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Stability of the 3D Navier–Stokes equations with anisotropic dissipation

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Abstract
This paper establishes the global existence and stability of the Navier–Stokes equations with dissipation acting in the vertical direction and on the vertical average in one of the horizontal directions. This Navier–Stokes model arises in various physical contexts such as strongly stratified fluids and anisotropic turbulence models.

Keywords: anisotropic Navier–Stokes equations, global existence, stability

Mathematics subject classification: 35B35, 35B40, 35Q35, 76D03

1. Introduction

This paper examines the following 3D Navier–Stokes model with dissipation acting in the vertical direction and on the vertical average in the x_1-direction

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\partial_3^2 u + \nu\partial_1^2 \bar{u}, & x\in\mathbb{T}^3,\ t>0,\\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

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where the spatial domain is taken to the 3D periodic box $\mathbb{T}^3 = [0,1]^3$, u is the velocity filed, p is the pressure and ν is the viscosity. Here $\nu\partial_3^2 u$ is dissipation acting only in the vertical direction and $\nu\partial_1^2\bar{u}$ is dissipation in the x_1 direction applied only to the vertical average of the velocity field, where $\bar{u}$ is defined as

$$\bar{u}(x_1,x_2) = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u(x_1,x_2,x_3)\,\mathrm{d}x_3.$$

For simplicity, ∂_{x_i} will be written as ∂_i.

This type of dissipation structure appears naturally in various physical systems. In strongly stratified fluids (e.g. oceans, atmospheres), vertical mixing is suppressed, but horizontal dissipation may still act on larger scales, especially on the vertically averaged velocity, as in large-scale geophysical models (see, e.g. [15, 16]). Some turbulence models use anisotropic viscosity, where vertical viscosity is higher due to strong stratification, while horizontal mixing operates at a larger scale (see, e.g. [14]).

A natural and important question is the non-linear stability problem: given small initial data, does (1.1) admit a unique global solution that remains uniformly small for all time? This is a non-trivial issue. Mathematically, the dissipation present in (1.1) lies between one-directional and two-directional dissipation.

To illustrate, consider the 3D Navier–Stokes equations with dissipation in two directions:

$$\begin{cases}\partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta_{\mathrm{h}} u,\\ \nabla\cdot u = 0,\end{cases}$$

where Δ_{h} denotes the horizontal Laplacian operator. In this setting, the small-data well-posedness problem has been extensively studied, and global-in-time solutions have been obtained in various functional frameworks (see, e.g. [2, 6, 7, 10–12]). Broadly speaking, dissipation in two directions, together with the divergence-free condition, is sufficient to control the non-linearity.

Very recently, new approaches have been developed to better understand the precise large-time behavior of these global solutions ([7, 17]). Classical methods for studying decay rates in the Navier–Stokes equations with full dissipation such as the Fourier splitting method are no longer effective in the anisotropic setting.

However, when the dissipation of the 3D Navier–Stokes is only in a single direction, that is,

$$\begin{cases}\partial_t u + u\cdot\nabla u = -\nabla p + \nu\partial_3^2 u,\\ \nabla\cdot u = 0.\end{cases}$$

The small data global well-posedness problem in the periodic domain $\mathbb{T}^3$ or the whole space $\mathbb{R}^3$ is open. The difficulty is immediate. Dissipation in a single direction is simply not sufficient to bound the non-linearity.

This paper intends to study the global existence and stability problem on an anisotropic 3D Navier–Stokes equations. As the spatial domain we use the 3D periodic box $\mathbb{T}^3 = [0,1]^3$.

The goal is to establish the global existence as well as nonlinear stability of (1.1). As aforementioned, $\bar{f}$ denotes the vertical average of a function f and $\widetilde{f}$ the remainder, namely

$$\bar{f}(x_1,x_2) = \int_{\mathbb{T}} f(x_1,x_2,x_3)\,\mathrm{d}x_3, \quad \widetilde{f} = f - \bar{f}. \tag{1.2}$$

We establish the following theorem.

Theorem 1.1. *Consider (1.1) with $\nu > 0$. Assume $u_0 \in H^3(\mathbb{T}^3)$ with $\nabla \cdot u_0 = 0$. There exists a suitable constant $C_0 > 0$ such that, if*

$$\|u_0\|_{H^3} \leqslant C_0\,\nu,$$

then (1.1) has a unique global solution $u \in L^\infty([0,\infty); H^3(\mathbb{T}^3))$. Furthermore, u remains uniformly bounded, for any $t > 0$,

$$\|u(\cdot,t)\|_{H^2(\mathbb{T}^3)} \leqslant C_0\nu.$$

This result appears to be the first small data global existence, regularity and stability result on the 3D Navier–Stokes with the least dissipation in the periodic setting. The existing small data global well-posedness results for the periodic case all require dissipation in two directions.

The proof exploits the control of non-linearity via the vertical dissipation and the dissipation of the vertical average in the x_1-direction. Several useful tools are involved. First, we utilize the orthogonal decomposition of a function into its vertical average and the remainder part (called the oscillatory part). The oscillatory part satisfies a strong version of a Poincaré-type inequality, allowing us to bound the Sobolev norm of a function via the corresponding Sobolev norm of its vertical derivative. Second, we employ anisotropic Sobolev upper bounds for triple products. These inequalities enable us to distribute directional derivatives to suitable terms. Additionally, to make use of the dissipation in the x_1-direction of the vertical average, we further decompose the vertical average into two parts:

$$\bar{u} = \bar{\bar{u}} + \widetilde{\bar{u}},$$

where $\bar{\bar{u}}$ is the average of $\bar{u}$ in the x_1-direction. This decomposition allows us to exploit the x_1-directional dissipation of the vertical average effectively. More technical details can be found in section 3.

The remainder of the paper is organized as follows. Section 2 introduces several technical lemmas that will be used in the proof of theorem 1.1. In particular, we establish key properties of the orthogonal decomposition given in (1.2), a strong Poincaré-type inequality for $\widetilde{f}$, and several anisotropic upper bounds for triple products. Section 3 contains the detailed proof of theorem 1.1.

2. Technical lemmas

This section presents several technical lemmas to be used the proof of theorem 1.1.

The proof of theorem 1.1 makes use of the following orthogonal decomposition

$$f = \bar{f} + \widetilde{f},$$

where $\bar{f}$ denotes the vertical average of f and $\widetilde{f}$ denotes the oscillatory part,

$$\bar{f}(x_1,x_2) = \int_{\mathbb{T}} f(x_1,x_2,x_3)\,\mathrm{d}x_3 \quad \text{and} \quad \widetilde{f} = f - \bar{f}. \tag{2.1}$$

The advantage of this decomposition is that the oscillatory part $\widetilde{f}$ enjoys a strong version of the Poincare type inequalities, which allows us to control the Sobolev norm of a function by that of its derivative in x_3 direction.

In the process of estimating the non-linearity, we deal with a triple product term involving all averages by further decomposition the vertical average into two parts,

$$\bar{f} = \bar{\bar{f}} + \widetilde{\bar{f}},$$

where $\bar{\bar{f}}$ denotes the average of $\bar{f}$ in the x_1-direction, namely

$$\bar{\bar{f}}(x_2) = \int_{\mathbb{T}^2} f(x_1, x_2, x_3)\, \mathrm{d}x_3 \mathrm{d}x_1, \quad \widetilde{\bar{f}} = \bar{f} - \bar{\bar{f}}.$$

The following lemma states this fact and some other properties to be used in the proof of theorem 1.1.

Lemma 2.1. *Let $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1). The following properties hold:*

(1) *The average and oscillation commute with any derivatives, namely*

$$\overline{\partial_i f} = \partial_i \bar{f}, \quad \widetilde{\partial_i f} = \partial_i \widetilde{f}$$

As a special consequence, if u is divergence-free, $\nabla \cdot u = 0$, then $\bar{u}$ and $\widetilde{u}$ are also divergence-free, $\nabla \cdot \bar{u} = 0$ and $\nabla \cdot \widetilde{u} = 0$.

(2) *$\bar{f}$ and $\widetilde{f}$ are orthogonal. More precisely, for $f \in H^k(\mathbb{T}^3)$ with any non-negative integer k, the inner product of $\bar{f}$ and $\widetilde{f}$ in H^k is zero,*

$$\int_{\mathbb{T}^3} \partial^\alpha \bar{f}(x)\, \partial^\alpha \widetilde{f}(x)\, \mathrm{d}x = 0$$

for any multi-index α with $|\alpha| \leqslant k$. As a special consequence,

$$\|f\|_{\dot{H}^k}^2 = \|\bar{f}\|_{\dot{H}^k}^2 + \|\tilde{f}\|_{\dot{H}^k}^2$$

and

$$\|\bar{f}\|_{H^k} \leqslant \|f\|_{H^k} \quad \text{and} \quad \|\tilde{f}\|_{H^k} \leqslant \|f\|_{H^k}.$$

(3) *$\widetilde{f}$ satisfies the strong Poincaré type inequality*

$$\|\widetilde{f}\|_{L^2} \leqslant C \|\partial_3 \widetilde{f}\|_{L^2} \tag{2.2}$$

A sharp version of (2.2) needs only fractional derivative in x_3-direction, that is, for any $\sigma > 0$

$$\|\widetilde{f}\|_{L^2} \leqslant C \|\Lambda_3^\sigma \widetilde{f}\|_{L^2},$$

where $\Lambda_3^\sigma f$ is defined through its Fourier transform $|k_3|^\sigma \widehat{f}(k_1, k_2, k_3)$.

The proof of this lemma can be found in [3–5].

Throughout the rest of this paper, we will use the following anisotropic Lebesgue space notation

$$\|f\|_{L^p_{x_1} L^q_{x_2} L^r_{x_3}} := \| \| \| f \|_{L^p_{x_1}(\mathbb{T})} \|_{L^q_{x_2}(\mathbb{T})} \|_{L^r_{x_3}(\mathbb{T})}.$$

The sub-indices x_1, x_2 and x_3 are used to distinguish in which direction the norm is taken. We will also use $\|f\|_{L^p_{x_1 x_2} L^q_{x_3}}$ as

$$\|f\|_{L^p_{x_1 x_2} L^q_{x_3}} := \|f\|_{L^p_{x_1} L^p_{x_2} L^q_{x_3}}.$$

The notation for anisotropic Lebesgue and Sobolev norms should be understood similarly.

The following lemma provides an anisotropic upper bound on the integral of triple products. It is an extremely useful when we estimate the nonlinear terms of PDEs with anisotropic dissipation. Several different versions of lemma 2.2 for different type of spatial domains can be found in [1, 3, 8, 13].

Lemma 2.2. *Assume that $f, \partial_1 f, g, \partial_2 g, h, \partial_3 h$ are all in $L^2(\mathbb{T}^3)$. Then, for a constant C independent of f, g and h,*

$$\left|\int_{\mathbb{T}^3} f(x)\, g(x)\, h(x)\, \mathrm{d}x\right| \leqslant C \|f\|_{L^2}^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \, \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2} + \|\partial_2 g\|_{L^2})^{\frac{1}{2}} \times \|h\|_{L^2}^{\frac{1}{2}} (\|h\|_{L^2} + \|\partial_3 h\|_{L^2})^{\frac{1}{2}}.$$

As a special consequence, if h just has the vertical oscillatory part, then

$$\left|\int_{\mathbb{T}^3} f(x)\, g(x)\, \widetilde{h}(x)\, \mathrm{d}x\right| \leqslant C \|f\|_{L^2}^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \, \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2} + \|\partial_2 g\|_{L^2})^{\frac{1}{2}} \times \|\widetilde{h}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \widetilde{h}\|_{L^2}^{\frac{1}{2}}. \tag{2.3}$$

The proof of lemma 2.2 follows from applying Hölder's inequality in each direction and invoking the following 1D Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant C \|f\|_{H^1(\mathbb{T})}.$$

We will also use the following 2D version of the anisotropic upper bound.

Lemma 2.3. *Assume that $f, \partial_1 f, g, \partial_2 g, h$ are all in $L^2(\mathbb{T}^2)$. Then, for a constant C independent of f, g and h,*

$$\left|\int_{\mathbb{T}^2} f(x)\, g(x)\, h(x)\, \mathrm{d}x\right| \leqslant C \|f\|_{L^2}^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \, \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2} + \|\partial_2 g\|_{L^2})^{\frac{1}{2}} \, \|h\|_{L^2}.$$

3. Proof of theorem 1.1

This section proves theorem 1.1.

Proof of theorem 1.1. We remark that the local-in-time well-posedness of (1.1) in H^3 can be established using similar arguments as those for the Navier–Stokes and Euler equations. Since the detailed procedure is well documented in the book by Majda and Bertozzi [9], we focus our attention on deriving global-in-time bounds for u.

We first take the L^2-inner product of (1.1) with u to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \nu\|\partial_3 u(t)\|_{L^2}^2 + \nu\|\partial_1 \bar{u}\|_{L^2}^2 = 0, \tag{3.1}$$

where we have invoked the standard identities due to $\nabla \cdot u = 0$,

$$\int (u \cdot \nabla u) \cdot u \, \mathrm{d}x = 0, \qquad \int \nabla p \cdot u \, \mathrm{d}x = 0.$$

In addition, we have also used lemma 2.1 to obtain

$$\int \partial_1^2 \bar{u} \cdot u \, \mathrm{d}x = \int \partial_1^2 \bar{u} \cdot (\bar{u} + \widetilde{u})\, \mathrm{d}x = \int \partial_1^2 \bar{u} \cdot \bar{u} \, \mathrm{d}x = -\|\partial_1 \bar{u}\|_{L^2}^2.$$

Due to the equivalence of the two norms $\|u\|_{H^3}$ and $\|u\|_{L^2}+\|D^3u\|_{L^2}$, we just need to estimate $\|D^3u\|_{L^2}$. Recalling the norm $\|D^3u\|_{L^2}$ is comparable to $\|\Delta\omega\|_{L^2}$, where ω denotes the corresponding vorticity $\omega=\nabla\times u$. Taking the curl of (1.1), we find that ω satisfies

$$\partial_t\omega+u\cdot\nabla\omega-\omega\cdot\nabla u-\nu\partial_3^2\omega-\nu\partial_1^2\bar{\omega}=0. \tag{3.2}$$

where we have used the fact in lemma 2.1 that, the curl commutes with the average. Applying Δ to (3.2) and taking the inner product with $\Delta\omega$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\omega\|_{L^2}^2+\nu\|\partial_3\Delta\omega\|_{L^2}^2+\nu\|\partial_1\Delta\bar{\omega}\|_{L^2}^2=I_1+I_2,$$

where

$$I_1=-\int\Delta\left(u\cdot\nabla\omega\right)\cdot\Delta\omega\,\mathrm{d}x,$$
$$I_2=\int\Delta\left(\omega\cdot\nabla u\right)\cdot\Delta\omega\,\mathrm{d}x.$$

Due to $\nabla\cdot u=0$,

$$I_1=-\int\Delta u\cdot\nabla\omega\cdot\Delta\omega\,\mathrm{d}x-2\int\nabla u\cdot\nabla\nabla\omega\cdot\Delta\omega\,\mathrm{d}x:=I_{11}+I_{12}$$

and

$$I_2=\int\Delta\omega\cdot\nabla u\cdot\Delta\omega\,\mathrm{d}x+2\int\nabla\omega\cdot\nabla\nabla u\cdot\Delta\omega\,\mathrm{d}x+\int\omega\cdot\nabla\Delta u\cdot\Delta\omega\,\mathrm{d}x$$
$$:=I_{21}+I_{22}+I_{23}.$$

We will use the following estimates to bound all these terms.

Lemma 3.1. *Assume that f,g,h are all elements of matrix $(\partial_ju^k)_{3\times3}$. Then, for a constant C independent of f, g and h,*

$$\left|\int_{\mathbb{T}^3}f(x)\,\partial_{lj}g(x)\,\partial_{ik}h(x)\,\mathrm{d}x\right|+\left|\int_{\mathbb{T}^3}\partial_lf(x)\,\partial_jg(x)\,\partial_{ik}h(x)\,\mathrm{d}x\right|$$
$$\leqslant C\|\omega\|_{H^2}\left(\|\partial_3\Delta\omega\|_{L^2}^2+\|\partial_1\Delta\bar{\omega}\|_{L^2}^2\right),$$

as long as

$$\int_{\mathbb{T}^3}\partial_l\bar{\bar{f}}\partial_j\bar{\bar{g}}\,\partial_{ik}\bar{\bar{h}}\,\mathrm{d}x_2=\int_{\mathbb{T}^3}\bar{\bar{f}}\partial_{lj}\bar{\bar{g}}\,\partial_{ik}\bar{\bar{h}}\,\mathrm{d}x_2=0.$$

Proof of lemma 3.1. We first write

$$f=\bar{f}+\widetilde{f},\quad g=\bar{g}+\widetilde{g}\quad\text{and}\quad h=\bar{h}+\widetilde{h}.$$

The integral can be written

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^3} f(x)\,\partial_{lj}g(x)\,\partial_{ik}h(x)\,\mathrm{d}x\right| + \left|\int_{\mathbb{T}^3} \partial_l f(x)\,\partial_j g(x)\,\partial_{ik}h(x)\,\mathrm{d}x\right| \\
&\quad \leqslant \int_{\mathbb{T}^3} \left[|f(x)\,\partial_{lj}g(x)| + |\partial_l f(x)\,\partial_j g(x)|\right] |\partial_{ik}h(x)|\,\mathrm{d}x \\
&\quad \leqslant E_1 + E_2 + E_3 + E_4 + E_{51} + E_{52} + E_{53} + E_{54},
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= \int \left(|\partial_l \widetilde{f}\partial_j \widetilde{g}| + |\widetilde{f}\partial_{lj}\widetilde{g}|\right)|\partial_{ki}\widetilde{h}|\,\mathrm{d}x, &\quad E_2 &= \int \left(|\partial_l \bar{f}\partial_j \widetilde{g}| + |\bar{f}\partial_{lj}\widetilde{g}|\right)|\partial_{ki}\widetilde{h}|\,\mathrm{d}x, \\
E_3 &= \int \left(|\partial_l \widetilde{f}\partial_j \bar{g}| + |\widetilde{f}\partial_{lj}\bar{g}|\right)|\partial_{ki}\widetilde{h}|\,\mathrm{d}x, &\quad E_4 &= \int \left(|\partial_l \widetilde{f}\partial_j \widetilde{g}| + |\widetilde{f}\partial_{lj}\widetilde{g}|\right)|\partial_{ki}\bar{h}|\,\mathrm{d}x, \\
E_{51} &= \int \left(|\partial_l \widetilde{\widetilde{f}}\partial_j \widetilde{\widetilde{g}}| + |\widetilde{\widetilde{f}}\partial_{lj}\widetilde{\widetilde{g}}|\right)|\partial_{ki}\widetilde{\widetilde{h}}|\,\mathrm{d}x, &\quad E_{52} &= \int \left(|\partial_l \bar{\bar{f}}\partial_j \widetilde{\widetilde{g}}| + |\bar{\bar{f}}\partial_{lj}\widetilde{\widetilde{g}}|\right)|\partial_{ki}\widetilde{\widetilde{h}}|\,\mathrm{d}x, \\
E_{53} &= \int \left(|\partial_l \widetilde{\widetilde{f}}\partial_j \bar{\bar{g}}| + |\widetilde{\widetilde{f}}\partial_{lj}\bar{\bar{g}}|\right)|\partial_{ki}\widetilde{\widetilde{h}}|\,\mathrm{d}x, &\quad E_{54} &= \int \left(|\partial_l \widetilde{\widetilde{f}}\partial_j \widetilde{\widetilde{g}}| + |\widetilde{\widetilde{f}}\partial_{lj}\widetilde{\widetilde{g}}|\right)|\partial_{ki}\bar{\bar{h}}|\,\mathrm{d}x.
\end{aligned}
$$

We will use lemma 2.2 and then lemma 2.1 to obtain bound for these terms. By (2.3) in lemma 2.2 and Hölder's inequality,

$$
|E_1| \leqslant C\|\partial_l\widetilde{f}\|_{L^2}^{\frac12}\|\partial_l\widetilde{f}\|_{H^1}^{\frac12}\|\partial_j\widetilde{g}\|_{L^2}^{\frac12}\|\partial_j\widetilde{g}\|_{H^1}^{\frac12}\|\partial_{ki}\widetilde{h}\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde{h}\|_{L^2}^{\frac12} + C\|\widetilde{f}\|_{L^\infty}\|\partial_{lj}\widetilde{g}\|_{L^2}\|\partial_{ki}\widetilde{h}\|_{L^2}.
$$

By (2.2) in lemma 2.1,

$$
|E_1| \leqslant C\|\partial_l\widetilde{f}\|_{H^1}\|\partial_{3j}\widetilde{g}\|_{H^1}\|\partial_{3ki}\widetilde{h}\|_{L^2} + C\|\widetilde{f}\|_{H^2}\|\partial_{3lj}\widetilde{g}\|_{L^2}\|\partial_{3ki}\widetilde{h}\|_{L^2}.
$$

Recall the fact that $\|\omega\|_{L^2} = \|\nabla u\|_{L^2}$ for $\omega = \nabla\times u$ with $\nabla\cdot u = 0$. Since f, g, h are all elements of matrix $(\partial_j u^k)_{3\times3}$,

$$
\|\partial_l\widetilde{f}\|_{H^1} \leqslant C\|\omega\|_{H^2}, \quad \|\partial_{3j}\widetilde{g}\|_{H^1}, \|\partial_{3ki}\widetilde{h}\|_{L^2} \leqslant C\|\partial_3\Delta\omega\|_{L^2}.
$$

Thus,

$$
\begin{aligned}
|E_1| &\leqslant C\|\omega\|_{H^2}\|\partial_3\Delta\omega\|_{L^2}^2 \\
&\leqslant C\|\omega\|_{H^2}\left(\|\partial_3\Delta\omega\|_{L^2}^2 + \|\partial_1\Delta\bar{\omega}\|_{L^2}^2\right).
\end{aligned}
$$

The estimates of E_2 and E_3 are very similar to that for E_1. More specifically,

$$
\begin{aligned}
|E_2| &\leqslant C\|\partial_l\bar{f}\|_{L^2}^{\frac12}\|\partial_l\bar{f}\|_{H^1}^{\frac12}\|\partial_j\widetilde{g}\|_{L^2}^{\frac12}\|\partial_j\widetilde{g}\|_{H^1}^{\frac12}\|\partial_{ki}\widetilde{h}\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde{h}\|_{L^2}^{\frac12} + C\|\bar{f}\|_{L^\infty}\|\partial_{lj}\widetilde{g}\|_{L^2}\|\partial_{ki}\widetilde{h}\|_{L^2} \\
&\leqslant C\|\partial_l\bar{f}\|_{H^1}\|\partial_{3j}\widetilde{g}\|_{H^1}\|\partial_{3ki}\widetilde{h}\|_{L^2} + C\|\bar{f}\|_{L^\infty}\|\partial_{lj}\widetilde{g}\|_{L^2}\|\partial_{ki}\widetilde{h}\|_{L^2} \\
&\leqslant C\|\omega\|_{H^2}\left(\|\partial_3\Delta\omega\|_{L^2}^2 + \|\partial_1\Delta\bar{\omega}\|_{L^2}^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|E_3| &\leqslant C\|\partial_l\widetilde{f}\|_{L^2}^{\frac12}\|\partial_l\widetilde{f}\|_{H^1}^{\frac12}\|\partial_j\bar{g}\|_{L^2}^{\frac12}\|\partial_j\bar{g}\|_{H^1}^{\frac12}\|\partial_{ki}\widetilde{h}\|_{L^2}^{\frac12}\|\partial_{3ki}\widetilde{h}\|_{L^2}^{\frac12} + C\|\widetilde{f}\|_{L^\infty}\|\partial_{lj}\bar{g}\|_{L^2}\|\partial_{ki}\widetilde{h}\|_{L^2} \\
&\leqslant C\|\partial_{3l}\widetilde{f}\|_{H^1}\|\partial_j\bar{g}\|_{H^1}\|\partial_{3ki}\widetilde{h}\|_{L^2} + C\|\partial_3\widetilde{f}\|_{H^2}\|\partial_{lj}\bar{g}\|_{L^2}\|\partial_{3ki}\widetilde{h}\|_{L^2} \\
&\leqslant C\|\omega\|_{H^2}\left(\|\partial_3\Delta\omega\|_{L^2}^2 + \|\partial_1\Delta\bar{\omega}\|_{L^2}^2\right).
\end{aligned}
$$

The estimate for E_4 differs slightly. Using Hölder's inequality and the Sobolev inequality

$$\|F\|_{L^4(\mathbb{R}^3)} \leqslant C\|F\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|\nabla F\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}},$$

we have, after applying (2.2) in lemma 2.1,

$$\begin{aligned}|E_4| &\leqslant C\|\partial_l \widetilde{f}\|_{L^2}^{\frac{1}{4}} \|\partial_l \widetilde{f}\|_{H^1}^{\frac{3}{4}} \|\partial_j \widetilde{g}\|_{L^2}^{\frac{1}{4}} \|\partial_j \widetilde{g}\|_{H^1}^{\frac{3}{4}} \|\partial_{ki}\bar{h}\|_{L^2} + C\|\widetilde{f}\|_{L^\infty} \|\partial_{lj}\widetilde{g}\|_{L^2} \|\partial_{ki}\bar{h}\|_{L^2} \\ &\leqslant C\|\partial_{3l}\widetilde{f}\|_{H^1} \|\partial_{3j}\widetilde{g}\|_{H^1} \|\partial_{ki}\bar{h}\|_{L^2} + C\|\partial_3 \widetilde{f}\|_{H^2} \|\partial_{3lj}\widetilde{g}\|_{L^2} \|\partial_{ki}\bar{h}\|_{L^2} \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right).\end{aligned}$$

Since $\widetilde{\bar{f}}$, $\widetilde{\bar{g}}$, and $\widetilde{\bar{h}}$ depend only on the two variables x_1 and x_2, we first apply the triple product estimate for 2D functions from lemma 2.3, followed by lemma 2.1, to obtain

$$\begin{aligned}|E_{51}| &\leqslant C\|\partial_l \widetilde{\bar{f}}\|_{L^2}^{\frac{1}{2}} \|\partial_l \widetilde{\bar{f}}\|_{H^1}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{L^2}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{H^1}^{\frac{1}{2}} \|\partial_{ki}\widetilde{\bar{h}}\|_{L^2} + C\|\widetilde{\bar{f}}\|_{L^\infty} \|\partial_{lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\partial_l \widetilde{\bar{f}}\|_{H^1} \|\partial_{1j}\widetilde{\bar{g}}\|_{H^1} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} + C\|\widetilde{\bar{f}}\|_{H^2} \|\partial_{1lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right).\end{aligned}$$

The estimates for E_{52}, E_{52} and E_{54} are very similar to that for E_{51}. More specifically,

$$\begin{aligned}|E_{52}| &\leqslant C\|\partial_l \bar{\bar{f}}\|_{L^2}^{\frac{1}{2}} \|\partial_l \bar{\bar{f}}\|_{H^1}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{L^2}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{H^1}^{\frac{1}{2}} \|\partial_{ki}\widetilde{\bar{h}}\|_{L^2} + C\|\bar{\bar{f}}\|_{L^\infty} \|\partial_{lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\partial_l \bar{\bar{f}}\|_{H^1} \|\partial_{1j}\widetilde{\bar{g}}\|_{H^1} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} + C\|\bar{\bar{f}}\|_{H^2} \|\partial_{1lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}|E_{53}| &\leqslant C\|\partial_l \widetilde{\bar{f}}\|_{L^2}^{\frac{1}{2}} \|\partial_l \widetilde{\bar{f}}\|_{H^1}^{\frac{1}{2}} \|\partial_j \bar{\bar{g}}\|_{L^2}^{\frac{1}{2}} \|\partial_j \bar{\bar{g}}\|_{H^1}^{\frac{1}{2}} \|\partial_{ki}\widetilde{h}\|_{L^2} + C\|\widetilde{\bar{f}}\|_{L^\infty} \|\partial_{lj}\bar{\bar{g}}\|_{L^2} \|\partial_{ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\partial_{1l}\widetilde{\bar{f}}\|_{H^1} \|\partial_j \bar{\bar{g}}\|_{H^1} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} + C\|\partial_1 \widetilde{\bar{f}}\|_{H^2} \|\partial_{lj}\bar{\bar{g}}\|_{L^2} \|\partial_{1ki}\widetilde{\bar{h}}\|_{L^2} \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right)\end{aligned}$$

and

$$\begin{aligned}|E_{54}| &\leqslant C\|\partial_l \widetilde{\bar{f}}\|_{L^2}^{\frac{1}{2}} \|\partial_l \widetilde{\bar{f}}\|_{H^1}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{L^2}^{\frac{1}{2}} \|\partial_j \widetilde{\bar{g}}\|_{H^1}^{\frac{1}{2}} \|\partial_{ki}\bar{\bar{h}}\|_{L^2} + C\|\widetilde{f}\|_{L^\infty} \|\partial_{lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{ki}\bar{\bar{h}}\|_{L^2} \\ &\leqslant C\|\partial_{1l}\widetilde{\bar{f}}\|_{H^1} \|\partial_{1j}\widetilde{\bar{g}}\|_{H^1} \|\partial_{ki}\bar{\bar{h}}\|_{L^2} + C\|\partial_1 \widetilde{\bar{f}}\|_{H^2} \|\partial_{1lj}\widetilde{\bar{g}}\|_{L^2} \|\partial_{ki}\bar{\bar{h}}\|_{L^2} \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right).\end{aligned}$$

This completes the proof of lemma 3.1. □

Observe that the terms $I_{11}, I_{12}, I_{21}, I_{22}, I_{23}$ all share the same structural form as the one considered in lemma 3.1. Consequently, applying lemma 3.1, we obtain the estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\omega\|_{L^2}^2 + \nu\|\partial_3 \Delta\omega\|_{L^2}^2 + \nu\|\partial_1 \Delta\bar{\omega}\|_{L^2}^2 \\ &\leqslant C\|\omega\|_{H^2} \left(\|\partial_3 \Delta\omega\|_{L^2}^2 + \|\partial_1 \Delta\bar{\omega}\|_{L^2}^2\right).\end{aligned} \tag{3.3}$$

Adding (3.1) and (3.3), and then integrating in time, we obtain

$$\|u(t)\|_{H^3}^2 + \int_0^t (2\nu - C\|u(\tau)\|_{H^3}) (\|\partial_3 u(\tau)\|_{H^3} + \|\partial_1 \bar{u}(\tau)\|_{H^3}) \,\mathrm{d}\tau \leqslant \|u_0\|_{H^3}^2.$$

In particular, if the initial data satisfies

$$2\nu - C\|u_0\|_{H^3} \leqslant 0,$$

then $\|u(t)\|_{H^3}$ decreases in time, and hence the inequality

$$2\nu - C\|u(t)\|_{H^3} \leqslant 0$$

holds for all $t \geqslant 0$. This completes the proof of theorem 1.1. □

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Cao C and Wu J 2013 Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation *Arch. Ration. Mech. Anal.* **208** 985–1004
[2] Chemin J and Zhang P 2007 On the global wellposedness to the 3-D incompressible anisotropic Navier–Stokes equations *Commun. Math. Phys.* **272** 529–66
[3] Dong B, Wu J, Xu X and Zhu N 2021 Stability and exponential decay for the 2D anisotropic Navier-Stokes equations with horizontal dissipation *J. Math. Fluid Mech.* **23** 100
[4] Feng W, Wang W and Wu J 2023 Nonlinear stability for the 2D incompressible MHD system with fractional dissipation in the horizontal direction *J. Evol. Equ.* **23** 37
[5] Feng W, Wang W and Wu J 2024 Stability for a system of the 2D incompressible MHD equations with fractional dissipation *J. Math. Fluid Mech.* **26** 57
[6] Ji R, Luo W and Jiang L 2023 Stability of the 3D incompressible Navier–Stokes equations with fractional horizontal dissipation *Appl. Math. Comput.* **448** 127934
[7] Ji R, Wu J and Yang W 2021 Stability and optimal decay for the 3D Navier-Stokes equations with horizontal dissipation *J. Differ. Equ.* **290** 57–77
[8] Lin H, Wu J and Zhu Y 2023 Global solutions to 3D incompressible MHD system with dissipation in only one dirction *SIAM J. Math. Anal.* **55** 4570–98
[9] Majda A and Bertozzi A 2002 *Vorticity and Incompressible Flow* (Cambridge University Press)
[10] Paicu M 2005 Équation de Navier–Stokes dans des espaces critiques *Rev. Mat. Iberoamericana* **21** 179–235
[11] Paicu M and Zhang P 2011 Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces *Comm. Math. Phys.* **307** 713–59
[12] Shang H and Zhai Y 2022 Stability and large time decay for the three-dimensional anisotropic magnetohydrodynamic equations *Z. Angew. Math. Phys.* **73** 71
[13] Wu J and Zhu Y 2021 Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium *Adv. Math.* **377** 107466

[14] Majda A J and Kramer P R 1999 Simplified models for turbulent diffusion: theory, numerical modeling and applications *Phys. Rep.* **314** 237–574
[15] Pedlosky J 1987 *Geophysical Fluid Dynamics* (Springer)
[16] Salmon R 1998 *Lectures on Geophysical Fluid Dynamics* (Oxford University Press)
[17] Xu L and Zhang P 2022 Enhanced dissipation for the third component of 3D anisotropic Navier-Stokes equations *J. Differ. Equ.* **335** 464–96