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Stability and large-time behavior for Euler-like equations

Jiahong Wu [b], Xiaojing Xu [c], Yueyuan Zhong [a,*], Ning Zhu [d]

[a] *School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou, Jiangsu, 215009, PR China*
[b] *Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, United States*
[c] *School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China*
[d] *School of Mathematical Sciences, Shandong University, Jinan, Shandong, 250100, PR China*

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Abstract

This paper intends to understand the long-time existence and stability of solutions to an Euler-like equation. An Euler-like equation is the 2D incompressible Euler equation with an extra singular integral operator (SIO) type term. In contrast to the 2D Euler equation, the vorticity to the 2D Euler-like equation is not known to be bounded due to the unboundedness of the SIO on the space L^∞. As a consequence, classical Yudovich theory fails on the Euler-like equation. The global existence, regularity and stability problems on the Euler-like equation are generally open. This paper makes progress on an Euler-like equation arising in the study of several fluids. We establish a long-time existence and stability result. When the Sobolev size of the initial data is of order ε, the solution is shown to live on a time interval of the size $1/\varepsilon^2$. When the initial data is restricted to a class with special symmetry, we obtain the global existence and nonlinear stability.

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Keywords: Euler-like equations; Large-time behavior; Stability

* Corresponding author.
E-mail addresses: jwu29@nd.edu (J. Wu), xjxu@bnu.edu.cn (X. Xu), yyzh@usts.edu.cn (Y. Zhong), ning.zhu@sdu.edu.cn (N. Zhu).

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1. Introduction

An Euler-like equation refers to the incompressible 2D Euler vorticity form with an extra term involving a singular integral operator. Attention here is focused on the following Euler-like equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \mathcal{R}_1^2 \omega, \quad x \in \mathbb{T}^2, \\ u = \nabla^{\perp} \Delta^{-1} \omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is a parameter, $\mathcal{R}_i = \partial_i (-\Delta)^{-\frac{1}{2}}$ with $i = 1, 2$ are the Riesz transforms, $\mathbb{T}^2 = [-\frac{1}{2}, \frac{1}{2}]^2$ denotes the 2D torus, and $\nabla^{\perp} = (-\partial_2, \partial_1)$. The fractional Laplacian operator is defined by the Fourier transform, for any $\beta \in \mathbb{R}$,

$$\widehat{(-\Delta)^{\beta} f}(k) = |k|^{2\beta} \widehat{f}(k).$$

For the sake of convenience, we denote

$$\Lambda = (-\Delta)^{\frac{1}{2}}.$$

(1.1) can be formulated in the following velocity form

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + \begin{bmatrix} 0 \\ \nu u_2 \end{bmatrix} = 0, \quad x \in \mathbb{T}^2, \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

(1.2) represents a partially damped Euler equation. It is easy to check that taking the *curl* to (1.2) yields (1.1). Even though the velocity formulation (1.2) appears to be asymmetric in the two directions, they can be converted into a symmetric form. Applying the divergence operator $\nabla \cdot$ to (1.2) yields

$$p = -\nu \partial_2 \Delta^{-1} u_2 - \Delta^{-1} \nabla \cdot (u \cdot \nabla u).$$

Separating the linear and nonlinear parts, we obtain

$$\begin{cases} \partial_t u_1 - \nu \partial_1 \partial_2 \Delta^{-1} u_2 = \partial_1 \Delta^{-1} \nabla \cdot (u \cdot \nabla u) - u \cdot \nabla u_1, \\ \partial_t u_2 + u_2 - \nu \partial_2 \partial_2 \Delta^{-1} u_2 = \partial_2 \Delta^{-1} \nabla \cdot (u \cdot \nabla u) - u \cdot \nabla u_2. \end{cases} \tag{1.3}$$

Invoking the divergence-free condition $\nabla \cdot u = 0$ in (1.3) leads to

$$\begin{cases} \partial_t u_1 = \nu \mathcal{R}_1^2 u_1 + \partial_1 \Delta^{-1} \nabla \cdot (u \cdot \nabla u) - u \cdot \nabla u_1, \\ \partial_t u_2 = \nu \mathcal{R}_1^2 u_2 + \partial_2 \Delta^{-1} \nabla \cdot (u \cdot \nabla u) - u \cdot \nabla u_2, \end{cases} \tag{1.4}$$

which reveals the symmetric structure due to the coupling and interaction of the two velocity components u_1 and u_2. It is easy to check from (1.4) that the divergence-free condition $\nabla \cdot u = 0$ is preserved in time.

The Euler-like equation arises in many physical applications such as fluid mechanics, general relativity, and mathematics biology. They also emerge in the study of several open regularity and stability problems on the Boussinesq equations and the magnetohydrodynamic (MHD) equations. The reduction of the Boussinesq equations in the zero-Prandtl-number limit to the Euler-like models can be found in [8]. The global regularity problem on the 2D MHD equations with only magnetic diffusion depends crucially on one of the Euler-like equations in [6,9,10].

Our goal here is to advance our understanding on the fundamental issues. The global-in-time existence, regularity and stability problems on this Euler-like equation are extremely challenging. The main difficulty is due to the lack of global bound on the vorticity. In the case of the Euler equation, namely (1.1) with $\nu = 0$, the vorticity is transported by a divergence-free velocity field and is thus bounded uniformly for all time. This boundedness is the essential ingredient of the celebrated Yudovich theory on the global existence and uniqueness of weak solutions to the 2D incompressible Euler equation [11].

Unfortunately, the situation is different from the Euler-like equation. Due to the appearance of the singular integral operator (SIO) and the fact that SIOs are generally not bounded on L^∞, the vorticity of (1.1) is not known to be bounded for all time. In fact, the work of Masmoudi and Elgindi showed that solutions to Euler-like equations can experience normal inflation in the L^∞ setting [3,4]. Therefore, we do not expect their solutions to be bounded for all time.

Without the boundedness of the vorticity, it then appears that the global regularity and stability problems are beyond reach. In fact, very few results are currently available for the Euler-like equations. This paper presents two results. The first one assesses the long-time stability of perturbations in the Sobolev setting. It says that, when the initial perturbation in a sufficiently regular Sobolev space is of the order ε, then (1.2) has a unique solution on a time interval with the length of the order ε^{-2}. This solution remains of the size ε.

To give a more precise account of our result, we introduce a few notations. We use $\bar{f}$ to denote the horizontal average of f and $\widetilde{f}$ the corresponding oscillation part, namely

$$\bar{f} \triangleq \int_{\mathbb{T}} f(x_1, x_2)dx_1, \qquad \widetilde{f} = f - \bar{f}.$$

More details on this decomposition can be found in the following section.

Theorem 1.1. *Consider the Euler-like equation (1.2) on the torus $\mathbb{T}^2$. Assume $u_0 \in H^{10}(\mathbb{T}^2)$ with $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon > 0$, which can be taken as $\varepsilon = c_0 \min\{\nu, \nu^{-\frac{5}{18}}\}$ for some universal constant $c_0 > 0$ such that, if*

$$\|u_0\|_{H^{10}(\mathbb{T}^2)} \le \varepsilon,$$

then (1.2) has a unique solution $u(t) \in H^{10}(\mathbb{T}^2)$ on the time interval $[0, C_0\,\nu\varepsilon^{-2}]$ for some constant $C_0 > 0$. Moreover, for some constants $C_1 > 0$ and $C_2 > 0$, and any $t \in [0,\ C_0\,\nu\varepsilon^{-2}]$,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \le C_1\varepsilon$$

and

$$\|\widetilde{u}(t)\|_{H^5(\mathbb{T}^2)} \le \frac{C_2\,\nu^{\frac{5}{18}}\varepsilon}{(1+t)^{\frac{5}{4}}}.$$

Since the coefficient ν is typically taken to be small (e.g., $\nu \le 1$), the quantity $\nu^{-\frac{5}{18}}$ is large. Therefore, we may choose $\varepsilon = c_0\nu$, a suitable multiple of ν. Even though this result doesn't fully resolve the global existence and stability problem, it does reflects the stabilization effort of the term involving the SIO. For the 2D incompressible Euler equation, the vorticity gradient and more general derivatives can grow rather rapidly in time. More precise growth type results can be found in [7] for bounded domains, and in [12] for the periodic setting.

We remark that the time interval is of the order ε^{-2}, much larger than the standard local existence time interval ε^{-1}. Therefore, this result is not just a local existence theory.

Our second main theorem presents a global existence and stability result for the case when the initial perturbation has some symmetry properties. More precisely, the following theorem holds.

Theorem 1.2. *Consider the Euler-like equation (1.2) on the torus $\mathbb{T}^2$. Assume $u_0 \in H^{10}(\mathbb{T}^2)$ with $\nabla \cdot u_0 = 0$. In addition, we assume the following symmetries,*

$$u_{0,1} \text{ is odd in } x_1, \quad \text{and } u_{0,2} \text{ is even in } x_1. \tag{1.5}$$

Then there exists $\varepsilon > 0$, which can be taken as $\varepsilon = c_0 \min\{\nu, \nu^{-\frac{5}{18}}\}$ for some universal constant $c_0 > 0$ such that, if

$$\|u_0\|_{H^{10}(\mathbb{T}^2)} \le \varepsilon,$$

then (1.2) has a unique global solution $u \in L^\infty(0, \infty;\ H^{10}(\mathbb{T}^2))$ satisfying

$$\|u(t)\|^2_{H^{10}(\mathbb{T}^2)} + \nu \int_0^t \|u_2(\tau)\|^2_{H^{10}(\mathbb{T}^2)}\, d\tau \le C\varepsilon^2,$$

and for some $C > 0$,

$$\|\widetilde{u}(t)\|_{H^5(\mathbb{T}^2)} \le \frac{C\,\nu^{\frac{5}{18}}\varepsilon}{(1+t)^{\frac{5}{4}}}.$$

Again, since ν is generally assumed to be small (e.g., $\nu \le 1$), we may choose $\varepsilon = c_0\nu$, where $c_0 > 0$ is a suitable constant. Due to the regularity and uniqueness of solutions in the Sobolev setting of the theorem, the symmetries specified in (1.5) are preserved in time. Theorem 1.2 states the stability and long-time behavior of perturbations restricted to the symmetry class in (1.5).

We briefly explain the difficulties and the main ideas in the proofs of our main results. The proof of Theorem 1.1 invokes the bootstrapping argument. We make the ansatz that, for two suitable constants $C_1 > 2$ and $C_0 > 0$,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \le C_1\varepsilon \quad \text{for } t \in [0, C_0\nu\varepsilon^{-2}]. \tag{1.6}$$

We then show that, for t on the same time interval,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \le \frac{C_1}{2}\varepsilon. \tag{1.7}$$

The local theory of existence and regularity guarantees that, for $t \in [0, C_2 \nu \varepsilon^{-1}]$ with a suitable constant $C_2 > 0$,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \leq C_1 \varepsilon, \tag{1.8}$$

when the initial norm is taken to be sufficiently small. To extend (1.8) to the longer-time interval $[0, C_0 \nu \varepsilon^{-2}]$, we estimate $\|u(t)\|_{H^{10}(\mathbb{T}^2)}$ via the vorticity formulation, which eliminates the pressure.

It then suffices to obtain suitable *a priori* bounds on $\|\omega\|_{H^9}$ due to the equivalence $\|u\|_{H^{10}} = \|u\|_{L^2} + \|\omega\|_{H^9}$. Because of the anisotropic nature of this Euler-like equation, we treat the good horizontal derivative differently from the vertical derivative. In addition, quantities are decomposed into their horizontal averages and the corresponding oscillation parts. To bound the oscillations parts, the following Poincaré type inequality is used to create the favorable horizontal derivative,

$$\|\widetilde{f}\|_{L^2} \leq C \|\partial_1 \widetilde{f}\|_{L^2}.$$

More details on this Poincaré type inequality can be found in Section 2. After implementing these two key points and various other suitable inequalities, we obtain the following inequality on the Sobolev norm of the vorticity (4.24), for some constant $C > 0$,

$$\frac{d}{dt} \|\omega(t)\|_{H^9}^2 + \nu \|\mathcal{R}_1 \omega\|_{H^9}^2 \leq C \|\widetilde{\omega}\|_{H^4} \|\omega\|_{H^9}^2 + C \nu^{-1} \|\omega\|_{H^9}^4. \tag{1.9}$$

To obtain a suitable upper bound on the norm $\|\omega(t)\|_{H^9}^2$, a natural next step is to explore the decay property of $\|\widetilde{\omega}\|_{H^4}$ or $\|\widetilde{u}\|_{H^5}$. We are able to show that

$$\|\widetilde{u}(t)\|_{H^5(\mathbb{T}^2)} \leq \frac{C \nu^{\frac{5}{18}} \varepsilon}{(1+t)^{\frac{5}{4}}}, \tag{1.10}$$

where C is just a constant independent of ν and ε. Where does this decay come from? It is mainly due to the fast L^2 decay of the vorticity oscillation part $\widetilde{\omega}$ (see Proposition 3.1). The decay estimate in (1.10) is a consequence of the interpolation inequality

$$\|\nabla^4 \widetilde{\omega}(t)\|_{L^2} \leq C \|\widetilde{\omega}\|_{L^2}^{\frac{5}{9}} \|\widetilde{\omega}\|_{H^9}^{\frac{4}{9}} \leq C \left(\nu^{\frac{1}{2}} \varepsilon (1+t)^{-\frac{9}{4}} \right)^{\frac{5}{9}} \varepsilon^{\frac{4}{9}} \leq C \nu^{\frac{5}{18}} \varepsilon \, (1+t)^{-\frac{5}{4}}.$$

Inserting (1.10) in (1.9), applying Gronwall's inequality and invoking the ansatz, we find that, for $t \in [0, C_0 \nu \varepsilon^{-2}]$,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \leq \|u_0\|_{H^{10}(\mathbb{T}^2)} \, e^{C_3 \nu^{-1} \varepsilon^2 t + C_4 \nu^{\frac{5}{18}} \varepsilon} \leq \frac{C_1}{2} \varepsilon.$$

Then the bootstrapping argument implies the desired upper bound for $\|u(t)\|_{H^{10}(\mathbb{T}^2)}$.

The proof of Theorem 1.2 relies crucially on the symmetries in (1.5). Due to the uniqueness of solutions in the regularity class $H^{10}(\mathbb{T}^2)$, it is easy to check that the symmetries in (1.5) are preserved in time. Making use of these symmetries, we are able to establish the following estimate

$$\frac{d}{dt}\|u(t)\|_{H^{10}}^2 + 2\nu\|u_2\|_{H^{10}}^2 \le C_5\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2 + C_6\|u\|_{H^{10}}\|u_2\|_{H^{10}}^2,$$

which yields the desired global stability of Theorem 1.2.

The rest of this paper is divided into four sections. The second section provides the properties of the aforementioned orthogonal decomposition and a decay lemma to be used in the proofs of theorems. The third section proves the decay property of the lower-order Sobolev norms. The fourth section details the proof of Theorem 1.1 while the last section proves Theorem 1.2.

2. Preliminaries

This section presents two tool lemmas to be used in the proofs of our main results. The first one states properties related to the orthogonal decomposition of a function into its horizontal average and the corresponding oscillation part.

Recall that, for a continuous function f on $\mathbb{T}^2$, the average part of f in x_1 is given by

$$\bar{f} \triangleq \int_{\mathbb{T}} f(x_1, x_2)dx_1, \tag{2.1}$$

and the corresponding oscillation part

$$\widetilde{f} = f - \bar{f}. \tag{2.2}$$

The following lemma states several properties of this decomposition. These properties and their proofs can be found in [1,2,5].

Lemma 2.1. *Let $f\colon \mathbb{T}^2 \to \mathbb{R}$ be a C^1 function, and $\overline{f}$ and $\widetilde{f}$ be defined as in (2.1) and (2.2).*

(1) *If f satisfies divergence-free condition, that is $\nabla \cdot f = 0$, then so do $\widetilde{f}$ and $\overline{f}$, namely*

$$\nabla \cdot \widetilde{f} = 0, \quad \nabla \cdot \overline{f} = 0.$$

(2) *The following basic properties on $\widetilde{f}$ and $\overline{f}$ are used frequently,*

$$\overline{\partial_1 f} = \partial_1 \overline{f} = 0, \quad \partial_1 f = \partial_1 \widetilde{f}.$$

(3) *For three continuous functions f, g and h from $\mathbb{T}^2$ to $\mathbb{R}$,*

$$\int_{\mathbb{T}^2} \overline{f}\widetilde{g}\overline{h}\, dx = 0, \tag{2.3}$$

and

$$\overline{\overline{f}\,\widetilde{g}} = 0. \tag{2.4}$$

(4) *The following Poincaré inequality holds for* $\widetilde{f}$,

$$\|\widetilde{f}\|_{L^2} \leq C\|\partial_1 \widetilde{f}\|_{L^2}. \tag{2.5}$$

A sharper version of (2.5) *is given by*

$$\|\widetilde{f}\|_{L^2} \leq C\||\partial_1|^\gamma \widetilde{f}\|_{L^2},$$

where $\gamma \geq 0$ *and* $\widehat{|\partial_1|^\gamma f} = |k_1|^\gamma \widehat{f}$.

The second lemma deduces a precise decay rate from a differential inequality.

Lemma 2.2. *Let* $g \in C^1[0,\infty)$ *and* $g \geq 0$ *on* $[0,\infty)$. *Assume that for two constants* $A > 0$ *and* $r > 1$,

$$\partial_t g \leq -\frac{g}{\sqrt{t+1}} + \frac{A}{(t+1)^r}$$

for $t \in [0,\infty)$. *Then, for some constant* $c_* \geq 2^r + 1$ *independent of* $g(0)$ *and* A,

$$g(t) \leq \frac{g(0) + c_* A}{(t+1)^{r-\frac{1}{2}}}.$$

Proof of Lemma 2.2. It is easy to see that

$$\partial_t \left(\mathbf{e}^{2\sqrt{t+1}} g\right) \leq \frac{A\mathbf{e}^{2\sqrt{t+1}}}{(t+1)^r}.$$

Integrating in time from 0 to t yields

$$\begin{aligned}
g(t) &\leq \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{-2\sqrt{t+1}} \int_0^t \frac{A\mathbf{e}^{2\sqrt{s+1}}}{(s+1)^r}\, ds \\
&\leq \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{-2\sqrt{t+1}} \left\{ \int_0^{\frac{t}{2}} \frac{\mathbf{e}^{2\sqrt{s+1}}}{(s+1)^r}\, ds + \int_{\frac{t}{2}}^t \frac{\mathbf{e}^{2\sqrt{s+1}}}{(s+1)^r}\, ds \right\} \\
&\leq \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{2\left(\sqrt{\frac{t}{2}+1}-\sqrt{t+1}\right)} \int_0^{\frac{t}{2}} \frac{1}{(s+1)^r}\, ds \\
&\quad + \frac{A\mathbf{e}^{-2\sqrt{t+1}}}{(\frac{t}{2}+1)^r} \int_{\frac{t}{2}}^t \mathbf{e}^{2\sqrt{s+1}}\, ds
\end{aligned}$$

$$\le \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{2\left(\sqrt{\frac{t}{2}+1}-\sqrt{t+1}\right)} + \frac{A\mathbf{e}^{-2\sqrt{t+1}}}{(\frac{t}{2}+1)^r}\int_{\frac{t}{2}}^{t} \mathbf{e}^{2\sqrt{s+1}}\,ds. \tag{2.6}$$

Let $\tau = 2\sqrt{s+1}$, then

$$\begin{aligned}
\int_{\frac{t}{2}}^{t} \mathbf{e}^{2\sqrt{s+1}}\,ds &= \int_{2\sqrt{\frac{t}{2}+1}}^{2\sqrt{t+1}} \mathbf{e}^{\tau} \times \left(\frac{1}{2}\tau\right) d\tau \\
&= \frac{1}{2} \times \left(\tau\mathbf{e}^{\tau}\Big|_{\tau=2\sqrt{\frac{t}{2}+1}}^{\tau=2\sqrt{t+1}} - \int_{2\sqrt{\frac{t}{2}+1}}^{2\sqrt{t+1}} \mathbf{e}^{\tau}\,d\tau \right) \\
&\le \sqrt{t+1} \times \mathbf{e}^{2\sqrt{t+1}}.
\end{aligned} \tag{2.7}$$

Inserting (2.7) in (2.6), we get

$$\begin{aligned}
g(t) &\le \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{2\left(\sqrt{\frac{t}{2}+1}-\sqrt{t+1}\right)} + \frac{A}{(\frac{t}{2}+1)^r} \times (1+t)^{\frac{1}{2}} \\
&\le \mathbf{e}^{-2\sqrt{t+1}}\mathbf{e}^2 g(0) + A\mathbf{e}^{2\left(\sqrt{\frac{t}{2}+1}-\sqrt{t+1}\right)} + \frac{2^r \times A}{(t+2)^{r-\frac{1}{2}}} \\
&\le \frac{g(0)}{(t+1)^{r-\frac{1}{2}}} + \frac{A}{(t+1)^{r-\frac{1}{2}}} + \frac{2^r \times A}{(t+1)^{r-\frac{1}{2}}} \\
&\le \frac{g(0) + c_* A}{(t+1)^{r-\frac{1}{2}}}
\end{aligned}$$

for some $c_* \ge 2^r + 1$. This finishes the proof of Lemma 2.2. □

3. Decay estimate of $\widetilde{u}$

The next two sections are devoted to the proof of Theorem 1.1. As aforementioned in the introduction, the proof invokes the bootstrapping argument. The main lines include making the ansatz (1.6) and prove (1.7).

This section serves as part of the proof. It shows that, under the bootstrapping ansatz (1.6), lower-order norms of the solution decay in time. More precisely, we prove the following proposition.

Proposition 3.1. *Assume that the solution u of (1.2) satisfies the ansatz (1.6). Then its oscillation part $\widetilde{u}$ satisfies, for some constant $C > 0$ independent of ν and ε,*

$$\|\widetilde{u}(t)\|_{H^5(\mathbb{T}^2)} \le \frac{C\nu^{\frac{5}{18}}\varepsilon}{(1+t)^{\frac{5}{4}}} \quad \text{for } t \in [0, C_0\nu\varepsilon^{-2}]. \tag{3.1}$$

Proof of Proposition 3.1. We use the vorticity formulation. The oscillation part of ω satisfies

$$\partial_t \widetilde{\omega} + \widetilde{u \cdot \nabla \omega} = \nu \mathcal{R}_1^2 \widetilde{\omega}.$$

Due to the divergence-free condition $\nabla \cdot u_0 = \nabla \cdot u = 0$, we write

$$u = \nabla^{\perp} \psi = (-\partial_2 \psi, \partial_1 \psi).$$

As a consequence, $\bar{u}_2 = 0$ and

$$\overline{u_2 \partial_2 \bar{\omega}} = \overline{u_2} \partial_2 \bar{\omega} = 0.$$

Therefore,

$$\begin{aligned}
\widetilde{u \cdot \nabla \omega} &= \widetilde{u \cdot \nabla \widetilde{\omega}} + \widetilde{u \cdot \nabla \bar{\omega}} = \widetilde{u \cdot \nabla \widetilde{\omega}} + \widetilde{u_2 \partial_2 \bar{\omega}} \\
&= \widetilde{u \cdot \nabla \widetilde{\omega}} + u_2 \partial_2 \bar{\omega}
\end{aligned}$$

and

$$\partial_t \widetilde{\omega} + \widetilde{u \cdot \nabla \widetilde{\omega}} + u_2 \partial_2 \bar{\omega} = \nu \mathcal{R}_1^2 \widetilde{\omega}. \tag{3.2}$$

Taking the L^2-inner product with $\widetilde{\omega}$ in (3.2) yields

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\widetilde{\omega}(t)\|_{L^2}^2 &= -\nu \int_{\mathbb{T}^2} |\mathcal{R}_1 \widetilde{\omega}|^2 \, dx - \int_{\mathbb{T}^2} \widetilde{u \cdot \nabla \widetilde{\omega}} \widetilde{\omega} \, dx - \int_{\mathbb{T}^2} u_2 \partial_2 \bar{\omega} \widetilde{\omega} \, dx \\
&\triangleq I + J + K.
\end{aligned}$$

We first estimate J and K. By integration by parts, $\nabla \cdot u = 0$ and (2.4),

$$\begin{aligned}
J &= -\int_{\mathbb{T}^2} \widetilde{u \cdot \nabla \widetilde{\omega}} \widetilde{\omega} \, dx \\
&= -\int_{\mathbb{T}^2} u \cdot \nabla \widetilde{\omega} \widetilde{\omega} \, dx + \int_{\mathbb{T}^2} \overline{u \cdot \nabla \widetilde{\omega}} \widetilde{\omega} \, dx \\
&= \int_{\mathbb{T}^2} \overline{\widetilde{u}_1 \partial_1 \widetilde{\omega}} \widetilde{\omega} \, dx + \int_{\mathbb{T}^2} \overline{\widetilde{u}_2 \partial_2 \widetilde{\omega}} \widetilde{\omega} \, dx \\
&\triangleq J_1 + J_2.
\end{aligned}$$

By Lemma 2.1, Hölder's, Sobolev's and Poincaré inequalities,

$$
\begin{aligned}
J_1 &= \int_{\mathbb{T}^2} \overline{\widetilde{u}_1 \partial_1 \widetilde{\omega}} \widetilde{\omega}\, dx = \int_{\mathbb{T}^2} \Lambda \left(\overline{\widetilde{u}_1 \partial_1 \widetilde{\omega}} \right) \Lambda^{-1} \widetilde{\omega}\, dx \\
&\le \left\| \Lambda \left(\overline{\widetilde{u}_1 \partial_1 \widetilde{\omega}} \right) \right\|_{L^2} \left\| \Lambda^{-1} \widetilde{\omega} \right\|_{L^2} \le \left\| \partial_2 \left(\overline{\widetilde{u}_1 \partial_1 \widetilde{\omega}} \right) \right\|_{L^2} \left\| \Lambda^{-1} \partial_1 \widetilde{\omega} \right\|_{L^2} \\
&\le \left(\| \overline{\partial_2 \widetilde{u}_1 \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega}} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&= \left(\| \overline{(\widetilde{\omega} + \partial_1 \widetilde{u}_2)\, \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega}} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le \left(\| \overline{\widetilde{\omega} \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\partial_1 \widetilde{u}_2 \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega}} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&= \left(\frac{1}{2} \| \overline{\partial_1 \, (\widetilde{\omega})^2} \|_{L^2} + \| \overline{\partial_1 \widetilde{u}_2 \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega}} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&= \left(\| \overline{\partial_1 \widetilde{u}_2 \partial_1 \widetilde{\omega}} \|_{L^2} + \| \overline{\widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega}} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le \left(\| \partial_1 \widetilde{u}_2 \partial_1 \widetilde{\omega} \|_{L^2} + \| \widetilde{u}_1 \partial_1 \partial_2 \widetilde{\omega} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le \left(\| \partial_1 \widetilde{u}_2 \|_{L^2} \| \partial_1 \widetilde{\omega} \|_{L^\infty} + \| \widetilde{u}_1 \|_{L^2} \| \partial_1 \partial_2 \widetilde{\omega} \|_{L^\infty} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le C \| \partial_1 \widetilde{u} \|_{L^2} \| \widetilde{\omega} \|_{H^4} \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le C \| \widetilde{\omega} \|_{H^4} \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2}^2 .
\end{aligned}
$$

Similarly, J_2 is bounded by

$$
\begin{aligned}
J_2 &= \int_{\mathbb{T}^2} \overline{\widetilde{u}_2 \partial_2 \widetilde{\omega}} \widetilde{\omega}\, dx = \int_{\mathbb{T}^2} \Lambda \left(\overline{\widetilde{u}_2 \partial_2 \widetilde{\omega}} \right) \Lambda^{-1} \widetilde{\omega}\, dx \\
&\le \left\| \Lambda \left(\overline{\widetilde{u}_2 \partial_2 \widetilde{\omega}} \right) \right\|_{L^2} \left\| \Lambda^{-1} \widetilde{\omega} \right\|_{L^2} \\
&\le \left(\| \partial_2 \widetilde{u}_2 \partial_2 \widetilde{\omega} \|_{L^2} + \| \widetilde{u}_2 \partial_2 \partial_2 \widetilde{\omega} \|_{L^2} \right) \times \| \partial_1 \Lambda^{-1} \widetilde{\omega} \|_{L^2} \\
&\le \left(\| \partial_1 \widetilde{u}_1 \|_{L^2} \| \partial_2 \widetilde{\omega} \|_{L^\infty} + \| \widetilde{u}_2 \|_{L^2} \| \partial_2 \partial_2 \widetilde{\omega} \|_{L^\infty} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le C \| \widetilde{\omega} \|_{H^4} \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2}^2 .
\end{aligned}
$$

Thus

$$
J \le 2 \| \widetilde{\omega} \|_{H^4} \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2}^2 .
$$

Applying Hölder's inequality and Sobolev embedding, we have

$$
\begin{aligned}
K &= -\int_{\mathbb{T}^2} u_2 \partial_2 \bar{\omega} \widetilde{\omega}\, dx = -\int_{\mathbb{T}^2} \Lambda \left(\widetilde{u}_2 \partial_2 \bar{\omega} \right) \Lambda^{-1} \widetilde{\omega}\, dx \\
&\le \| \Lambda \left(\widetilde{u}_2 \partial_2 \bar{\omega} \right) \|_{L^2} \| \Lambda^{-1} \widetilde{\omega} \|_{L^2} \\
&\le C \left(\| \nabla \widetilde{u}_2 \partial_2 \bar{\omega} \|_{L^2} + \| \widetilde{u}_2 \partial_2 \partial_2 \bar{\omega} \|_{L^2} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le C \left(\| \nabla \widetilde{u}_2 \|_{L^2} \| \partial_2 \bar{\omega} \|_{L^\infty} + \| \widetilde{u}_2 \|_{L^2} \| \partial_2 \partial_2 \bar{\omega} \|_{L^\infty} \right) \times \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2} \\
&\le C \| \omega \|_{H^4} \| \mathcal{R}_1 \widetilde{\omega} \|_{L^2}^2 .
\end{aligned}
$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\omega}(t)\|_{L^2}^2 \le -\nu\int\limits_{\mathbb{T}^2} |\mathcal{R}_1\widetilde{\omega}|^2\,dx + 3C\|\omega\|_{H^4}\|\mathcal{R}_1\widetilde{\omega}\|_{L^2}^2.$$

By the ansatz (1.6) and the choice of ε, we have

$$3C\|\omega\|_{H^4}\|\mathcal{R}_1\widetilde{\omega}\|_{L^2}^2 \le 3CC_1\varepsilon\|\mathcal{R}_1\widetilde{\omega}\|_{L^2}^2 \le 3CC_1c_0\nu\|\mathcal{R}_1\widetilde{\omega}\|_{L^2}^2 \le \frac{1}{2}\nu\|\mathcal{R}_1\widetilde{\omega}\|_{L^2}^2$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{\omega}(t)\|_{L^2}^2\\
&\le -\nu\int\limits_{\mathbb{T}^2}|\mathcal{R}_1\widetilde{\omega}|^2\,dx\\
&\le -\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0}\frac{\nu n^2}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0}\frac{\nu}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0,n^2+k^2\le N}\frac{\nu}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2-\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0,n^2+k^2>N}\frac{\nu}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\frac{\nu}{N}\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0,n^2+k^2\le N}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2-\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0,n^2+k^2>N}\frac{\nu}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\frac{\nu}{N}\left(\|\widetilde{\omega}\|_{L^2}^2-\sum_{n^2+k^2>N}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\right)-\sum_{(n,k)\in\mathbb{Z}^2,n\neq 0,n^2+k^2>N}\frac{\nu}{n^2+k^2}\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\frac{\nu}{N}\|\widetilde{\omega}\|_{L^2}^2+\sum_{n^2+k^2>N}\nu\left(\frac{1}{N}-\frac{1}{n^2+k^2}\right)\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&\le -\frac{\nu}{N}\|\widetilde{\omega}\|_{L^2}^2+\frac{\nu}{N}\sum_{n^2+k^2>N}\left(\frac{n^2+k^2}{N}\right)^9\left|\widehat{\widetilde{\omega}_{n,k}}\right|^2\\
&= -\frac{\nu}{N}\|\widetilde{\omega}\|_{L^2}^2+\frac{\nu}{N^{10}}\|\widetilde{\omega}\|_{H^9}^2.
\end{aligned}$$

Taking $N=(1+t)^{1/2}$ yields

$$\frac{d}{dt}\|\widetilde{\omega}(t)\|_{L^2}^2 \le -\frac{\nu\|\widetilde{\omega}\|_{L^2}^2}{\sqrt{t+1}}+\frac{\nu\|\widetilde{\omega}\|_{L_t^\infty H^9}^2}{(t+1)^5} \le -\frac{\nu\|\widetilde{\omega}\|_{L^2}^2}{\sqrt{t+1}}+\frac{\nu C_1^2\varepsilon^2}{(t+1)^5}.$$

By Lemma 2.2,

$$\|\widetilde{\omega}(t)\|_{L^2}^2 \leq \frac{\nu\|\omega_0\|_{L^2}^2 + c_* \nu C_1^2 \varepsilon^2}{(t+1)^{\frac{9}{2}}}.$$

By interpolation

$$\begin{aligned}\|\nabla^4 \widetilde{\omega}(t)\|_{L^2} &\leq C\|\widetilde{\omega}\|_{L^2}^{\frac{5}{9}} \|\widetilde{\omega}\|_{H^9}^{\frac{4}{9}} \leq C\left((1+c_*C_1^2)^{\frac{1}{2}} \nu^{\frac{1}{2}} \varepsilon (1+t)^{-\frac{9}{4}}\right)^{\frac{5}{9}} (C_1\varepsilon)^{\frac{4}{9}} \\ &\leq C\nu^{\frac{5}{18}} \varepsilon (1+t)^{-\frac{5}{4}},\end{aligned}$$

where we have written C for $C(1+c_*C_1^2)^{\frac{5}{18}} C_1^{\frac{4}{9}}$. Clearly it is independent of ν and ε. By Poincaré inequality stated in Lemma 2.1,

$$\|\widetilde{\omega}(t)\|_{H^4} \leq C\nu^{\frac{5}{18}} \varepsilon (1+t)^{-\frac{5}{4}}.$$

Therefore,

$$\|\widetilde{u}(t)\|_{H^5} \leq C\nu^{\frac{5}{18}} \varepsilon (1+t)^{-\frac{5}{4}},$$

This completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.1

With the decay estimate in the previous section at our disposal, this section completes the proof of Theorem 1.1. The center piece of the proof is the following estimate.

Proposition 4.1. *Assume that u solves (1.2). Then u obeys the a priori differential inequality, for any $0 < t \leq T$ and some constant $\widetilde{C} > 0$,*

$$\begin{aligned}&\|u(t)\|_{H^{10}}^2 + \nu \int_0^T \|u_2(\tau)\|_{H^{10}}^2 \, d\tau \\ &\leq \|u_0\|_{H^{10}}^2 \exp\left\{\int_0^T \widetilde{C}\left(\|\widetilde{u}(\tau)\|_{H^5} + \nu^{-1}\|u(\tau)\|_{H^{10}}^2\right) d\tau\right\}. \qquad (4.1)\end{aligned}$$

We first give the proof of Theorem 1.1 with the help of Proposition 4.1. We will then return to prove Proposition 4.1.

Proof of Theorem 1.1. The proof invokes the bootstrapping argument. We make the ansatz that, for two suitable positive constant C_0 and C_1,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \leq C_1\varepsilon \quad \text{for } t \in [0, C_0\nu\varepsilon^{-2}]. \qquad (4.2)$$

We then show that, for t on the same time interval,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \le \frac{C_1}{2}\varepsilon.$$

The local theory of existence and regularity guarantees that, for $t \in [0, C_2\nu\varepsilon^{-1}]$ with a suitable constant $C_2 > 0$,

$$\|u(t)\|_{H^{10}(\mathbb{T}^2)} \le C_1\varepsilon,$$

when the initial norm is taken to be sufficiently small. To extend (1.8) to the longer-time interval $[0, C_0\nu\varepsilon^{-2}]$, we estimate $\|u(t)\|_{H^{10}(\mathbb{T}^2)}$. By Proposition 4.1,

$$\begin{aligned} &\|u(t)\|_{H^{10}}^2 + \nu\int_0^T \|u_2(\tau)\|_{H^{10}}^2\,d\tau \\ &\le \|u_0\|_{H^{10}}^2 \exp\left\{\int_0^T \widetilde{C}\left(\|\widetilde{u}(\tau)\|_{H^5} + \nu^{-1}\|u(\tau)\|_{H^{10}}^2\right)d\tau\right\}. \end{aligned} \tag{4.3}$$

Inserting (3.1) and (4.2) in (4.3), for any $t \in [0, C_0\nu\varepsilon^{-2}]$, we obtain

$$\begin{aligned} \|u(t)\|_{H^{10}}^2 &\le \|u_0\|_{H^{10}}^2 \exp\left\{\int_0^T \widetilde{C}C\,\nu^{\frac{5}{18}}\varepsilon(1+\tau)^{-\frac{5}{4}}\,d\tau + \widetilde{C}C_1^2\nu^{-1}\varepsilon^2 T\right\} \\ &\le \|u_0\|_{H^{10}}^2 \times \mathbf{e}^{\left(\widetilde{C}C\nu^{\frac{5}{18}}\varepsilon + \widetilde{C}C_0C_1^2\right)}, \end{aligned}$$

where $T = C_0\nu\varepsilon^{-2}$. By the choice of ε and the fact that $C_1 > 2$,

$$\mathbf{e}^{\left(\widetilde{C}C\nu^{\frac{5}{18}}\varepsilon + \widetilde{C}C_0C_1^2\right)} \le \mathbf{e}^{(c_0\widetilde{C}C + \widetilde{C}C_0C_1^2)} \le \frac{C_1^2}{4}$$

for suitable $c_0 > 0$ and $C_0 > 0$. Therefore,

$$\|u(t)\|_{H^{10}}^2 \le \frac{C_1^2}{4}\varepsilon^2,$$

or

$$\|u(t)\|_{H^{10}} \le \frac{C_1}{2}\varepsilon.$$

This completes the proof of the Theorem 1.1. □

The rest of this section proves Proposition 4.1.

Proof of Proposition 4.1. Taking H^9-inner product with ω in $(1.1)_1$, due to

$$\|f\|_{H^s}^2 \approx \|f\|_{L^2}^2 + \|f\|_{\dot{H}^s}^2,$$

and by the divergence-free condition of u, we get

$$\begin{aligned}
&\frac{d}{dt}\|\omega(t)\|_{H^9}^2 + 2\nu\|\mathcal{R}_1\omega\|_{H^9}^2 \\
&\approx -2\int_{\mathbb{T}^2} u\cdot\nabla\omega\omega\,dx - 2\int_{\mathbb{T}^2} \nabla^9(u\cdot\nabla\omega)\,\nabla^9\omega\,dx \\
&= -2\int_{\mathbb{T}^2} \partial_1^9(u\cdot\nabla\omega)\,\partial_1^9\omega\,dx - 2\int_{\mathbb{T}^2} \partial_2^9(u\cdot\nabla\omega)\,\partial_2^9\omega\,dx \\
&= -2\int_{\mathbb{T}^2} \partial_1^9(u_1\partial_1\omega)\,\partial_1^9\omega\,dx - 2\int_{\mathbb{T}^2} \partial_1^9(u_2\partial_2\omega)\,\partial_1^9\omega\,dx \\
&\quad - 2\int_{\mathbb{T}^2} \partial_2^9(u_1\partial_1\omega)\,\partial_2^9\omega\,dx - 2\int_{\mathbb{T}^2} \partial_2^9(u_2\partial_2\omega)\,\partial_2^9\omega\,dx \\
&= -2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx - 2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_1^i u_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx \\
&\quad - 2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_2^i u_1\partial_1\partial_2^{9-i}\omega\partial_2^9\omega\,dx - 2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_2^i u_2\partial_2\partial_2^{9-i}\omega\partial_2^9\omega\,dx \\
&\triangleq N_1 + N_2 + N_3 + N_4.
\end{aligned} \tag{4.4}$$

For N_1, by Hölder's inequality and Sobolev embedding, we obtain

$$\begin{aligned}
N_1 &= -2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx \\
&= -2\sum_{i=1}^{5}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx - 2\sum_{i=6}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx \\
&\le 2\sum_{i=1}^{5}\mathcal{C}_9^i\|\partial_1\partial_1^{9-i}\omega\|_{L^2}\|\partial_1^i u_1\|_{L^\infty}\|\partial_1^9\omega\|_{L^2} \\
&\quad + 2\sum_{i=6}^{9}\mathcal{C}_9^i\|\partial_1\partial_1^{9-i}\omega\|_{L^\infty}\|\partial_1^i u_1\|_{L^2}\|\partial_1^9\omega\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le 2\sum_{i=1}^{5}\mathcal{C}_9^i\left\|\frac{\partial_1}{\Lambda}\Lambda\partial_1^{9-i}\omega\right\|_{L^2}\|\partial_1^i u_1\|_{L^\infty}\left\|\frac{\partial_1}{\Lambda}\Lambda\partial_1^8\omega\right\|_{L^2}\\
&\quad+2\sum_{i=6}^{9}\mathcal{C}_9^i\|\partial_1\partial_1^{9-i}\omega\|_{L^\infty}\left\|\frac{\partial_1}{\Lambda}\Lambda\partial_1^{i-1}u_1\right\|_{L^2}\left\|\frac{\partial_1}{\Lambda}\Lambda\partial_1^8\omega\right\|_{L^2}\\
&\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.
\end{aligned}
$$

For N_2, we have

$$
\begin{aligned}
N_2&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_1^i u_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx\\
&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_1^i \widetilde{u}_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx\\
&=-2\sum_{i=1}^{3}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_1^i \widetilde{u}_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx-2\sum_{i=4}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_1^i \widetilde{u}_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx\\
&\le 2\sum_{i=1}^{2}\mathcal{C}_9^i\|\partial_2\partial_1^{9-i}\omega\|_{L^2}\|\partial_1^i\widetilde{u}_2\|_{L^\infty}\|\partial_1^9\omega\|_{L^2}\\
&\quad+2\sum_{i=3}^{9}\mathcal{C}_9^i\|\partial_1^i\widetilde{u}_2\|_{L^2}\|\partial_2\partial_1^{9-i}\omega\|_{L^\infty}\|\partial_1^9\omega\|_{L^2}\\
&\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.
\end{aligned}
$$

The estimate of N_3 is more complex. We split N_3 into the following two part

$$
\begin{aligned}
N_3&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i u_1\partial_1\partial_2^{9-i}\omega\partial_2^9\omega\,dx\\
&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i \widetilde{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\omega\,dx-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i \overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\omega\,dx\\
&\triangleq N_{31}+N_{32}.
\end{aligned}
$$

N_{31} is bounded by

$$
N_{31}=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i \widetilde{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\omega\,dx
$$

$$
\begin{aligned}
&\le 2\sum_{i=1}^{6}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^2}\|\partial_2^i\widetilde{u}_1\|_{L^\infty}\|\partial_2^9\omega\|_{L^2}\\
&\quad+2\sum_{i=7}^{9}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^\infty}\|\partial_2^i\widetilde{u}_1\|_{L^2}\|\partial_2^9\omega\|_{L^2}\\
&\le 2\sum_{i=1}^{6}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^2}\|\partial_2^i\widetilde{u}_1\|_{H^2}\|\partial_2^9\omega\|_{L^2}\\
&\quad+2\sum_{i=7}^{9}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{H^2}\|\partial_2^i\widetilde{u}_1\|_{L^2}\|\partial_2^9\omega\|_{L^2}\\
&\le 2\sum_{i=1}^{6}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^2}\|\partial_1\partial_2^i\widetilde{u}_1\|_{H^2}\|\partial_2^9\omega\|_{L^2}\\
&\quad+2\sum_{i=7}^{9}\mathcal{C}_9^i\|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{H^2}\|\partial_1\partial_2^i\widetilde{u}_1\|_{L^2}\|\partial_2^9\omega\|_{L^2}\\
&\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \qquad (4.5)
\end{aligned}
$$

For N_{32}, we split N_{32} into two parts and use (2.3) to obtain

$$
\begin{aligned}
N_{32}&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\omega\,dx\\
&=-2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\widetilde{\omega}\,dx\\
&=-2\int_{\mathbb{T}^2}\partial_2\overline{u}_1\partial_1\partial_2^{8}\widetilde{\omega}\partial_2^9\widetilde{\omega}\,dx-2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\widetilde{\omega}\,dx\\
&\triangleq N_{321}+N_{322}. \qquad (4.6)
\end{aligned}
$$

Integrating by part for N_{322} and applying Hölder's inequality and Sobolev embedding, we have

$$
\begin{aligned}
N_{322}&=-2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^9\widetilde{\omega}\,dx\\
&=2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2\left(\partial_2^i\overline{u}_1\partial_1\partial_2^{9-i}\widetilde{\omega}\right)\partial_2^8\widetilde{\omega}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{i=2}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2\partial_2^i \overline{u}_1 \partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^8\widetilde{\omega}\, dx + 2\sum_{i=2}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i \overline{u}_1 \partial_2\partial_1\partial_2^{9-i}\widetilde{\omega}\partial_2^8\widetilde{\omega}\, dx \\
&\le 2\sum_{i=2}^{7} \mathcal{C}_9^i \|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^2}\|\partial_2\partial_2^i\overline{u}_1\|_{L^\infty}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\quad + 2\sum_{i=8}^{9} \mathcal{C}_9^i \|\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^\infty}\|\partial_2\partial_2^i\overline{u}_1\|_{L^2}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\quad + 2\sum_{i=2}^{3} \mathcal{C}_9^i \|\partial_2\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^2}\|\partial_2^i\overline{u}_1\|_{L^\infty}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\quad + 2\sum_{i=4}^{9} \mathcal{C}_9^i \|\partial_2\partial_1\partial_2^{9-i}\widetilde{\omega}\|_{L^\infty}\|\partial_2^i\overline{u}_1\|_{L^2}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\le C\|\omega\|_{H^9}\|\partial_1\widetilde{\omega}\|_{H^8}\|\partial_1\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.
\end{aligned} \tag{4.7}
$$

For N_{321}, by Hölder's inequality, Sobolev embedding and Young's inequality, we obtain

$$
\begin{aligned}
N_{321} &= -2\int_{\mathbb{T}^2} \partial_2\overline{u}_1\partial_1\partial_2^8\widetilde{\omega}\partial_2^9\widetilde{\omega}\, dx \\
&\le 2\|\partial_2\overline{u}_1\|_{L^\infty}\|\partial_1\partial_2^8\widetilde{\omega}\|_{L^2}\|\partial_2^9\widetilde{\omega}\|_{L^2} \\
&\le C\|\mathcal{R}_1\omega\|_{H^9}\|\omega\|_{H^9}^2 \\
&\le \frac{\nu}{2}\|\mathcal{R}_1\omega\|_{H^9}^2 + C\nu^{-1}\|\omega\|_{H^9}^4.
\end{aligned} \tag{4.8}
$$

Inserting (4.7) and (4.8) into (4.6), N_{32} is bounded by

$$
N_{32} \le \frac{\nu}{2}\|\mathcal{R}_1\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2 + C\nu^{-1}\|\omega\|_{H^9}^4. \tag{4.9}
$$

Furthermore, collecting the bound for N_{31} in (4.5) and the bound for N_{32} in (4.9), N_3 is bounded by

$$
N_3 \le \frac{\nu}{2}\|\mathcal{R}_1\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2 + C\nu^{-1}\|\omega\|_{H^9}^4.
$$

Then we estimate N_4. Similar as in N_3, N_4 can be written as

$$
N_4 = -2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i u_2 \partial_2\partial_2^{9-i}\omega\partial_2^9\omega\, dx
$$

$$
\begin{aligned}
&= -2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \omega \partial_2^9 \omega \, dx \\
&= -2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \widetilde{\omega} \partial_2^9 \widetilde{\omega} \, dx - 2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \widetilde{\omega} \partial_2^9 \overline{\omega} \, dx \\
&\quad - 2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \overline{\omega} \partial_2^9 \widetilde{\omega} \, dx - 2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \overline{\omega} \partial_2^9 \overline{\omega} \, dx \\
&\triangleq N_{41} + N_{42} + N_{43} + N_{44}.
\end{aligned} \tag{4.10}
$$

According to (2.3), we get

$$
N_{44} = 0. \tag{4.11}
$$

Next we estimate N_{41}, N_{42} and N_{43} in order. For N_{41}, by Hölder's inequality, Sobolev embedding, we obtain

$$
\begin{aligned}
N_{41} &= -2\sum_{i=1}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \widetilde{\omega} \partial_2^9 \widetilde{\omega} \, dx \\
&= -2 \int_{\mathbb{T}^2} \partial_2 \widetilde{u}_2 \partial_2^9 \widetilde{\omega} \partial_2^9 \widetilde{\omega} \, dx - 2\sum_{i=2}^{7} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \widetilde{\omega} \partial_2^9 \widetilde{\omega} \, dx \\
&\quad - 2\sum_{i=8}^{9} C_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \widetilde{\omega} \partial_2^9 \widetilde{\omega} \, dx \\
&\le 2\|\partial_2 \widetilde{u}_2\|_{L^\infty} \|\partial_2^9 \widetilde{\omega}\|_{L^2}^2 + 2\sum_{i=2}^{7} C_9^i \|\partial_2^i \widetilde{u}_2\|_{L^\infty} \|\partial_2 \partial_2^{9-i} \widetilde{\omega}\|_{L^2} \|\partial_2^9 \widetilde{\omega}\|_{L^2} \\
&\quad + 2\sum_{i=8}^{9} C_9^i \|\partial_2 \partial_2^{9-i} \widetilde{\omega}\|_{L^\infty} \|\partial_2^i \widetilde{u}_2\|_{L^2} \|\partial_2^9 \widetilde{\omega}\|_{L^2} \\
&\le C\|\widetilde{\omega}\|_{H^2} \|\omega\|_{H^9}^2 + C\|\omega\|_{H^9} \|\widetilde{\omega}\|_{H^8}^2 \\
&\le C\|\widetilde{\omega}\|_{H^4} \|\omega\|_{H^9}^2 + C\|\omega\|_{H^9} \|\partial_1 \widetilde{\omega}\|_{H^8}^2 \\
&\le C\|\widetilde{\omega}\|_{H^4} \|\omega\|_{H^9}^2 + C\|\omega\|_{H^9} \|\mathcal{R}_1 \omega\|_{H^9}^2,
\end{aligned}
$$

that is

$$
N_{41} \le C\|\widetilde{\omega}\|_{H^4} \|\omega\|_{H^9}^2 + C\|\omega\|_{H^9} \|\mathcal{R}_1 \omega\|_{H^9}^2. \tag{4.12}
$$

Similarly, N_{42} is bounded by

$$N_{42} \le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \tag{4.13}$$

For N_{43}, we split it two parts,

$$\begin{aligned} N_{43} &= -2\sum_{i=1}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^9\widetilde{\omega}\,dx \\ &= -2\int_{\mathbb{T}^2}\partial_2\widetilde{u}_2\partial_2^9\overline{\omega}\partial_2^9\widetilde{\omega}\,dx - 2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^9\widetilde{\omega}\,dx \\ &\triangleq N_{431} + N_{432}. \end{aligned} \tag{4.14}$$

N_{431} is bounded by

$$\begin{aligned} N_{431} &= -2\int_{\mathbb{T}^2}\partial_2\widetilde{u}_2\partial_2^9\overline{\omega}\partial_2^9\widetilde{\omega}\,dx \\ &\le 2\|\partial_2\widetilde{u}_2\|_{L^\infty}\|\partial_2^9\overline{\omega}\|_{L^2}\|\partial_2^9\widetilde{\omega}\|_{L^2} \\ &\le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2. \end{aligned} \tag{4.15}$$

N_{432} is slightly different, using the integration by parts, we obtain

$$\begin{aligned} N_{432} &= -2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^9\widetilde{\omega}\,dx \\ &= 2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2\left(\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\right)\partial_2^8\widetilde{\omega}\,dx \\ &= 2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^8\widetilde{\omega}\,dx + 2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2^i\widetilde{u}_2\partial_2^2\partial_2^{9-i}\overline{\omega}\partial_2^8\widetilde{\omega}\,dx \\ &\triangleq N_{4321} + N_{4322}. \end{aligned} \tag{4.16}$$

We split N_{4321} into two parts,

$$\begin{aligned} N_{4321} &= 2\sum_{i=2}^{9}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^8\widetilde{\omega}\,dx \\ &= 2\sum_{i=2}^{8}\mathcal{C}_9^i\int_{\mathbb{T}^2}\partial_2\partial_2^i\widetilde{u}_2\partial_2\partial_2^{9-i}\overline{\omega}\partial_2^8\widetilde{\omega}\,dx + 2\int_{\mathbb{T}^2}\partial_2\partial_2^9\widetilde{u}_2\partial_2\overline{\omega}\partial_2^8\widetilde{\omega}\,dx \\ &\triangleq N_{43211} + N_{43212}. \end{aligned} \tag{4.17}$$

By Poincaré inequality, we get

$$
\begin{aligned}
N_{43211} &= 2\sum_{i=2}^{8} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2 \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx \\
&= 2\int_{\mathbb{T}^2} \partial_2 \partial_2^2 \widetilde{u}_2 \partial_2 \partial_2^7 \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx + 2\sum_{i=3}^{8} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2 \partial_2^i \widetilde{u}_2 \partial_2 \partial_2^{9-i} \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx \\
&\le 2\|\partial_2 \partial_2^2 \widetilde{u}_2\|_{L^4} \|\partial_2 \partial_2^7 \overline{\omega}\|_{L^4} \|\partial_2^8 \widetilde{\omega}\|_{L^2} \\
&\quad + 2\sum_{i=3}^{8} \mathcal{C}_9^i \|\partial_2 \partial_2^i \widetilde{u}_2\|_{L^2} \|\partial_2 \partial_2^{9-i} \overline{\omega}\|_{L^\infty} \|\partial_2^8 \widetilde{\omega}\|_{L^2} \\
&\le C\|\omega\|_{H^9} \|\widetilde{\omega}\|_{H^8}^2 \le C\|\omega\|_{H^9} \|\partial_1 \widetilde{\omega}\|_{H^8}^2 \le C\|\omega\|_{H^9} \|\mathcal{R}_1 \omega\|_{H^9}^2.
\end{aligned} \tag{4.18}
$$

By integration by parts, the divergence-free condition of u and Poincaré inequality,

$$
\begin{aligned}
N_{43212} &= 2\int_{\mathbb{T}^2} \partial_2 \partial_2^9 \widetilde{u}_2 \partial_2 \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx \\
&= -2\int_{\mathbb{T}^2} \partial_1 \partial_2^9 \widetilde{u}_1 \partial_2 \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx \\
&= 2\int_{\mathbb{T}^2} \partial_2^9 \widetilde{u}_1 \partial_2 \partial_1 \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx + 2\int_{\mathbb{T}^2} \partial_2^9 \widetilde{u}_1 \partial_2 \overline{\omega} \partial_1 \partial_2^8 \widetilde{\omega}\, dx \\
&\le 2\|\partial_2^9 \widetilde{u}_1\|_{L^2} \|\partial_2 \partial_1 \overline{\omega}\|_{L^\infty} \|\partial_2^8 \widetilde{\omega}\|_{L^2} + 2\|\partial_2^9 \widetilde{u}_1\|_{L^2} \|\partial_2 \overline{\omega}\|_{L^\infty} \|\partial_1 \partial_2^8 \widetilde{\omega}\|_{L^2} \\
&\le C\|\omega\|_{H^9} \|\widetilde{\omega}\|_{H^8}^2 + C\|\omega\|_{H^9} \|\widetilde{\omega}\|_{H^8} \|\partial_1 \widetilde{\omega}\|_{H^8} \\
&\le C\|\omega\|_{H^9} \|\partial_1 \widetilde{\omega}\|_{H^8}^2 \le C\|\omega\|_{H^9} \|\mathcal{R}_1 \omega\|_{H^9}^2.
\end{aligned} \tag{4.19}
$$

Inserting (4.18) and (4.19) into (4.17), N_{4321} is bounded by

$$
N_{4321} \le C\|\omega\|_{H^9} \|\mathcal{R}_1 \omega\|_{H^9}^2. \tag{4.20}
$$

Now we turn to estimate N_{4322},

$$
\begin{aligned}
N_{4322} &= 2\sum_{i=2}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2^2 \partial_2^{9-i} \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx \\
&= 2\int_{\mathbb{T}^2} \partial_2^2 \widetilde{u}_2 \partial_2^2 \partial_2^7 \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx + 2\sum_{i=3}^{8} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i \widetilde{u}_2 \partial_2^2 \partial_2^{9-i} \overline{\omega} \partial_2^8 \widetilde{\omega}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+2\int_{\mathbb{T}^2} \partial_2^9\widetilde{u}_2\partial_2^2\overline{\omega}\partial_2^8\widetilde{\omega}\,dx \\
&\le 2\|\partial_2^2\widetilde{u}_2\|_{L^\infty}\|\partial_2^2\partial_2^7\overline{\omega}\|_{L^2}\|\partial_2^8\widetilde{\omega}\|_{L^2}+2\sum_{i=3}^{8}C_9^i\|\partial_2^i\widetilde{u}_2\|_{L^4}\|\partial_2^2\partial_2^{9-i}\overline{\omega}\|_{L^4}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\quad+2\|\partial_2^9\widetilde{u}_2\|_{L^2}\|\partial_2^2\overline{\omega}\|_{L^\infty}\|\partial_2^8\widetilde{\omega}\|_{L^2} \\
&\le C\|\omega\|_{H^9}\|\widetilde{\omega}\|_{H^8}\le C\|\omega\|_{H^9}\|\partial_1\widetilde{\omega}\|_{H^8}^2\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \qquad (4.21)
\end{aligned}
$$

Inserting (4.20) and (4.21) into (4.16), N_{432} is bounded by

$$N_{432}\le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \qquad (4.22)$$

Again, inserting (4.15) and (4.22) into (4.14), N_{43} can be bounded by

$$N_{43}\le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2+C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \qquad (4.23)$$

Inserting (4.11), (4.12), (4.13) and (4.23) into (4.10), we obtain

$$N_4\le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2+C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.$$

Collecting the bounds for N_1 through N_4 and inserting these inequalities in (4.4), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\omega(t)\|_{H^9}^2+2\nu\|\mathcal{R}_1\omega\|_{H^9}^2 \\
&\le\frac{\nu}{2}\|\mathcal{R}_1\omega\|_{H^9}^2+C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2+C\nu^{-1}\|\omega\|_{H^9}^4+C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9} \\
&\le\nu\|\mathcal{R}_1\omega\|_{H^9}^2+C\nu^{-1}\|\omega\|_{H^9}^2\|\mathcal{R}_1\omega\|_{H^9}^2+C\nu^{-1}\|\omega\|_{H^9}^4+C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2 \qquad (4.24)\\
&\le\nu\|\mathcal{R}_1\omega\|_{H^9}^2+C\nu^{-1}\|u\|_{H^{10}}^2\|u_2\|_{H^{10}}^2+C\nu^{-1}\|u\|_{H^{10}}^4+C\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dt}\|\omega(t)\|_{H^9}^2+\nu\|\mathcal{R}_1\omega\|_{H^9}^2&\le C\nu^{-1}\|u\|_{H^{10}}^2\|u_2\|_{H^{10}}^2+C\nu^{-1}\|u\|_{H^{10}}^4+C\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2 \\
&\le\widetilde{C}\nu^{-1}\|u\|_{H^{10}}^4+\widetilde{C}\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2, \qquad (4.25)
\end{aligned}
$$

for some constant $\widetilde{C}>0$.

In addition, taking L^2-estimate for the velocity u in equation $(1.2)_1$, we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2+2\nu\|u_2\|_{L^2}^2=0. \qquad (4.26)$$

Combining (4.25) with (4.26), we obtain

$$\frac{d}{dt}\|u(t)\|_{H^{10}}^2+\nu\|u_2\|_{H^{10}}^2\le\widetilde{C}\nu^{-1}\|u\|_{H^{10}}^4+\widetilde{C}\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2.$$

By Grönwall's lemma,

$$\|u(t)\|_{H^{10}}^2 + \nu \int_0^T \|u_2(\tau)\|_{H^{10}}^2 \, d\tau$$

$$\le \|u_0\|_{H^{10}}^2 \exp\left\{ \int_0^T \widetilde{C} \left(\|\widetilde{u}(\tau)\|_{H^5} + \nu^{-1} \|u(\tau)\|_{H^{10}}^2 \right) d\tau \right\},$$

for some constant $\widetilde{C} > 0$. Which verifies (4.1). This completes the proof of Proposition 4.1. □

5. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The symmetry assumptions in (1.5) on the initial data allow us to prove a differential inequality that leads to the global stability.

The framework of the proof is still the bootstrapping argument. A crucial part is the energy inequality stated in the following proposition.

Proposition 5.1. *Assume the initial data satisfies the symmetry conditions of Theorem 1.2. Let u be the corresponding solution of (1.2). Then u satisfies*

$$\frac{d}{dt} \|u(t)\|_{H^{10}}^2 + 2\nu \|u_2\|_{H^{10}}^2 \le C_1 \|\widetilde{u}\|_{H^5} \|u\|_{H^{10}}^2 + C_2 \|u\|_{H^{10}} \|u_2\|_{H^{10}}^2. \tag{5.1}$$

We first prove Theorem 1.2 making use of Proposition 5.1. Then we verify (5.1).

Proof of Theorem 1.2. The framework of the proof is still the bootstrapping argument. We make the ansatz that, for $t > 0$,

$$\|u(t)\|_{H^{10}} \le 4\varepsilon,$$

where ε is sufficiently small such that

$$4C_2\varepsilon \le \nu \quad \text{or} \quad \varepsilon \le \frac{1}{4C_2}\nu.$$

Then (5.1) leads to

$$\frac{d}{dt} \|u(t)\|_{H^{10}}^2 + \nu \|u_2\|_{H^{10}}^2 \le C_1 \|\widetilde{u}\|_{H^5} \|u\|_{H^{10}}^2.$$

By Grönwall's Lemma,

$$\|u(t)\|_{H^{10}}^2 + \nu \int_0^t \|u_2(\tau)\|_{H^{10}}^2 \, d\tau \le \|u_0\|_{H^{10}}^2 \exp\left\{ C_1 \int_0^t \|\widetilde{u}(\tau)\|_{H^5} \, d\tau \right\}. \tag{5.2}$$

Inserting (3.1) in (5.2), one has

$$\begin{aligned}\|u(t)\|_{H^{10}}^2+\nu\int_0^t\|u_2(\tau)\|_{H^{10}}^2\,d\tau &\le \|u_0\|_{H^{10}}^2\exp\left\{C_1\int_0^t\|\widetilde{u}(\tau)\|_{H^5}\,d\tau\right\}\\ &\le \|u_0\|_{H^{10}}^2\exp\left\{C_1C\,\nu^{\frac{5}{18}}\int_0^t\varepsilon(1+\tau)^{-\frac{5}{4}}\,d\tau\right\}\\ &\le \|u_0\|_{H^{10}}^2\times \mathbf{e}^{C_3\nu^{\frac{5}{18}}\varepsilon}\\ &\le 4\|u_0\|_{H^{10}}^2,\end{aligned}$$

by further requiring ε obey

$$\mathbf{e}^{C_3\nu^{\frac{5}{18}}\varepsilon}\le 4 \quad \text{or} \quad 0<\varepsilon\le C\,\nu^{-\frac{5}{18}}$$

for suitable $C>0$. Then

$$\|u(t)\|_{H^{10}}\le 2\|u_0\|_{H^{10}}\le 2\varepsilon.$$

This completes the bootstrapping argument and thus the proof of Theorem 1.2. □

The rest of this section is to prove (5.1). We make use of the simple fact that the symmetries in (1.5) are preserved in time.

Lemma 5.2. *Assume $u_0\in H^{10}(\mathbb{T}^2)$ satisfies $\nabla\cdot u_0=0$ and the symmetries*

$$u_{0,1} \text{ is odd in } x_1, \quad \text{and } u_{0,2} \text{ is even in } x_1.$$

Let $u\in L^\infty(0,T;\,H^{10}(\mathbb{T}^2))$ be the corresponding solution of (1.2) for some $T>0$. Then for any $t\le T$, $u(t)$ obeys the same symmetries as in (1.5), which are

$$u_1 \text{ is odd in } x_1, \quad u_2 \text{ and } p \text{ are even in } x_1.$$

Proof of Lemma 5.2. This lemma is a simple consequence of the uniqueness of solutions in the Sobolev space H^{10}. Trivially the solutions to (1.2) are unique in H^{10}.

If $(u,p)=(u_1,u_2,p)$ is a solution of (1.2), then (U,P) with

$$U_1=-u_1(-x_1,x_2,t),\quad U_2=u_2(-x_1,x_2,t),\quad P=p(-x_1,x_2,t)$$

also satisfies the same equation (1.2) with the initial datum $U_0=(U_{0,1},U_{0,2})$ given by

$$U_{0,1}=-u_{0,1}(-x_1,x_2),\quad U_{0,2}=u_{0,2}(-x_1,x_2).$$

Due to the symmetries of the initial data, we have

$$U_0 = u_0.$$

By the uniqueness of solutons to (1.2), we have

$$(U, P) = (u, p),$$

which is

$$\begin{aligned}
u_1(x_1, x_2, t) &= -u_1(-x_1, x_2, t), \\
u_2(x_1, x_2, t) &= u_2(-x_1, x_2, t), \\
p(x_1, x_2, t) &= p(-x_1, x_2, t).
\end{aligned}$$

Therefore, (u, p) has the desired symmetries. □

Finally we give the proof of (5.1).

Proof of Proposition 5.1. Taking the H^9-inner product with ω in $(1.1)_1$, we obtain, after a similar process as in (4.4),

$$\begin{aligned}
&\frac{d}{dt}\|\omega(t)\|_{H^9}^2 + 2\|\mathcal{R}_1\omega\|_{H^9}^2 \\
&\approx -2\int_{\mathbb{T}^2} u\cdot\nabla\omega\omega\,dx - 2\int_{\mathbb{T}^2} \nabla^9(u\cdot\nabla\omega)\nabla^9\omega\,dx \\
&= -2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx - 2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_1^i u_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx \\
&\quad - 2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i u_1\partial_1\partial_2^{9-i}\omega\partial_2^9\omega\,dx - 2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_2^i u_2\partial_2\partial_2^{9-i}\omega\partial_2^9\omega\,dx \\
&\triangleq M_1 + M_2 + M_3 + M_4.
\end{aligned} \tag{5.3}$$

M_1, M_2 and M_4 can be similarly bounded as N_1, N_2 and N_4, respectively. That is,

$$M_1 = -2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_1^i u_1\partial_1\partial_1^{9-i}\omega\partial_1^9\omega\,dx \le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2,$$

$$M_2 = -2\sum_{i=1}^{9} \mathcal{C}_9^i \int_{\mathbb{T}^2} \partial_1^i u_2\partial_2\partial_1^{9-i}\omega\partial_1^9\omega\,dx \le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2$$

and

$$M_4 = 2\sum_{i=1}^{9} C_9^i \int\limits_{\mathbb{T}^2} \partial_2^i u_2 \partial_2 \partial_2^{9-i}\omega \partial_2^9 \omega\, dx \le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.$$

M_3 is bounded differently. Splitting u_1 into its average part $\overline{u}_1$ and its corresponding oscillation part $\widetilde{u}_1$, we have

$$\begin{aligned} M_3 &= -2\sum_{i=1}^{9} C_9^i \int\limits_{\mathbb{T}^2} \partial_2^i u_1 \partial_1 \partial_2^{9-i}\omega \partial_2^9 \omega\, dx \\ &= -2\sum_{i=1}^{9} C_9^i \int\limits_{\mathbb{T}^2} \partial_2^i \widetilde{u}_1 \partial_1 \partial_2^{9-i}\widetilde{\omega} \partial_2^9 \omega\, dx - 2\sum_{i=1}^{9} C_9^i \int\limits_{\mathbb{T}^2} \partial_2^i \overline{u}_1 \partial_1 \partial_2^{9-i}\widetilde{\omega} \partial_2^9 \omega\, dx \\ &= -2\sum_{i=1}^{9} C_9^i \int\limits_{\mathbb{T}^2} \partial_2^i \widetilde{u}_1 \partial_1 \partial_2^{9-i}\widetilde{\omega} \partial_2^9 \omega\, dx, \end{aligned}$$

where we have used that $\partial_2 \overline{u}_1 = 0$ due to $u_{0,1}$ is odd in x_1. In fact, since $u_{0,1}$ is odd in x_1 and $u_{0,2}$ is even in x_1, by Corollary 5.2, we get u_1 is odd in x_1, thus

$$\overline{u}_1 = \int\limits_{\mathbb{T}} u_1(x_1, x_2)\, dx_1 = 0$$

and

$$\partial_2 \overline{u}_1 = 0.$$

As in N_{31}, M_3 is bounded by

$$M_3 \le C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2.$$

Collecting the bounds for M_1, M_2, M_3, M_4 and inserting them in (5.3), we obtain

$$\frac{d}{dt}\|\omega(t)\|_{H^9}^2 + 2\nu\|\mathcal{R}_1\omega\|_{H^9}^2 \le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2. \tag{5.4}$$

Combining (5.4) and the bound for $\|u\|_{L^2}$ in (4.26), we have

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{H^{10}}^2 + 2\nu\|u_2\|_{H^{10}}^2 &\le C\|\widetilde{\omega}\|_{H^4}\|\omega\|_{H^9}^2 + C\|\omega\|_{H^9}\|\mathcal{R}_1\omega\|_{H^9}^2 \\ &\le C_1\|\widetilde{u}\|_{H^5}\|u\|_{H^{10}}^2 + C_2\|u\|_{H^{10}}\|u_2\|_{H^{10}}^2, \end{aligned}$$

for some constants $C_1, C_2 > 0$. This completes the proof of Proposition 5.1. □

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Data availability

No data was used for the research described in the article.

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