 23 (2021) 100.
[12] H. Lin, J. Wu, Y. Zhu, Global solutions to 3D incompressible MHD system with dissipation in only one dirction, SIAM J. Math. Anal. 55 (2023) 4570–4598.
[13] J. Wu, Y. Zhu, Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium, Adv. Math. 377 (2021) 107466.