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Calculus of Variations

Stability and large-time behavior for the 2D FENE dumbbell model near an equilibrium

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Abstract

The FENE dumbbell model combines the Navier–Stokes equations of the fluid velocity with a Fokker-Planck equation for the dynamics of polymer distribution in the fluid medium. The FENE model admits a special equilibrium solution $(0, \psi_\infty)$. This paper explores two types of enhanced dissipation associated with the system governing the perturbations near this steady state, one due to the equilibrium and one due to the coupling and interaction. Mathematically the linearized perturbation system admits a hidden wave structure. Making use of the smoothing and stabilizing effects in this wave structure, we are able to establish the global existence and stability of a 2D anisotropic FENE model in $\mathbb{R}^2$ with the velocity equation involving only horizontal dissipation. Without the coupling, the corresponding 2D Navier–Stokes is not known to be stable. When the spatial domain is $\mathbb{T} \times \mathbb{R}$, the FENE model with even less dissipation is shown to be stable, and the solution is shown to decay exponentially to its horizontal average. This result also relies on the above enhanced dissipation. The last part of our paper illustrates the importance of enhanced dissipation in the study of inviscid limit for a partially dissipated FENE system.

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1 Introduction

Due to their importance in physics, chemistry, and biology, PDE systems modeling the interaction of fluids and polymers have recently attracted considerable attention [5, 33]. One of these models is the finite extensible nonlinear elastic (FENE) dumbbell model. In this model, a polymer is represented as an "elastic dumbbell" consisting of two "beads" connected by a spring that can be modeled by a vector *R* [5, 6, 13, 35]. At the fluid level, the FENE dumbbell model combines the Navier–Stokes equations describing fluid velocity with a Fokker-Planck equation governing the dynamics of polymer distribution in the liquid medium. The micro–macro FENE dumbbell model reads as follows:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \mu\Delta u - \nabla p + \operatorname{div}\tau, \quad \operatorname{div} u = 0, \\ \partial_t\Psi + u\cdot\nabla\Psi = \varepsilon\Delta\Psi + \operatorname{div}_R\left[-\sigma(u)\cdot R\Psi + \beta\nabla_R\Psi + \nabla_R\mathcal{U}\Psi\right], \\ \tau_{j,k} = \int_B (R_j\partial_{R_k}\mathcal{U})\Psi(x,R,t)\,\mathrm{d}R, \\ u(x,0) = u_0, \quad \Psi(x,R,0) = \Psi_0, \\ (\nabla_R\Psi + \nabla_R\mathcal{U}\Psi)\cdot n = 0 \quad \text{on } \partial B(0,R_0). \end{cases} \tag{1}$$

In (1), *u*(*x*, *t*) denotes the velocity of the polymeric liquid, *p*(*x*, *t*) denotes the pressure, and $\Psi(x,R,t)$ denotes the distribution function for the internal configuration, where $x\in\mathbb{R}^2$ or $x\in\mathbb{T}\times\mathbb{R}$. The polymer elongation *R* is bounded in a ball $B(0,R_0)$, which indicates that the extensibility of the polymers is finite. Also, the potential $\mathcal{U}$ is given by

$$\mathcal{U}(R) = -k\log\left(1-\frac{|R|^2}{R_0^2}\right)$$

for some $k>0$, the magnitude of *k* is an important parameter as it portrays the strength of singularity of the equilibrium state ψ_∞ on the boundary $\partial B(0,R_0)$. In addition, μ is the viscosity of the fluid, $\varepsilon\ge 0$ is the center-of-mass diffusion rate of the polymer, and β relates to the Boltzmann constant and temperature. In general, $\sigma(u)=\nabla u$. For the co-rotation case, $\sigma(u)=\frac{\nabla u-(\nabla u)^{\mathrm{T}}}{2}$.

As in [33], to ensure the conservation of Ψ, we add an additional boundary condition, namely

$$(-\nabla u\cdot R\Psi + \beta\nabla_R\Psi + \nabla_R\mathcal{U}\Psi)\cdot n = 0 \qquad \text{on } \partial B(0,R_0).$$

This boundary condition implies that $\Psi=0$ on $\partial B(0,R_0)$, and if $\int_B \Psi_0(x,R)\,\mathrm{d}R = 1$, then for all (*x*, *t*), we have $\int_B \Psi(x,R,t)\,\mathrm{d}R=1$. In fact, by integrating the equation of Ψ in (3) in *R*-variable and using the above boundary condition, we find

$$\partial_t\int_B \Psi\,\mathrm{d}R + u\cdot\nabla\int_B\Psi\,\mathrm{d}R = \varepsilon\Delta\int_B\Psi\,\mathrm{d}R.$$

Hence, if $\varepsilon=0$, the above equation is a transport equation, and the conservation of the polymer density comes from the property of trajectory; if $\varepsilon>0$, the above equation is a parabolic equation with initial data $\int_B\psi(\cdot,R,0)\,\mathrm{d}R=1$, and the conservation of the polymer density comes from the uniqueness of the solution.

In this paper, we will set $\mu=1$, $\beta=1$ and $R_0=1$. Notice that (u,ψ) with $u=0$ and

$$\psi=\psi_\infty(R):=\frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R)}\,\mathrm{d}R}=\frac{(1-|R|^2)^k}{\int_B(1-|R|^2)^k\,\mathrm{d}R}$$

is a stationary solution of (1), and direct computation shows that

$$\nabla_R\Psi+\nabla_R\mathcal{U}\Psi=\psi_\infty\nabla_R\frac{\Psi}{\psi_\infty},$$

and we denote

$$\mathcal{L}\Psi=\mathrm{div}_R(\nabla_R\Psi+\nabla_R\mathcal{U}\Psi)=\mathrm{div}_R\left(\psi_\infty\nabla_R\frac{\Psi}{\psi_\infty}\right).$$

The well-posedness of various types of viscoelastic models has been widely studied. For the FENE dumbbell model with general drag term, in the case that the effect of center-of-mass diffusion of Ψ is ignored, namely, $\varepsilon=0$, Renardy [36] established the local well-posedness in Sobolev space with potential $\mathcal{U}(R)=(1-|R|^2)^{1-\sigma}$ for some $\sigma>1$. Later, Jourdain et al. [20] proved the local existence of a stochastic differential equation with potential $\mathcal{U}(R)=-k\log(1-|R|^2)$ in the case $b=2k>6$ for a Couette flow. Zhang and Zhang [41] proved the local well-posedness for the FENE model in three dimensions when $b=2k>76$ in weighted Sobolev spaces. Masmoudi [33] discovered some useful Hardy-type inequalities to handle the singular term $\mathrm{div}\,\tau$, and proved local and global well-posedness under small assumption near the equilibrium for the FENE model when $b=2k>0$. Also near the equilibrium, Lin et al. [25] proved the global existence under certain constraints on the potential. Masmoudi [34] proved the global existence of weak solutions in L^2 under some entropy conditions. On the other hand, the center-of-mass diffusion of polymer is physically justifiable, and if this diffusion term is not neglected, there is also a number of relevant research. The global existence of weak solution of the FENE model with center-of-mass diffusion is established in [4]. Also, Barrett and Süli [3] inserted a "microscopic" cut-off function in the drag term of the Fokker-Planck equation, and established the existence of global-in-time weak solutions to a mollification model with a general class of spring-force potentials, including the FENE potential. Later, Barrett and Süli [2] removed the cut-off in [3] and extended the results to the case of bead-spring chain models. For the FENE dumbbell model with co-rotation drag term, for instance, see [26, 29, 31, 33]. For the Hookean dumbbell model, one can consult [9, 18, 19, 23–25, 28] and the references therein.

Moreover, substantial research has been devoted to exploring the decay of viscoelastic fluid models. For incompressible viscoelastic dumbbell models, Schonbek [37] studied the L^2 decay of the velocity for the co-rotation FENE dumbbell model, and obtained the decay rate $(1+t)^{-\frac{N}{4}+\frac{1}{2}}$. Later, Luo and Yin [30, 31] improved the L^2 decay results developed in [37] by Fourier splitting methods, and obtained the decay rate $(1+t)^{-\frac{1}{4}}$ with $N=2$. Ai et al. [1] proved the optimal decay rates of the solution and its higher order derivatives of 3D general rate-type viscoelastic fluids, in their assumptions, the L^2 norm of the higher order derivatives $(n\geq 4)$ of the initial data can be arbitrarily large. Chen et al. [10] showed the sharp decay rates of the 3D incompressible Phan-Thien-Tanner system with large data. While for compressible viscoelastic dumbbell models, Hu and Wu [19] obtained the optimal L^p decay rate of the compressible Hookean-type viscoelastic flows. Recently, Deng et al.

[12] studied the micro–macro compressible polymeric fluids near the equilibrium with one type of general potential terms, and obtained the decay result.

For the inviscid limit of viscoelastic fluids, there have been a number of important results. For incompressible viscoelastic dumbbell models, Zi [42] considered the vanishing viscosity limit of the 3D incompressible Oldroyd-B model with small analytic data. Luo et al. [32] gave a result of local in time vanishing viscosity limit to the FENE model in Besov space. For compressible viscoelastic dumbbell models, Cai et al. [7] justified the vanishing viscosity limit for multi-dimensional incompressible viscoelasticity. Cui and Hu [11] proved the global existence of solutions of three-dimensional compressible viscoelastic systems around the equilibrium when the shear viscosity is arbitrarily small and the volume viscosity is arbitrarily large. Very recently, Wang and Xie [39] studied the inviscid limit of 2D compressible viscoelastic equations with the no-slip boundary condition by conormal derivatives framework. Gu et al. [17] investigated the inviscid limit of 3D half-space compressible viscoelastic systems with no-slip or Navier-slip boundary conditions, their result indicates that the deformation gradient can prevent the formation of strong boundary layers.

In this paper, we focus on the following 2D anisotropic FENE dumbbell model with partial dissipation

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \partial_1^2 u - \nabla p + \operatorname{div}\tau, \quad \operatorname{div} u = 0,\\ \partial_t\Psi + u\cdot\nabla\Psi = \upsilon\partial_2^2\Psi + \operatorname{div}_R(-\nabla u\cdot R\Psi) + \mathcal{L}\Psi,\\ \tau_{l,m} = \int_B (R_l\partial_{R_m}\mathcal{U})\Psi(x,R,t)\,\mathrm{d}R, \end{cases} \tag{2}$$

where $\upsilon = 0$ or 1 represents whether we have the vertical center of mass diffusion of ψ.

Next, we derive the perturbation form of (2) near the stationary solution $(0, \psi_\infty)$. For this purpose, we denote $\psi = \Psi - \psi_\infty$. Since $\nabla_x\psi_\infty = 0$, we have

$$\operatorname{div}\tau = \operatorname{div}\int_B (R\otimes\nabla_R\mathcal{U})\Psi\,\mathrm{d}R = \operatorname{div}\int_B (R\otimes\nabla_R\mathcal{U})\psi\,\mathrm{d}R.$$

Hence, we may assume that

$$\tau = \int_B (R\otimes\nabla_R\mathcal{U})\psi\,\mathrm{d}R.$$

Now, we write $\psi_0 := \Psi_0 - \psi_\infty$, and by direct computation, the equation governing the perturbation (u,ψ) of 2D FENE dumbbell model with partial dissipation reads as follows

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \partial_1^2 u - \nabla p + \operatorname{div}\tau, \quad \operatorname{div} u = 0,\\ \partial_t\psi + u\cdot\nabla\psi = \upsilon\partial_2^2\psi + \operatorname{div}_R(-\nabla u\cdot R(\psi+\psi_\infty)) + \mathcal{L}\psi,\\ \tau_{l,m} = \int_B (R_l\partial_{R_m}\mathcal{U})\psi(x,R,t)\,\mathrm{d}R. \end{cases} \tag{3}$$

(3) will be our focus. We intend to understand its three aspects: small data global existence and stability, asymptotic behavior, and vanishing viscosity problem.

Hydrodynamic problems with partial dissipation are widely studied [8, 15, 21, 40]. Dissipation in some systems of partial differential equations (PDEs) modeling fluids reduces to partial cases in certain physical regimes and after appropriate scaling. One of the significant examples is Prandtl's boundary layer equation. Mathematically, compared to the FENE dumbbell model with full dissipation, the involvement of only one directional dissipation makes the existence and stability problem much more challenging.

For the viscoelastic model with mixed partial dissipation, Feng et al. [16] obtained the small data global existence and stability for the 2D Oldroyd-B model in Sobolev space $H^2(\mathbb{R}^2)$. In contrast to the Oldroyd-B model, the FENE model incorporates the finite extensibility of real polymers and plays a pivotal role in elucidating complex and more realistic molecular behaviors [20]. In addition, the FENE dumbbell model presents heightened intricacy due to its nature as a micro–macro model with singularities at the elongation limit. These considerations partially motivated our study on the well-posedness and stability of the anisotropic dumbbell model (3).

We remark that the stability and long-time behavior problems are generally not trivial for partially dissipated PDE systems. Many classical approaches such as the Fourier splitting method no longer work. Due to the lack of full dissipation, the schemes to obtain the exponential decay for the co-rotation FENE dumbbell ([30, 31]) now fail since the drag term in the equation of ψ in (3) cannot be eliminated directly. To overcome these difficulties, this paper introduces new ideas to solve the large-time behavior problem on the anisotropic FENE dumbbell model (3). We explore the enhanced dissipation generated by the steady-state and the coupling and interaction. Mathematically we derive the hidden wave structure in the linearized system governing the dynamics of perturbations. In addition, we make use of the orthogonal decomposition associated with the horizontal periodic setting of the spatial domain.

The last part of this paper illustrates the power of the enhanced dissipation and wave structure in the study of the inviscid limits for partially dissipated FENE models. Without making use of the smoothing effect of the wave structure, it is impossible to obtain suitable upper bounds independent of the viscosity κ as κ approaches zero. Exploiting the wave structure helps us overcome this difficulty.

1.1 Statement of results

Our first topic is about the global existence and stability of small data solution of 2D anisotropic FENE dumbbell model near the steady solution $(0, \psi_\infty)$. For brevity, in this paper, we use *c* or *C* to denote constants independent of ε, η and *t*.

We begin with the global stability of perturbations satisfying the 2D FENE dumbbell model with mixed partial dissipation,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \partial_1^2 u - \nabla p + \operatorname{div}\tau, \quad \operatorname{div} u = 0, \\ \partial_t\psi + u\cdot\nabla\psi = \partial_2^2\psi + \operatorname{div}_R(-\nabla u\cdot R(\psi+\psi_\infty)) + \mathcal{L}\psi, \\ u(x,0) = u_0, \quad \psi(x,R,0) = \psi_0, \\ \psi_\infty \nabla_R \frac{\psi}{\psi_\infty}\cdot n = 0 \quad \text{on } \partial B. \end{cases} \tag{4}$$

We have the following result of small data global well-posedness.

Theorem 1.1 *Suppose that $k > 1$, there exists a small constant $\varepsilon_1 > 0$ such that for $u_0 \in H^2(\mathbb{R}^2)$, $\operatorname{div} u_0 = 0$ and $\psi_0 \in H^2(\mathbb{R}^2; \mathcal{L}^2)$, if*

$$||u_0||_{H^2} + ||\psi_0||_{H^2(\mathcal{L}^2)} \le \varepsilon_1, \tag{5}$$

then (4) has a unique global solution (u, ψ). In addition, for all $t > 0$, (u, ψ) satisfies

$$\begin{aligned} &||u(t)||^2_{H^2} + ||\psi(t)||^2_{H^2(\mathcal{L}^2)} \\ &+ \int_0^t \left\{ ||\partial_1 u(s)||^2_{H^2} + ||\partial_2 u(s)||^2_{H^1} + ||\partial_2 \psi(s)||^2_{H^2(\mathcal{L}^2)} + ||\psi(s)||^2_{H^2(\dot{\mathcal{H}}^1)} \right\} \mathrm{d}s \le C\varepsilon_1^2. \end{aligned}$$

Remark 1.1 Without coupling the equation of polymer flow, the uniform in time stability of the 2D Navier–Stokes equation in $\mathbb{R}^2$ with only horizontal dissipation is still open.

Remark 1.2 If the initial data ψ_0 is even in R_1 or R_2, then $k > 1$ can be removed.

Since physically, the center-of-mass diffusion of ψ is much weaker than the diffusion in *R*-variable, it is natural to neglect the vertical center-of-mass diffusion of ψ, and consider the following 2D FENE dumbbell system with merely horizontal dissipation of *u*:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \partial_1^2 u - \nabla p + \operatorname{div} \tau, \quad \operatorname{div} u = 0, \\ \partial_t \psi + u \cdot \nabla \psi = \operatorname{div}_R(-\nabla u \cdot R(\psi + \psi_\infty)) + \mathcal{L}\psi, \\ u(x, 0) = u_0, \quad \psi(x, R, 0) = \psi_0, \\ \psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n = 0 \quad \text{on } \partial B. \end{cases} \tag{6}$$

The absence of vertical dissipation complicates the problem. By imposing a symmetry condition in the *R*-variable, we can effectively eliminate the originally unmanageable terms arising from this lack of vertical dissipation. More precisely, we have the following result:

Theorem 1.2 *Suppose that $u_0 \in H^2(\mathbb{R}^2)$, $\operatorname{div} u_0 = 0$, $\psi_0 \in H^2(\mathbb{R}^2; \mathcal{L}^2)$ and ψ_0 is even in R_2. There exists a small constant $\varepsilon_2 > 0$ such that if*

$$||u_0||_{H^2} + ||\psi_0||_{H^2(\mathcal{L}^2)} \le \varepsilon_2, \tag{7}$$

then (6) has a unique global solution (u, ψ), and ψ is even in R_2. In addition, for all $t > 0$, (u, ψ) satisfies

$$||u(t)||^2_{H^2} + ||\psi(t)||^2_{H^2(\mathcal{L}^2)} + \int_0^t \left\{ ||\partial_1 u(s)||^2_{H^2} + ||\partial_2 u(s)||^2_{H^1} + ||\psi(s)||^2_{H^2(\dot{\mathcal{H}}^1)} \right\} \mathrm{d}s \le C\varepsilon_2^2. \tag{8}$$

Remark 1.3 As stated in Theorem 1.2, we can establish the global well-posedness and stability results even in the absence of vertical dissipation. In this case, the symmetry condition in *R*-variable needs to be introduced.

Remark 1.4 We can replace the symmetry in R_2 with the symmetry in R_1, which will have the same eliminating effect.

Once the global existence result is established, we can investigate the asymptotic behavior of the solution of system (6). Here, we consider the system in $\Omega = \mathbb{T} \times \mathbb{R}$. Our result states that ψ and the oscillation part $\tilde{u}$ decay to zero exponentially in time.

Theorem 1.3 *Suppose that $\Omega = \mathbb{T} \times \mathbb{R}$, $div\, u_0 = 0$ and ψ_0 is even in R_2. There exists a small constant $\varepsilon_3 > 0$ such that if*

$$||u_0||_{H^2} + ||\psi_0||_{H^2(\mathcal{L}^2)} \leq \varepsilon_3,$$

then (6) has a unique global solution (u, ψ). Moreover, ψ and the oscillation part $\tilde{u}$ decays exponentially in time in the sense that

$$||\tilde{u}||_{H^1} + ||\psi||_{H^1(\mathcal{L}^2)} \leq C(||u_0||_{H^1} + ||\psi_0||_{H^1(\mathcal{L}^2)})e^{-c't},$$

for some constant $c' > 0$ and for all $t > 0$.

Remark 1.5 This is the first result of the exponential decay of the polymer distribution ψ for FENE dumbbell model with general drag term rather than co-rotation drag term.

As a corollary of the above theorem, we deduce that the solution (u, ψ) approaches to $(\bar{u}, 0)$, where $(\bar{u}, \bar{\psi})$ is governed by

$$\begin{cases} \partial_t \bar{u} + \overline{\tilde{u} \cdot \nabla \tilde{u}} = \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} + \partial_2 \begin{pmatrix} 0 \\ \bar{\tau}_{2,2} \end{pmatrix}, \\ \partial_t \bar{\psi} + \overline{\tilde{u} \cdot \nabla \tilde{\psi}} = -\mathrm{div}_R (\overline{\nabla \tilde{u} \cdot R \tilde{\psi}}) - \partial_2 \bar{u}_1 R_2 \partial_{R_1} (\bar{\psi} + \psi_\infty) + \mathcal{L} \bar{\psi}. \end{cases}$$

The final part of this paper focuses on the vertical vanishing viscosity limit problem. Having already established the global existence of the 2D FENE dumbbell model with both full dissipation (1) (with $\varepsilon = 0$) and only horizontal dissipation (6), an important question is that if the vertical dissipation of *u* in (1) approaches zero, whether the solution of this system will converge to the solution of (6).

More precisely, we consider the vertical vanishing viscosity limit of the following anisotropic FENE dumbbell model:

$$\begin{cases} \partial_t u^\kappa + (u^\kappa \cdot \nabla) u^\kappa = \partial_1^2 u^\kappa + \kappa \partial_2^2 u^\kappa - \nabla p^\kappa + \mathrm{div}\, \tau^\kappa, \quad \mathrm{div}\, u^\kappa = 0, \\ \partial_t \psi^\kappa + u^\kappa \cdot \nabla \psi^\kappa = \mathrm{div}_R \left[-\nabla u^\kappa \cdot R(\psi^\kappa + \psi_\infty) + \psi_\infty \nabla_R \frac{\psi^\kappa}{\psi_\infty} \right], \\ u^\kappa(x, 0) = u_0, \quad \psi^\kappa(x, R, 0) = \psi_0, \\ \psi_\infty \nabla_R \frac{\psi^\kappa}{\psi_\infty} \cdot n = 0 \quad \text{on } \partial B. \end{cases} \tag{9}$$

We investigate the behavior of the solution when $\kappa \to 0$. Our result indicates that the solution of 2D FENE dumbbell model (6) is the global in time vanishing viscosity limit solution of the FENE dumbbell model with anisotropic full velocity dissipation (9).

Theorem 1.4 *Suppose that $\Omega = \mathbb{T} \times \mathbb{R}$, $div\, u_0 = 0$ and ψ_0 is even in R_2. Let $u_0 \in H^2(\Omega)$ and $\psi_0 \in H^2(\Omega; \mathcal{L}^2)$ and are sufficiently small, namely*

$$||u_0||_{H^2} + ||\psi_0||_{H^2(\mathcal{L}^2)} \le \varepsilon_4$$

such that (6) and (9) each has a unique global solution. Let (u^κ, ψ^κ) and (u, ψ) be the solutions of (6) and (9), respectively. Then, the following estimate holds:

$$||u^\kappa - u||^2_{H^1} + ||\psi^\kappa - \psi||^2_{H^1(\mathcal{L}^2)} \le C\kappa, \quad \forall t > 0, \tag{10}$$

where C is a constant independent of t and κ.

1.2 Main difficulties and strategies

We now explain the main difficulties we encountered during our proofs and demonstrate our main strategies.

1.2.1 Global existence and stability

When proving the global existence and stability results in Theorems 1.1 and 1.2, a major obstacle arises from the absence of vertical dissipation of u. This greatly complicates the analysis and makes the issues of global existence and stability non-trivial.

Without coupling the equation of polymer density, system (3) becomes the Navier–Stokes equation with only horizontal dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \partial_1^2 u - \nabla p, \quad x \in \mathbb{R}^2, \\ \operatorname{div} u = 0, \end{cases} \tag{11}$$

and the H^2-stability problem on perturbations near the trivial solution $u = 0$ of (11) still remains open. When there is no dissipation at all, (11) becomes the 2D Euler equation

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

Kiselev and Šverák [22] proved that the gradient of vorticity could grow double exponentially in time. This result on the growth of the Euler equations suggests that the stability of (11) near the trivial solution may not be expected in Sobolev settings.

Based on the reasoning above, to achieve the global existence result of the FENE dumbbell model with partial dissipation, we do not expect the energy estimate could be closed only by the following conventional H^2-energy structure:

$$\begin{aligned} E_{\upsilon,1}(t) &= \sup_{0\le s\le t} ||u(s)||^2_{H^2} + \sup_{0\le s\le t} ||\psi(s)||^2_{H^2(\mathcal{L}^2)} \\ &\quad + \int_0^t \left\{ ||\partial_1 u(s)||^2_{H^2} + \upsilon||\partial_2 \psi(s)||^2_{H^2(\mathcal{L}^2)} + ||\psi(s)||^2_{H^2(\dot{\mathcal{H}}^1)} \right\} \mathrm{d}s. \end{aligned} \tag{12}$$

Instead, we must investigate some special structure of system (3), and discover additional vertical regularity of u. Let $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ be the Leray projection onto divergence free vector fields. By applying the Leray projection $\mathbb{P}$ to the first equation of (3), we obtain

$$\partial_t u = \partial_1^2 u + \mathbb{P}\mathrm{div}\,\tau + N_1, \quad N_1 = -\mathbb{P}(u\cdot\nabla u). \tag{13}$$

Now, by integrating the second equation of (3) with $R\otimes\nabla_R\mathcal{U}$, and applying $\mathbb{P}\mathrm{div}$, we have

$$\begin{aligned}\partial_t\mathbb{P}\mathrm{div}\,\tau =& \upsilon\partial_2^2\mathbb{P}\mathrm{div}\,\tau + \mathbb{P}\mathrm{div}\int_B \mathrm{div}_R(-\nabla u\cdot R\psi_\infty)R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R\\ &+\mathbb{P}\mathrm{div}\int_B \mathcal{L}\psi R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R + N_2,\end{aligned} \tag{14}$$

where

$$N_2 = -\mathbb{P}\mathrm{div}\left\{u\cdot\nabla\int_B \psi R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R\right\} + \mathbb{P}\mathrm{div}\int_B \mathrm{div}_R(-\nabla u\cdot R\psi)R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R.$$

Using the symmetry of ψ_∞, direct computation shows that

$$\begin{aligned}&\mathbb{P}\mathrm{div}\int_B \mathrm{div}_R(-\nabla u\cdot R\psi_\infty)R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R\\ &\quad = \mathbb{P}\begin{pmatrix} c_1(k)\partial_1^2u_1 - 2c_2(k)\partial_1^2u_1 + c_2(k)\partial_2^2u_2\\ c_2(k)\partial_1^2u_2 - 2c_2(k)\partial_2^2u_2 + c_1(k)\partial_2^2u_2\end{pmatrix} = c_2(k)\Delta u,\end{aligned}$$

where

$$c_1(k) = -2k\int_B \frac{R_1^3\partial_{R_1}\psi_\infty}{1-|R|^2}\,\mathrm{d}R, \quad c_2(k) = -2k\int_B \frac{R_1^2R_2\partial_{R_2}\psi_\infty}{1-|R|^2}\,\mathrm{d}R,$$

and $c_1(k)$, $c_2(k)$ are integrable if $k>1$, and satisfy $c_1(k) = 3c_2(k) > 0$.

Hence, by differentiating (13) and (14) in time and making several substitutions, with the notation $\Upsilon = \mathbb{P}\mathrm{div}\,\tau$, we obtain that

$$\begin{cases}\partial_t^2u - (\partial_1^2+\upsilon\partial_2^2)\partial_t u + \upsilon\partial_1^2\partial_2^2 u - c_2(k)\Delta u + \mathbb{P}\mathrm{div}\int_B \mathcal{L}\psi R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R = N_3,\\ \partial_t^2\Upsilon - (\partial_1^2+\upsilon\partial_2^2)\partial_t\Upsilon + \upsilon\partial_1^2\partial_2^2\Upsilon - c_2(k)\Delta\Upsilon - (\partial_t-\partial_1^2)\mathbb{P}\mathrm{div}\int_B \mathcal{L}\psi R\otimes\nabla_R\mathcal{U}\,\mathrm{d}R = N_4,\end{cases} \tag{15}$$

where

$$N_3 = (\partial_t - \upsilon\partial_2^2)N_1 + N_2, \quad N_4 = c_2(k)\Delta N_1 + (\partial_t - \partial_1^2)N_2.$$

In particular, (15) provides the crucial vertical regularity of u, which seems to be missing in the original system (3). Thanks to this wave structure, we can extract the enhanced regularity of time integral of $\partial_2 u$, and this makes it promising to give the uniform bound of the following energy structure

$$E_2(t) = \int_0^t ||\partial_2 u(s)||_{H^1}^2\,\mathrm{d}s. \tag{16}$$

Besides understanding the additional vertical regularity of u from the wave structure, there is another way to derive (16) without introducing the assumption $k>1$. In fact, by integrating the second equation of (3) with R_kR_l, where $k,l=1,2$, we obtain

$$\begin{aligned}&\int_B \partial_t\psi R_kR_l\,\mathrm{d}R+\int_B u\cdot\nabla\psi R_kR_l\,\mathrm{d}R\\&\quad=\upsilon\int_B \partial_2^2\psi R_kR_l\,\mathrm{d}R+\int_B \mathrm{div}_R(\psi_\infty\nabla_R\frac{\psi}{\psi_\infty})R_kR_l\,\mathrm{d}R\\&\qquad+\int_B \mathrm{div}_R(-\nabla u\cdot R\psi)R_kR_l\,\mathrm{d}R+\int_B \mathrm{div}_R(-\nabla u\cdot R\psi_\infty)R_kR_l\,\mathrm{d}R.\end{aligned}\tag{17}$$

Due to the symmetry of ψ_∞, we deduce that

$$\begin{aligned}&\int_B \mathrm{div}_R(-\nabla u\cdot R\psi_\infty)R_kR_l\,\mathrm{d}R=\sum_{i,j=1}^2\int_B \partial_ju_iR_j\psi_\infty\partial_{R_i}(R_kR_l)\,\mathrm{d}R\\&=\sum_{j=1}^2\int_B\{\partial_ju_kR_j\psi_\infty R_l+\partial_ju_lR_j\psi_\infty R_k\}\,\mathrm{d}R=\Big(2\int_B R_1^2\psi_\infty\,\mathrm{d}R\Big)[\mathbb{D}u]_{k,l},\end{aligned}$$

where $[\mathbb{D}u]_{k,l}=\frac{1}{2}(\partial_ku_l+\partial_lu_k)$. By plugging the above equation into (17), we find

$$\begin{aligned}\mathcal{C}[\mathbb{D}u]_{j,k}=&\int_B\partial_t\psi R_jR_k\,\mathrm{d}R+\int_B u\cdot\nabla\psi R_jR_k\,\mathrm{d}R-\upsilon\int_B\partial_2^2\psi R_jR_k\,\mathrm{d}R\\&-\int_B\mathcal{L}\psi R_jR_k\,\mathrm{d}R-\int_B\mathrm{div}_R(-\nabla u\cdot R\psi)R_jR_k\,\mathrm{d}R,\end{aligned}\tag{18}$$

where $\mathcal{C}:=2\int_B R_1^2\psi_\infty\,\mathrm{d}R$.

Since $||\mathbb{D}u||_{H^1}=||\nabla u||_{H^1}$, the time integrability of ∂_2u can be estimated by the right-hand side of (18). The reasoning above explains our strategy on how to prevent the growth of the Sobolev norms of velocity by exploiting the stabilizing effect of ψ on the fluid.

With suitable energy structures $E_{\upsilon,1}(t)$ and $E_2(t)$(defined in (12) and (16)) at hand, we now introduce the most difficult term that will be encountered when performing the $\dot{H}^2$ -type energy estimate. If we apply ∂_2^2 to the second equation of (3) and multiply it by $\partial_2^2\psi$ in $L^2(\mathcal{L}^2)$, we have to deal with the following nonlinear term

$$\begin{aligned}&\iint_{\mathbb{R}^2 B}\partial_2^2\nabla u\cdot R\psi\nabla_R\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x\\&=\sum_{\substack{i,j=1,2,\\(i,j)\neq(2,1)}}\iint_{\mathbb{R}^2 B}\partial_2^2\partial_ju_iR_j\psi\partial_{R_i}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x+\iint_{\mathbb{R}^2 B}\partial_2^3u_1R_2\psi\partial_{R_1}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x.\end{aligned}$$

Although the first term on the right-hand side of the above equation can be estimated by utilizing the dissipation ∂_1^2u or the divergence free condition $\partial_2u_2=-\partial_1u_1$, the second term can not be treated directly due to the lack of vertical dissipation ∂_2^2u.

System (4) **in Theorem** 1.1 In this case, since we have the vertical center-of-mass diffusion of ψ, we can handle this term by integrating by parts. However, this will introduce a new singular term $\psi/(1-|R|^2)^{k/2+1}$. To handle this singularity, we take advantage of Hardy inequality of *R*-variable with $k > 1$.

System (6) **in Theorem**1.2 In this case, we don't have any vertical dissipation in *x*-variable to work with. To overcome this difficulty, we introduce the symmetry condition of ψ in *R*-variable. Roughly speaking, by setting the initial data ψ_0 even in R_2, and if ψ can keep this property for all $t > 0$, then $\partial_{R_1}(\partial_2^2\psi/\psi_\infty)$ is also even in R_2, and

$$\int_B R_2\psi\partial_{R_1}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R = 0.$$

Nevertheless, it is fascinating that we introduce the condition on *R*-variable to handle the problem raised in *x*-variable.

1.2.2 Exponential decay

When proving the exponential decay result in Theorem 1.3, the Fourier splitting method used in [30, 31] no longer works due to the loss of vertical dissipation. On the other hand, different from the co-rotation case, the drag term in the equation of ψ in (3) cannot be eliminated directly, this makes the problem much more difficult.

To overcome the above difficulties, we consider the decay of *u* and ψ simultaneously to counteract the linear part of the drag term. Also, we consider the system in the domain periodic in x_1, namely $\Omega = \mathbb{T}\times\mathbb{R} = [0,1]\times\mathbb{R}$. One of the significant advantages of the periodic domain Ω is that it allows us to separate the physical domain into its horizontal average and the corresponding oscillation part.

More precisely, we define the horizontal average

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1,x_2)\,\mathrm{d}x_1,$$

then, *f* can decomposed into horizontal average $\bar{f}$ and the corresponding oscillation part $\tilde{f}$, namely

$$f = \bar{f} + \tilde{f}.$$

We know that the horizontal average $\bar{f}$ represents the zeroth horizontal Fourier mode while the oscillation $\tilde{f}$ consists of all the rest non-zero horizontal frequencies. Mathematically, the decay of the horizontal average of *u* is hardly to be expected. In fact, it is associated with the zeroth horizontal Fourier mode, and the dissipative effect in this mode vanishes. Hence, we will focus on the decay of ψ, and the oscillation part $\tilde{u}$. The vital mathematical ingredient for obtaining the exponential decay of the oscillation part is the strong version of the Poincaré inequality,

$$||\tilde{f}||_{L^2(\Omega)} \le C||\partial_1\tilde{f}||_{L^2(\Omega)}. \tag{19}$$

Thanks to this Poincaré inequality, we can handle the convection term $\widetilde{u \cdot \nabla u}$ by the horizontal dissipation $\partial_1^2 \tilde{u}$ under the smallness assumption.

1.2.3 Inviscid limit

When proving the vanishing viscosity result in Theorem 1.4, the main challenge is to make the estimates independent of time rather than growing over time. Our strategy is to consider this problem in $\Omega = \mathbb{T} \times \mathbb{R}$, and use the extra dissipation term discovered from the wave structure (16). By decomposing the nonlinear term in the average part and the corresponding oscillation part, we can take advantage of the strong version of Poincaré inequality (19) in Ω to obtain the desired dissipation terms. By the key ingredients listed above, we are able to bound the H^1-type norm of the difference of the solution of (6) and (9) by $C\kappa$, and C is a constant independent of time.

The rest of this paper is organized as follows. In Sect. 2, we introduce some notations and list several Lemmas that will be frequently used; Sect. 3 prepares the local existence result for the bootstrap argument; Section 4 devotes to the global existence and stability results and proves Theorems 1.1 and 1.2; Sect. 5 considers the exponential decay and proves Theorem 1.3; Sect. 6 focuses on the vanishing viscosity limit problem and proves Theorem 1.4.

2 Preliminaries

In this section, we introduce some notations and useful lemmas that we shall use throughout the paper.

2.1 Notations

In this paper, we will use the following notations. We use $f \lesssim g$ to denote $f \le Cg$. Also, we use the abbreviation $B = B(0,1)$. ∂_{R_i}, div_R and ∇_R denote the R_i-derivative, divergence and gradient in *R*-variable, respectively. We define the following Hilbert spaces in *R*-variable:

$$
\begin{aligned}
\mathcal{L}^2 &= L^2(\mathrm{d}R/\psi_\infty) = \left\{ \psi \,\middle|\, \|\psi\|_{\mathcal{L}^2}^2 = \int_B |\psi|^2 \frac{\mathrm{d}R}{\psi_\infty} < \infty \right\}, \\
\dot{\mathcal{H}}^1 &= \left\{ \psi \,\middle|\, \|\psi\|_{\dot{\mathcal{H}}^1}^2 = \int_B \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 \mathrm{d}R < \infty \right\}, \\
\mathcal{H}^1 &= \left\{ \psi \in \mathcal{L}^2 \,\middle|\, \int_B \frac{\psi^2}{\psi_\infty} + \psi_\infty \left| \nabla_R \frac{\psi}{\psi_\infty} \right|^2 \mathrm{d}R < \infty \right\}.
\end{aligned}
$$

Next, we define the norm involving *x* and *R*. For $s \ge 0$,

$$
\begin{aligned}
||\psi||^2_{H^s(\mathcal{L}^2)} &= \sum_{|\alpha|\le s} \iint_{\mathbb{R}^2 B} |\partial^\alpha \psi|^2 \, \frac{\mathrm{d}R}{\psi_\infty}\, \mathrm{d}x,\\
||\psi||^2_{H^s(\dot{\mathcal{H}}^1)} &= \sum_{|\alpha|\le s} \iint_{\mathbb{R}^2 B} \psi_\infty \left| \nabla_R \frac{\partial^\alpha \psi}{\psi_\infty}\right|^2 \, \mathrm{d}R\, \mathrm{d}x,
\end{aligned}
$$

here, for $\alpha \in \mathbb{N}^2$, ∂^α denotes α_1 times derivatives in x_1 and α_2 times derivatives in x_2.

We also define the linear operator in *R*:

$$
\mathcal{L}\psi = \mathrm{div}_R \left(\psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \right)
$$

with the domain

$$
D(\mathcal{L}) = \left\{ \psi \in \mathcal{L}^2 \;\middle|\; \psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \in \mathcal{L}^2,\ \mathrm{div}_R\left(\psi_\infty \nabla_R \frac{\psi}{\psi_\infty}\right) \in \mathcal{L}^2,\ \psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n\Big|_{\partial B} = 0 \right\}.
$$

The boundary condition $\psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n\Big|_{\partial B} = 0$ should be understood in the sense that for any $\phi \in \dot{\mathcal{H}}^1$,

$$
\int_B \phi \mathcal{L}\psi \, \frac{\mathrm{d}R}{\psi_\infty} = -\int_B \psi_\infty \nabla_R \frac{\phi}{\psi_\infty} \nabla_R \frac{\psi}{\psi_\infty} \, \mathrm{d}R.
$$

2.2 Horizontal average and oscillation

To study the decay property and the inviscid limit of (u, ψ), we define the horizontal average

$$
\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, \mathrm{d}x_1. \tag{20}
$$

Then, we can decompose *f* into horizontal average $\bar{f}$ and the corresponding oscillation part $\tilde{f}$:

$$
f = \bar{f} + \tilde{f}. \tag{21}
$$

The following lemma contains some properties of $\tilde{f}$ and $\bar{f}$, which are frequently used in the proofs of Theorems 1.3 and 1.4.

Lemma 2.1 *([14]) Let f be a 2D function defined on $\Omega = \mathbb{T} \times \mathbb{R}$ and $f \in H^2(\Omega)$, $\bar{f}$ and $\tilde{f}$ are defined as in (20) and (21).*

i. $\bar{f}$ and $\tilde{f}$ satisfy the following properties

$$
\overline{\partial_1 f} = \partial_1 \bar{f} = 0, \quad \widetilde{\partial_1 f} = \partial_1 \tilde{f}, \quad \overline{\partial_2 f} = \partial_2 \bar{f}, \quad \widetilde{\partial_2 f} = \partial_2 \tilde{f}, \quad \bar{\tilde{f}} = 0.
$$

ii. If $\operatorname{div} f = 0$, then $\bar{f}$ and $\tilde{f}$ are also divergence free, namely

$$\operatorname{div} \bar{f} = 0 \quad \text{and} \quad \operatorname{div} \tilde{f} = 0.$$

iii. $\bar{f}$ and $\tilde{f}$ are orthogonal in L^2, namely

$$(\bar{f}, \tilde{f}) = \int_\Omega \bar{f} \tilde{f} \, \mathrm{d}x = 0, \quad ||f||_{L^2}^2 = ||\bar{f}||_{L^2}^2 + ||\tilde{f}||_{L^2}^2.$$

iv. If *f* and *g* are defined in Ω, then

$$(\bar{f}, \tilde{g}) = \int_\Omega \bar{f} \tilde{g} \, \mathrm{d}x = 0.$$

Also, an important property of $\tilde{f}$ is that it satisfies a strong version of Poincaré inequality, which is crucial for proving both exponential decay and the inviscid limit.

Lemma 2.2 *([14]) Let f be a 2D function defined on $\Omega = \mathbb{T} \times \mathbb{R}$. $\tilde{f}$ is defined as in (21), and $\tilde{f} \in H^1(\Omega)$. Then*

$$||\tilde{f}||_{L^2(\Omega)} \leq C||\partial_1 \tilde{f}||_{L^2(\Omega)}.$$

2.3 Anisotropic inequalities

To optimize the utilization of anisotropic dissipation, we introduce a series of anisotropic inequalities that will play a crucial role in the proofs presented in the subsequent sections. These anisotropic bounds are very powerful tools when investigating anisotropic systems.

The first anisotropic inequality of $\mathbb{R}^2$ is for the triple product, which is a useful tool in bounding the nonlinear terms.

Lemma 2.3 *([8]) Assume that f, g, $\partial_1 f$, $\partial_2 g$ are all in $L^2(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} fgh \, \mathrm{d}x \lesssim ||f||_{L^2}^{\frac{1}{2}} ||\partial_1 f||_{L^2}^{\frac{1}{2}} ||g||_{L^2}^{\frac{1}{2}} ||\partial_2 g||_{L^2}^{\frac{1}{2}} ||h||_{L^2}.$$

Sometimes we take L^∞ norm for the part with lower order derivatives in nonlinear terms, so the following anisotropic inequality is required.

Lemma 2.4 *([27]) Assume that f, $\partial_1 f$, $\partial_2 f$, $\partial_1 \partial_2 f$ are all in $L^2(\mathbb{R}^2)$, then*

$$||f||_{L^\infty(\mathbb{R}^2)} \lesssim ||f||_{L^2}^{\frac{1}{4}} ||\partial_1 f||_{L^2}^{\frac{1}{4}} ||\partial_2 f||_{L^2}^{\frac{1}{4}} ||\partial_1 \partial_2 f||_{L^2}^{\frac{1}{4}} \lesssim ||f||_{H^1}^{\frac{1}{2}} ||\partial_1 f||_{H^1}^{\frac{1}{2}}.$$

For $\Omega = \mathbb{T} \times \mathbb{R}$, the corresponding anisotropic inequalities read as follows.

Lemma 2.5 *([14]) Let $\Omega = \mathbb{T} \times \mathbb{R}$.*

i. Assume that *f*, *g*, *h*, $\partial_1 f$, $\partial_2 g$ are all in $L^2(\Omega)$, then

$$\int_\Omega fgh\,\mathrm{d}x \lesssim ||f||_{L^2}^{\frac{1}{2}}(||f||_{L^2}+||\partial_1 f||_{L^2})^{\frac{1}{2}}||g||_{L^2}^{\frac{1}{2}}||\partial_2 g||_{L^2}^{\frac{1}{2}}||h||_{L^2},$$

$$\int_\Omega \tilde{f}gh\,\mathrm{d}x \lesssim ||\tilde{f}||_{L^2}^{\frac{1}{2}}||\partial_1 \tilde{f}||_{L^2}^{\frac{1}{2}}||g||_{L^2}^{\frac{1}{2}}||\partial_2 g||_{L^2}^{\frac{1}{2}}||h||_{L^2}.$$

ii. Assume that *f*, $\partial_1 f$, $\partial_2 f$, $\partial_1\partial_2 f$ are all in $L^2(\Omega)$, we have

$$\begin{aligned}||f||_{L^\infty(\Omega)} &\lesssim ||f||_{L^2}^{\frac{1}{4}}(||f||_{L^2}+||\partial_1 f||_{L^2})^{\frac{1}{4}}||\partial_2 f||_{L^2}^{\frac{1}{4}}(||\partial_2 f||_{L^2}+||\partial_1\partial_2 f||_{L^2})^{\frac{1}{4}}\\ &\lesssim ||f||_{H^1}^{\frac{1}{2}}(||f||_{H^1}+||\partial_1 f||_{H^1})^{\frac{1}{2}},\\ ||\tilde{f}||_{L^\infty(\Omega)} &\lesssim ||\tilde{f}||_{L^2}^{\frac{1}{4}}||\partial_1\tilde{f}||_{L^2}^{\frac{1}{4}}||\partial_2\tilde{f}||_{L^2}^{\frac{1}{4}}||\partial_1\partial_2\tilde{f}||_{L^2}^{\frac{1}{4}} \lesssim ||\tilde{f}||_{H^1}^{\frac{1}{2}}||\partial_1\tilde{f}||_{H^1}^{\frac{1}{2}}.\end{aligned}$$

2.4 Inequalities in *R*-variable

The first inequality in *R* is the Poincaré inequality, which is frequently used in Section 4 to identify the dissipation terms. Additionally, it plays a pivotal role in establishing the exponential decay.

Lemma 2.6 *([33]) Assume that $\psi \in \mathcal{H}^1$ and $\int_B \psi = 0$, then*

$$\int_B \frac{\psi^2}{\psi_\infty}\,\mathrm{d}R \lesssim \int \psi_\infty \left|\nabla_R \frac{\psi}{\psi_\infty}\right|^2 \mathrm{d}R.$$

To deal with the singular term $\mathrm{div}\,\tau$, the main tool is the Hardy type inequality innovatively developed in [33]. We denote $x = 1 - |R|$.

Lemma 2.7 *([33]) For all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\left(\int_B \frac{|\psi|}{x}\,\mathrm{d}R\right)^2 \le \varepsilon \int_B \psi_\infty \left|\nabla_R \frac{\psi}{\psi_\infty}\right|^2 \mathrm{d}R + C_\varepsilon \int_B \frac{\psi^2}{\psi_\infty}\,\mathrm{d}R.$$

The following Hardy inequality is restricted in the case that $k > 1$, and it can deal with stronger singularity.

Lemma 2.8 *([33]) Assume that $k > 1$, $\int_B \psi = 0$, and $\psi \in \dot{\mathcal{H}}^1$, then*

$$\int_B \frac{\psi^2}{\psi_\infty x^2}\,\mathrm{d}R \lesssim |\psi|_{\dot{\mathcal{H}}^1}^2.$$

3 Local existence

In this section, we devote to proving the large data local existence of 2D FENE dumbbell model (6) with symmetric initial data, and demonstrating the symmetry-preserving property of ψ. Then, with a similar procedure, we can obtain the local existence results of the models mentioned in Theorems 1.1, 1.3 and 1.4. Given that we account for the local existence of large solutions, this section examines the original form of the equation rather than the perturbed form (6). When $\nu = 0$, the original form of system (2) reads as follows

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \partial_1^2 u - \nabla p + \operatorname{div} \tau, \quad \operatorname{div} u = 0, \\ \partial_t \psi + u \cdot \nabla \psi = \operatorname{div}_R(-\nabla u \cdot R\psi) + \mathcal{L}\psi, \end{cases} \tag{22}$$

here we use the same notation for simplicity.

Before stating the local existence result, we should first clarify the definition of even and odd in this paper.

Definition 3.1 Suppose f is defined on $B \times [0, T]$ and $f \in L^2([0, T]; \mathcal{L}^2)$, then f is even (odd) in R_2 if for all $\xi \in L^\infty([0, T]; \mathcal{L}^2)$ and ξ is odd (even) in R_2, we have

$$\int_B f\xi \, \mathrm{d}R = 0 \quad \text{for almost every} \quad t \in [0, T].$$

Definition 3.2 Suppose g is defined on $B \times \mathbb{R}^2 \times [0, T]$ and $g \in L^2([0, T]; L^2(\mathcal{L}^2))$, then g is even (odd) in R_2 if for all $\xi \in L^\infty([0, T]; L^2(\mathcal{L}^2))$ and ξ is odd (even) in R_2, we have

$$\int_B g\xi \, \mathrm{d}R = 0 \quad \text{for almost every} \quad (x, t) \in \mathbb{R}^2 \times [0, T].$$

With the above definitions, the local existence result reads as follows.

Proposition 3.1 *Suppose that $u_0 \in H^2(\mathbb{R}^2)$, $div\, u_0 = 0$, $\psi_0 \in H^2(\mathcal{L}^2)$ and is even in R_2, then there exists $T > 0$ such that there exists a unique solution (u, ψ) of (22) in the sense that*

$$u \in L^\infty([0, T]; H^2(\mathbb{R}^2)), \quad \partial_1 u \in L^2([0, T]; H^2(\mathbb{R}^2)),$$
$$\psi \in L^\infty([0, T]; H^2(\mathbb{R}^2; \mathcal{L}^2)) \cap L^2([0, T]; H^2(\mathbb{R}^2, \dot{\mathcal{H}}^1)).$$

Moreover, for $s = 0, 1, 2$ and $j = 1, 2$,

$$\partial_j^s \psi,\ \partial_{R_1} \frac{\partial_j^s \psi}{\psi_\infty} are even in R_2, \quad \partial_{R_2} \frac{\partial_j^s \psi}{\psi_\infty} is odd in R_2.$$

Due to the involvement of the singular term τ, the local existence of (3) is not a standard result. To overcome this singularity, Masmoudi [33] established the crucial Hardy type inequality in Lemma 2.7, and discovered that the linear operator $-\mathcal{L}$ in R variable has a similar dissipative effect as $-\Delta$.

Compared to [33], the local existence problem we consider is more difficult. On the one hand, we no longer have the vertical dissipation of u; on the other hand, we have to additionally verify that the symmetry is maintained.

3.1 Linear solution in R

To prove the local existence result, we first show the global existence of a linear evolution equation in (R, t).

Lemma 3.2 *Assume that $\Lambda(t) \in L^2([0,\infty))$ is a matrix-valued function, $f \in L^2([0, T]; \mathcal{L}^2)$ and $\psi_0(R) \in \mathcal{L}^2$ is even in R_2, then for any $T > 0$,*

$$\begin{cases} \partial_t \psi = -div_R(\Lambda(t) \cdot R\psi) + \mathcal{L}\psi + div_R f, \\ \psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n\Big|_{\partial_B} = 0, \quad \psi(0) = \psi_0, \end{cases} \tag{23}$$

has a unique weak solution in $L^\infty([0, T]; \mathcal{L}^2) \cap L^2([0, T]; \dot{\mathcal{H}}^1)$, and

$$\psi,\ \partial_{R_1} \frac{\psi}{\psi_\infty} areeveninR_2, \quad \partial_{R_2} \frac{\psi}{\psi_\infty} isoddinR_2.$$

Proof The proof of the existence and uniqueness of the solution is parallel to [33, Proposition 3.9]. Hence, it suffices to consider the symmetry property of ψ and its derivatives. First, we can deduce the symmetry of ψ by the uniqueness of solution. Suppose that ψ is a solution of (23), then φ with

$$\varphi = \psi(R_1, -R_2, t)$$

also satisfies (23) with initial data φ_0 defined as

$$\varphi_0 = \psi_0(R_1, -R_2).$$

Since ψ_0 is even in R_2, we have $\varphi_0 = \psi_0$. By the uniqueness of the solution, for all $0 \le t \le T$, we have $\varphi(t) = \psi(t)$, or

$$\psi(R_1, R_2, t) = \psi(R_1, -R_2, t).$$

Next, we prove the symmetry of the derivatives of ψ. We consider $\partial_{R_2} \frac{\psi}{\psi_\infty}$ as an example.

Notice that $C_c^\infty(B)$ is dense in $\mathcal{L}^2(B)$, for all $\zeta \in L^\infty([0, T]; \mathcal{L}^2)$ and ζ is even in R_2, we can use standard mollification to obtain a sequence $\{\zeta_N\}_{N\ge1}$ and each ζ_N is even in R_2, such that $\zeta_N \to \zeta$ in $L^\infty([0, T]; \mathcal{L}^2)$ as $N \to \infty$. Therefore, for any $0 < t_1 < t_2$,

$$\int_{t_1}^{t_2} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta \,\mathrm{d}R \,\mathrm{d}t = H_1 + H_2,$$

where

$$H_1 := \int_{t_1}^{t_2} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta \, \mathrm{d}R \, \mathrm{d}t - \int_{t_1}^{t_2} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta_N \, \mathrm{d}R \, \mathrm{d}t,$$
$$H_2 := \int_{t_1}^{t_2} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta_N \, \mathrm{d}R \, \mathrm{d}t.$$

For H_1, by Hölder's inequality,

$$|H_1| \leq ||\psi||_{L^2([0,T];\dot{\mathcal{H}}^1)} ||\zeta - \zeta_N||_{L^2([0,T];\mathcal{L}^2)} \to 0 \quad \text{as} \quad N \to \infty.$$

For H_2, since $\zeta_N \in C_c^\infty(B)$, we can obtain directly from the definition of classical derivative that $\partial_{R_2} \zeta_N$ is odd in R_2. Also, since ζ_N has compact support, we can use integration by parts and ψ is even in R_2 to obtain that

$$H_2 = -\int_{t_1}^{t_2} \int_B \frac{\psi}{\psi_\infty} \partial_{R_2} \zeta_N \, \mathrm{d}R \, \mathrm{d}t = 0.$$

Therefore, by passing the limit, we have

$$\int_{t_1}^{t_2} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta \, \mathrm{d}R \, \mathrm{d}t = 0.$$

Since t_1 and t_2 are arbitrary, we can deduce from Lebesgue's Lemma that for almost every $t > 0$,

$$\int_B \partial_{R_2} \frac{\psi(t)}{\psi_\infty} \zeta(t) \, \mathrm{d}R = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{t+\delta}^{t-\delta} \int_B \partial_{R_2} \frac{\psi}{\psi_\infty} \zeta \, \mathrm{d}R \, \mathrm{d}t = 0.$$

Since the above equality holds for all ζ even in R_2, we deduce from Definition 3.1 that $\partial_{R_2} \frac{\psi(t)}{\psi_\infty}$ is odd in R_2. Using the same density argument, we can obtain that $\partial_{R_1} \frac{\psi(t)}{\psi_\infty}$ is even in R_2. □

Next, we introduce the dependence of ψ on the *x*-variable through the involvement of a given *u*(*x*, *t*), and show the regularity in the *x*-variable as well as the symmetry of ψ.

Proposition 3.3 *Assume that for some* $T > 0$, $u \in L^\infty([0,T]; H^2)$, $\partial_t u \in L^2([0,T]; H^2)$. *Also, suppose that* $\psi_0(x, R) \in H^2(\mathcal{L}^2)$ *and is even in* R_2 *for almost every* $x \in \mathbb{R}^2$. *Then*

$$\begin{cases} \partial_t \psi + u \cdot \nabla \psi = -div_R(\nabla u \cdot R\psi) + \mathcal{L}\psi, \\ \psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n \Big|_{\partial B} = 0, \quad \psi(0) = \psi_0, \end{cases} \tag{24}$$

has a unique solution

$$\psi \in L^\infty([0,T]; H^2(\mathbb{R}^2;\mathcal{L}^2)) \cap L^2([0,T]; H^2(\mathbb{R}^2;\dot{\mathcal{H}}^1)).$$

Moreover, for $s = 0, 1, 2$ and $j = 1, 2$,

$$\partial_j^s \psi,\ \partial_{R_1} \frac{\partial_j^s \psi}{\psi_\infty} are even in R_2, \quad \partial_{R_2} \frac{\partial_j^s \psi}{\psi_\infty} is odd in R_2.$$

Proof The proof of the spatial regularity is parallel to [33, Proposition 3.10], except that Lemma 2.4 is utilized when necessary.

Regarding the symmetry of ψ, similar to the corresponding part in Lemma 3.2, we utilize mollification and Lebesgue's Lemma. For any $\zeta \in L^\infty([0,T]; L^2(\mathcal{L}^2))$ and is even in R_2, we have

$$\int_B \partial_{R_2} \frac{\partial_k \psi}{\psi_\infty} \zeta \, \mathrm{d}R = 0 \quad \text{for almost every} \quad (x,t) \in \mathbb{R}^2 \times [0,T],$$

and by the Definition 3.2, $\partial_{R_2} \frac{\partial_k \psi}{\psi_\infty}$ is odd in R_2.

Using a similar argument, we can establish the remaining desired spatial regularity and symmetry as stated in Proposition 3.3. □

3.2 A priori estimates

In this part, we prepare the a priori energy estimate of ψ for the subsequent fixed-point argument. The proof of the energy inequality will utilize the symmetry property of ψ, as established in Proposition 3.3.

Proposition 3.4 *Suppose that the assumptions of Proposition 3.3 hold. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\sup_{0\le s\le T} ||\psi(s)||^2_{H^2(\mathcal{L}^2)} + \int_0^T ||\psi(s)||^2_{H^2(\dot{\mathcal{H}}^1)} \, \mathrm{d}s \\ &\le \exp\left\{ C \sup_{0\le s\le T} ||u(s)||^2_{H^2} T + C \int_0^T ||\partial_1 u(s)||^2_{H^2} \, \mathrm{d}s \right\} ||\psi_0||^2_{H^2(\mathcal{L}^2)}. \end{aligned} \tag{25}$$

Proof First, we deduce the L^2-type energy estimate of ψ. Integrating the second equation of (6) with ψ, then using $\operatorname{div} u = 0$, integrating by parts, Lemma 2.4 and Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} ||\psi||^2_{L^2(\mathcal{L}^2)} + ||\psi||^2_{L^2(\dot{\mathcal{H}}^1)} \le C(||\nabla u||^2_{H^1} + ||\partial_1 \nabla u||^2_{H^1}) ||\psi||^2_{L^2(\mathcal{L}^2)}. \tag{26}$$

Now, we consider the $\dot{H}^2$-type estimate. By applying $\partial_k^2 (k = 1, 2)$ to the second equation of (6), and integrating it with $\partial_k^2 \psi$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_k^2\psi||^2_{L^2(\mathcal{L}^2)} + ||\partial_k^2\psi||^2_{L^2(\dot{\mathcal{H}}^1)} = H_3 + H_4, \tag{27}$$

where

$$H_3 := -\iint\limits_{\mathbb{R}^2 B} \partial_k^2(u\cdot\nabla\psi)\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,$$
$$H_4 := \iint\limits_{\mathbb{R}^2 B} \partial_k^2\mathrm{div}_R(-\nabla u\cdot R\psi)\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.$$

Thanks to Lemmas 2.3, 2.4, and Hölder inequality, we can bound H_3 by

$$\begin{aligned}H_3 = &-\iint\limits_{\mathbb{R}^2 B} \partial_k^2 u\cdot\nabla\psi\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x - \iint\limits_{\mathbb{R}^2 B}\partial_k u\cdot\nabla\partial_k\psi\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x\\
\lesssim&||\partial_k^2 u||^{\frac12}_{L^2}||\partial_1\partial_k^2 u||^{\frac12}_{L^2}||\nabla\psi||^{\frac12}_{L^2(\mathcal{L}^2)}||\partial_2\ \nabla\psi||^{\frac12}_{L^2(\mathcal{L}^2)}||\partial_k^2\psi||_{L^2(\mathcal{L}^2)}\\
&+||\partial_k u||^{\frac12}_{H^1}||\partial_1\partial_k u||^{\frac12}_{H^1}||\nabla\partial_k\psi||_{L^2(\mathcal{L}^2)}||\partial_k^2\psi||_{L^2(\mathcal{L}^2)}\\
\lesssim&(||u||^2_{H^2}+||\partial_1 u||^2_{H^2}+1)||\psi||^2_{H^2(\mathcal{L}^2)}.\end{aligned}$$

For H_4, after integration by parts in *R*-variable and using Lemmas 2.3, 2.4 and $\mathrm{div}\,u = 0$, we find

$$\begin{aligned}H_4 =&\sum_{l=0}^{2}\iint\limits_{\mathbb{R}^2 B}\partial_k^l\partial_1 u_1\left(R_1\partial_k^{2-l}\psi\partial_{R_1}\frac{\partial_k^2\psi}{\psi_\infty} - R_2\partial_k^{2-l}\psi\partial_{R_2}\frac{\partial_k^2\psi}{\psi_\infty}\right)\mathrm{d}R\,\mathrm{d}x\\
\leq&C||\nabla u||^{\frac12}_{H^1}||\partial_1\ \nabla u||^{\frac12}_{H^1}||\partial_k^2\psi||_{L^2(\mathcal{L}^2)}||\partial_k^2\psi||_{L^2(\dot{\mathcal{H}}^1)}\\
&+C||\partial_k\partial_1 u||^{\frac12}_{L^2}||\partial_1\ \partial_k\partial_1 u||^{\frac12}_{L^2}||\partial_k\psi||^{\frac12}_{L^2(\mathcal{L}^2)}||\partial_2\ \partial_k\psi||^{\frac12}_{L^2(\mathcal{L}^2)}||\partial_k^2\psi||_{L^2(\dot{\mathcal{H}}^1)}\\
&+C||\partial_k^2\partial_1 u||_{L^2}||\psi||_{H^2(\mathcal{L}^2)}||\partial_k^2\psi||_{L^2(\dot{\mathcal{H}}^1)}\\
\leq&\frac12||\partial_k^2\psi||^2_{L^2(\dot{\mathcal{H}}^1)} + C(||u||^2_{H^2}+||\partial_1 u||^2_{H^2})||\psi||^2_{H^2(\mathcal{L}^2)},\end{aligned}$$

where for the first equality, we use the symmetry property in Proposition 3.3 to deduce

$$\iint\limits_{\mathbb{R}^2 B}\partial_k^l\partial_2 u_1 R_2\partial_k^{2-l}\psi\partial_{R_1}\frac{\partial_k^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x = \iint\limits_{\mathbb{R}^2 B}\partial_k^l\partial_1 u_2 R_1\partial_k^{2-l}\psi\partial_{R_2}\frac{\partial_k^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x = 0.$$

Substituting the estimates of H_5 and H_6 into (27) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}||\partial_k^2\psi||^2_{L^2(\mathcal{L}^2)} + ||\partial_k^2\psi||^2_{L^2(\dot{\mathcal{H}}^1)}\\
&\quad\leq C(||u||^2_{H^2}+||\partial_1 u||^2_{H^2}+1)||\partial_k^2\psi||^2_{L^2(\mathcal{L}^2)}.\end{aligned} \tag{28}$$

Due to the norm equivalence

$$||\psi||_{H^2(\mathcal{L}^2)\cap H^2(\dot{\mathcal{H}}^1)} \sim ||\psi||_{L^2(\mathcal{L}^2)\cap L^2(\dot{\mathcal{H}}^1)} + \sum_{k=1}^{2} ||\partial_k^2\psi||_{L^2(\mathcal{L}^2)\cap L^2(\dot{\mathcal{H}}^1)},$$

we can deduce from (26) and (28) that

$$\frac{\mathrm{d}}{\mathrm{d}t}||\psi||^2_{H^2(\mathcal{L}^2)} + ||\psi||^2_{H^2(\dot{\mathcal{H}}^1)} \le C(||u||^2_{H^2} + ||\partial_1 u||^2_{H^2} + 1)||\psi||^2_{H^2(\mathcal{L}^2)},$$

and (25) comes directly after integrating the above inequality in time. □

3.3 Existence proof

Now, we are ready to prove Proposition 3.1. In the energy estimates of *u*, due to the absence of vertical dissipation, it is necessary for us to utilize the symmetry of ψ to handle the singular term $\operatorname{div}\tau$.

Proof of Proposition 3.1 In order to use a fixed-point argument, for $0 < T \le 1$, we set

$$\mathcal{X} = \Big\{(u,\psi)|u \in L^\infty([0,T];H^2),\ \partial_1 u \in L^2([0,T];H^2),\ \operatorname{div} u = 0,\ \psi \in L^\infty([0,T];H^2(\mathcal{L}^2))\Big\}.$$

Also, we define operator $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ by $\mathcal{F}(v,\xi) = (u,\psi)$, where ψ is the unique solution of

$$\begin{cases} \partial_t\psi + v\cdot\nabla\psi = \operatorname{div}_R(-\nabla v\cdot R\psi) + \mathcal{L}\psi, \\ \psi(x,R,0) = \psi_0, \\ \psi_\infty\nabla_R\frac{\psi}{\psi_\infty}\cdot n = 0 \quad \text{on } \partial B, \end{cases}$$

and *u* is the unique solution of the following linear equation

$$\begin{cases} \partial_t u + (v\cdot\nabla)u = \partial_1^2 u - \nabla p + \operatorname{div}\tau, \\ \operatorname{div} u = 0, \quad u(x,0) = u_0, \end{cases}$$

where $\tau = \int_B (R\otimes\nabla\mathcal{U})\psi\,\mathrm{d}R$.

Let $\mathcal{X}_0$ be a closed subset of $\mathcal{X}$ defined by

$$\mathcal{X}_0 = \Big\{(u,\psi)\in\mathcal{X}\,\Big|\sup_{0\le t\le T}||u(t)||^2_{H^2} + \int_0^T ||\partial_1 u(t)||^2_{H^2}\,\mathrm{d}t \le 6||u_0||^2_{H^2}+1,\ \sup_{0\le t\le T}||\psi(t)||^2_{H^2(\mathcal{L}^2)} \le A\Big\},$$

where

$$A = \exp\left\{2C(3||u_0||^2_{H^2}+1)\right\}||\psi_0||^2_{H^2(\mathcal{L}^2)}.$$

Now, we show that for *T* small enough, $\mathcal{F}$ maps $\mathcal{X}_0$ into $\mathcal{X}_0$. Suppose that $(v,\xi)\in\mathcal{X}_0$, one first deduces from Proposition 3.4 that

$$\sup_{0\le t\le T}||\psi(t)||^2_{H^2(\mathcal{L}^2)}+\int_0^T||\psi(t)||^2_{H^2(\dot{\mathcal{H}}^1)}\,\mathrm{d}t\le e^{2C(3||u_0||^2_{H^2}+1)}||\psi_0||^2_{H^2(\mathcal{L}^2)}=A. \tag{29}$$

Then we estimate $\dot{H}^2$ norm of u. For $k=1,2$, direct energy estimates show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_k^2 u||^2_{L^2}+||\partial_1\partial_k^2 u||^2_{L^2}=H_5+H_6, \tag{30}$$

where

$$\begin{aligned}H_5&:=-\int_{\mathbb{R}^2}\partial_k^2(v\cdot\nabla u)\partial_k^2 u\,\mathrm{d}x,\\ H_6&:=\int_{\mathbb{R}^2}\operatorname{div}\partial_k^2\tau\partial_k^2 u\,\mathrm{d}x.\end{aligned}$$

By using Lemmas 2.3, 2.4, $\operatorname{div} v=0$ and Young's inequality, we have

$$\begin{aligned}H_5=&-\int_{\mathbb{R}^2}\partial_k v\cdot\nabla\partial_k u\partial_k^2 u\,\mathrm{d}x-\int_{\mathbb{R}^2}\partial_k^2 v\cdot\nabla u\partial_k^2 u\,\mathrm{d}x\\ \le& C||\partial_k v||^{\frac12}_{L^2}||\partial_2\partial_k v||^{\frac12}_{L^2}||\partial_k\nabla u||^{\frac12}_{L^2}||\partial_1\partial_k\nabla u||^{\frac12}_{L^2}||\partial_k^2 u||_{L^2}\\ &+C||\partial_k^2 v||_{L^2}||\nabla u||^{\frac12}_{H^1}||\partial_1\ \nabla u||^{\frac12}_{H^1}||\partial_k^2 u||_{L^2}\\ \le&\frac14||\partial_1\partial_k^2 u||^2_{L^2}+C(||v||^2_{H^2}+1)||u||^2_{H^2}.\end{aligned}$$

Next, thanks to the symmetry property, by integrating by parts, $\operatorname{div} u=0$ and Lemma 2.7, we have

$$\begin{aligned}H_6&=-2k\sum_{l=1}^{2}\iint\limits_{\mathbb{R}^2 B}\frac{R_l^2}{1-|R|^2}\partial_k^2\psi\partial_l\partial_k^2 u_l\,\mathrm{d}R\,\mathrm{d}x\\ &=2k\iint\limits_{\mathbb{R}^2 B}\partial_1\partial_k^2 u_1\frac{R_1^2-R_2^2}{1-|R|^2}\partial_k^2\psi\,\mathrm{d}R\,\mathrm{d}x\\ &\le\frac14||\partial_1\partial_k^2 u||^2_{L^2}+a||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}+C_a||\psi||^2_{H^2(\mathcal{L}^2)},\end{aligned}$$

where $a>0$ is a small constant that will be determined later. Inserting the bounds of H_7 and H_8 into (30), together with the norm equivalence $||u||_{H^2}\sim||u||_{L^2}+\sum_k||\partial_k^2 u||_{L^2}$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}||u||^2_{H^2}+||\partial_1 u||^2_{H^2}\le C(||v||^2_{H^2}+1)||u||^2_{H^2}+a||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}+C_a||\psi||^2_{H^2(\mathcal{L}^2)}.$$

Integrating the above inequality in time, and recalling the bound of ψ in (29), we obtain that

$$\sup_{0\le t\le T}||u(t)||^2_{H^2}+\int_0^T||\partial_1 u(t)||^2_{H^2}\,\mathrm{d}t\le \big(||u_0||^2_{H^2}+A(a+C_aT)\big)e^{2C(3||u_0||^2_{H^2}+1)T}. \tag{31}$$

Hence, if we take

$$a\le\frac{1}{2eA}\quad\text{and}\quad T\le\max\left\{\frac{1}{2C(3||u_0||^2_{H^2}+1)},\frac{1}{2eC_a}\right\},$$

then the right hand side of (31) is bounded by $6||u_0||^2_{H^2}+1$. Together with (29), $\mathcal{F}$ maps $\mathcal{X}_0$ into $\mathcal{X}_0$.

Next, we prove that $\mathcal{F}$ is a contraction under the following norm:

$$||(u,\psi)||^2_{\mathcal{X}_0}=\sup_{0\le t\le T}||u(t)||^2_{L^2}+\int_0^T||\partial_1 u(t)||^2_{L^2}\,\mathrm{d}t+\theta\sup_{0\le t\le T}||\psi(t)||^2_{L^2(\mathcal{L}^2)},$$

namely, by choosing θ and T suitably small, there is

$$||\mathcal{F}(v_1,\xi_1)-\mathcal{F}(v_2,\xi_2)||_{\mathcal{X}_0}\le\frac{1}{2}||(v_1,\xi_1)-F(v_2,\xi_2)||_{\mathcal{X}_0}. \tag{32}$$

For convenience, we denote $(u_i,\psi_i)=\mathcal{F}(v_i,\xi_i)$, $i=1,2$, then $(\delta u,\delta\psi):=(u_1-u_2,\psi_1-\psi_2)$ satisfies

$$\begin{cases}\partial_t\delta u+v_1\cdot\nabla\delta u+(v_1-v_2)\cdot\nabla u_2=\partial_1^2\delta u+\nabla P+\mathrm{div}\,(\tau_1-\tau_2),\\ \partial_t\delta\psi+v_1\cdot\nabla\delta\psi+(v_1-v_2)\cdot\nabla\psi_2=\mathrm{div}_R(-\nabla u_1R\delta\psi)+\mathrm{div}_R(-\nabla(v_1-v_2)R\psi_1)+\mathcal{L}\delta\psi,\\ \delta u|_{t=0}=0,\ \delta\psi|_{t=0}=0.\end{cases}$$

Similar to the previous energy estimates, by choosing θ small enough, it is not difficult to obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(||\delta u||^2_{L^2}+\theta||\delta\psi||^2_{L^2(\mathcal{L}^2)})+||\partial_1\delta u||^2_{L^2}\\&\quad\le C(||\delta u||^2_{L^2}+\theta||\delta\psi||^2_{L^2(\mathcal{L}^2)}+||v_1-v_2||^2_{L^2}+\theta||\xi_1-\xi_2||^2_{L^2})+\frac{1}{2}||\partial_1(v_1-v_2)||^2_{L^2}.\end{aligned}$$

Hence, by integrating the above inequality in time and choosing T necessarily small, we obtain (32). Finally, by standard fixed-point argument, there exists a unique solution (u,ψ) in $\mathcal{X}$. □

4 Global existence

This section focuses on proving Theorems 1.1 and 1.2. The framework for proving the uniform global bound is the bootstrap argument (see e.g. p.21 of [38]). Our goal is to select the appropriate energy functionals and verify that they satisfy the required energy inequalities. In this section, we will give the proof of Theorem 1.1 in Section 4.1 and the proof of Theorem 1.2 in Section 4.2.

4.1 Mixed partial dissipation

As described in the introduction, our energy functional consists of two parts, the first part is the natural H^2-type energy functional, and the second part incorporates the extra regularizing property revealed by the special wave structure (15). More precisely, we recall that

$$E(t) = E_1(t) + E_2(t),$$

where

$$\begin{aligned} E_1(t) = &\sup_{0\le s\le t} ||u(s)||_{H^2}^2 + \sup_{0\le s\le t} ||\psi(s)||_{H^2(\mathcal{L}^2)}^2 \\ &+ \int_0^t \left\{ ||\partial_1 u(s)||_{H^2}^2 + ||\partial_2 \psi(s)||_{H^2(\mathcal{L}^2)}^2 + ||\psi(s)||_{H^2(\dot{\mathcal{H}}^1)}^2 \right\} \mathrm{d}s, \end{aligned} \tag{33}$$

$$E_2(t) = \int_0^t ||\partial_2 u(s)||_{H^1}^2 \,\mathrm{d}s. \tag{34}$$

The main ingredient of the proof of Theorem 1.1 is the following energy inequalities.

Proposition 4.1 *Let $E_1(t)$ and $E_2(t)$ be the ones as (33) and (34), then there exists a constant $C > 0$ independent of t such that*

$$E_1(t) \le E_1(0) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}, \tag{35}$$

$$E_2(t) \le CE_1(0) + CE_1(t) + CE_1(t)^{\frac{3}{2}} + CE_2(t)^{\frac{3}{2}}. \tag{36}$$

With the above two energy inequalities at hand, we are able to prove Theorem 1.1.

Proof (Proof of Theorem 1.1) We employ the bootstrap argument. It follows from (35) and (36) that

$$E_1(t) + E_2(t) \le C_1 E_1(0) + C_2 (E_1(t) + E_2(t))^{\frac{3}{2}},$$

where C_1 and C_2 are some pure constants. If we make the ansatz that

$$E_1(t) + E_2(t) \le \frac{1}{4C_2^2},$$

then we have

$$E_1(t) + E_2(t) \le 2C_1 E_1(0).$$

Hence, if we take the initial data (u_0, ψ_0) sufficiently small such that

$$E_1(0) = ||u_0||_{H^2}^2 + ||\psi_0||_{H^2(\mathcal{L}^2)}^2 \leq \frac{1}{16C_1C_2^2} := \varepsilon_1^2, \tag{37}$$

then

$$E_1(t) + E_2(t) \leq \frac{1}{8C_2^2}, \tag{38}$$

which is exactly half of our ansatz. Following the bootstrap argument, (38) holds for all $t > 0$ if the initial data satisfies (37), and this completes our proof. □

Now, it suffices to prove Proposition 4.1.

Proof of Proposition 4.1 We first prove (35). Due to the equivalence

$$||u||_{H^2}^2 + ||\psi||_{H^2(\mathcal{L}^2)}^2 \sim ||u||_{L^2}^2 + \sum_{k=1}^{2} ||\partial_k^2 u||_{L^2}^2 + ||\psi||_{L^2(\mathcal{L}^2)}^2 + \sum_{k=1}^{2} ||\partial_k^2 \psi||_{L^2(\mathcal{L}^2)}^2, \tag{39}$$

we consider the L^2-type norm and the homogeneous $\dot{H}^2$-type norm of (u, ψ). We first consider the L^2 estimate. By Standard energy estimates and $\operatorname{div} u = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u||_{L^2}^2 + ||\psi||_{L^2(\mathcal{L}^2)}^2) + ||\partial_1 u||_{L^2}^2 + ||\partial_2 \psi||_{L^2(\mathcal{L}^2)}^2 + ||\psi||_{L^2(\dot{\mathcal{H}}^1)}^2 \\
&\quad = \int_{\mathbb{R}^2} \operatorname{div} \tau \cdot u \,\mathrm{d}x + \iint_{\mathbb{R}^2 B} \operatorname{div}_R(-\nabla u \cdot R\psi_\infty)\psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \\
&\qquad + \iint_{\mathbb{R}^2 B} \operatorname{div}_R(-\nabla u \cdot R\psi)\psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}$$

We start by analyzing the first two terms on the right-hand side of the equation. By integrating by parts and using the fact that $-\frac{\partial_{R_i}\psi_\infty}{\psi_\infty} = \partial_{R_i}\mathcal{U}$ and $\operatorname{div} u = 0$, we obtain that

$$\begin{aligned}
&\iint_{\mathbb{R}^2 B} \operatorname{div}_R(-\nabla u \cdot R\psi_\infty)\psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \\
&\quad = -\iint_{\mathbb{R}^2 B} \operatorname{div} u\, \psi_\infty \psi \,\mathrm{d}R\,\mathrm{d}x + \sum_{i,j=1}^{2} \iint_{\mathbb{R}^2 B} \partial_j u_i R_j \partial_{R_i}\mathcal{U}\psi \,\mathrm{d}R dx \\
&\quad = -\int_{\mathbb{R}^2} \operatorname{div} \tau \cdot u \,\mathrm{d}x.
\end{aligned} \tag{40}$$

Thanks to the above equality, by using integrating by parts in *R* and Sobolev embedding, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u||_{L^2}^2+||\psi||_{L^2(\mathcal{L}^2)}^2)+||\partial_1 u||_{L^2}^2+||\partial_2\psi||_{L^2(\mathcal{L}^2)}^2+||\psi||_{L^2(\dot{\mathcal{H}}^1)}^2\\
&\quad=\iint\limits_{\mathbb{R}^2 B}\nabla u\cdot R\psi\cdot\nabla_R\frac{\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x\\
&\quad\lesssim||\psi||_{H^2(\mathcal{L}^2)}(||\nabla u||_{L^2}^2+||\psi||_{L^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned}\tag{41}$$

Next, we consider the $\dot{H}^2$-type estimates. By applying $\partial_k^2(k=1,2)$ to (4) and multiplying them by $(\partial_k^2 u,\partial_k^2\psi)$ in L^2 and $L^2(\mathcal{L}^2)$ respectively, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\partial_k^2 u||_{L^2}^2+||\partial_k^2\psi||_{L^2(\mathcal{L}^2)}^2)\\
&\quad+||\partial_1\partial_k^2 u||_{L^2}^2+||\partial_2\partial_k^2\psi||_{L^2(\mathcal{L}^2)}^2+||\partial_k^2\psi||_{L^2(\dot{\mathcal{H}}^1)}^2=\sum_{i=1}^{5}I_i,
\end{aligned}\tag{42}$$

where

$$\begin{aligned}
I_1&:=\int_{\mathbb{R}^2}\partial_k^2\mathrm{div}\,\tau\cdot\partial_k^2 u\,\mathrm{d}x,\\
I_2&:=\iint\limits_{\mathbb{R}^2 B}\partial_k^2\mathrm{div}_R(-\nabla u\cdot R\psi_\infty)\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
I_3&:=-\int_{\mathbb{R}^2}\partial_k^2(u\cdot\nabla u)\cdot\partial_k^2 u\,\mathrm{d}x,\\
I_4&:=-\iint\limits_{\mathbb{R}^2 B}\partial_k^2(u\cdot\nabla\psi)\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
I_5&:=\iint\limits_{\mathbb{R}^2 B}\partial_k^2\mathrm{div}_R(-\nabla u\cdot R\psi)\partial_k^2\psi\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}$$

First of all, similar to (40), we directly obtain $I_1+I_2=0$.

Thanks to $\mathrm{div}\,u=0$, we can bound I_3 directly by using Lemma 2.3:

$$\begin{aligned}
I_3&=-\sum_{i=1}^{2}\int_{\mathbb{R}^2}\partial_k^2 u_i\partial_i u\cdot\partial_k^2 u\,\mathrm{d}x-\sum_{i=1}^{2}\int_{\mathbb{R}^2}\partial_k u_i\partial_i\partial_k u\cdot\partial_k^2 u\,\mathrm{d}x\\
&\lesssim||\partial_k^2 u||_{L^2}^{\frac12}||\partial_1\partial_k^2 u||_{L^2}^{\frac12}||\partial_i u||_{L^2}^{\frac12}||\partial_2\ \partial_i u||_{L^2}^{\frac12}||\partial_k^2 u||_{L^2}\\
&\quad+||\partial_k u||_{L^2}^{\frac12}||\partial_2\partial_k u||_{L^2}^{\frac12}||\partial_i\partial_k u||_{L^2}^{\frac12}||\partial_1\ \partial_i\partial_k u||_{L^2}^{\frac12}||\partial_k^2 u||_{L^2}\\
&\lesssim||u||_{H^2}(||\partial_2 u||_{H^1}^2+||\partial_1 u||_{H^2}^2).
\end{aligned}\tag{43}$$

In order to bound I_4, we take advantage of Lemmas 2.3, 2.4, 2.6 and $\mathrm{div}\,u=0$ to obtain

$$
\begin{aligned}
I_4 &= -\sum_{i=1}^{2}\iint_{\mathbb{R}^2 B} \partial_k^2 u_i \partial_i \psi \partial_k^2 \psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x - \sum_{i=1}^{2}\iint_{\mathbb{R}^2 B} \partial_k u_i \partial_i \partial_k \psi \partial_k^2 \psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \\
&\lesssim ||\partial_k^2 u||_{L^2}^{\frac{1}{2}} ||\partial_1 \partial_k^2 u||_{L^2}^{\frac{1}{2}} ||\partial_i \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2\ \partial_i \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_k^2 \psi||_{L^2(\mathcal{L}^2)} \\
&\quad + ||\partial_k u||_{H^1(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_1 \partial_k u||_{H^1(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_i \partial_k \psi||_{L^2(\mathcal{L}^2)} ||\partial_k^2 \psi||_{L^2(\mathcal{L}^2)} \\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_2 u||_{H^1}^2 + ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned} \tag{44}
$$

Now we consider I_5, which is the most complex one to deal with. We let $I_5 = I_{5,1} + I_{5,2}$, where

$$
\begin{aligned}
I_{5,1} &:= \iint_{\mathbb{R}^2 B} \partial_1^2 \mathrm{div}_R(-\nabla u \cdot R\psi)\partial_1^2 \psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x, \\
I_{5,2} &:= \iint_{\mathbb{R}^2 B} \partial_2^2 \mathrm{div}_R(-\nabla u \cdot R\psi)\partial_2^2 \psi \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}
$$

We first consider $I_{5,1}$. By integrating by parts in R and divergence free condition, We further divide $I_{5,1}$ into three parts, namely $I_{5,1} = I_{5,1,1} + I_{5,1,2} + I_{5,1,3}$, where

$$
\begin{aligned}
I_{5,1,1} &:= \sum_{i,j=1}^{2}\iint_{\mathbb{R}^2 B} \partial_j u_i R_j \partial_1^2 \psi \partial_{R_i} \frac{\partial_1^2 \psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x, \\
I_{5,1,2} &:= \sum_{i,j=1}^{2}\iint_{\mathbb{R}^2 B} \partial_1 \partial_j u_i R_j \partial_1 \psi \partial_{R_i} \frac{\partial_1^2 \psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x, \\
I_{5,1,3} &:= \sum_{i,j=1}^{2}\iint_{\mathbb{R}^2 B} \partial_1^2 \partial_j u_i R_j \psi \partial_{R_i} \frac{\partial_1^2 \psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x.
\end{aligned}
$$

Since we have the horizontal dissipation $\partial_1^2 u$, by Lemmas 2.3, 2.4, and Sobolev embedding, we can directly bound the above terms as follows

$$
\begin{aligned}
I_{5,1,1} &\lesssim \sum_{j=1}^{2} ||\partial_j u||_{H^1}^{\frac{1}{2}} ||\partial_1\ \partial_j u||_{H^1}^{\frac{1}{2}} ||\partial_1^2 \psi||_{L^2(\mathcal{L}^2)} ||\partial_1^2 \psi||_{L^2(\dot{\mathcal{H}}^1)} \\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_2 u||_{H^1}^2 + ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2), \\
I_{5,1,2} &\lesssim \sum_{j=1}^{2} ||\partial_1 \partial_j u||_{L^2}^{\frac{1}{2}} ||\partial_1\ \partial_1 \partial_j u||_{L^2}^{\frac{1}{2}} ||\partial_1 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2\ \partial_1 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_1^2 \psi||_{L^2(\dot{\mathcal{H}}^1)} \\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2), \\
I_{5,1,3} &\lesssim \sum_{j=1}^{2} ||\partial_1^2 \partial_j u||_{L^2} ||\psi||_{L^\infty(\mathcal{L}^2)} ||\partial_1^2 \psi||_{L^2(\dot{\mathcal{H}}^1)} \\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned}
$$

Next, we consider $I_{5,2}$. We use integrating by parts and $\operatorname{div} u = 0$ to write $I_{5,2} = I_{5,2,1} + I_{5,2,2}$, where

$$I_{5,2,1} := \sum_{r=0}^{1} \sum_{i,j=1}^{2} \iint_{\mathbb{R}^2 B} \partial_2^r \partial_j u_i R_j \partial_2^{2-r} \psi \partial_{R_i} \frac{\partial_2^2 \psi}{\psi_\infty} \, \mathrm{d}R \, \mathrm{d}x,$$

$$I_{5,2,2} := -\sum_{i,j=1}^{2} \iint_{\mathbb{R}^2 B} \partial_2^2 \partial_j u_i R_j \partial_{R_i} \psi \partial_2^2 \psi \, \frac{\mathrm{d}R}{\psi_\infty} \, \mathrm{d}x.$$

Similar to $I_{5,1,1}$ and $I_{5,1,2}$, we can bound $I_{5,2,1}$ as follows

$$\begin{aligned}
I_{5,2,1} \lesssim & \sum_{j=1}^{2} ||\partial_j u||_{H^1}^{\frac{1}{2}} ||\partial_1 \partial_j u||_{H^1}^{\frac{1}{2}} ||\partial_2^2 \psi||_{L^2(\mathcal{L}^2)} ||\partial_2^2 \psi||_{L^2(\dot{\mathcal{H}}^1)} \\
& + ||\partial_2 \partial_j u||_{L^2}^{\frac{1}{2}} ||\partial_1 \ \partial_2 \partial_j u||_{L^2}^{\frac{1}{2}} ||\partial_2 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2 \ \partial_2 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2^2 \psi||_{L^2(\dot{\mathcal{H}}^1)} \\
\lesssim & ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_2 u||_{H^1}^2 + ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned}$$

We turn our attention to $I_{5,2,2}$, which presents greater challenges due to insufficient vertical dissipation of *u*. Though it is natural to consider integrating by parts and utilize the vertical dissipation of ψ, this approach prevents us from transferring ∂_{R_i} to $\partial_2^3 \psi$ as we did previously, or it will introduce a new term $\partial_{R_i}(\partial_2^3 \psi / \psi_\infty)$, which cannot be bounded by $E_1(t)$ and $E_2(t)$. However, converting $\partial_{R_i} \psi$ into the form of $\psi_\infty^{1/2} \partial_{R_i}(\psi / \psi_\infty)$ introduces a high degree of singularity $\psi / (x \psi_\infty^{1/2})$. To effectively address this term, we will utilize Hardy inequality Lemma 2.8. Following the methodology discussed above, We deal with $I_{5,2,2}$ as follows

$$\begin{aligned}
I_{5,2,2} = & \sum_{i,j=1}^{2} \iint_{\mathbb{R}^2 B} \left\{ \partial_2 \partial_j u_i R_j \left(\psi_\infty^{\frac{1}{2}} \partial_{R_i} \frac{\psi}{\psi_\infty} \right) \psi_\infty^{-\frac{1}{2}} \partial_2^3 \psi - 2k \partial_2 \partial_j u_i R_j \left(\psi_\infty^{-\frac{1}{2}} \frac{\psi}{x} \right) \psi_\infty^{-\frac{1}{2}} \partial_2^3 \psi \right\} \mathrm{d}R \, \mathrm{d}x \\
& + \sum_{i,j=1}^{2} \iint_{\mathbb{R}^2 B} \left\{ \partial_2 \partial_j u_i R_j \left(\psi_\infty^{\frac{1}{2}} \partial_{R_i} \frac{\partial_2 \psi}{\psi_\infty} \right) \psi_\infty^{-\frac{1}{2}} \partial_2^2 \psi - 2k \partial_2 \partial_j u_i R_j \left(\psi_\infty^{-\frac{1}{2}} \frac{\partial_2 \psi}{x} \right) \psi_\infty^{-\frac{1}{2}} \partial_2^2 \psi \right\} \mathrm{d}R \, \mathrm{d}x \\
\lesssim & \sum_{j=1}^{2} ||\partial_2 \partial_j u||_{L^2} ||\partial_2 \psi||_{H^1(\dot{\mathcal{H}}^1)}^{\frac{1}{2}} ||\partial_1 \ \partial_2 \psi||_{H^1(\dot{\mathcal{H}}^1)}^{\frac{1}{2}} ||\partial_2^3 \psi||_{L^2(\mathcal{L}^2)} \\
& + \sum_{j=1}^{2} ||\partial_2 \partial_j u||_{L^2} ||\partial_2 \psi||_{L^2(\dot{\mathcal{H}}^1)}^{\frac{1}{2}} ||\partial_1 \ \partial_2 \psi||_{L^2(\dot{\mathcal{H}}^1)}^{\frac{1}{2}} ||\partial_2^2 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2 \partial_2^2 \psi||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} \\
\lesssim & ||u||_{H^2} (||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2 + ||\partial_2 \psi||_{H^2(\mathcal{L}^2)}^2),
\end{aligned}$$

where in the first equation, we have used integrating by parts and $\operatorname{div} u = 0$, in the first inequality, we have used anisotropic inequalities in Lemmas 2.3, 2.4, as well as Hardy inequality in Lemma 2.8. Adding up all the components of I_5, we obtain that

$$I_5 \lesssim (||u||_{H^2} + ||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1 u||_{H^2}^2 + ||\partial_2 u||_{H^1}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2 + ||\partial_2 \psi||_{H^2(\mathcal{L}^2)}^2). \quad (45)$$

Inserting the upper bounds (43)-(45) into (42), integrating in time and invoking the norm equivalence (39), we find

$$
\begin{aligned}
&||u(t)||_{H^2}^2 + ||\psi(t)||_{H^2(\mathcal{L}^2)}^2 \\
&\quad + \int_0^t \Big\{ ||\partial_1 u(s)||_{H^2}^2 + ||\psi(s)||_{H^2(\dot{\mathcal{H}}^1)}^2 + ||\partial_2 \psi(s)||_{H^2(\mathcal{L}^2)}^2 \Big\} \, \mathrm{d}s \\
&\le ||u(0)||_{H^2}^2 + ||\psi(0)||_{H^2(\mathcal{L}^2)}^2 \\
&\quad + C\big(\sup_{0\le s\le t} ||u||_{H^2} + \sup_{0\le s\le t} ||\psi||_{H^2(\mathcal{L}^2)} \big) \\
&\quad \times \int_0^t \Big\{ ||\partial_1 u||_{H^2}^2 + ||\partial_2 u||_{H^1}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2 + ||\partial_2 \psi||_{H^2(\mathcal{L}^2)}^2 \Big\} \, \mathrm{d}s \\
&\le E_1(0) + C E_1^{\frac{3}{2}}(t) + C E_2^{\frac{3}{2}}(t),
\end{aligned}
$$

and this proves (35).

Next, we take advantage of wave structure and prove (36). Thanks to $\operatorname{div} u = 0$, we have $||\nabla u||_{H_1} = ||\mathbb{D}u||_{H_1}$. We deduce from the equation of $\mathbb{D}u$ in (18) that

$$
||\mathbb{D}u||_{H^1}^2 = (\mathbb{D}u, \mathbb{D}u) + \sum_{l=1}^{2} (\partial_l \mathbb{D}u, \partial_l \mathbb{D}u) = \sum_{k=1}^{5} J_k, \tag{46}
$$

where

$$
\begin{aligned}
J_1 &:= \sum_{j,k=1}^{2} \iint_{\mathbb{R}^2 B} \frac{\mathrm{d}}{\mathrm{d}t} \psi R_j R_k [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x + \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \frac{\mathrm{d}}{\mathrm{d}t} \partial_l \psi R_j R_k \partial_l [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x, \\
J_2 &:= \sum_{j,k=1}^{2} \iint_{\mathbb{R}^2 B} u \cdot \nabla \psi R_j R_k [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x + \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \partial_l (u \cdot \nabla \psi R_j R_k) \partial_l [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x, \\
J_3 &:= - \sum_{j,k=1}^{2} \iint_{\mathbb{R}^2 B} \partial_2^2 \psi R_j R_k [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x - \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \partial_2^2 \partial_l \psi R_j R_k \partial_l [\mathbb{D}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x, \\
J_4 &:= \iint_{\mathbb{R}^2 B} \Big\{ \sum_{j,k=1}^{2} \operatorname{div}_R (\nabla u \cdot R\psi) R_j R_k [\mathbb{D}u]_{j,k} + \sum_{j,k,l=1}^{2} \partial_l \operatorname{div}_R (\nabla u \cdot R\psi) R_j R_k \partial_l [\mathbb{D}u]_{j,k} \, \mathrm{d}R \Big\} \, \mathrm{d}x, \\
J_5 &:= - \iint_{\mathbb{R}^2 B} \Big\{ \sum_{j,k=1}^{2} \mathcal{L}\psi R_j R_k [\mathbb{D}u]_{j,k} + \sum_{j,k,l=1}^{2} \partial_l \mathcal{L}\psi R_j R_k \partial_l [\mathbb{D}u]_{j,k} \Big\} \, \mathrm{d}R \, \mathrm{d}x.
\end{aligned}
$$

To bound J_1, we set $J_1 = J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4}$, where

$$J_{1,1} := \sum_{j,k=1}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^2 B} \psi R_j R_k [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x,$$

$$J_{1,2} := \sum_{j,k,l=1}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^2 B} \partial_l \psi R_j R_k \partial_l [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x,$$

$$J_{1,3} := - \sum_{j,k=1}^{2} \iint_{\mathbb{R}^2 B} \psi R_j R_k \frac{\mathrm{d}}{\mathrm{d}t} [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x,$$

$$J_{1,4} := - \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \partial_l \psi R_j R_k \frac{\mathrm{d}}{\mathrm{d}t} \partial_l [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x.$$

By integrating in time, we have

$$\begin{aligned}
&\int_0^t J_{1,1} \, \mathrm{d}s + \int_0^t J_{1,2} \, \mathrm{d}s \\
&\lesssim ||\psi_0||_{H^1(\mathcal{L}^2)} ||u_0||_{H^2} + \sum_{j,k=1}^{2} \left| \iint_{\mathbb{R}^2 B} \psi R_j R_k [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x \right| \\
&\quad + \sum_{j,k,l=1}^{2} \left| \iint_{\mathbb{R}^2 B} \partial_l \psi R_j R_k \partial_l [\mathbb{D}\mathbb{P}u]_{j,k} \, \mathrm{d}R \, \mathrm{d}x \right| \\
&\lesssim ||u||_{H^2}^2 + ||\psi||_{H^2(\mathcal{L}^2)}^2 + ||u_0||_{H^2}^2 + ||\psi_0||_{H^2(\mathcal{L}^2)}^2.
\end{aligned}$$

By substituting the second equation of (4) into $J_{1,3}$ and $J_{1,4}$, then using Lemmas 2.4 and 2.6, we obtain

$$\begin{aligned}
J_{1,3} + J_{1,4} &\lesssim \sum_{j,k=1}^{2} \left| \iint_{\mathbb{R}^2 B} \psi R_j R_k [\mathbb{D}\mathbb{P}(u \cdot \nabla u)]_{j,k} \, \mathrm{d}R \, \mathrm{d}x \right| \\
&\quad + \sum_{j,k,l=1}^{2} \left| \iint_{\mathbb{R}^2 B} \partial_l^2 \psi R_j R_k [\mathbb{D}(u \cdot \nabla u)]_{j,k} \, \mathrm{d}R \, \mathrm{d}x \right| \\
&\quad + ||\partial_1 u||_{H^2} ||\psi||_{H^2(\dot{\mathcal{H}}^1)} + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2 \\
&\lesssim (||\psi||_{H^2(\mathcal{L}^2)} + ||u||_{H^2})(||\partial_2 u||_{H^1}^2 + ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2) \\
&\quad + ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2.
\end{aligned}$$

Summing up the bounds of $J_{1,i} (i = 1, 2, 3, 4)$ and integrating in t yields that

$$\int_0^t J_1(s)\,\mathrm{d}s \lesssim ||u_0||_{H^2}^2 + ||\psi_0||_{H^2(\mathcal{L}^2)}^2 + \sup_{0\le s\le t} ||u(s)||_{H^2}^2 + \sup_{0\le s\le t} ||\psi(s)||_{H^2(\mathcal{L}^2)}^2$$
$$+ \big(1 + \sup_{0\le s\le t} ||u||_{H^2} + \sup_{0\le s\le t} ||\psi||_{H^2(\mathcal{L}^2)}\big) \int_0^t ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2\,\mathrm{d}s \quad (47)$$
$$+ \big(\sup_{0\le s\le t} ||u||_{H^2} + \sup_{0\le s\le t} ||\psi||_{H^2(\mathcal{L}^2)}\big) \int_0^t ||\partial_2 u||_{H^1}^2\,\mathrm{d}s.$$

By integrating by parts and Lemmas 2.3 and 2.4, we can bound J_2 and J_4 as follows

$$\begin{aligned} J_2 \lesssim & ||u||_{L^\infty} ||\nabla\psi||_{L^2(\mathcal{L}^2)} ||\nabla u||_{L^2} \\ & + ||\nabla u||_{L^2}^{\frac12} ||\partial_1 \nabla u||_{L^2}^{\frac12} ||\nabla\psi||_{L^2(\mathcal{L}^2)}^{\frac12} ||\partial_2\, \nabla\psi||_{L^2(\mathcal{L}^2)}^{\frac12} ||\nabla^2 u||_{L^2} \\ \lesssim & ||u||_{H^2} (||\nabla u||_{H^1}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2), \end{aligned} \quad (48)$$

and

$$\begin{aligned} J_4 = & - \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \nabla u \cdot R\psi \cdot \nabla_R (R_j R_k) [\mathbb{D}u]_{j,k}\,\mathrm{d}R\,\mathrm{d}x \\ & - \sum_{j,k,l=1}^{2} \iint_{\mathbb{R}^2 B} \partial_l (\nabla u \cdot R\psi) \cdot \nabla_R (R_j R_k) \partial_l [\mathbb{D}u]_{j,k}\,\mathrm{d}R\,\mathrm{d}x \\ \lesssim & ||\nabla u||_{L^2}^2 ||\psi||_{L^\infty} + ||\psi||_{L^\infty} ||\nabla^2 u||_{L^2}^2 \\ & + ||\nabla u||_{H^1}^{\frac12} ||\partial_1 \nabla u||_{H^1}^{\frac12} ||\nabla\psi||_{L^2(\mathcal{L}^2)} ||\nabla^2 u||_{L^2} \\ \lesssim & ||\psi|_{H^2(\mathcal{L}^2)} (||\partial_1 u||_{H^2}^2 + ||\partial_2 u||_{H^1}^2). \end{aligned} \quad (49)$$

Finally, by Hölder inequality,

$$J_3 + J_5 \le \frac12 ||\nabla u||_{H^1}^2 + C||\partial_2\psi||_{H^2(\mathcal{L}^2)}^2 + C||\psi||_{H^1(\dot{\mathcal{H}}^1)}^2. \quad (50)$$

Integrating (46) in time and invoking upper bounds in (47)-(50), we obtain

$$\begin{aligned} E_2(t) \le & C||u_0||_{H^2}^2 + C||\psi_0||_{H^2(\mathcal{L}^2)}^2 + C \sup_{0\le s\le t} ||\partial_1 u(s)||_{H^2}^2 + C \sup_{0\le s\le t} ||\psi(s)||_{H^2(\dot{\mathcal{H}}^1)}^2 \\ & + C \int_0^t ||\partial_1 u||_{H^2}^2 + ||\psi||_{H^1(\dot{\mathcal{H}}^1)}^2 + ||\partial_2\psi||_{H^2(\mathcal{L}^2)}^2\,\mathrm{d}s \\ & + C\big(\sup_{0\le s\le t} ||u||_{H^2} + \sup_{0\le s\le t} ||\psi||_{H^2(\mathcal{L}^2)}\big) \int_0^t ||\partial_1 u||_{H^2}^2 + ||\partial_2 u||_{H^1}^2 + ||\psi||_{H^2(\dot{\mathcal{H}}^1)}^2\,\mathrm{d}s \\ \le & CE_1(0) + CE_1(t) + CE_1^{\frac32}(t) + CE_2^{\frac32}(t), \end{aligned}$$

which is exactly (36), and our proof is accomplished. □

4.2 Merely horizontal dissipation

In this part, we consider the global existence and stability of the 2D FENE dumbbell model without any vertical dissipation. To address the challenge posed by the loss of derivative, we introduce the symmetry condition of initial data, namely, ψ_0 is even in R_2. In Section 3, we have shown that this symmetry property holds as long as the solution exists.

Due to the absence of vertical dissipation $\partial_2^2\psi$, the corresponding energy functionals of (6) read as follows

$$\begin{aligned}\mathcal{E}_1(t) &= \sup_{0\le s\le t}||u(s)||_{H^2}^2 + \sup_{0\le s\le t}||\psi(s)||_{H^2(\mathcal{L}^2)}^2 \\ &\quad + \int_0^t \left\{||\partial_1 u(s)||_{H^2}^2 + ||\psi(s)||_{H^2(\dot{\mathcal{H}}^1)}^2\right\} \mathrm{d}s, \\ \mathcal{E}_2(t) &= \int_0^t ||\partial_2 u(s)||_{H^1}^2 \,\mathrm{d}s.\end{aligned}$$

With this suitable energy structure at hand, we are ready to prove Theorem 1.2.

Proof (Proof of Theorem 1.2) In a manner akin to the bootstrap argument used in the proof of Theorem 1.1, we proceed with the L^2-type and $\dot{H}^2$-type estimate to establish Proposition 4.1, replacing E_1 and E_2 with $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively:

$$\mathcal{E}_1(t) \le \mathcal{E}_1(0) + C\mathcal{E}_1(t)^{\frac{3}{2}} + C\mathcal{E}_2(t)^{\frac{3}{2}}, \tag{51}$$

$$\mathcal{E}_2(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1(t)^{\frac{3}{2}} + C\mathcal{E}_2(t)^{\frac{3}{2}}. \tag{52}$$

Similar to (41) and (42), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u||_{L^2}^2 + ||\psi||_{L^2(\mathcal{L}^2)}^2) + ||\partial_1 u||_{L^2}^2 + ||\psi||_{L^2(\dot{\mathcal{H}}^1)}^2 \\ &\quad \lesssim ||\psi||_{H^2(\mathcal{L}^2)}(||\nabla u||_{L^2}^2 + ||\psi||_{L^2(\dot{\mathcal{H}}^1)}^2),\end{aligned} \tag{53}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\partial_k^2 u||_{L^2}^2 + ||\partial_k^2\psi||_{L^2(\mathcal{L}^2)}^2) \\ &\quad + \sum_{k=1}^{2}||\partial_1\partial_k^2 u||_{L^2}^2 + \sum_{k=1}^{2}||\partial_k^2\psi||_{L^2(\dot{\mathcal{H}}^1)}^2 := \sum_{k=1}^{2}\sum_{i=1}^{5} I_i',\end{aligned} \tag{54}$$

where I_1' to I_5' are identical to the corresponding I_1 to I_5 in (42). On the one hand, $I_1' + I_2' = 0$. On the other hand, since the bounds of I_3, I_4 and $I_{5,1}$ in (43)-(45) do not involve the vertical dissipation term $||\partial_2\psi||_{H^2(\mathcal{L}^2)}$, these bounds also can be used to control I_3', I_4' and $I_{5,1}'$. Hence, it suffices to consider $I_{5,2}'$. Using integrating by parts in *R*, we have $I_{5,2}' = I_{5,2,1}' + I_{5,2,2}' + I_{5,2,3}'$, where

$$I'_{5,2,1} := \sum_{r=0}^{2}\sum_{j=1}^{2}\iint_{\mathbb{R}^2 B} \partial_1\partial_2^r u_j R_j \partial_2^{2-r}\psi \partial_{R_1}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x,$$
$$I'_{5,2,2} := \sum_{r=0}^{2}\iint_{\mathbb{R}^2 B} \partial_2\partial_2^r u_1 R_2 \partial_2^{2-r}\psi \partial_{R_1}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x,$$
$$I'_{5,2,3} := \sum_{r=0}^{2}\iint_{\mathbb{R}^2 B} \partial_2\partial_2^r u_2 R_2 \partial_2^{2-r}\psi \partial_{R_2}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x.$$

By using Lemmas 2.3 and 2.4, $I'_{5,2,1}$ can be bounded by

$$I'_{5,2,1} \lesssim (||u||_{H^2} + ||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1 u||^2_{H^2} + ||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}).$$

By $\partial_2 u_2 = -\partial_1 u_1$, $I'_{5,2,3}$ shares the same bound as $I'_{5,2,1}$.

Now, we take advantage of the symmetry property of ψ to eliminate $I'_{5,2,2}$. Thanks to the symmetry property stated in Proposition 3.1,

we know that for $i = 0, 1, 2$, $\partial_2^i\psi$ and $\partial_{R_1}\frac{\partial_2^i\psi}{\psi_\infty}$ are even in R_2. Hence, for almost every $x \in \mathbb{R}^2$ and $t \in [0, T]$,

$$\int_B R_2\partial_2^{2-r}\psi\partial_{R_1}\frac{\partial_2^2\psi}{\psi_\infty}\,\mathrm{d}R = 0,$$

and consequently $I'_{5,2,2} = 0$. Hence, $I'_{5,2}$ can be bounded by

$$I'_{5,2} \lesssim (||u||_{H^2} + ||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1 u||^2_{H^2} + ||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}).$$

Using the arguments parallel to the proof of Theorem 1.1, it is not difficult to achieve (51). Also, since estimate (52) can be treated identically to (36) except for some harmless details, we choose not to repeat them tediously. □

5 Exponential decay

In this section, we deal with the equation of oscillation part $(\tilde{u}, \tilde{\psi})$ and the equation of $\bar{\psi}$, focusing on proving Theorem 1.3. Throughout our proof, the properties of the orthogonal decomposition in Lemma 2.1, Poincaré inequality in Lemma 2.2 and anisotropic inequalities in Lemma 2.5 will be frequently used.

By the definition of $\bar{u}$, we know that $\partial_1\bar{u} = 0$ and

$$\overline{u\cdot\nabla\bar{u}} = \overline{u_1\partial_1\bar{u}} + \overline{u_2\partial_2\bar{u}} = \bar{u}_2\partial_2\bar{u}.$$

Due to the divergence free condition, there exists a stream function $\Psi(x, t)$ associated with u such that

$$u = \nabla^\perp\Psi := (-\partial_2\Psi, \partial_1\Psi),$$

then

$$\bar{u}_2 = \overline{\partial_1 \Psi} = 0.$$

Therefore,

$$\overline{u \cdot \nabla \bar{u}} = 0,$$

and

$$\overline{u \cdot \nabla u} = \overline{u \cdot \nabla \tilde{u}} = \bar{u} \cdot \nabla \bar{\tilde{u}} + \overline{\tilde{u} \cdot \nabla \tilde{u}} = \overline{\tilde{u} \cdot \nabla \tilde{u}}.$$

Similarly,

$$\begin{aligned} &\overline{u \cdot \nabla \psi} = \overline{\tilde{u} \cdot \nabla \tilde{\psi}}, \\ &\mathrm{div}_R(\overline{-\nabla u \cdot R(\widetilde{\psi + \psi_\infty})}) = \mathrm{div}_R(\overline{-\nabla \tilde{u} \cdot R\tilde{\psi}}). \end{aligned}$$

On the other hand, using $\operatorname{div} u = 0$, $\partial_1 \bar{u} = 0$ and $\bar{u}_2 = 0$, we have

$$\begin{aligned} \mathrm{div}_R(\overline{-\nabla u \cdot R\bar{\psi}}) &= -\sum_{i,j=1}^{2} \partial_j \bar{u}_i R_j \partial_{R_i} \bar{\psi} = -\partial_2 \bar{u}_1 R_2 \partial_{R_1} \bar{\psi}, \\ \mathrm{div}_R(\overline{-\nabla u \cdot R\psi_\infty}) &= -\sum_{i,j=1}^{2} \partial_j \bar{u}_i R_j \partial_{R_i} \psi_\infty = -\partial_2 \bar{u}_1 R_2 \partial_{R_1} \psi_\infty. \end{aligned}$$

Now, we deduce the equation of $(\bar{u}, \bar{\psi})$. Since ψ is even in R_2, we have $\tau_{1,2} = \tau_{2,1} = 0$. Taking the x_1-average of (6) and using the above properties, we deduce that

$$\begin{cases} \partial_t \bar{u} + \overline{\tilde{u} \cdot \nabla \tilde{u}} = \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} + \partial_2 \begin{pmatrix} 0 \\ \bar{\tau}_{2,2} \end{pmatrix}, \\ \partial_t \bar{\psi} + \overline{\tilde{u} \cdot \nabla \tilde{\psi}} = -\mathrm{div}_R(\overline{\nabla \tilde{u} \cdot R\tilde{\psi}}) - \partial_2 \bar{u}_1 R_2 \partial_{R_1}(\bar{\psi} + \psi_\infty) + \mathcal{L}\bar{\psi}. \end{cases} \tag{55}$$

By the result stated in Theorem 1.3, the oscillation part $(\tilde{u}, \tilde{\psi})$ will eventually decay to (0, 0), and the velocity *u* in system (6) will turn to the first equation of the above one-dimensional system of average part. To establish the decay result, we first prove the decay of the oscillation part $(\tilde{u}, \tilde{\psi})$, as shown in Proposition 5.2. Then, we use this result to prove the decay of the average part $\bar{\psi}$ in Proposition 5.3.

Before proving the exponential decay, we first state the global existence and stability of the solution in $\Omega = \mathbb{T} \times \mathbb{R}$.

Proposition 5.1 *Suppose that the assumptions of Theorem 1.3 hold, then (6) has a unique solution (u, ψ), and ψ is even in R_2. In addition, for all $t > 0$, (u, ψ) satisfies*

$$||u(t)||_{H^2}^2 + ||\psi(t)||_{H^2(\mathcal{L}^2)}^2 + \int_0^t \left\{ ||\partial_1 u(s)||_{H^2}^2 + ||\partial_2 u(s)||_{H^1}^2 + ||\psi(s)||_{H^2(\dot{\mathcal{H}}^1)}^2 \right\} \mathrm{d}s \leq C\varepsilon_3^2.$$

Since the existence result can be proven in a manner almost identical to the $\mathbb{R}^2$ case, we choose to omit the proof.

5.1 Decay of oscillation part

The decay of the oscillation part $(\tilde{u}, \tilde{\psi})$ reads as follows.

Proposition 5.2 *Suppose that the assumptions of Theorem 1.3 hold, then* $(\tilde{u}, \tilde{\psi})$ *satisfies*

$$||\tilde{u}||_{H^1} + ||\tilde{\psi}||_{H^1(\mathcal{L}^2)} \leq (||\tilde{u}_0||_{H^1} + ||\tilde{\psi}_0||_{H^1(\mathcal{L}^2)})e^{-c_1' t},$$

for some constant $c_1' > 0$ and for all $t > 0$.

Proof To demonstrate the exponential decay of the oscillation part, we first deduce the equations of $(\tilde{u}, \tilde{\psi})$. Taking the average of (6) yields

$$\begin{cases} \partial_t \bar{u} + \overline{(u \cdot \nabla)u} = \partial_1^2 \bar{u} - \nabla \bar{p} + \operatorname{div} \bar{\tau}, \quad \operatorname{div} \bar{u} = 0, \\ \partial_t \bar{\psi} + \overline{u \cdot \nabla \psi} = \operatorname{div}_R \left[-\overline{\nabla u \cdot R(\psi + \psi_\infty)} + \psi_\infty \nabla_R \frac{\bar{\psi}}{\psi_\infty} \right]. \end{cases} \tag{56}$$

Taking the difference of (6) and (56), we obtain that

$$\begin{cases} \partial_t \tilde{u} + \widetilde{u \cdot \nabla u} = \partial_1^2 \tilde{u} - \nabla \tilde{p} + \operatorname{div} \tilde{\tau}, \quad \operatorname{div} \tilde{u} = 0, \\ \partial_t \tilde{\psi} + \widetilde{u \cdot \nabla \psi} = \operatorname{div}_R \left[-\big(\nabla u \cdot R(\psi + \psi_\infty)\big)^{\sim} + \psi_\infty \nabla_R \frac{\tilde{\psi}}{\psi_\infty} \right]. \end{cases} \tag{57}$$

By the standard L^2 energy method, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (||\tilde{u}||_{L^2}^2 + ||\tilde{\psi}||_{L^2(\mathcal{L}^2)}^2) + ||\tilde{\partial}_1 u||_{L^2}^2 + ||\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2 = \sum_{i=1}^5 R_i, \tag{58}$$

where

$$R_1 := -\int_\Omega \widetilde{u\cdot\nabla u}\tilde{u}\,\mathrm{d}x,$$
$$R_2 := \int_\Omega \operatorname{div}\tilde{\tau}\tilde{u}\,\mathrm{d}x,$$
$$R_3 := -\iint\limits_{\Omega B} \widetilde{u\cdot\nabla\psi}\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,$$
$$R_4 := -\iint\limits_{\Omega B} \operatorname{div}_R(\nabla u\cdot R\psi)^{\sim}\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,$$
$$R_5 := -\iint\limits_{\Omega B} \operatorname{div}_R(\nabla\tilde{u}\cdot R\psi_\infty)\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.$$

By divergence free condition and the orthogonal property in Lemma 2.1,

$$R_1 = \int_\Omega -u\cdot\nabla\tilde{u}\tilde{u}\,\mathrm{d}x + \int_\Omega \overline{u\cdot\nabla\tilde{u}}\tilde{u}\,\mathrm{d}x - \int_\Omega \widetilde{u\cdot\nabla\bar{u}}\tilde{u}\,\mathrm{d}x = -\int_\Omega \widetilde{u\cdot\nabla\bar{u}}\tilde{u}\,\mathrm{d}x.$$

Since $\nabla\bar{u}$ and $\partial_1\bar{u}$ are independent of x_1, we have

$$R_1 = -\int_\Omega \widetilde{u\cdot\nabla\bar{u}}\tilde{u}\,\mathrm{d}x = -\int_\Omega \tilde{u}\cdot\nabla\bar{u}\tilde{u}\,\mathrm{d}x = -\int_\Omega \tilde{u}_2\partial_2\bar{u}\tilde{u}\,\mathrm{d}x.$$

Hence, using Lemmas 2.2 and 2.5, we can bound R_1 by

$$\begin{aligned} R_1 &\lesssim ||\tilde{u}_2||_{L^2}^{\frac{1}{2}}||\partial_1\tilde{u}||_{L^2}^{\frac{1}{2}}||\partial_2\bar{u}||_{L^2}^{\frac{1}{2}}||\partial_2\partial_2\bar{u}||_{L^2}^{\frac{1}{2}}||\tilde{u}||_{L^2} \\ &\lesssim ||u||_{H^2}||\partial_1\tilde{u}||_{L^2}^2. \end{aligned} \tag{59}$$

For R_2 and R_5, base on the argument in (40), we have

$$R_2 + R_5 = 0. \tag{60}$$

Similar to the argument in R_1, we can bound R_3 by

$$\begin{aligned} R_3 &= -\iint\limits_{\Omega B} \tilde{u}\cdot\nabla\bar{\psi}\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \\ &\lesssim ||\tilde{u}||_{L^2}^{\frac{1}{2}}||\partial_1\tilde{u}||_{L^2}^{\frac{1}{2}}||\nabla\bar{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}}||\partial_2\,\nabla\bar{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}}||\tilde{\psi}||_{L^2(\mathcal{L}^2)} \\ &\lesssim ||\psi||_{H^2(\mathcal{L}^2)}(||\partial_1\tilde{u}||_{L^2}^2 + ||\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2). \end{aligned} \tag{61}$$

To bound R_4, we use integrating by parts and Lemma 2.1 to decompose R_4 into two parts

$$R_4 = \iint\limits_{\Omega B} \nabla u\cdot R\tilde{\psi}\nabla_R\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x + \iint\limits_{\Omega B} \nabla\tilde{u}\cdot R\bar{\psi}\nabla_R\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x.$$

Recall the symmetry property stated in Proposition 5.1, for almost every *x* and *t*,

$$\int_B R_1\tilde{\psi}\partial_{R_2}\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R=\int_B R_2\tilde{\psi}\partial_{R_1}\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R=\int_B R_1\bar{\psi}\partial_{R_2}\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R=\int_B R_2\bar{\psi}\partial_{R_1}\frac{\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R=0.$$

The above equation together with divergence free condition $\partial_2 u_2=-\partial_1 u_1$, $\partial_1\bar{u}_1=0$ and Sobolev embedding yields

$$\begin{aligned}R_4=&\iint\limits_{\Omega B}\partial_1\tilde{u}_1(R_1\tilde{\psi}\partial_{R_1}\frac{\tilde{\psi}}{\psi_\infty}-R_2\tilde{\psi}\partial_{R_2}\frac{\tilde{\psi}}{\psi_\infty})\,\mathrm{d}R\,\mathrm{d}x\\&+\iint\limits_{\Omega B}\partial_1\tilde{u}_1(R_1\bar{\psi}\partial_{R_1}\frac{\tilde{\psi}}{\psi_\infty}-R_2\bar{\psi}\partial_{R_2}\frac{\tilde{\psi}}{\psi_\infty})\,\mathrm{d}R\,\mathrm{d}x\\&\lesssim||\psi||_{H^2(\mathcal{L}^2)}(||\partial_1\tilde{u}||_{L^2}^2+||\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2).\end{aligned}\tag{62}$$

Substituting (59)-(62) into (58), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(||\tilde{u}||_{L^2}^2+||\tilde{\psi}||_{L^2(\mathcal{L}^2)}^2)+2(1-C||u||_{H^2}-C||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1\tilde{u}||_{L^2}^2+||\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2)\leq 0.\tag{63}$$

We can deduce from Proposition 5.1 that if $\varepsilon_3>0$ is sufficiently small and $||u_0||_{H^2}^2+||\psi_0||_{H^2(\mathcal{L}^2)}\leq\varepsilon_3$, then

$$C||u||_{H^2}^2+C||\psi||_{H^2(\mathcal{L}^2)}\leq c<1.$$

Therefore, we can use Poincaré inequalities in Lemmas 2.2 and 2.6 to obtain

$$||\tilde{u}(t)||_{L^2}+||\tilde{\psi}(t)||_{L^2(\mathcal{L}^2)}\leq(||\tilde{u}(0)||_{L^2}+||\tilde{\psi}(0)||_{L^2(\mathcal{L}^2)})e^{-c_{11}'t},$$

where $c_{11}'=c_{11}'(\varepsilon_3)>0$.

Next we consider the exponential decay of $||\nabla\tilde{u}||_{L^2}$ and $||\nabla\tilde{\psi}||_{L^2(\mathcal{L}^2)}$. By applying ∂_k $(k=1,2)$ to (57), we have

$$\begin{cases}\partial_t\partial_k\tilde{u}+\partial_k\widetilde{u\cdot\nabla u}=\partial_k\partial_1^2\tilde{u}-\partial_k\nabla\tilde{p}+\partial_k\mathrm{div}\,\tilde{\tau}, \quad \mathrm{div}\,\tilde{u}=0,\\ \partial_t\partial_k\tilde{\psi}+\partial_k\widetilde{u\cdot\nabla\psi}=\partial_k\mathrm{div}_R\left[-\left(\nabla u\cdot R(\psi+\psi_\infty)\right)^{\sim}+\psi_\infty\nabla_R\frac{\tilde{\psi}}{\psi_\infty}\right].\end{cases}\tag{64}$$

Integrating (64) with $(\partial_k\tilde{u},\partial_k\tilde{\psi})$, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\partial_k\tilde{u}||_{L^2}^2+||\partial_k\tilde{\psi}||_{L^2(\mathcal{L}^2)}^2)+||\partial_k\partial_1\tilde{u}||_{L^2}^2+||\partial_k\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2=\sum_{i=6}^{10}R_i,\tag{65}$$

where

$$
\begin{aligned}
R_6 &:= -\int_\Omega \partial_k(\widetilde{u\cdot\nabla u})\partial_k\tilde{u}\,\mathrm{d}x,\\
R_7 &:= \int_\Omega \partial_k \mathrm{div}\,\tilde{\tau}\partial_k\tilde{u}\,\mathrm{d}x,\\
R_8 &:= -\iint_{\Omega B} \partial_k(\widetilde{u\cdot\nabla\psi})\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
R_9 &:= -\iint_{\Omega B} \partial_k \mathrm{div}_R(\nabla u\cdot R\psi)^{\sim}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
R_{10} &:= -\iint_{\Omega B} \mathrm{div}_R(\partial_k\nabla\tilde{u}\cdot R\psi_\infty)\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}
$$

Similar to the reasoning in R_1, we bound R_6 by

$$
\begin{aligned}
R_6 =& -\int_\Omega \partial_k u\cdot\nabla\tilde{u}\partial_k\tilde{u}\,\mathrm{d}x - \int_\Omega \partial_k\tilde{u}\cdot\nabla\bar{u}\partial_k\tilde{u}\,\mathrm{d}x - \int_\Omega \tilde{u}\cdot\nabla\partial_k\bar{u}\partial_k\tilde{u}\,\mathrm{d}x\\
\lesssim& ||u||_{H^2}||\partial_1\nabla\tilde{u}||_{L^2}^2.
\end{aligned}
\tag{66}
$$

For R_7 and R_{10}, we use the argument in (40) to obtain

$$
R_7 + R_{10} = 0. \tag{67}
$$

To bound R_8, we use Lemma 2.1 and divergence free condition to rewrite it as $R_8 = R_{8,1} + R_{8,2}$, where

$$
\begin{aligned}
R_{8,1} &:= \iint_{\Omega B} \partial_k u\cdot\nabla\tilde{\psi}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
R_{8,2} &:= \iint_{\Omega B} \partial_k(\tilde{u}\cdot\nabla\bar{\psi})\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}
$$

Since $\tilde{\psi}$ does not have any center-of-mass diffusion, to bound $R_{8,1}$, we further rewrite it as $R_{8,1} = R_{8,1,1} + R_{8,1,2}$ where

$$
\begin{aligned}
R_{8,1,1} &:= \iint_{\Omega B} \partial_k\tilde{u}\cdot\nabla\tilde{\psi}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
R_{8,1,2} &:= \iint_{\Omega B} \partial_k\bar{u}\cdot\nabla\tilde{\psi}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}
$$

Using Lemmas 2.3 and 2.6, $R_{8,1,1}$ can be bounded directly as

$$
\begin{aligned}
R_{8,1,1} &\lesssim ||\partial_k\tilde{u}||_{L^2}^{\frac{1}{2}}||\partial_1\partial_k\tilde{u}||_{L^2}^{\frac{1}{2}}||\nabla\tilde{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}}||\partial_2\nabla\tilde{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}}||\partial_k\tilde{\psi}||_{L^2(\mathcal{L}^2)}\\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)}(||\partial_1\partial_k\tilde{u}||_{L^2}^2 + ||\partial_k\tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned}
$$

For $R_{8,1,2}$, notice that $\bar{u}$ is only dependent on x_2, and for any one dimensional function $f \in H^1(\mathbb{R})$,

$$||f||_{L^\infty(\mathbb{R})} \leq \sqrt{2}||f||^{\frac{1}{2}}_{L^2(\mathbb{R})}||f'||^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{68}$$

Hence, we apply Lemma 2.6 and (68) to obtain

$$\begin{aligned} R_{8,1,2} &\leq ||\partial_k \bar{u}||_{L^\infty}||\nabla\tilde{\psi}||_{L^2(\mathcal{L}^2)}||\partial_k\tilde{\psi}||_{L^2(\mathcal{L}^2)} \\ &\leq ||u||_{H^2}||\partial_k\tilde{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}. \end{aligned}$$

Collecting the above bounds of $R_{8,1,1}$ and $R_{8,1,2}$ yields that

$$R_{8,1} \lesssim (||u||_{H^2} + ||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1\partial_k\tilde{u}||^2_{L^2} + ||\partial_k\tilde{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}). \tag{69}$$

For $R_{8,2}$, we infer from Lemmas 2.2, 2.5 and (68) that

$$\begin{aligned} R_{8,2} &= \iint\limits_{\Omega B} \partial_k\tilde{u}\cdot\nabla\bar{\psi}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x + \iint\limits_{\Omega B} \tilde{u}\cdot\nabla\partial_k\bar{\psi}\partial_k\tilde{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \\ &\lesssim ||\partial_k\tilde{u}||^{\frac{1}{2}}_{L^2}||\partial_1\partial_k\tilde{u}||^{\frac{1}{2}}_{L^2}||\nabla\bar{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||\partial_2\nabla\bar{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||||\partial_k\tilde{\psi}||_{L^2(\mathcal{L}^2)} \\ &\quad + ||\tilde{u}||^{\frac{1}{2}}_{L^2}||\partial_2\tilde{u}||^{\frac{1}{2}}_{L^2}||\partial_k\nabla\bar{\psi}||_{L^2(\mathcal{L}^2)}||\partial_k\tilde{\psi}||_{L^2(\mathcal{L}^2)} \\ &\lesssim ||\psi||_{H^2(\mathcal{L}^2)}(||\partial_1\partial_k\tilde{u}||^2_{L^2} + ||\partial_k\tilde{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}). \end{aligned} \tag{70}$$

Combining (69) with (70), we find

$$R_8 \lesssim (||u||_{H^2} + ||\psi||_{H^2(\mathcal{L}^2)})(||\partial_1\partial_k\tilde{u}||^2_{L^2} + ||\partial_k\tilde{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}). \tag{71}$$

We now proceed to estimate R_9, which presents greater difficulties. Using Lemma 2.1 and integrating by parts, we divide R_9 by

$$R_9 = R_{9,1} + R_{9,2} + R_{9,3}, \tag{72}$$

where

$$\begin{aligned} R_{9,1} &:= \iint\limits_{\Omega B} \partial_k(\nabla\tilde{u}\cdot R\tilde{\psi})\nabla_R\frac{\partial_k\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x, \\ R_{9,2} &:= \iint\limits_{\Omega B} \partial_k(\nabla\bar{u}\cdot R\tilde{\psi})\nabla_R\frac{\partial_k\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x, \\ R_{9,3} &:= \iint\limits_{\Omega B} \partial_k(\nabla\tilde{u}\cdot R\bar{\psi})\nabla_R\frac{\partial_k\tilde{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x. \end{aligned} \tag{73}$$

First, we Lemma 2.1 and symmetry of ψ to proof $R_{9,2} = 0$. Recall that $\bar{u}_2 = 0$, then

$$
\begin{aligned}
R_{9,2} &= \sum_{j=1}^{2} \iint_{\Omega B} \partial_k(\partial_2 \bar{u}_j R_j \tilde{\psi}) \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty} \, \mathrm{d}R \, \mathrm{d}x \\
&= \iint_{\Omega B} \partial_k(\partial_2 \bar{u}_1 R_1 \tilde{\psi}) \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty} \, \mathrm{d}R \, \mathrm{d}x.
\end{aligned}
$$

However, we obtain from the symmetry property of ψ that

$$
\int_B R_1 \tilde{\psi} \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty} \, \mathrm{d}R = \int_B R_1 \partial_k \tilde{\psi} \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty} \, \mathrm{d}R = 0.
$$

Hence, $R_{9,2} = 0$.

Next, by using the symmetry property of ψ, $\partial_2 u_2 = -\partial_1 u_1$ and Lemma 2.5, we can directly bound $R_{9,1}$ as

$$
\begin{aligned}
R_{9,1} &= \iint_{\Omega B} \partial_1 \partial_k \tilde{u}_1 \tilde{\psi} (R_1 \partial_{R_1} \frac{\partial_k \tilde{\psi}}{\psi_\infty} - R_2 \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty}) \, \mathrm{d}R \, \mathrm{d}x \\
&\quad + \iint_{\Omega B} \partial_1 \tilde{u}_1 \partial_k \tilde{\psi} (R_1 \partial_{R_1} \frac{\partial_k \tilde{\psi}}{\psi_\infty} - R_2 \partial_{R_2} \frac{\partial_k \tilde{\psi}}{\psi_\infty}) \, \mathrm{d}R \, \mathrm{d}x \\
&\lesssim ||\partial_1 \partial_k \tilde{u}||_{L^2} ||\tilde{\psi}||_{L^\infty(\mathcal{L}^2)} ||\partial_k \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)} \\
&\quad + ||\partial_1 \tilde{u}||_{L^2}^{\frac{1}{2}} ||\partial_1 \partial_1 \tilde{u}||_{L^2}^{\frac{1}{2}} ||\partial_k \tilde{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_2 \partial_k \tilde{\psi}||_{L^2(\mathcal{L}^2)}^{\frac{1}{2}} ||\partial_k \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)} \\
&\lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_1 \nabla \tilde{u}||_{L^2}^2 + ||\partial_k \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2).
\end{aligned}
\tag{74}
$$

Also, parallel to $R_{9,1}$, we can bound $R_{9,3}$ by

$$
R_{9,3} \lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_1 \nabla \tilde{u}||_{L^2}^2 + ||\partial_k \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2). \tag{75}
$$

Plugging the upper bounds in (74) and (75) into (72) gives

$$
R_9 \lesssim ||\psi||_{H^2(\mathcal{L}^2)} (||\partial_1 \nabla \tilde{u}||_{L^2}^2 + ||\partial_k \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2). \tag{76}
$$

Substituting (66), (67), (71) and (76) into (65), we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} (||\nabla \tilde{u}||_{L^2}^2 + ||\nabla \tilde{\psi}||_{L^2(\mathcal{L}^2)}^2) \\
&+ 2(1 - C||u||_{H^2} - C||\psi||_{H^2(\mathcal{L}^2)}) (||\partial_1 \nabla \tilde{u}||_{L^2}^2 + ||\nabla \tilde{\psi}||_{L^2(\dot{\mathcal{H}}^1)}^2) \leq 0.
\end{aligned}
\tag{77}
$$

Similar to the previous L^2 argument, if the initial data is sufficiently small, we can use the Poincaré inequalities in Lemmas 2.2 and 2.6 to obtain

$$
||\nabla \tilde{u}(t)||_{L^2} + ||\nabla \tilde{\psi}(t)||_{L^2(\mathcal{L}^2)} \leq (||\nabla \tilde{u}_0||_{H^1} + ||\nabla \tilde{\psi}_0||_{H^1(\mathcal{L}^2)}) e^{-c'_{12} t},
$$

where $c'_{12} = c'_{12}(\varepsilon_3) > 0$. By taking $c'_1 = \min\{c'_{11}, c'_{12}\}$, and the proof is accomplished. □

5.2 Decay of average part

Once we have established the decay of the oscillation part $(\tilde{u},\tilde{\psi})$, we can leverage this property, in conjunction with the dissipative effect of the operator $\mathcal{L}$, to demonstrate the decay of the average component $\bar{\psi}$. Here, the symmetry of ψ is pivotal in eliminating the linear term.

Proposition 5.3 *Suppose that the assumptions of Theorem 1.3 hold, then $\bar{\psi}$ satisfies*

$$||\bar{\psi}||_{H^1(\mathcal{L}^2)} \le (||\psi_0||_{H^1(\mathcal{L}^2)})e^{-c_2' t}, \tag{78}$$

for some constant $c_2' > 0$ and for all $t > 0$.

Proof Recall from the second equation of (55) that the average part $\bar{\psi}$ satisfies

$$\partial_t\bar{\psi} + \overline{\tilde{u}\cdot\nabla\tilde{\psi}} = \mathrm{div}_R(\overline{-\nabla\tilde{u}\cdot R\tilde{\psi}}) - \partial_2\bar{u}_1 R_2\partial_{R_1}(\bar{\psi}+\psi_\infty) + \mathcal{L}\bar{\psi}.$$

Integrating the above equation with $\bar{\psi}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\bar{\psi}||^2_{L^2(\mathcal{L}^2)} + ||\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} = S_7 + S_8 + S_9, \tag{79}$$

where

$$\begin{aligned}
S_7 &:= -\iint_{\Omega B} \overline{\tilde{u}\cdot\nabla\tilde{\psi}}\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
S_8 &:= \iint_{\Omega B} \mathrm{div}_R(\overline{-\nabla\tilde{u}\cdot R\tilde{\psi}})\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
S_9 &:= \iint_{\Omega B} -\partial_2\bar{u}_1 R_2\partial_{R_1}(\bar{\psi}+\psi_\infty)\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}$$

Using Fubini-Tonelli Theorem, integrating by parts, Lemmas 2.5 and 2.6,

$$\begin{aligned}
S_7 &= \iint_{\Omega B} \tilde{u}\cdot\nabla\tilde{\psi}\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x \lesssim ||\tilde{u}||^2_{L^2} + ||\tilde{\psi}||^2_{H^2(\mathcal{L}^2)}||\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)},\\
S_8 &= \iint_{\Omega B} -\nabla\tilde{u}\cdot R\tilde{\psi}\nabla_R\frac{\bar{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x \lesssim ||\nabla\tilde{u}||^2_{L^2} + ||\tilde{\psi}||^2_{H^1(\mathcal{L}^2)}||\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}.
\end{aligned}$$

For S_9, we deduce from the symmetry property of ψ that

$$S_9 = \iint_{\Omega B} \partial_2\bar{u}_1 R_2(\bar{\psi}+\psi_\infty)\partial_1\frac{\bar{\psi}}{\psi_\infty}\,\mathrm{d}R\,\mathrm{d}x = 0.$$

Hence, substituting the estimates of S_7, S_8 and S_9 into (79), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}||\bar{\psi}||^2_{L^2(\mathcal{L}^2)} \leq -(2 - C||\tilde{\psi}||^2_{H^1(\mathcal{L}^2)})||\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} + C||\tilde{u}||^2_{H^1}. \tag{80}$$

Thanks to Propositions 5.1 and 5.2, we have

$$-(2 - C||\tilde{\psi}||^2_{H^1(\mathcal{L}^2)})||\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} \leq -c||\bar{\psi}||^2_{L^2(\mathcal{L}^2)},$$

and

$$||\tilde{u}(t)||^2_{H^1} \leq (||\tilde{u}_0||^2_{H^1} + ||\tilde{\psi}_0||^2_{H^1(\mathcal{L}^2)})e^{-c_1' t},$$

together with Lemmas 2.1 and 2.6, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}||\bar{\psi}||^2_{L^2(\mathcal{L}^2)} + c||\bar{\psi}||^2_{L^2(\mathcal{L}^2)} \leq C(||\tilde{u}_0||^2_{H^1} + ||\tilde{\psi}_0||^2_{H^1(\mathcal{L}^2)})e^{-c_1' t}.$$

Hence, by the comparison principle of ODE, there exists a constant $c_{21}' > 0$ such that

$$||\bar{\psi}||_{L^2(\mathcal{L}^2)} \leq C(||u_0||_{H^1} + ||\psi_0||_{H^1(\mathcal{L}^2)})e^{-c_{21}' t}.$$

Next, we consider the $\dot{H}^1$-type estimate of $\bar{\psi}$. By applying ∂_2 to (79), and integrating with $\partial_2\bar{\psi}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial_2\bar{\psi}||^2_{L^2(\mathcal{L}^2)} + ||\partial_2\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} = S_{10} + S_{11} + S_{12}, \tag{81}$$

where

$$\begin{aligned}
S_{10} &:= -\iint\limits_{\Omega B} \partial_2(\overline{\tilde{u}\cdot\nabla\tilde{\psi}})\partial_2\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
S_{11} &:= \iint\limits_{\Omega B} \partial_2 \mathrm{div}_R(\overline{-\nabla\tilde{u}\cdot R\tilde{\psi}})\partial_2\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
S_{12} &:= \iint\limits_{\Omega B} -\partial_2(\partial_2\bar{u}_1 R_2 \partial_{R_1}(\bar{\psi}+\psi_\infty))\partial_2\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}$$

For S_{10}, we use $\mathrm{div}\, u = 0$, Fubini-Tonelli Theorem, Lemmas 2.5 and 2.6 to obtain

$$\begin{aligned}
S_{10} &= \iint\limits_{\Omega B} \partial_2\tilde{u}\cdot\nabla\tilde{\psi}\partial_2\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x + \iint\limits_{\Omega B} \tilde{u}\cdot\nabla\partial_2\tilde{\psi}\partial_2\bar{\psi}\,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x\\
&\lesssim ||\partial_2\tilde{u}||_{L^2}||\nabla\tilde{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||\partial_2\nabla\tilde{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||\partial_2\bar{\psi}||_{L^2(\mathcal{L}^2)}\\
&\quad + ||\tilde{u}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||\partial_2\tilde{u}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)}||\tilde{\psi}||_{H^2(\mathcal{L}^2)}||\partial_2\bar{\psi}||_{L^2(\mathcal{L}^2)}\\
&\lesssim ||\tilde{u}||^2_{H^1} + ||\tilde{\psi}||^2_{H^2(\mathcal{L}^2)}||\partial_2\bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)},
\end{aligned}$$

when applying the anisotropic inequality, we use the fact that $\partial_1\partial_2\bar{\psi} = 0$.

For S_{11}, by $\operatorname{div} u = 0$, Fubini-Tonelli Theorem, symmetry property of ψ and Lemma 2.5

$$\begin{aligned} S_{11} =& \iint\limits_{\Omega B} \partial_2 \nabla \tilde{u} \cdot R \tilde{\psi} \nabla_R \frac{\partial_2 \bar{\psi}}{\psi_\infty} \,\mathrm{d}R\,\mathrm{d}x + \iint\limits_{\Omega B} \nabla \tilde{u} \cdot R \partial_2 \tilde{\psi} \nabla_R \frac{\partial_2 \bar{\psi}}{\psi_\infty} \,\mathrm{d}R\,\mathrm{d}x \\ \lesssim& ||\partial_2\partial_1 \tilde{u}||_{L^2(\mathcal{L}^2)} ||\tilde{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)} ||\partial_2 \tilde{\psi}||^{\frac{1}{2}}_{L^2(\mathcal{L}^2)} ||\partial_2 \bar{\psi}||_{L^2(\dot{\mathcal{H}}^1)} \\ &+ ||\partial_1 \tilde{u}||^{\frac{1}{2}}_{L^2} ||\partial_2\partial_1 \tilde{u}||^{\frac{1}{2}}_{L^2} ||\partial_2 \tilde{\psi}||_{L^2(\mathcal{L}^2)} ||\partial_2 \bar{\psi}||_{L^2(\dot{\mathcal{H}}^1)} \\ \lesssim& ||\tilde{\psi}||^2_{H^1(\mathcal{L}^2)} + ||\tilde{\psi}||^2_{H^2(\mathcal{L}^2)} ||\partial_2 \bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)}. \end{aligned}$$

By the symmetry of ψ, we have $S_{12} = 0$.

We plug the estimates of S_{10}, S_{11} and S_{12} into (81) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} ||\partial_2 \bar{\psi}||^2_{L^2(\mathcal{L}^2)} \leq -(2 - C||\tilde{\psi}||^2_{H^2(\mathcal{L}^2)}) ||\partial_2 \bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} + C(||\tilde{u}||^2_{H^1} + ||\tilde{\psi}||^2_{H^1(\mathcal{L}^2)}). \quad (82)$$

By Lemma 2.6, Propositions 5.1 and 5.2, for sufficiently small ε_3, we have

$$\begin{aligned} &-(2 - C||\tilde{\psi}||^2_{H^2(\mathcal{L}^2)}) ||\partial_2 \bar{\psi}||^2_{L^2(\dot{\mathcal{H}}^1)} \leq -c||\partial_2 \bar{\psi}||^2_{L^2(\mathcal{L}^2)}, \\ &||\tilde{u}(t)||^2_{H^1} + ||\tilde{\psi}(t)||^2_{H^1(\mathcal{L}^2)} \leq (||\tilde{u}_0||^2_{H^1} + ||\tilde{\psi}_0||^2_{H^1(\mathcal{L}^2)}) e^{-c_1' t}. \end{aligned}$$

Thus, following a similar argument as before, we establish the exponential decay of $||\partial_2 \bar{\psi}||_{L^2(\mathcal{L}^2)}$. Then, by the equivalence of norm $||\bar{f}||_{H^1} \sim ||\bar{f}||_{L^2} + ||\partial_2 \bar{f}||_{L^2}$, we arrive at the result of (78). □

With Propositions 5.2 and 5.3 at hand, the decay property stated in Theorem 1.3 comes immediately.

Proof of Theorem 1.3 By combining Proposition 5.2 with Proposition 5.3, and choosing $c' = \min\{c_1', c_2'\}$, we can directly derive the result stated in Theorem 1.3. □

6 Vanishing viscosity limit

In this section, we focus on the vanishing viscosity limit problem of (9), our result indicates that when $\kappa \to 0$, system (9) converges to (6) in H^1 sense. Throughout the proof, the enhanced dissipation (16) arises from the wave structure, as well as the strong type Poincaré inequality in Lemma 2.2 play critical roles.

Through arguments similar to Theorem 1.2, we have the global existence and stability of the solution of (9) in $\Omega = \mathbb{T} \times \mathbb{R}$.

Proposition 6.1 *Suppose that $\kappa \in (0, 1)$, $div\, u_0 = 0$ and ψ_0 is even in R_2. Assume that $u_0 \in H^2(\mathbb{R}^2)$ and $\psi_0 \in H^2(\mathbb{R}^2; \mathcal{L}^2)$. There exists a small constant $\varepsilon_5 > 0$ such that if*

$$||u_0||_{H^2} + ||\psi_0||_{H^2(\mathcal{L}^2)} \leq \varepsilon_5,$$

then (9) has a unique global solution (u^κ, ψ^κ), and ψ is even in R_2. In addition, for all $t > 0$, (u^κ, ψ^κ) satisfies

$$
\begin{aligned}
&||u^\kappa(t)||_{H^2}^2 + ||\psi^\kappa(t)||_{H^2(\mathcal{L}^2)}^2 \\
&\quad + \int_0^t \left\{||\partial_1 u^\kappa(s)||_{H^2}^2 + \kappa||\partial_2 u^\kappa(s)||_{H^2}^2 + ||\partial_2 u^\kappa(s)||_{H^1}^2 + ||\psi^\kappa(s)||_{H^2(\dot{\mathcal{H}}^1)}^2\right\} \mathrm{d}s \le C\varepsilon_5^2.
\end{aligned}
$$

It is worth emphasizing that the ε_5 is independent of viscosity parameter κ, this implies that the energy estimate in Theorem is independent of κ, which is crucial to the following proof of vanishing viscosity limit result.

Proof of Theorem 1.4 Define

$$
U^\kappa = u^\kappa - u, \quad \Psi^\kappa = \psi^\kappa - \psi, \quad \Pi^\kappa = \mathbb{P}^k - P,
$$

then, from (6) and (9), we have

$$
\begin{cases}
\partial_t U^\kappa + u^\kappa \cdot \nabla U^\kappa = -U^\kappa \cdot \nabla u + \partial_1^2 U^\kappa + \kappa \partial_2^2 U^\kappa + \nabla \Pi^\kappa + \mathrm{div}\,(\tau^\kappa - \tau) + \kappa \partial_2^2 u, \\
\partial_t \Psi^\kappa + u^\kappa \cdot \nabla \Psi^\kappa = -U^\kappa \cdot \nabla \psi + \mathrm{div}_R(-\nabla u^\kappa \cdot R\Psi^\kappa) + \mathrm{div}_R(-\nabla U^\kappa \cdot R(\psi + \psi_\infty)) + \mathcal{L}\Psi^\kappa.
\end{cases} \tag{83}
$$

Integrating the above system with (U^κ, Ψ^κ) yields

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||U^\kappa||_{L^2}^2 + ||\Psi^\kappa||_{L^2(\mathcal{L}^2)}^2) + ||\partial_1 U^\kappa||_{L^2}^2, + \kappa||\partial_2 U^\kappa||_{L^2}^2 + ||\Psi^\kappa||_{L^2(\dot{\mathcal{H}}^1)}^2 = \sum_{i=1}^5 T_i, \tag{84}
$$

where

$$
\begin{aligned}
T_1 :=& -\int_\Omega (U^\kappa \cdot \nabla u) U^\kappa \,\mathrm{d}x, \\
T_2 :=& \kappa \int_\Omega \partial_2^2 u U^\kappa \,\mathrm{d}x, \\
T_3 :=& -\iint_{\Omega B} (U^\kappa \cdot \nabla \psi) \Psi^\kappa \frac{\mathrm{d}R}{\psi_\infty} \,\mathrm{d}x, \\
T_4 :=& \iint_{\Omega B} \mathrm{div}_R(-\nabla u^\kappa \cdot R\Psi^\kappa) \Psi^\kappa \frac{\mathrm{d}R}{\psi_\infty} \,\mathrm{d}x \\
T_5 :=& \iint_{\Omega B} \mathrm{div}_R(-\nabla U^\kappa \cdot R\psi) \Psi^\kappa \frac{\mathrm{d}R}{\psi_\infty} \,\mathrm{d}x.
\end{aligned}
$$

Here we use the fact that

$$
\int_\Omega \mathrm{div}\,(\tau^\kappa - \tau) U^\kappa \,\mathrm{d}x + \iint_{\Omega B} \mathrm{div}_R(-\nabla U^\kappa \cdot R\psi_\infty) \Psi^\kappa \frac{\mathrm{d}R}{\psi_\infty} \,\mathrm{d}x = 0.
$$

For T_1, to take advantage of Lemma 2.2, we use Lemma 2.1 and split T_1 into the average part and the oscillation part $T_1 = T_{1,1} + T_{1,2} + T_{1,3}$, where

$$T_{1,1} := -\int_\Omega \widetilde{U^\kappa} \cdot \nabla \tilde{u}\, \overline{U^\kappa}\, \mathrm{d}x,$$
$$T_{1,2} := -\int_\Omega \overline{U^\kappa} \cdot \nabla \tilde{u} \widetilde{U^\kappa}\, \mathrm{d}x,$$
$$T_{1,1} := -\int_\Omega \widetilde{U^\kappa} \cdot \nabla u \widetilde{U^\kappa}\, \mathrm{d}x.$$

Using Lemmas 2.2 and 2.5, we can bound $T_{1,1}$ by

$$\begin{aligned} T_{1,1} \leq & C||\widetilde{U^\kappa}||_{L^2}^{\frac{1}{2}}||\partial_1\widetilde{U^\kappa}||_{L^2}^{\frac{1}{2}}||\nabla\tilde{u}||_{L^2}^{\frac{1}{2}}||\partial_2\nabla\tilde{u}||_{L^2}^{\frac{1}{2}}||\overline{U^\kappa}||_{L^2} \\ \leq & C||\nabla u||_{H^1}^2||U^\kappa||_{L^2}^2 + \frac{1}{6}||\partial_1 U^\kappa||_{L^2}^2, \end{aligned}$$

and $T_{1,2}$, $T_{1,3}$ can be bounded similarly. Hence,

$$T_1 \leq C||\nabla u||_{H^1}^2||U^\kappa||_{L^2}^2 + \frac{1}{2}||\partial_1 U^\kappa||_{L^2}^2.$$

For T_2-T_5, by integrating by parts, Lemmas 2.5, 2.6 and symmetry of ψ,

$$\begin{aligned} T_2 &\leq \frac{\kappa}{2}||\partial_2 u||_{L^2(\mathcal{L}^2)}^2 + \frac{\kappa}{2}||\partial_2 U^\kappa||_{L^2(\mathcal{L}^2)}^2, \\ T_3 &\leq C||\psi||_{H^2(\mathcal{L}^2)}^2(||U^\kappa||_{L^2}^2 + ||\partial_1 U^\kappa||_{L^2}^2) + \frac{1}{3}||\Psi^\kappa||_{L^2(\dot{\mathcal{H}}^1)}^2, \\ T_4 &\leq C(||\partial_2 u^\kappa||_{H^1}^2 + ||\partial_1 u^\kappa||_{H^2}^2)||\Psi^\kappa||_{L^2(\mathcal{L}^2)}^2 + \frac{1}{3}||\Psi^\kappa||_{L^2(\dot{\mathcal{H}}^1)}^2, \\ T_5 &\leq C||\psi||_{H^2(\mathcal{L}^2)}^2||\partial_1 U^\kappa||_{L^2(\mathcal{L}^2)}^2 + \frac{1}{3}||\Psi^\kappa||_{L^2(\dot{\mathcal{H}}^1)}^2. \end{aligned}$$

Combining (84) with the bounds of T_1-T_5 yields that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(||U^\kappa||_{L^2}^2 + ||\Psi^\kappa||_{L^2(\mathcal{L}^2)}^2) \\ &\quad \leq -(1 - 2C||\psi||_{H^2(\mathcal{L}^2)})||\partial_1 U^\kappa||_{L^2}^2 + \kappa||\partial_2 u||_{L^2}^2 \\ &\qquad + 2C(||\partial_2(u^\kappa, u)||_{H^1}^2 + ||\partial_1(u^\kappa, u)||_{H^2}^2 + ||\psi||_{H^2(\mathcal{L}^2)}^2)(||U^\kappa||_{L^2}^2 + ||\Psi^\kappa||_{L^2(\mathcal{L}^2)}^2). \end{aligned} \tag{85}$$

Next, we consider the $\dot{H}^1$-type estimate. By applying $\partial_k (k = 1, 2)$ to (83) and multiplying them by $(\partial_k U^\kappa, \partial_k \Psi^\kappa)$ in L^2 and $L^2(\mathcal{L}^2)$ respectively, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\partial_k U^\kappa||_{L^2}^2 + ||\partial_k \Psi^\kappa||_{L^2(\mathcal{L}^2)}^2) \\ &\quad + ||\partial_1\partial_k U^\kappa||_{L^2}^2 + \kappa||\partial_2\partial_k U^\kappa||_{L^2}^2 + ||\partial_k\Psi^\kappa||_{L^2(\dot{\mathcal{H}}^1)}^2 = \sum_{i=6}^{12} T_i, \end{aligned} \tag{86}$$

where

$$
\begin{aligned}
T_6 &:= -\int_\Omega \partial_k(u^\kappa \cdot \nabla U^\kappa)\partial_k U^\kappa \,\mathrm{d}x,\\
T_7 &:= -\int_\Omega \partial_k(U^\kappa \cdot \nabla u)\partial_k U^\kappa \,\mathrm{d}x,\\
T_8 &:= \kappa\int_\Omega \partial_2^2\partial_k u\partial_k U^\kappa \,\mathrm{d}x,\\
T_9 &:= -\iint_{\Omega B} \partial_k(u^\kappa \cdot \nabla \Psi^\kappa)\partial_k \Psi^\kappa \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
T_{10} &:= -\iint_{\Omega B} \partial_k(U^\kappa \cdot \nabla \psi)\partial_k \Psi^\kappa \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
T_{11} &:= \iint_{\Omega B} \partial_k \mathrm{div}_R(-\nabla u^\kappa \cdot R\Psi^\kappa)\partial_k \Psi^\kappa \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x,\\
T_{12} &:= \iint_{\Omega B} \partial_k \mathrm{div}_R(-\nabla U^\kappa \cdot R\psi)\partial_k \Psi^\kappa \,\frac{\mathrm{d}R}{\psi_\infty}\,\mathrm{d}x.
\end{aligned}
$$

Similar to T_1, we split T_6 into the average parts and oscillation parts as $T_6 = T_{6,1} + T_{6,2} + T_{6,3}$, where

$$
\begin{aligned}
T_{6,1} &:= \int_\Omega (\partial_k \widetilde{u^\kappa} \cdot \nabla \widetilde{U^\kappa})\partial_k \overline{U^\kappa}\,\mathrm{d}x,\\
T_{6,2} &:= -\int_\Omega \partial_k \widetilde{u^\kappa} \cdot \nabla \overline{U^\kappa}\partial_k \widetilde{U^\kappa}\,\mathrm{d}x,\\
T_{6,3} &:= -\int_\Omega \partial_k u^\kappa \cdot \nabla \widetilde{U^\kappa}\partial_k \widetilde{U^\kappa}\,\mathrm{d}x.
\end{aligned}
$$

When $k = 1$, $T_{6,1} = 0$. When $k = 2$, by $\bar{U}_2 = 0$, Lemmas 2.2 and 2.5, we can bound $T_{6,1}$ by

$$
\begin{aligned}
T_{6,1} &\le C||\partial_2\widetilde{u^\kappa}||_{L^2}^{\frac{1}{2}}||\partial_2\partial_2\widetilde{u^\kappa}||_{L^2}^{\frac{1}{2}}||\partial_2\widetilde{U^\kappa}||_{L^2}^{\frac{1}{2}}||\partial_1\ \partial_2\widetilde{U^\kappa}||_{L^2}^{\frac{1}{2}}||\partial_2 U^\kappa||_{L^2}\\
&\le C||\nabla u^\kappa||_{H^1}^2||\nabla U^\kappa||_{L^2}^2 + \frac{1}{12}||\partial_1\partial_2 U^\kappa||_{L^2}^2,
\end{aligned}
$$

and $T_{6,2}, T_{6,3}$ shares similar bounds with $T_{6,1}$. Hence,

$$
T_6 \le C||\nabla u^\kappa||_{H^1}^2||\nabla U^\kappa||_{L^2}^2 + \frac{1}{4}||\partial_1\partial_k U^\kappa||_{L^2}^2.
$$

By the reasoning identical to T_6,

$$
T_7 \le C(||\partial_2 u||_{H^1}^2 + ||\partial_1 u||_{H^2}^2)||U^\kappa||_{H^1}^2 + \frac{1}{4}||\partial_1\nabla U^\kappa||_{L^2}^2.
$$

For T_8, we directly use integrating by parts and Young's inequality to bound it as

$$T_8 \leq \frac{\kappa}{2}||\partial_2\partial_k u||^2_{L^2(\mathcal{L}^2)} + \frac{\kappa}{2}||\partial_2\partial_k U^\kappa||^2_{L^2(\mathcal{L}^2)}.$$

For T_9 and T_{10}, we can apply $\operatorname{div} u = 0$, Lemmas 2.5 and 2.6 to obtain

$$T_9 \leq C(||\partial_2 u||^2_{H^1} + ||\partial_1 u||^2_{H^2})||\nabla \Psi^\kappa||^2_{L^2(\mathcal{L}^2)} + \frac{1}{4}||\partial_k \Psi^\kappa||^2_{L^2(\dot{\mathcal{H}}^1)},$$

$$T_{10} \leq C||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}||U^\kappa||^2_{H^1} + C||\psi||^2_{H^2(\mathcal{L}^2)}||\partial_1 U^\kappa||^2_{H^1} + \frac{1}{4}||\partial_k \Psi^\kappa||^2_{L^2(\dot{\mathcal{H}}^1)}.$$

For T_{11} and T_{12}, after integrating by parts, we can bound them by using Lemmas 2.5, 2.6 and symmetry of ψ:

$$\begin{aligned}
T_{11} \leq& C(||\partial_2 u^\kappa||^2_{H^1} + ||\partial_1 u^\kappa||^2_{H^2})||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)} + \frac{1}{4}||\partial_k \Psi^\kappa||^2_{L^2(\dot{\mathcal{H}}^1)},\\
T_{12} =& \sum_{r=0,1}\iint_{\Omega B} \partial_1\partial_k^r U^\kappa \left(R_1\partial_k^{1-r}\psi\partial_{R_1}\frac{\partial_k\Psi^\kappa}{\psi_\infty} - R_2\partial_k^{1-r}\psi\partial_{R_2}\frac{\partial_k\Psi^\kappa}{\psi_\infty}\right)\mathrm{d}R\,\mathrm{d}x\\
\leq& C||\psi||^2_{H^2(\dot{\mathcal{H}}^1)}||U^\kappa||^2_{H^1} + C||\psi||^2_{H^2(\mathcal{L}^2)}||\partial_1 U^\kappa||^2_{H^1} + \frac{1}{4}||\partial_k \Psi^\kappa||^2_{L^2(\dot{\mathcal{H}}^1)}.
\end{aligned}$$

Inserting the bounds of T_6-T_{12} into (86), we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(||\nabla U^\kappa||^2_{L^2} + ||\nabla\Psi^\kappa||^2_{L^2(\mathcal{L}^2)})\\
&\quad\leq -(1-2C||\psi||^2_{H^2(\mathcal{L}^2)})||\partial_1\nabla U^\kappa||^2_{L^2} + \kappa||\partial_2\nabla u||^2_{L^2}\\
&+ 2C(||\partial_2(u^\kappa,u)||^2_{H^1} + ||\partial_1(u^\kappa,u)||^2_{H^2} + ||\psi||^2_{H^2(\dot{\mathcal{H}}^1)})(||\nabla U^\kappa||^2_{L^2} + ||\nabla\Psi^\kappa||^2_{L^2(\mathcal{L}^2)}).
\end{aligned}\tag{87}$$

Due to the norm equivalence, we can deduce from (85) and (87) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(||U^\kappa||^2_{H^1} + ||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)})\\
&\quad\leq -(1-2C||\psi||^2_{H^2(\mathcal{L}^2)})||\partial_1 U^\kappa||^2_{H^1} + \kappa||\partial_2 u||^2_{H^1}\\
&\qquad+ 2C(||\partial_2(u^\kappa,u)||^2_{H^1} + ||\partial_1(u^\kappa,u)||^2_{H^2} + ||\psi||^2_{H^2(\dot{\mathcal{H}}^1)})(||U^\kappa||^2_{H^1} + ||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)}).
\end{aligned}$$

Thanks to the stability stated in (8), we can choose $\varepsilon_4 \leq \min\{\varepsilon_3, \varepsilon_5\}$ small enough such that

$$-(1-2C||\psi||^2_{H^2(\mathcal{L}^2)}) \leq 0,$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(||U^\kappa||^2_{H^1} + ||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)})\\
&\quad\leq C\mathcal{D}(t)(||U^\kappa||^2_{H^1} + ||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)}) + \kappa\mathcal{D}(t),
\end{aligned}$$

where

$$\mathcal{D}(t) := ||\partial_2(u^\kappa, u)(t)||^2_{H^1} + ||\partial_1(u^\kappa, u)(t)||^2_{H^2} + ||\psi(t)||^2_{H^2(\dot{\mathcal{H}}^1)}.$$

Since $U^\kappa(0) = \Psi^\kappa(0) = 0$ and $\mathcal{D}(t)$ is integrable in time, by Gronwall's inequality,

$$||U^\kappa||^2_{H^1} + ||\Psi^\kappa||^2_{H^1(\mathcal{L}^2)} \le e^{C\int_0^t \mathcal{D}(s)\,\mathrm{d}s} \int_0^t \kappa\mathcal{D}(s)\,\mathrm{d}s \le e^{C\varepsilon_4^2} C\varepsilon_4^2\kappa,$$

which is exactly (10). □

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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