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RESEARCH ARTICLE

Global Solutions of a General Hyperbolic Navier–Stokes Equations

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Abstract Whether or not classical solutions to the hyperbolic Navier–Stokes equations (NSE) can develop finite-time singularities remains a challenging open problem. For general data without smallness condition, even the L^2-norm of solutions is not known to be globally bounded in time. This paper presents a systematic approach to the global existence and stability problem by examining the difference between a general hyperbolic NSE and its corresponding Navier–Stokes counterpart. We make use of the integral representations. The functional setting is taken to be critical Sobolev spaces for the NSE. As a special consequence, any d-dimensional ($d \geq 2$) hyperbolic NSE with general fractional dissipation is shown to possess a unique global solution if the coefficient of the double-time derivative and the initial data obey a suitable constraint.

Keywords Critical Sobolev space, global well-posedness, hyperbolic Navier–Stokes equations

MSC2020 35A05, 35Q35, 76D03

1 Introduction

This paper intends to develop an effective approach to the global existence and stability problem on the following general incompressible hyperbolic Navier–Stokes equations (NSE) in $\mathbb{R}^d$,

$$\begin{cases} \gamma\partial_{tt}u + \eta(-\Delta)^\beta\partial_t u + \nu(-\Delta)^\alpha u + u\cdot\nabla u + \nabla p = 0, \quad x\in\mathbb{R}^d,\, t>0,\\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^d,\, t>0,\\ u(x,0)=u_0(x), \quad \partial_t u(x,0)=u_1(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

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where $d \geq 2$ is an integer, $\gamma > 0$, $\eta > 0$, $\beta \geq 0$, $\alpha \geq 0$ and $\nu > 0$ are real parameters, $u = u(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ denotes the velocity field and $p = p(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ the pressure. The fractional Laplacian operator $(-\Delta)^\sigma$ is defined in terms of the Fourier transform

$$\widehat{(-\Delta)^\sigma f}(\xi) = |\xi|^{2\sigma}\, \widehat{f}(\xi).$$

For notational convenience, we use $\Lambda = (-\Delta)^{\frac{1}{2}}$. Our attention will be mostly focused on the case when $\beta = 0$ and $\eta = 1$, namely the incompressible hyperbolic NSE with fractional dissipation

$$\begin{cases} \gamma\partial_{tt}u + \partial_t u + \nu(-\Delta)^\alpha u + u\cdot\nabla u + \nabla p = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ \nabla\cdot u = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

The hyperbolic NSE (1.2) is a natural modification of the corresponding NSE

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + u\cdot\nabla u + \nabla p = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ \nabla\cdot u = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

To make the propagation speed of heat transfer finite, Cattaneo [6, 7] and Vernotte [22] originally proposed replacing the heat operator $\partial_t u - \nu\Delta u$ by the damped wave operator $\gamma\partial_{tt}u + \partial_t u - \nu\Delta u$. This idea was later extended to fluid-dynamics by Carrassi and Morro and others (see [5]). The hyperbolic NSE can be derived from the NSE by replacing the the Fourier law by the law proposed by Cattaneo. One advantage of the hyperbolic NSE is that it no longer has infinite propagation speed. The paper of Coulaud, Hachicha and Raugel [9] contains a very nice description on the physical background of the hyperbolic NSE.

A general fractional Laplacian operator $(-\Delta)^\alpha$ in (1.2) and (1.3) is considered here for two reasons. The first is physical. The fractional diffusion operators can model the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., [1, 12]). Especially, (1.2) and (1.3) allow us to study long-range diffusive interactions. The second is mathematical. When α is a general fractional power, (1.2) and (1.3) allow us to examine families of equations simultaneously and the process is certainly more efficient than that with $\alpha = 1$ fixed. The study of (1.2) also helps reveal the criticality of certain fractional powers. For the fractional NSE (1.3), extensive investigations on the global regularity problem have identified $\alpha = \frac{1}{2} + \frac{d}{4}$ as the critical index (see [13–16, 18, 21, 23]). Recent extraordinary non-uniqueness results on weak solutions obtained via convex integration also revealed the criticality of certain fractional indices [4, 8, 10, 17].

Many fundamental issues on the hyperbolic NSE are far from being understood. Attention here is focused on two problems. The first is the global existence and stability problem. Whether or not solutions to the hyperbolic NSE can develop finite-time singularities is an outstanding open problem. For general data without any constraint, even the L^2-norm of solutions to the hyperbolic NSE is not known to be globally bounded in time. Furthermore, we are also interested in the global nonlinear stability. The second problem is to understand the difference between solutions to the hyperbolic NSE and the corresponding ones to the NSE. In particular, we solve the singular limit problem when $\gamma \to 0$.

We first give a simple explanation on why the global existence problem on hyperbolic NSE (1.2) is hard. Due to the presence of the term $\gamma\partial_{tt}u$ in (1.2), direct energy estimates do not lead to any global *a priori* bound on solutions to (1.2). In fact, we do not even know if the L^2-norm is bounded for all time in the two-dimensional case. If we perform the standard energy estimate on (1.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + 2\gamma\langle u, \partial_t u\rangle\right) + \nu\|\Lambda^\alpha u\|_{L^2}^2 - \gamma\|\partial_t u\|_{L^2}^2 = 0, \tag{1.4}$$

where $\langle f, g\rangle$ denotes the L^2-inner product. To eliminate the bad term $-\gamma\|\partial_t u\|_{L^2}^2$, we take the L^2-inner product of (1.2) with $\partial_t u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\gamma\|\partial_t u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2\right) + \|\partial_t u\|_{L^2}^2 = -\langle\partial_t u, u\cdot\nabla u\rangle. \tag{1.5}$$

Multiplying (1.5) by 2γ and adding to (1.4), we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|u\|_{L^2}^2 + 2\gamma\langle u, \partial_t u\rangle + 2\gamma^2\|\partial_t u\|_{L^2}^2 + 2\gamma\nu\|\nabla u\|_{L^2}^2\right) + \nu\|\Lambda^\alpha u\|_{L^2}^2 + \gamma\|\partial_t u\|_{L^2}^2\\ &= -2\gamma\langle\partial_t u, u\cdot\nabla u\rangle.\end{aligned} \tag{1.6}$$

Trivially,

$$\|u\|_{L^2}^2 + 2\gamma\langle u, \partial_t u\rangle + 2\gamma^2\|\partial_t u\|_{L^2}^2 = \|u + \gamma\partial_t u\|_{L^2}^2 + \gamma^2\|\partial_t u\|_{L^2}^2 \geq 0.$$

It then suffices to bound the right-hand side. Unfortunately, we need to control some higher-order derivatives in order to bound the term on the right-hand side of (1.6). As a consequence, we do not even know if the L^2-norm is global in time for general solutions.

The lack of global L^2-bound suggests that we should not expect general large-data global well-posedness for the hyperbolic NSE even in the 2D case. In addition, the process described above also reveals that the approach of energy estimates treats $\gamma\partial_{tt}u$ as a bad term and thus leads to the involvement of higher derivatives. A more natural idea is to treat the damped wave part in (1.2), namely

$$\gamma\partial_{tt}u + \partial_t u + \nu(-\Delta)^\alpha u$$

as a whole, and solve the damped wave equation to represent (1.2) in an integral form. This is exactly the motivation behind the approach presented in this paper.

As our first step, we solve the general linearized wave equation and convert (1.1) into an integral form. As presented in Lemma 2.1 and Corollary 2.1, (1.1) can be recast as

$$u(t) = \left(K_0 + \frac{\eta}{2}(-\Delta)^{\beta} K_1\right) u_0 + \gamma K_1 u_1 - \int_0^t K_1(t-\tau)\, \mathbb{P}(u\cdot\nabla u)(\tau)\, d\tau, \quad (1.7)$$

where K_0 and K_1 are Fourier multiplier operators defined in (2.3) and (2.4). Setting $\beta = 0$ in (1.7) yields the representation of (1.2),

$$u_\gamma(t) = \left(K_0 + \frac{1}{2}K_1\right) u_0 + \gamma K_1 u_1 - \int_0^t K_1(t-s)\, \mathbb{P}(u_\gamma\cdot\nabla u_\gamma)(s)\, ds. \quad (1.8)$$

Instead of working with the integral representation of the hyperbolic NSE (1.2) alone, we study the evolution of the difference between the hyperbolic NSE and the corresponding NSE counterpart (1.3). This practice would allow us to simultaneously tackle the global existence problem on (1.2) and assess the closeness between (1.2) and (1.3). We invert the fractional heat operator to write the NSE (1.3) in the integral form

$$u(t) = e^{-\nu(-\Delta)^{\alpha} t} u_0 - \int_0^t e^{-\nu(-\Delta)^{\alpha}(t-s)}\, \mathbb{P}(u\cdot\nabla u)(s)\, ds. \quad (1.9)$$

Taking the difference of (1.8) and (1.9) yields

$$\begin{aligned} u_\gamma - u = {} & \left(K_0 + \frac{1}{2}K_1 - e^{-\nu(-\Delta)^{\alpha} t}\right) u_0 + \gamma K_1 u_1 \\ & - \int_0^t (K_1(t-s) - e^{-\nu(-\Delta)^{\alpha}(t-s)})\mathbb{P}(u\cdot\nabla u)(s)\, ds \\ & - \int_0^t K_1(t-s)\nabla\cdot((u_\gamma - u)\otimes u_\gamma + u\otimes(u_\gamma - u))(s)\, ds. \end{aligned} \quad (1.10)$$

We evaluate this difference in the space-time space

$$X := L^4(0,T; \dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha}(\mathbb{R}^d)).$$

X is a critical space for the fractional NSE in the sense that any natural rescaling in the solution does not change the norm in X.

A few preparations are made to help facilitate the evaluation of $u_\gamma - u$ in X. We first provide pointwise estimates for $\widehat{K}_0(\xi,t)$ and $\widehat{K}_1(\xi,t)$, the kernel functions in frequency space. $\widehat{K}_0(\xi,t)$ and $\widehat{K}_1(\xi,t)$ are shown to admit different upper bounds for ξ in different frequency ranges. A precise statement is given in

Proposition 3.1. Next we assess the boundedness of the fractional heat operator $e^{-\nu(-\Delta)^\alpha t}$, $K_0(t)$ and $K_1(t)$ on the space-time space $L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}}(\mathbb{R}^d))$ with $T>0$, $2\le p\le\infty$ and $s\in\mathbb{R}$. The upper bounds are presented in Propositions 4.1, 4.2 and 4.3. These bounds reveal a remarkable fact that the heat operator $e^{-\nu(-\Delta)^\alpha t}$ and the kernel function K_1 from the wave equation actually share almost identical upper bounds.

To evaluate (1.10) in X, we take advantage of the preparations made above to derive an explicit upper bound for the difference between the fractional heat solution and the solution to the wave equation, for any $s\in\mathbb{R}$ and $2\le q\le\infty$,

$$\begin{aligned}\left\|\left(K_0+\frac{1}{2}K_1-e^{-\nu(-\Delta)^\alpha t}\right)u_0+\gamma K_1 u_1\right\|_{L^q(0,T;\dot{H}^{s+\frac{2\alpha}{q}}(\mathbb{R}^d))}\\ \le C\,\gamma^{\frac{1}{q}}\|u_0\|_{\dot{H}^s\cap\dot{H}^{s+\alpha}(\mathbb{R}^d)}+C\,\gamma\,\nu^{-\frac{1}{q}}\|u_1\|_{\dot{H}^s(\mathbb{R}^d)}.\end{aligned}\tag{1.11}$$

Note that the upper bound is explicit in γ and the initial data. A detailed derivation of (1.11) is given in the proof of Proposition 5.1. With all these preparations at our disposal, we are able to evaluate the nonlinear parts in (1.10) to obtain the following upper bound

$$\begin{aligned}\|u_\gamma-u\|_X\le\ & C\,\gamma^{\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}\cap\dot{H}^{\frac{d}{2}+1-\alpha}}+C\,\gamma\nu^{-\frac{1}{4}}\|u_1\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}}\\ &+C\,\gamma^{\frac{1}{4}}(\nu^{-1}+\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}})\|u_0\|^2_{\dot{H}^{\frac{d}{2}+1-2\alpha}\cap\dot{H}^{\frac{d}{2}+1-\alpha}}\\ &+C_1\,\nu^{-\frac{3}{4}}\|u\|_X\,\|u_\gamma-u\|_X+C_1\,\nu^{-\frac{3}{4}}\|u_\gamma-u\|_X^2.\end{aligned}\tag{1.12}$$

All constants in (1.12) are independent of T, γ and ν. Our idea is to mount a bootstrapping argument on (1.12). The presence of the linear term

$$C_1\,\nu^{-\frac{3}{4}}\|u\|_X\,\|u_\gamma-u\|_X$$

makes the process a little bit more involved. By decomposing the time interval $[0,T)$ into a finite number of sub-intervals $[0,T_1)$, $[T_1,2T_1),\ldots,[k_0T_1,T)$ for an integer $k_0>0$, and making use of the fact that the norm of u is small on each sub-interval, we obtain the following upper bound after going through an iterative process,

$$\|u_\gamma-u\|_X\le C\,\gamma^{\frac{1}{4}}\,H(u_0,u_1)\tag{1.13}$$

if, for some sufficiently small C_0,

$$\gamma^{\frac{1}{4}}\,H(u_0,u_1)\le C_0.\tag{1.14}$$

Here $H(u_0,u_1)$ is explicitly given by

$$H(u_0,u_1) := \|u_0\|_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}} + \gamma^{\frac34}\nu^{-\frac14}\|u_1\|_{\dot H^{\frac d2+1-2\alpha}} + (\nu^{-1}+\gamma^{\frac14}\nu^{-\frac12})\|u_0\|^2_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}}.$$

(1.13) and (1.14) assert that, when γ and the initial norms of u_0 and u_1 obey the constraint (1.14), then the solution u_γ to the hyperbolic NSE (1.2) and the solution u to (1.3) remain close in the sense of (1.13). This conclusion, together with the global existence result on the fractional NSE stated in Proposition 6.1, allows us to obtain the global result on the hyperbolic NSE (1.2).

Theorem 1.1. *Let $d \geq 2$ be an integer. Assume that $\nu > 0$ and α is in the range*

$$\frac23 < \alpha < \frac23 + \frac d3.$$

Consider the hyperbolic NSE (1.2) *with*

$$u_0 \in \dot H^{\frac d2+1-2\alpha}(\mathbb{R}^d) \cap \dot H^{\frac d2+1-\alpha}(\mathbb{R}^d), \quad u_1 \in \dot H^{\frac d2+1-2\alpha}(\mathbb{R}^d)$$

and $\nabla\cdot u_0 = \nabla\cdot u_1 = 0$. Define

$$H(u_0,u_1) :=\|u_0\|_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}} + \gamma^{\frac34}\nu^{-\frac14}\|u_1\|_{\dot H^{\frac d2+1-2\alpha}} + (\nu^{-1}+\gamma^{\frac14}\nu^{-\frac12})\|u_0\|^2_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}}.$$

Assume that γ and the norms of u_0 and u_1 obey the following constraint, for sufficiently small constant $C_0 > 0$,

$$\gamma^{\frac14}\, H(u_0,u_1) \leq C_0. \tag{1.15}$$

For $\frac23 < \alpha < \frac12 + \frac d4$, we further assume

$$\|u_0\|_{\dot H^{\frac d2+1-2\alpha}} \leq C_0\,\nu. \tag{1.16}$$

Then (1.2) *has a unique global solution u_γ satisfying*

$$u_\gamma \in \bigcap_{p=2}^{\infty} L^p(0,\infty;\dot H^{\frac d2+1-2\alpha+\frac{2\alpha}{p}}(\mathbb{R}^d)) \cap C([0,\infty);\dot H^{\frac d2+1-2\alpha}(\mathbb{R}^d)).$$

Furthermore, the difference between u_γ and the corresponding solution u to the NSE (1.3) *satisfies*

$$\|u_\gamma - u\|_{L^4(0,\infty;\dot H^{\frac d2+1-\frac32\alpha}(\mathbb{R}^d))} \leq C\,\gamma^{\frac14}\, H(u_0,u_1).$$

As a consequence, u_γ admits the following uniform upper bound,

$$\|u_\gamma\|_{L^4(0,\infty;\dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}(\mathbb{R}^d))} \le C\,\gamma^{\frac14}\,H(u_0,u_1) + 2C_0\,\nu,$$

which yields the global nonlinear stability.

The smallness condition (1.16) imposed for the index range $\frac{2}{3} < \alpha < \frac{1}{2} + \frac{d}{4}$ is to ensure the global existence of solutions to the NSE (1.3). A precise statement of this fact and more is presented in Proposition 6.1 in Section 6. For $\alpha \ge \frac{1}{2} + \frac{d}{4}$, the NSE (1.3) always possesses global solutions without the smallness condition (1.16), due to a well-known fact for the fractional NSE (1.3) (see, e.g., [13–16, 18, 21, 23]). We include below a lemma stating this simple fact.

Lemma 1.1. *Let $d \ge 2$ and $\alpha \ge \frac{1}{2} + \frac{d}{4}$. Then any initial data $u_0 \in L^2(\mathbb{R}^d)$ leads to a unique global solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^d))$ to (1.3). In addition, $u = u(x,t)$ is infinitely smooth for any $x \in \mathbb{R}^d$ and $t > 0$.*

To place our result in a proper perspective, we describe a few related works on the well-posedness of hyperbolic Navier–Stokes type equations. The hyperbolic NSE with standard Laplacian dissipation, namely (1.2) with $\alpha = 1$, has been studied by a number of authors. Brenier, Natalini and Puel [3] considered a 2D system of Euler equations in the 2D periodic box $\mathbb{T}^2$, which can be converted to a hyperbolic NSE in the leading order. They proved its global existence and convergence to the 2D NSE in the functional setting $H^2 \times H^1$. Based on a refinement of the energy method in [3], Hachicha [11] obtained the global existence and uniqueness of the hyperbolic NSE with a large class of initial data and the convergence to the corresponding NSE. The papers of Racke and Saal [19, 20] established the local existence and uniqueness of 2D and 3D hyperbolic Navier–Stokes type equations in $C(0,T; H^{m+2}(\mathbb{R}^d))$ for $m > \frac{d}{2}$, and the global existence for initial data $(u_0,u_1) \in (H^{m+2} \cap L^1 \cap W^{m_1+6,p}) \times (H^{m+1} \cap L^1 \cap W^{m_1+5,p})$ with $m_1 > 3$, $m \ge m_1 + 9$ and $p < \frac{4}{3}$. A very recent work of Coulaud, Hachicha and Raugel [9] shows that a 2D hyperbolic quasilinear NSE always have a unique global solution when the initial data $(u_0,u_1) \in H^{2+\eta}(\mathbb{R}^2) \times H^{1+\eta}(\mathbb{R}^2)$ with $0 < \eta < 1$. The list of results described here is by no means exhaustive.

The rest of this paper is divided into seven sections. Section 2 solves a general linear damped wave equation and derives an integral representation for (1.1). Section 3 provides pointwise upper bounds for the kernel functions K_0 and K_1 representing the solution of the linear damped wave equation in the frequency space. Section 4 shows that the fractional heat operator, K_1 and K_2 are bounded on the space $L^p(0,T; \dot H^{s+\frac{2\alpha}{p}})$ for any $\alpha \ge 0$, $2 \le p \le \infty$ and $s \in \mathbb{R}$, and explicit upper bounds are presented. Section 5 estimates the difference between the fractional heat equation and the linear damped wave equation. Section 6 states the global well-posedness result for the generalized NSE. Section 7 proves the existence part of Theorem 1.1. Section 8 proves the uniqueness part of Theorem 1.1 via the integral representation in the functional setting $L^4(0,T; \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}(\mathbb{R}^d))$.

2 Solution Representation

This section represents the solution of the general hyperbolic Navier–Stokes equations (1.1) in an integral form. To serve this purpose, we first solve a nonhomogeneous linear wave equation. Let $\gamma > 0$, $\eta > 0$, $\nu > 0$, $\alpha \geq 0$ and $\beta \geq 0$ be real numbers. Consider

$$\begin{cases} \gamma\partial_t^2 v + \eta(-\Delta)^\beta \partial_t v + \nu(-\Delta)^\alpha v = f(x,t), \quad x \in \mathbb{R}^d,\ t > 0, \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x). \end{cases} \tag{2.1}$$

(2.1) can be solved explicitly, as stated in the following lemma.

Lemma 2.1. *Let $\gamma > 0$, $\eta > 0$, $\nu > 0$, $\alpha \geq 0$ and $\beta \geq 0$ be real numbers. The solution of* (2.1) *is given by*

$$v(t) = \left(K_0(t) + \frac{\eta}{2}(-\Delta)^\beta K_1\right) v_0 + \gamma K_1(t) v_1 + \int_0^t K_1(t-\tau) f(\tau)\, d\tau \tag{2.2}$$

where K_0 and K_1 are Fourier multiplier operators defined by

$$\widehat{K_0} = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}), \tag{2.3}$$

$$\widehat{K_1} = \frac{1}{\gamma}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}} \tag{2.4}$$

with λ_+ and λ_- being the roots of the quadratic equation

$$\gamma\lambda^2 + \eta|\xi|^{2\beta}\lambda + \nu|\xi|^{2\alpha} = 0 \tag{2.5}$$

or

$$\lambda_\pm = \frac{-\eta|\xi|^{2\beta} \pm \sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}}{2\gamma}. \tag{2.6}$$

In addition, the operators K_0 and K_1 satisfy

$$K_0(0) = I, \quad K_1(0) = 0, \tag{2.7}$$

$$\partial_t K_0(0) = -\frac{\eta}{2\gamma}(-\Delta)^\beta, \quad \partial_t K_1(0) = \frac{1}{\gamma} I \tag{2.8}$$

with I being the identity operator, and

$$\partial_t K_0(t) = -\frac{\eta}{2\gamma}(-\Delta)^\beta K_0(t) + \left(\frac{\eta^2}{4\gamma}(-\Delta)^\beta - \nu(-\Delta)^{2\alpha}\right) K_1(t), \tag{2.9}$$

$$\partial_t K_1(t) = \frac{1}{\gamma} K_0(t) - \frac{\eta}{2\gamma}(-\Delta)^\beta K_1(t). \tag{2.10}$$

Proof. We first solve (2.1) in the special case when $f \equiv 0$. Taking the Fourier transform of (2.1) gives

$$\begin{cases} \gamma \partial_t^2 \widehat{v} + \eta |\xi|^{2\beta} \partial_t \widehat{v} + \nu |\xi|^{2\alpha} \widehat{v} = 0, \\ \widehat{v}(\xi, 0) = \widehat{v}_0(\xi), \quad \widehat{\partial_t v}(\xi, 0) = \widehat{v}_1(\xi). \end{cases} \tag{2.11}$$

For any fixed $\xi \in \mathbb{R}^d$, (2.11) is a second-order ODE of t. The corresponding characteristic equation is (2.5) with its roots explicitly given by (2.6). In terms of $\lambda_\pm$, the solution of (2.11) can be explicitly written as

$$\widehat{v}(\xi, t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \widehat{v}_0 + \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \widehat{v}_1.$$

For the conciseness of the representation, we write

$$\widehat{K}_1 := \frac{1}{\gamma} \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}$$

and it is easy to check that

$$\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}) + \frac{\eta}{2} |\xi|^{2\beta} \widehat{K}_1 = \widehat{K}_0 + \frac{\eta}{2} |\xi|^{2\beta} \widehat{K}_1.$$

Therefore,

$$\widehat{v}(\xi, t) = \left(\widehat{K}_0 + \frac{\eta}{2} |\xi|^{2\beta} \widehat{K}_1 \right) \widehat{v}_0 + \gamma \widehat{K}_1 \widehat{v}_1,$$

which can be identified with (2.2) when $f \equiv 0$. When f is not identically zero, we use Duhamel's principle to obtain (2.2). For the sake of clarity, we give an explanation on how Duhamel's principle is applied here. Duhamel's principle states that the solution of

$$\begin{cases} \gamma \partial_t^2 v + \eta (-\Delta)^\beta \partial_t v + \nu (-\Delta)^\alpha v = f, \quad x \in \mathbb{R}^d, \, t > 0, \\ v(x, 0) = 0, \quad \partial_t v(x, 0) = 0 \end{cases} \tag{2.12}$$

can be obtained by taking the time integral of the solution

$$\begin{cases} \gamma \partial_t^2 v_1 + \eta (-\Delta)^\beta \partial_t v_1 + \nu (-\Delta)^\alpha v_1 = 0, \quad x \in \mathbb{R}^d, \, t > 0, \\ v_1(x, \tau) = 0, \quad \partial_t v_1(x, \tau) = \frac{1}{\gamma} f(x, \tau). \end{cases} \tag{2.13}$$

According to the solution formula for the case when $f \equiv 0$, the solution of (2.13) is

$$v_1(t) = \gamma K_1(t - \tau) \left(\frac{1}{\gamma} f(x, \tau) \right).$$

Therefore, the solution of (2.12) is given by

$$v(x, t) = \int_0^t K_1(t - \tau) f(x, \tau) \, d\tau.$$

The supposition principle allows us to obtain the complete formula (2.2). Finally we verify the equations (2.7), (2.8), (2.9) and (2.10). Setting $t = 0$ in (2.2) and comparing with $v(0) = v_0$ yield (2.7). Differentiating (2.2) in time and setting $t = 0$ lead to (2.8). (2.9) and (2.10) can be checked directly. This completes the proof of Lemma 2.1. □

We apply the solution formula in Lemma 2.1 to provide the integral representation of (1.1), namely

$$\begin{cases}\gamma\partial_t^2 u + \eta(-\Delta)^\beta \partial_t u + \nu(-\Delta)^\alpha u + u\cdot\nabla u + \nabla p = 0, \quad x\in\mathbb{R}^d,\, t>0,\\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x),\end{cases} \tag{2.14}$$

where u denotes the velocity and p the pressure, and the parameters $\gamma > 0$, $\eta > 0$, $\nu > 0$, $\alpha \geq 0$ and $\beta \geq 0$. Applying the Leray projection operator

$$\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$$

to (2.14) yields

$$\begin{cases}\gamma\partial_t^2 u + \eta(-\Delta)^\beta \partial_t u + \nu(-\Delta)^\alpha u = -\mathbb{P}(u\cdot\nabla u), \quad x\in\mathbb{R}^d,\, t>0,\\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x).\end{cases} \tag{2.15}$$

Applying Lemma 2.1 to (2.15) allows us to obtain an integral representation of the general hyperbolic NSE stated in the following corollary.

Corollary 2.1. *The solution of* (2.14) *can be represented in the following integral form*

$$u(t) = \Big(K_0 + \frac{\eta}{2}(-\Delta)^\beta K_1\Big)u_0 + \gamma K_1 u_1 - \int_0^t K_1(t-\tau)\,\mathbb{P}(u\cdot\nabla u)(\tau)\,d\tau,$$

where K_0 and K_1 are defined in (2.3) *and* (2.4).

3 Upper Bounds for $\widehat{K_0}$ and $\widehat{K_1}$

The kernels $\widehat{K_0}$ and $\widehat{K_1}$ behave quite differently for low and high frequencies. This section provides precise upper bounds for K_0 and K_1 in the frequency space, as stated in the following proposition.

Proposition 3.1. *Let $\gamma > 0$, $\eta > 0$, $\nu > 0$, $\alpha \geq 0$ and $\beta \geq 0$. Assume $\alpha \geq 2\beta$. Let*

$$\lambda_\pm = \frac{-\eta|\xi|^{2\beta} \pm \sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}}{2\gamma}$$

and

$$\widehat{K_0} = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}), \quad \widehat{K_1} = \frac{1}{\gamma}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}}.$$

Define, for $\alpha > 2\beta$,

$$S_1 = \left\{ \xi \in \mathbb{R}^d :\ 4\gamma\nu|\xi|^{2\alpha} \geq \frac{3}{4}\eta^2|\xi|^{4\beta} \ \textit{or}\ |\xi| \geq \left(\frac{3\eta^2}{16\gamma\nu}\right)^{\frac{1}{2\alpha-4\beta}} \right\},$$
$$S_2 = \mathbb{R}^d \setminus S_1.$$

We consider three cases:

(1) $\alpha = 2\beta$*:*

$$\lambda_\pm = \left(-\frac{\eta}{2\gamma} \pm \frac{1}{2\gamma}\sqrt{\eta^2 - 4\gamma\nu}\right)|\xi|^{2\beta}, \quad \operatorname{Re}\lambda_\pm < 0,$$
$$|\widehat{K}_0(\xi,t)| \leq e^{-c_0\frac{\eta}{\gamma}|\xi|^{2\beta}t} \quad \textit{for } c_0 > 0,$$
$$|\widehat{K}_1(\xi,t)| \leq C\,\gamma\eta^{-1}\,|\xi|^{-2\beta}\,e^{-c_0\frac{\eta}{\gamma}|\xi|^{2\beta}t}.$$

(2) $\alpha > 2\beta$ *and* $\xi \in S_1$*:*

$$\operatorname{Re}\lambda_- \leq -\frac{\eta}{2\gamma}|\xi|^{2\beta}, \quad \operatorname{Re}\lambda_+ \leq -\frac{\eta}{4\gamma}|\xi|^{2\beta},$$
$$|\widehat{K}_0(\xi,t)| \leq C\,e^{-\frac{\eta}{8\gamma}|\xi|^{2\beta}t}, \quad \left|\widehat{K}_1(\xi,t)\right| \leq C\,\gamma\eta^{-1}|\xi|^{-2\beta}\,e^{-\frac{\eta}{8\gamma}|\xi|^{2\beta}t}. \tag{3.1}$$

Alternatively, for $\xi \in S_1$,

$$\left|\widehat{K}_1(\xi,t)\right| \leq C\,\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}\,|\xi|^{-\alpha}\,e^{-\frac{\eta}{8\gamma}|\xi|^{2\beta}t}, \tag{3.2}$$

or more generally, for any $0 \leq \theta \leq 1$,

$$\left|\widehat{K}_1(\xi,t)\right| \leq C\,\gamma^{-\frac{\theta}{2}}\nu^{-\frac{\theta}{2}}\,|\xi|^{-\theta\alpha}\,e^{-\frac{\eta}{8\gamma}|\xi|^{2\beta}t}. \tag{3.3}$$

(3) $\alpha > 2\beta$ *and* $\xi \in S_2$*:*

$$\lambda_- \leq -\frac{3\eta}{4\gamma}|\xi|^{2\beta}, \quad \lambda_+ \leq -\frac{2\nu|\xi|^{2\alpha}}{\eta|\xi|^2 + \sqrt{\eta^|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}} \leq -\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}$$

and

$$|\widehat{K}_0(\xi,t)| \leq C\left(e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t} + e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t}\right),$$
$$|\widehat{K}_1(\xi,t)| \leq C\,\eta^{-1}|\xi|^{-2\beta}\left(e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t} + e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t}\right).$$

Furthermore, for $\xi \in S_2$,

$$\left|\widehat{K}_1(\xi,t)\right| \leq C\,\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}\,|\xi|^{-\alpha}e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t} + C\,\eta^{-1}|\xi|^{-2\beta}\,e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t}. \tag{3.4}$$

We remark that the alternative upper bound for $\widehat{K}_1$ is a sharper estimate and allows us to gain one derivative. This fact is very useful in the proof of our main results.

Proof. The case $\alpha = 2\beta$ follows directly from the definition of $\lambda_\pm$. We now focus on the case when $\alpha > 2\beta$.

For $\xi \in S_1$, $\mathrm{Re}\,\lambda_- \le -\frac{\eta}{2\gamma}|\xi|^{2\beta}$ follows directly from the definition of λ_-. Using the fact that, for $\xi \in S_1$,

$$\mathrm{Re}\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}} \le \frac{1}{2}\eta|\xi|^{2\beta},$$

we have

$$\mathrm{Re}\,\lambda_+ = \frac{-\eta|\xi|^{2\beta} + \mathrm{Re}\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}}{2\gamma} \le -\frac{\eta}{4\gamma}\,|\xi|^{2\beta}.$$

Then

$$|\widehat{K}_0(\xi,t)| \le C\,e^{-\frac{\eta}{8\gamma}\,|\xi|^{2\beta}t}.$$

To prove the bound for $|K_1(\xi,t)|$, we consider two cases:

$$\text{(a) } 4\gamma\nu|\xi|^{2\alpha} > \eta^2|\xi|^{4\beta}, \quad \text{(b) } \frac{3}{4}\eta^2|\xi|^{4\beta} \le 4\gamma\nu|\xi|^{2\alpha} \le \eta^2|\xi|^{4\beta}.$$

In the first case (a), $\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}$ is imaginary and

$$\begin{aligned}
\widehat{K}_1(\xi,t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}} \\
&= e^{-\frac{\eta}{2\gamma}|\xi|^{2\beta}t}\,\frac{2\sin t\sqrt{4\gamma\nu|\xi|^{2\alpha} - \eta^2|\xi|^{4\beta}}}{\sqrt{4\gamma\nu|\xi|^{2\alpha} - \eta^2|\xi|^{4\beta}}} \\
&\le 2t\,e^{-\frac{\eta}{2\gamma}|\xi|^{2\beta}t} \\
&\le C\,\gamma\,\eta^{-1}|\xi|^{-2\beta}\,e^{-\frac{\eta}{4\gamma}|\xi|^{2\beta}t},
\end{aligned} \tag{3.5}$$

where we have used $|\sin\zeta| \le |\zeta|$ for any $\zeta \in \mathbb{R}$. In the second case (b), we use the mean-value theorem to obtain, for $\lambda_- \le c \le \lambda_+$,

$$\widehat{K}_1(\xi,t) = 2t\,e^{ct} \le C\,\gamma\,\eta^{-1}|\xi|^{-2\beta}\,e^{-\frac{\eta}{4\gamma}|\xi|^{2\beta}t},$$

which yields (3.1). To obtain the alternative estimate (3.2), we consider two cases:

$$4\gamma\nu|\xi|^{2\alpha} \ge 2\eta^2|\xi|^{4\beta}, \quad \frac{3}{4}\eta^2|\xi|^{4\beta} \le 4\gamma\nu|\xi|^{2\alpha} < 2\eta^2|\xi|^{4\beta}.$$

When $4\gamma\nu|\xi|^{2\alpha} \ge 2\eta^2|\xi|^{4\beta}$, the quantity $\sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}$ is imaginary and

$$\begin{aligned}|\widehat{K}_1(\xi,t)| &= e^{-\frac{\eta}{2\gamma}|\xi|^{2\beta}t}\,\frac{2|\sin t\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}|}{|\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}|}\\ &\le C\,e^{-\frac{\eta}{2\gamma}|\xi|^{2\beta}t}\,\frac{1}{\sqrt{2\gamma\nu|\xi|^{2\alpha}+2\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}}\\ &\le C\,\gamma^{-\frac12}\nu^{-\frac12}\,|\xi|^{-\alpha}\,e^{-\frac{\eta}{2\gamma}|\xi|^{2\beta}t}.\end{aligned}$$

For $\frac34\eta^2|\xi|^{4\beta}\le 4\gamma\nu|\xi|^{2\alpha}<2\eta^2|\xi|^{4\beta}$, we have

$$\frac{\sqrt3}{4}\gamma^{-\frac12}\nu^{-\frac12}|\xi|^{-\alpha}\le\eta^{-1}|\xi|^{-2\beta}<\frac{1}{\sqrt2}\gamma^{-\frac12}\nu^{-\frac12}|\xi|^{-\alpha}.$$

It then follows from the bound in (3.5) that

$$|\widehat{K}_1(\xi,t)|\le C\,\gamma\,\eta^{-1}|\xi|^{-2\beta}\,e^{-\frac{\eta}{4\gamma}|\xi|^{2\beta}t}\le C\,\gamma^{\frac12}\nu^{-\frac12}\,|\xi|^{-\alpha}\,e^{-\frac{\eta}{4\gamma}|\xi|^{2\beta}t}.$$

We now deal with the case (3), $\alpha>2\beta$ and $\xi\in S_2$. For $\xi\in S_2$ or $4\gamma\nu|\xi|^{2\alpha}\ge\frac34\eta^2|\xi|^{4\beta}$, we have

$$\sqrt{\eta^2|\xi|^{4\beta}-4\gamma\nu|\xi|^{2\alpha}}\ge\frac12\,\eta\,|\xi|^{2\beta}.$$

Clearly, $\lambda_-\le-\frac{3\eta}{4\gamma}|\xi|^{2\beta}$.

$$\begin{aligned}\lambda_+ &= -\frac{1}{2\gamma}\left(\eta\,|\xi|^{2\beta}-\sqrt{\eta^2\,|\xi|^{4\beta}-4\gamma\nu|\xi|^{2\alpha}}\right)\\ &= -\frac{1}{2\gamma}\left(\frac{4\gamma\nu|\xi|^{2\alpha}}{\eta\,|\xi|^{2\beta}+\sqrt{\eta^2\,|\xi|^{4\beta}-4\gamma\nu|\xi|^{2\alpha}}}\right)\\ &\le-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}.\end{aligned}$$

Then,

$$\begin{aligned}|\widehat{K}_0(\xi,t)| &\le C\left(e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t}+e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t}\right),\\ |\widehat{K}_1(\xi,t)| &\le C\,\eta^{-1}|\xi|^{-2\beta}\left(e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t}+e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t}\right).\end{aligned}\tag{3.6}$$

(3.4) is a consequence of (3.6) and the fact $4\gamma\nu|\xi|^{2\alpha}\ge\frac34\eta^2|\xi|^{4\beta}$ for $\xi\in S_2$. This completes the proof of Proposition 3.1. □

4 $e^{-\nu(-\Delta)^\alpha t}$, K_0 and K_1 Acting on Sobolev Functions

This section presents upper bounds on the operators $e^{-\nu(-\Delta)^\alpha t}$, K_0 and K_1 when they act on Sobolev functions. These bounds reflect how much space-time regularization these operators provide. In addition, a maximal regularity estimate is obtained for K_1.

We start with a regularization estimate of the heat operator on Sobolev functions.

Proposition 4.1. *Let $\nu > 0$, $\alpha \geq 0$, $s \in \mathbb{R}$ and $v_0 \in \dot{H}^s(\mathbb{R}^d)$. Let $T > 0$ and $f \in L^2(0,T;\dot{H}^{s-\alpha}(\mathbb{R}^d))$. Then, for any $2 \leq p \leq \infty$,*

$$\|e^{-\nu(-\Delta)^\alpha t} v_0\|_{L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}}(\mathbb{R}^d))} \leq \frac{1}{(2\nu)^{\frac{1}{p}}} \|v_0\|_{\dot{H}^s(\mathbb{R}^d)},$$

$$\left\| \int_0^t e^{-\nu(-\Delta)^\alpha (t-\tau)} f(\tau)\, d\tau \right\|_{L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}}(\mathbb{R}^d))} \leq \frac{1}{\nu^{\frac{1}{2}+\frac{1}{p}}} \|f\|_{L^2(0,T;\dot{H}^{s-\alpha}(\mathbb{R}^d))}.$$

In addition,

$$e^{-\nu(-\Delta)^\alpha t} v_0, \ \int_0^t e^{-\nu(-\Delta)^\alpha (t-\tau)} f(\tau)\, d\tau \in C([0,T];\dot{H}^s(\mathbb{R}^d)).$$

Proof. We first consider the two special cases $p = 2$ and $p = \infty$. The general case $2 \leq p \leq \infty$ can be shown via interpolation. When $p = 2$,

$$\begin{aligned}
\|e^{-\nu(-\Delta)^\alpha t} v_0\|^2_{L^2(0,T;\dot{H}^{s+\alpha}(\mathbb{R}^d))} &= \int_0^T \int_{\mathbb{R}^d} |\xi|^{2(s+\alpha)}\, e^{-2\nu t|\xi|^{2\alpha}}\, |\widehat{v}_0|^2\, d\xi dt \\
&= \int_{\mathbb{R}^d} |\xi|^{2(s+\alpha)}\, |\widehat{v}_0|^2 \int_0^T e^{-2\nu t|\xi|^{2\alpha}} dt d\xi \\
&\leq \frac{1}{2\nu} \int_{\mathbb{R}^d} |\xi|^{2(s+\alpha)-2\alpha}\, |\widehat{v}_0|^2\, d\xi = \frac{1}{2\nu} \|v_0\|^2_{\dot{H}^s(\mathbb{R}^d)}
\end{aligned}$$

or

$$\|e^{-\nu(-\Delta)^\alpha t} v_0\|_{L^2(0,T;\dot{H}^{s+\alpha}(\mathbb{R}^d))} \leq \frac{1}{(2\nu)^{1/2}} \|v_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$

When $p = \infty$,

$$\|e^{-\nu(-\Delta)^\alpha t} v_0\|_{L^\infty(0,T;\dot{H}^s(\mathbb{R}^d))} = \sup_{0\leq t\leq T} \|e^{-\nu t|\xi|^{2\alpha}} \widehat{v}_0\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|v_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$

By the interpolation inequality,

$$\begin{aligned}
\|e^{-\nu(-\Delta)^\alpha t} v_0\|_{L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}})} &\leq \|e^{-\nu(-\Delta)^\alpha t} v_0\|^{\frac{2}{p}}_{L^2(0,T;\dot{H}^{s+\alpha})} \|e^{-\nu(-\Delta)^\alpha t} v_0\|^{1-\frac{2}{p}}_{L^\infty(0,T;\dot{H}^s)} \\
&\leq \left(\frac{1}{(2\nu)^{1/2}} \|v_0\|_{\dot{H}^s} \right)^{\frac{2}{p}} \left(\|v_0\|_{\dot{H}^s} \right)^{1-\frac{2}{p}} \\
&= \frac{1}{(2\nu)^{1/p}} \|v_0\|_{\dot{H}^s}.
\end{aligned}$$

This establishes the first inequality. To prove the second one, we start with the case when $p = 2$. By Young's inequality for the time convolution,

$$\begin{aligned}
&\left\| \int_0^t e^{-\nu(-\Delta)^\alpha(t-\tau)} f(\tau)\,d\tau \right\|_{L^2(0,T;\dot H^{s+\alpha})} \\
&\quad\le \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^{2\alpha}} |\xi|^{s+\alpha} \widehat{f}(\xi,\tau)\,d\tau \right\|_{L^2(\mathbb{R}^d)} \right\|_{L^2(0,T)} \\
&\quad= \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^{2\alpha}} |\xi|^{s+\alpha} \widehat{f}(\xi,\tau)\,d\tau \right\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad\le \left\| \|e^{-\nu t|\xi|^{2\alpha}}\|_{L^1(0,T)} \, |\xi|^{s+\alpha} \|\widehat{f}(\xi,\tau)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad\le \left\| \frac{1}{\nu|\xi|^{2\alpha}} \, |\xi|^{s+\alpha} \|\widehat{f}(\xi,\tau)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad= \frac{1}{\nu} \left\| |\xi|^{s-\alpha} \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^d)} = \frac{1}{\nu} \|f\|_{L^2(0,T;\dot H^{s-\alpha})}.
\end{aligned}$$

For $p=\infty$, by Minkowski's inequality and Young's inequality,

$$\begin{aligned}
&\left\| \int_0^t e^{-\nu(t-\tau)|\xi|^{2\alpha}} f(\tau)\,d\tau \right\|_{L^\infty(0,T;\dot H^s)} \\
&\quad= \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^{2\alpha}} |\xi|^s \widehat{f}(\xi,\tau)\,d\tau \right\|_{L^2(\mathbb{R}^2)} \right\|_{L^\infty(0,T)} \\
&\quad\le \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^{2\alpha}} |\xi|^s \widehat{f}(\xi,\tau)\,d\tau \right\|_{L^\infty(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\quad\le \left\| \|e^{-\nu t|\xi|^{2\alpha}}\|_{L^2(0,T)} \, |\xi|^s \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\quad\le \left\| \frac{1}{(2\nu)^{1/2}|\xi|^\alpha} \, |\xi|^s \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\quad= \frac{1}{(2\nu)^{1/2}} \left\| |\xi|^{s-\alpha} \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} = \frac{1}{(2\nu)^{1/2}} \|f\|_{L^2(0,T;\dot H^{s-\alpha})}.
\end{aligned}$$

It then follows from an interpolation inequality that

$$\begin{aligned}
&\left\| \int_0^t e^{-\nu(-\Delta)^\alpha(t-\tau)} f(\tau)\,d\tau \right\|_{L^p(0,T;\dot H^{s+\frac{2\alpha}{p}})} \\
&\quad\le \left(\frac{1}{\nu} \|f\|_{L^2(0,T;\dot H^{s-\alpha})} \right)^{\frac{2}{p}} \left(\frac{1}{(2\nu)^{1/2}} \|f\|_{L^2(0,T;\dot H^{s-\alpha})} \right)^{1-\frac{2}{p}} \\
&\quad= \frac{1}{2^{\frac12-\frac1p}\nu^{\frac12+\frac1p}} \|f\|_{L^2(0,T;\dot H^{s-\alpha})}.
\end{aligned}$$

The continuity in time

$$e^{-\nu(-\Delta)^\alpha t} v_0, \ \int_0^t e^{-\nu(-\Delta)^\alpha(t-\tau)} f(\tau)\,d\tau \in C([0,T];\dot H^s)$$

follows from the dominated convergence theorem. This completes the proof of Proposition 4.1. □

The next proposition shows how much space-time regularity we gain when K_0 and K_1 act on Sobolev functions.

Proposition 4.2. *Let $\gamma > 0$, $\nu > 0$ and $s \in \mathbb{R}$.*

(1) *There is a constant $C > 0$ such that, for any $v_0 \in \dot{H}^s \cap \dot{H}^{s+\alpha}$,*

$$\|K_0 v_0\|_{L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}})} \le C\,\|v_0\|_{\dot{H}^s}^{1-\frac{2}{p}} \left(\gamma^{\frac{1}{p}} \|v_0\|_{\dot{H}^{s+\alpha}}^{\frac{2}{p}} + \nu^{-\frac{1}{p}} \|v_0\|_{\dot{H}^s}^{\frac{2}{p}}\right)$$
$$\le C\,\gamma^{\frac{1}{p}}\,\|v_0\|_{\dot{H}^s \cap \dot{H}^{s+\alpha}} + C\,\nu^{-\frac{1}{p}} \|v_0\|_{\dot{H}^s}. \tag{4.1}$$

(2) *There is a constant $C > 0$ such that, for any $v_0 \in \dot{H}^s$,*

$$\|K_1 v_0\|_{L^p(0,T;\dot{H}^{s+\frac{2\alpha}{p}})} \le C\,\nu^{-\frac{1}{p}}\,\|v_0\|_{\dot{H}^s}. \tag{4.2}$$

Proof. We first consider two special cases, $p = \infty$ and $p = 2$, and the general case follows from an interpolation inequality. For $p = \infty$, by the uniform upper bound for $\widehat{K_0}$ in Proposition 3.1,

$$\begin{aligned}\|K_0 v_0\|_{L^\infty(0,T;\dot{H}^s)} &= \left\| \left\| |\xi|^s \widehat{K_0} \widehat{v_0} \right\|_{L^2(\mathbb{R}^d)} \right\|_{L^\infty(0,T)} \\ &\le \left\| \left\| |\xi|^s \widehat{K_0} \widehat{v_0} \right\|_{L^2(S_1)} + \left\| |\xi|^s \widehat{K_0} \widehat{v_0} \right\|_{L^2(S_2)} \right\|_{L^\infty(0,T)} \\ &\le C\, \left\| |\xi|^s \widehat{v_0} \right\|_{L^2(\mathbb{R}^d)} = C\,\|v_0\|_{\dot{H}^s}.\end{aligned}$$

For $p = 2$, by the upper bound on K_0 in Proposition 3.1,

$$\begin{aligned}\|K_0 v_0\|_{L^2(0,T;\dot{H}^{s+\alpha})}^2 &= \int_0^T \int_{\mathbb{R}^d} \left| |\xi|^{s+\alpha} \widehat{K_0} \widehat{v_0} \right|^2 d\xi dt \\ &= \int_0^T \int_{S_1} \left| |\xi|^{s+\alpha} \widehat{K_0} \widehat{v_0} \right|^2 d\xi dt + \int_0^T \int_{S_2} \left| |\xi|^{s+\alpha} \widehat{K_0^\gamma} \widehat{v_0} \right|^2 d\xi dt \\ &\le \int_0^T \int_{S_1} |\xi|^{2(s+\alpha)} e^{-\frac{1}{4\gamma} t} \, |\widehat{v_0}|^2 \, d\xi dt \\ &\quad + \int_0^T \int_{S_2} |\xi|^{2(s+\alpha)} \big(e^{-\frac{3}{4\gamma} t} + e^{-\nu |\xi|^2 t}\big)^2 |\widehat{v_0}|^2 \, d\xi dt \\ &\le C\,\gamma \|v_0\|_{\dot{H}^{s+\alpha}}^2 + C\,\nu^{-1} \|v_0\|_{\dot{H}^s}^2.\end{aligned}$$

Therefore,

$$\|K_0 v_0\|_{L^2(0,T;\dot{H}^{s+\alpha})} \le C \gamma^{\frac{1}{2}} \|v_0\|_{\dot{H}^{s+\alpha}} + C\,\nu^{-\frac{1}{2}} \|v_0\|_{\dot{H}^s}.$$

(4.1) then follows from interpolation.

The proof of (4.2) is similar. For $p=\infty$,

$$\|K_1 v_0\|_{L^\infty(0,T;\dot H^s)} \le C\,\|v_0\|_{\dot H^s}.$$

For $p=2$,

$$\begin{aligned}
&\|K_1 v_0\|^2_{L^2(0,T;\dot H^{s+\alpha})}\\
&\quad\le C\,\gamma^{-1}\nu^{-1}\int_0^T\int_{S_1}|\xi|^{2(s+\alpha)}|\xi|^{-2\alpha}e^{-\frac{1}{4\gamma}t}\,|\widehat{v_0}|^2\,d\xi dt\\
&\qquad+\int_0^T\int_{S_2}|\xi|^{2(s+\alpha)}\big(\gamma^{-\frac12}\nu^{-\frac12}|\xi|^{-\alpha}e^{-\frac{3}{4\gamma}t}+e^{-\nu|\xi|^{2\alpha}t}\big)^2|\widehat{v_0}|^2\,d\xi dt\\
&\quad\le\; C\,\nu^{-1}\|v_0\|^2_{\dot H^s}.
\end{aligned}$$

Therefore,

$$\|K_1 v_0\|_{L^2(0,T;\dot H^{s+\alpha})} \le C\,\nu^{-\frac12}\|v_0\|_{\dot H^s}.$$

(4.2) then follows from interpolation. This completes the proof of Proposition 4.2. □

The following proposition assesses the maximal regularity on K_1.

Proposition 4.3. *Let $\gamma>0$, $\nu>0$, $\alpha>0$, $s\in\mathbb{R}$ and $2\le p\le\infty$. Then there is a constant $C>0$ such that, for any $T>0$ and $f\in L^2(0,T;\dot H^{s-\alpha})$,*

$$\left\|\int_0^t K_1(t-\tau)\,f(\tau)\,d\tau\right\|_{L^p(0,T;\dot H^{s+\frac{2\alpha}{p}})} \le \frac{C}{\nu^{\frac12+\frac1p}}\|f\|_{L^2(0,T;\dot H^{s-\alpha})}.$$

Proof. We start with the special case $p=\infty$,

$$\begin{aligned}
&\left\|\int_0^t K_1(t-\tau)\,f(\tau)\,d\tau\right\|_{L^\infty(0,T;\dot H^s)}\\
&\quad=\left\|\left\|\int_0^t |\xi|^s\widehat{K_1}(t-\tau)\widehat{f}(\xi,\tau)\,d\tau\right\|_{L^2(\mathbb{R}^d)}\right\|_{L^\infty(0,T)}\\
&\quad\le\left\|\left\|\int_0^t |\xi|^s\widehat{K_1}(t-\tau)\widehat{f}(\xi,\tau)\,d\tau\right\|_{L^\infty(0,T)}\right\|_{L^2(S_1)}\\
&\qquad+\left\|\left\|\int_0^t |\xi|^s\widehat{K_1}(t-\tau)\widehat{f}(\xi,\tau)\,d\tau\right\|_{L^\infty(0,T)}\right\|_{L^2(S_2)}\\
&\quad\le\left\||\xi|^s\|\widehat{K_1}(t)\|_{L^2(0,T)}\|\widehat{f}(\xi,t)\|_{L^2(0,T)}\right\|_{L^2(S_1)}\\
&\qquad+\left\||\xi|^s\|\widehat{K_1}(t)\|_{L^2(0,T)}\|\widehat{f}(\xi,t)\|_{L^2(0,T)}\right\|_{L^2(S_2)}.
\end{aligned}\tag{4.3}$$

We apply the upper bounds in Proposition 3.1. For $\xi \in S_1$,

$$\|\widehat{K_1}(t)\|_{L^2(0,T)} \le C\,\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}|\xi|^{-\alpha}\|e^{-\frac{1}{8\gamma}t}\|_{L^2(0,T)} \le C\,\nu^{-\frac{1}{2}}|\xi|^{-\alpha}.$$

For $\xi \in S_2$, by (3.4),

$$|\widehat{K_1}| \le C\,\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}|\xi|^{-\alpha}e^{-\frac{1}{8\gamma}t} + e^{-\nu|\xi|^{2\alpha}t}$$

and thus

$$\|\widehat{K_1}(t)\|_{L^2(0,T)} \le C\,\nu^{-\frac{1}{2}}|\xi|^{-\alpha}.$$

Inserting these upper bounds in (4.3), we obtain

$$\left\|\int_0^t K_1(t-\tau)\,f(\tau)\,d\tau\right\|_{L^\infty(0,T;\dot{H}^s)} \le C\,\nu^{-\frac{1}{2}}\|f\|_{L^2(0,T;\dot{H}^{s-\alpha})}.$$

In addition, for $p=2$,

$$\begin{aligned}
&\left\|\int_0^t K_1(t-\tau)\,f(\tau)\,d\tau\right\|_{L^2(0,T;\dot{H}^{s+\alpha})}\\
&= \left\|\left\||\xi|^{s+\alpha}\int_0^t \widehat{K_1}(t-\tau)\,\widehat{f}(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^d)}\right\|_{L^2(0,T)}\\
&= \left\|\left\|\int_0^t |\xi|^{2\alpha}\widehat{K_1}(t-\tau)\,|\xi|^{s-\alpha}\widehat{f}(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^d)}\right\|_{L^2(0,T)}\\
&\le \left\|\left\|\int_0^t |\xi|^{2\alpha}\widehat{K_1}(t-\tau)\,|\xi|^{s-\alpha}\widehat{f}(\tau)\,d\tau\right\|_{L^2(0,T)}\right\|_{L^2(\mathbb{R}^d)}\\
&\le \left\|\||\xi|^{2\alpha}\widehat{K_1}(t-\tau)\|_{L^1(0,T)}\||\xi|^{s-\alpha}\widehat{f}(\tau)\|_{L^2(0,T)}\right\|_{L^2(\mathbb{R}^d)}\\
&\le \left\|\||\xi|^{2\alpha}\widehat{K_1}(t-\tau)\|_{L^1(0,T)}\right\|_{L^\infty(\mathbb{R}^d)}\left\|\||\xi|^{s-\alpha}\widehat{f}(\tau)\|_{L^2(0,T)}\right\|_{L^2(\mathbb{R}^d)}\\
&\le \left\|\int_0^t |\xi|^{2\alpha}\big|\widehat{K_1}(t-\tau)\big|d\tau\right\|_{L^\infty(\mathbb{R}^d)}\|f\|_{L^2(0,T;\dot{H}^{s-\alpha})}\,. \qquad (4.4)
\end{aligned}$$

To estimate $\int_0^t |\xi|^{2\alpha}\big|\widehat{K_1}(t-\tau)\big|d\tau \le \frac{C}{\nu}$, we divide our consideration into two cases:

$$(1)\ 4\gamma\nu|\xi|^{2\alpha} > \eta^2|\xi|^{4\beta}, \qquad (2)\ 4\gamma\nu|\xi|^{2\alpha} \le \eta^2|\xi|^{4\beta}.$$

For the case (1),

$$\lambda_\pm = \frac{-\eta|\xi|^{2\beta} \pm \sqrt{\eta^2|\xi|^{4\beta} - 4\gamma\nu|\xi|^{2\alpha}}}{2\gamma} := m \pm ni \quad (m<0,\ n>0),$$

and

$$\begin{aligned}
\big|\widehat{K}_1(\xi,t)\big| &= \left|\frac{e^{\lambda_+ t}-e^{\lambda_- t}}{\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}i}\right| = \left|\frac{e^{mt}(e^{int}-e^{-int})}{\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}i}\right| \\
&= \left|\frac{2e^{mt}\sin nt}{\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}i}\right| = \left|\frac{e^{mt}\sin nt}{n\gamma}\right|.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^t |\xi|^{2\alpha}\big|\widehat{K_1}(t-\tau)\big|d\tau \\
&= \frac{|\xi|^{2\alpha}}{n\gamma}\int_0^t e^{m\tau}|\sin n\tau|d\tau \le \frac{|\xi|^{2\alpha}}{n\gamma}\int_0^\infty e^{m\tau}|\sin n\tau|d\tau \\
&= \frac{|\xi|^{2\alpha}}{n\gamma}\sum_{k=0}^\infty\left[\int_{\frac{2k\pi}{n}}^{\frac{(2k+1)\pi}{n}} e^{m\tau}|\sin n\tau|d\tau + \int_{\frac{(2k+1)\pi}{n}}^{\frac{(2k+2)\pi}{n}} e^{m\tau}|\sin n\tau|d\tau\right] \\
&= \frac{|\xi|^{2\alpha}}{n\gamma}\sum_{k=0}^\infty\left[\int_{\frac{2k\pi}{n}}^{\frac{(2k+1)\pi}{n}} e^{m\tau}\sin n\tau d\tau - \int_{\frac{(2k+1)\pi}{n}}^{\frac{(2k+2)\pi}{n}} e^{m\tau}\sin n\tau d\tau\right] \\
&= \frac{|\xi|^{2\alpha}}{(m^2+n^2)\gamma}\sum_{k=0}^\infty\left[e^{\frac{(2k+1)\pi m}{n}} - e^{\frac{2k\pi m}{n}} + e^{\frac{(2k+2)\pi m}{n}}\right] \\
&= \frac{|\xi|^{2\alpha}}{(m^2+n^2)\gamma}C_o \\
&= \frac{C_o|\xi|^{2\alpha}}{\gamma\left[\left(\frac{-\eta|\xi|^{2\beta}}{2\gamma}\right)^2 + \left(\frac{\sqrt{4\gamma\nu|\xi|^{2\alpha}-\eta^2|\xi|^{4\beta}}i}{2\gamma}\right)^2\right]} \\
&= \frac{C_0}{\nu},
\end{aligned} \tag{4.5}$$

where the series $\sum_{k=0}^\infty\left[e^{\frac{(2k+1)\pi m}{n}} - e^{\frac{2k\pi m}{n}} + e^{\frac{(2k+2)\pi m}{n}}\right]$ converges to C_0, and we also have used the fact that

$$\int_0^t e^{m\tau}\sin n\tau d\tau = \frac{e^{m\tau}(m\sin n\tau - n\cos n\tau)}{m^2+n^2}.$$

For the case (2) $4\gamma\nu|\xi|^{2\alpha} \le \eta^2|\xi|^{4\beta}$, we apply the upper bounds in Proposition 3.1,

$$\big|\widehat{K}_1(\xi,t)\big| \le C\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}|\xi|^{-\alpha}e^{-\frac{3\eta}{4\gamma}|\xi|^{2\beta}t} + C\,\eta^{-1}|\xi|^{-2\beta}e^{-\frac{\nu}{\eta}|\xi|^{2\alpha-2\beta}t},$$

and thus

$$\int_0^t |\xi|^{2\alpha}\big|\widehat{K_1}(t-\tau)\big|d\tau \le \frac{C}{\nu}. \tag{4.6}$$

Inserting these upper bounds in (4.4), (4.5) and (4.6), we obtain

$$\left\| \int_0^t K_1(t-\tau)\, f(\tau)\, d\tau \right\|_{L^2(0,T;\dot H^{s+\alpha})} \le \frac{C}{\nu} \|f\|_{L^2(0,T;\dot H^{s-\alpha})}.$$

The general case is obtained via interpolation. This proves Proposition 4.3. □

5 Difference Between the Linearized Equations

This section bounds the difference between the solution of the generalized heat equation and the corresponding solution of the damped wave equation. This upper bound is explicit in terms of the coefficient of the double time derivative term.

Proposition 5.1. *Let $\gamma > 0$, $\nu > 0$ and $\alpha \ge 0$. Let $s \in \mathbb{R}$. Assume that $v_0 \in \dot H^s \cap \dot H^{s+\alpha}(\mathbb{R}^d)$ and $v_1 \in \dot H^s(\mathbb{R}^d)$. Let v_γ be the solution of the damped wave equation*

$$\begin{cases} \gamma\partial_{tt}v_\gamma + \partial_t v_\gamma + \nu(-\Delta)^\alpha v_\gamma = 0, & x \in \mathbb{R}^d,\ t > 0, \\ v_\gamma(x,0) = v_0(x), \quad (\partial_t v_\gamma)(x,0) = v_1(x), & x \in \mathbb{R}^d. \end{cases}$$

Let v be the solution of the generalized heat equation

$$\begin{cases} \partial_t v + \nu(-\Delta)^\alpha v = 0, & x \in \mathbb{R}^d,\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Let $2 \le q \le \infty$. Then there exists a constant $C > 0$ independent of γ and ν such that, for any $T > 0$,

$$\begin{aligned} &\|v_\gamma - v\|_{L^q(0,T;\dot H^{s+\frac{2\alpha}{q}}(\mathbb{R}^d))} \\ &\quad = \left\| \left(K_0 + \frac{1}{2}K_1 - e^{-\nu(-\Delta)^\alpha t} \right) v_0 + K_1(\gamma v_1) \right\|_{L^q(0,T;\dot H^{s+\frac{2\alpha}{q}}(\mathbb{R}^d))} \\ &\quad \le C\,\gamma^{\frac{1}{q}} \|v_0\|_{\dot H^s \cap \dot H^{s+\alpha}(\mathbb{R}^d)} + C\,\gamma\,\nu^{-\frac{1}{q}} \|v_1\|_{\dot H^s(\mathbb{R}^d)}. \end{aligned}$$

Proof. Let $A := v_\gamma - v$ be the difference. By Plancherel's theorem and Minkowski's inequality, for $2 \le q \le \infty$,

$$\begin{aligned} \|A\|_{L^q(0,T;\dot H^{s+\frac{2\alpha}{q}})} &= \left\| \left\| |\xi|^{s+\frac{2\alpha}{q}} \left| \widehat{A}(\xi,t) \right| \right\|_{L^2} \right\|_{L^q(0,T)} \\ &\le \left\| |\xi|^{s+\frac{2\alpha}{q}} \left\| \widehat{A}(\xi,t) \right\|_{L^q(0,T)} \right\|_{L^2(\mathbb{R}^d)} \\ &= \left\| |\xi|^{s+\frac{2\alpha}{q}} \left\| \widehat{A}(\xi,t) \right\|_{L^q(0,T)} \right\|_{L^2(S_1)} \\ &\quad + \left\| |\xi|^{s+\frac{2\alpha}{q}} \left\| \widehat{A}(\xi,t) \right\|_{L^q(0,T)} \right\|_{L^2(S_2)}. \end{aligned} \tag{5.1}$$

To estimate $\|\widehat{A}(\xi,t)\|_{L^q(0,T)}$, we divide our consideration into two cases: $\xi \in S_1$ and $\xi \in S_2$. Here S_1 and S_2 are defined in Proposition 3.1. Setting $\beta = 0$ and $\eta = 1$ in Proposition 3.1, we have, for $\xi \in S_1$ (the high frequency case),

$$4\gamma\nu|\xi|^{2\alpha} \geq \frac{3}{4} \quad \text{or} \quad \nu^{-1}|\xi|^{-2\alpha} \leq \frac{16}{3}\gamma.$$

We take advantage of the explicit representation formula for A,

$$A = \left(K_0 + \frac{1}{2}K_1 - e^{-\nu(-\Delta)^\alpha t}\right) v_0 + K_1(\gamma v_1).$$

There is no need to reply on the difference in this case. By Proposition 3.1,

$$\begin{aligned}
\left\|\widehat{A}(\xi,t)\right\|_{L^q(0,T)} &\leq \left\|\widehat{K}_0 + \frac{1}{2}\widehat{K}_1 - e^{-\nu|\xi|^{2\alpha}t}\right\|_{L^q(0,T)} |\widehat{v}_0(\xi)| + \gamma\|\widehat{K}_1\|_{L^q(0,T)}\,|\widehat{v}_1(\xi)| \\
&\leq \left(\left\|\widehat{K}_0 + \frac{1}{2}\widehat{K}_1\right\|_{L^q(0,T)} + \|e^{-\nu|\xi|^{2\alpha}t}\|_{L^q(0,T)}\right)|\widehat{v}_0(\xi)| \\
&\quad + \gamma\|\widehat{K}_1\|_{L^q(0,T)}\,|\widehat{v}_1(\xi)| \\
&\leq C\left(\|e^{-\frac{1}{8\gamma}t}\|_{L^q(0,T)} + \|e^{-\nu|\xi|^{2\alpha}t}\|_{L^q(0,T)}\right)|\widehat{v}_0(\xi)| \\
&\quad + C\,\gamma\gamma^{-\frac{1}{q}}\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}\|e^{-\frac{1}{8\gamma}t}\|_{L^q(0,T)}\,|\widehat{v}_1(\xi)| \qquad (5.2)\\
&\leq C\left(\gamma^{\frac{1}{q}} + \nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}\right)|\widehat{v}_0(\xi)| + C\,\gamma\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}\,|\widehat{v}_1(\xi)| \\
&\leq C\,\gamma^{\frac{1}{q}}\,|\widehat{v}_0(\xi)| + C\,\gamma\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}\,|\widehat{v}_1(\xi)|, \qquad (5.3)
\end{aligned}$$

where (5.2) follows from the upper bound (3.3) with $\eta = 1$ and $\beta = 0$.

We now turn to the case when $\xi \in S_2$. We make use of the equation that A satisfies

$$\begin{cases} \gamma\partial_{tt}A + \partial_t A + \nu(-\Delta)^\alpha A = -\gamma\partial_{tt}v, \\ A(x,0) = 0, \quad (\partial_t A)(x,0) = v_1 - (\partial_t v)(x,0). \end{cases} \qquad (5.4)$$

Taking the Fourier transform of (5.4) and noticing that

$$\partial_{tt}\widehat{v}(\xi,t) = \partial_{tt}(e^{-\nu|\xi|^{2\alpha}t}\widehat{v}_0) = \nu^2|\xi|^{4\alpha}e^{-\nu|\xi|^{2\alpha}t}\widehat{v}_0, \quad \partial_t\widehat{v}(\xi,0) = -\nu|\xi|^{2\alpha}\widehat{v}_0,$$

we obtain

$$\begin{cases} \gamma\partial_{tt}\widehat{A} + \partial_t\widehat{A} + \nu|\xi|^{2\alpha}\widehat{A} = -\gamma\nu^2|\xi|^{4\alpha}e^{-\nu|\xi|^{2\alpha}t}\widehat{v}_0, \\ \widehat{A}(\xi,0) = 0, \quad \widehat{\partial_t A}(\xi,0) = \widehat{v}_1 + \nu|\xi|^{2\alpha}\widehat{v}_0. \end{cases} \qquad (5.5)$$

According to Lemma 2.1, the solution of (5.5) can be represented as

$$\widehat{A}(\xi,t) = \widehat{K}_1(\xi,t)(\gamma\widehat{v}_1 + \gamma\nu|\xi|^{2\alpha}\widehat{v}_0) - \gamma\nu^2|\xi|^{4\alpha}\int_0^t \widehat{K}_1(\xi,t-\tau)\,e^{-\nu|\xi|^{2\alpha}\tau}\widehat{v}_0\,d\tau.$$

Taking the L^q-norm in time and applying Young's inequality to the second term yield

$$\begin{aligned}\big\|\widehat{A}(\xi,t)\big\|_{L^q(0,T)} &\le \big(\gamma|\widehat{v}_1| + \gamma\nu|\xi|^{2\alpha}|\widehat{v}_0|\big)\|\widehat{K}_1\|_{L^q(0,T)} \\ &\quad + \gamma\nu^2|\xi|^{4\alpha}\|\widehat{K}_1\|_{L^q(0,T)}\,\big\|e^{-\nu|\xi|^{2\alpha}t}\big\|_{L^1}|\widehat{v}_0| \\ &= \big(\gamma|\widehat{v}_1| + 2\gamma\nu|\xi|^{2\alpha}|\widehat{v}_0|\big)\|\widehat{K}_1\|_{L^q(0,T)}. \end{aligned} \tag{5.6}$$

For $\xi \in S_2$, according to Proposition 3.1,

$$\widehat{K}_1 \le C\,e^{-\frac{3}{4\gamma}t} + C\,e^{-\nu|\xi|^{2\alpha}t}.$$

Thus

$$\|\widehat{K}_1\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + C\,(\nu|\xi|^{2\alpha})^{-\frac{1}{q}}$$

and, for $\xi \in S_2$ or $4\gamma\nu|\xi|^{2\alpha} < \frac{3}{4}$,

$$\gamma\nu|\xi|^{2\alpha}\|\widehat{K}_1\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + \gamma(\nu|\xi|^2)^{1-\frac{1}{q}} \le C\,\gamma^{\frac{1}{q}}. \tag{5.7}$$

In addition, by $4\gamma\nu|\xi|^{2\alpha} < \frac{3}{4}$ again,

$$\gamma\,\|\widehat{K}_1\|_{L^q(0,T)} \le C\,\gamma\,\gamma^{\frac{1}{q}} + C\,\gamma\,\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}} \le C\,\gamma\,\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}. \tag{5.8}$$

Inserting (5.7) and (5.8) in (5.6) yields

$$\big\|\widehat{A}(\xi,t)\big\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}}|\widehat{v}_0| + C\,\gamma\,\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}|\widehat{v}_1|. \tag{5.9}$$

Therefore, (5.3) and (5.9) together imply that, for any $\xi \in \mathbb{R}^d$,

$$\big\|\widehat{A}(\xi,t)\big\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}}|\widehat{v}_0| + C\,\gamma\,\nu^{-\frac{1}{q}}|\xi|^{-\frac{2\alpha}{q}}|\widehat{v}_1|. \tag{5.10}$$

Inserting (5.10) in (5.1) yields

$$\begin{aligned}\|A\|_{L^q(0,T;\dot{H}^{s+\frac{2\alpha}{q}})} &\le C\,\gamma^{\frac{1}{q}}\big\||\xi|^{s+\frac{2\alpha}{q}}|\widehat{v}_0|\big\|_{L^2} + C\,\gamma\nu^{-\frac{1}{q}}\,\||\xi|^s|\widehat{v}_1|\|_{L^2} \\ &= C\,\gamma^{\frac{1}{q}}\|v_0\|_{\dot{H}^s\cap\dot{H}^{s+\alpha}} + C\,\gamma\nu^{-\frac{1}{q}}\,\|v_1\|_{\dot{H}^s}.\end{aligned}$$

This completes the proof of Proposition 5.1. □

Lemma 5.1. *Let $\gamma > 0$ and $\nu > 0$. Let $2 \le q \le \infty$. There is a constant $C > 0$ independent of γ and ν such that, for any $\xi \in \mathbb{R}^d$ and any $0 < T \le \infty$,*

$$\big\|\widehat{K}_1(\xi,t) - e^{-\nu|\xi|^{2\alpha}t}\big\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}}.$$

Proof. For high frequencies, say $\xi \in S_1$, we do not need to make use of the difference since each part can be bounded suitably. For $\xi \in S_1$,

$$\begin{aligned}\left\|\widehat{K}_1(\xi,t) - e^{-\nu|\xi|^{2\alpha}t}\right\|_{L^q(0,T)} &\le \left\|\widehat{K}_1(\xi,t)\right\|_{L^q(0,T)} + \left\|e^{-\nu|\xi|^{2\alpha}t}\right\|_{L^q(0,T)} \\ &\le C\left\|e^{-\frac{1}{8\gamma}t}\right\|_{L^q(0,T)} + \left\|e^{-\nu|\xi|^{2\alpha}t}\right\|_{L^q(0,T)} \\ &\le C\gamma^{\frac{1}{q}} + C\left(\nu|\xi|^{2\alpha}\right)^{-\frac{1}{q}} \le C\,\gamma^{\frac{1}{q}}.\end{aligned}$$

Now we consider the low frequency case $\xi \in S_2$,

$$4\gamma\nu|\xi|^{2\alpha} < \frac{3}{4}.$$

We make use of the equation of

$$H = K_1 - e^{-\nu(-\Delta)^\alpha t} \quad \text{or} \quad \widehat{H} = \widehat{K}_1 - e^{-\nu|\xi|^{2\alpha}t}.$$

$\widehat{H}$ satisfies

$$\begin{aligned}&\gamma\partial_{tt}\widehat{H} + \partial_t\widehat{H} + \nu|\xi|^{2\alpha}\widehat{H} = -\gamma\nu^2|\xi|^{4\alpha}e^{-\nu|\xi|^{2\alpha}t}, \\ &\widehat{H}(\xi,0) = -1, \quad \partial_t\widehat{H}(\xi,0) = \frac{1}{\gamma} + \nu|\xi|^{2\alpha}.\end{aligned}$$

Solving this equation yields

$$\begin{aligned}\widehat{H} &= \left(\widehat{K}_0 + \frac{1}{2}\widehat{K}_1\right)(-1) + \widehat{K}_1(1+\gamma\nu|\xi|^{2\alpha}) - \gamma\nu^2|\xi|^{4\alpha}\int_0^t \widehat{K}_1(t-\tau)e^{-\nu|\xi|^{2\alpha}\tau}\,d\tau \\ &= \left(\frac{1}{2}\widehat{K}_1 - \widehat{K}_0\right) + \gamma\nu|\xi|^{2\alpha}\widehat{K}_1 - \gamma\nu^2|\xi|^{4\alpha}\int_0^t \widehat{K}_1(t-\tau)e^{-\nu|\xi|^{2\alpha}\tau}\,d\tau.\end{aligned}$$

Therefore,

$$\|\widehat{H}\|_{L^q(0,T)} \le \left\|\frac{1}{2}\widehat{K}_1 - \widehat{K}_0\right\|_{L^q(0,T)} + C\,\gamma\nu|\xi|^{2\alpha}\|\widehat{K}_1\|_{L^q(0,T)},$$

where we used Young's inequality in the estimate of the last part

$$\begin{aligned}&\gamma\nu^2|\xi|^{4\alpha}\left\|\int_0^t \widehat{K}_1(t-\tau)e^{-\nu|\xi|^{2\alpha}\tau}\,d\tau\right\|_{L^q(0,T)} \\ &\quad\le \gamma\nu^2|\xi|^{4\alpha}\|\widehat{K}_1\|_{L^q(0,\infty)}\left\|e^{-\nu|\xi|^{2\alpha}t}\right\|_{L^1(0,T)} \le \gamma\nu|\xi|^{2\alpha}\|\widehat{K}_1\|_{L^q(0,T)}.\end{aligned}$$

For $\xi \in S_2$, according to Proposition 3.1,

$$\widehat{K}_1 \le C\,e^{-\frac{3}{4\gamma}t} + C\,e^{-\nu|\xi|^{2\alpha}t}.$$

Thus

$$\|\widehat{K}_1\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + C\left(\nu|\xi|^{2\alpha}\right)^{-\frac{1}{q}}$$

and, for $\xi \in S_2$ or $4\gamma\nu|\xi|^{2\alpha} < \frac{3}{4}$,

$$\gamma\nu|\xi|^{2\alpha}\|\widehat{K}_1\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + \gamma(\nu|\xi|^{2\alpha})^{1-\frac{1}{q}} \le C\,\gamma^{\frac{1}{q}}.$$

Recall that

$$\begin{aligned}\frac{1}{2}\widehat{K}_1 - \widehat{K}_0 &= \frac{1}{2\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}}(e^{\lambda_+ t} - e^{\lambda_- t}) - \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}) \\ &= \frac{1}{2}\Big(\frac{1}{\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}} - 1\Big)e^{\lambda_+ t} - \frac{1}{2}\Big(\frac{1}{\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}} + 1\Big)e^{\lambda_- t}.\end{aligned}$$

For $\xi \in S_2$,

$$4\gamma\nu|\xi|^{2\alpha} < \frac{3}{4}, \quad \frac{1}{\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}} \le 2$$

and thus

$$\frac{1}{2}\Big(\frac{1}{\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}} - 1\Big) = \frac{2\gamma\nu|\xi|^{2\alpha}}{\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}(1+\sqrt{1-4\gamma\nu|\xi|^{2\alpha}})} \le 4\gamma\nu|\xi|^{2\alpha}.$$

Therefore, if we use the upper bounds for λ_+ and λ_- in Proposition 3.1, we have

$$\begin{aligned}&\lambda_+ = -\frac{1}{2\gamma} + \frac{1}{2\gamma}\sqrt{1-4\gamma\nu|\xi|^{2\alpha}} = -\frac{2\nu|\xi|^{2\alpha}}{1+\sqrt{1-4\gamma\nu|\xi|^{2\alpha}}} \le -\nu|\xi|^{2\alpha}, \\ &\lambda_- \le -\frac{1}{2\gamma}.\end{aligned}$$

Thus

$$\Big|\frac{1}{2}\widehat{K}_1 - \widehat{K}_0\Big| \le 4\gamma\nu|\xi|^{2\alpha}e^{-\nu|\xi|^{2\alpha}t} + \frac{3}{2}e^{-\frac{1}{2\gamma}t}$$

and

$$\begin{aligned}\Big\|\frac{1}{2}\widehat{K}_1 - \widehat{K}_0\Big\|_{L^q(0,T)} &\le C\,\gamma\nu|\xi|^{2\alpha}(\nu|\xi|^{2\alpha})^{-\frac{1}{q}} + C\,\gamma^{\frac{1}{q}} \le C\,\gamma\gamma^{-1+\frac{1}{q}} + C\,\gamma^{\frac{1}{q}} \\ &\le C\,\gamma^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of Lemma 5.1. □

6 Solutions to the Generalized Navier–Stokes Equations

For the purpose of comparing with the solution of the hyperbolic Navier–Stokes equations, this section provides a global existence and uniqueness result for the fractional Navier–Stokes equations (1.3) in the functional setting suitable for our purpose.

Let $\nu > 0$ and $\alpha \geq 0$ be real parameters. Consider the fractional Navier–Stokes equations in (1.3), namely

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = -\nu(-\Delta)^\alpha u, & x \in \mathbb{R}^d,\ t > 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{6.1}$$

For $\alpha \geq \frac{1}{2} + \frac{d}{4}$, any L^2 data leads to a unique global solution of (6.1).

Lemma 6.1. *Let $d \geq 2$ and $\alpha \geq \frac{1}{2} + \frac{d}{4}$. Then any initial data $u_0 \in L^2$ leads to a unique global solution $u \in L^\infty(0,\infty; L^2)$. In addition, $u = u(x,t)$ is infinitely smooth for any $x \in \mathbb{R}^d$ and $t > 0$.*

The following proposition asserts the global existence, uniqueness and stability of solutions to (6.1).

Proposition 6.1. *Let $d \geq 2$ be an integer. Assume that $\nu > 0$ and α in the range*

$$\frac{2}{3} < \alpha < \frac{2}{3} + \frac{d}{3}. \tag{6.2}$$

Consider (6.1) with $u_0 \in \dot{H}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ and $\nabla \cdot u_0 = 0$. For $\alpha < \frac{1}{2} + \frac{d}{4}$, we assume that u_0 satisfies, for a constant $C > 0$ independent of ν,

$$\|u_0\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}} \leq C\,\nu. \tag{6.3}$$

Then (6.1) has a unique global solution u satisfying

$$\|u(t)\|^2_{\dot{H}^{\frac{d}{2}+1-2\alpha}} + \nu \int_0^t \|u(\tau)\|^2_{\dot{H}^{\frac{d}{2}+1-\alpha}} d\tau \leq \|u_0\|^2_{\dot{H}^{\frac{d}{2}+1-2\alpha}}.$$

Furthermore, if u_0 is in the space

$$u_0 \in \dot{H}^{\frac{d}{2}+1-2\alpha} \cap \dot{H}^{\frac{d}{2}+1-\alpha}$$

and (6.3) holds for a suitable constant $C > 0$, then the solution u further satisfies

$$\|u(t)\|^2_{\dot{H}^{\frac{d}{2}+1-\alpha}} + \nu \int_0^t \|u(\tau)\|^2_{\dot{H}^{\frac{d}{2}+1}} d\tau \leq \|u_0\|^2_{\dot{H}^{\frac{d}{2}+1-\alpha}}. \tag{6.4}$$

We note that $\dot{H}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ is a critical space for (6.1). If (u,p) is a solution of (6.1), then (u_λ, p_λ) with

$$u_\lambda(x,t) = \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \quad p_\lambda(x,t) = \lambda^{2(2\alpha-1)} u(\lambda x, \lambda^{2\alpha} t)$$

is also a solution of (6.1) with the initial data $u_{0\lambda}(x) = \lambda^{2\alpha-1} u_0(\lambda x)$. The space $\dot{H}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ is critical in the sense that

$$\|u(\cdot, \lambda^{2\alpha} t)\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)} = \|u_\lambda(\cdot, t)\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)}.$$

More generally, for any $2 \le p \le \infty$,

$$\|u\|_{L^p\left(0,\infty;\dot{H}^{\frac{d}{2}+1-2\alpha+\frac{2\alpha}{p}}(\mathbb{R}^d)\right)} = \|u_\lambda\|_{L^p\left(0,\infty;\dot{H}^{\frac{d}{2}+1-2\alpha+\frac{2\alpha}{p}}(\mathbb{R}^d)\right)}.$$

The range requirement on α in (6.2) is originated from the controlling of the nonlinear term to form a closed inequality.

Proof. The proof for the global existence and uniqueness of solutions to (6.1) is similar to that for the standard Navier–Stokes equations (see [2]). The details are thus omitted. We shall only provide the proof for the high regularity estimate in (6.4). Applying $\Lambda^{\frac{d}{2}+1-\alpha}$ to (6.1) and then taking the L^2-inner product with $\Lambda^{\frac{d}{2}+1-\alpha}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{d}{2}+1-\alpha}u\|_{L^2}^2 + \nu\|\Lambda^{\frac{d}{2}+1}u\|_{L^2}^2 &= -\int \Lambda^{\frac{d}{2}+1-\alpha}(u\cdot\nabla u)\cdot\Lambda^{\frac{d}{2}+1-\alpha}u\,dx \\ &\le \left\|\Lambda^{\frac{d}{2}+1}u\right\|_{L^2}\left\|\Lambda^{\frac{d}{2}+1-2\alpha}(u\cdot\nabla u)\right\|_{L^2} \\ &\le C\left\|\Lambda^{\frac{d}{2}+1}u\right\|_{L^2}\left\|\Lambda^{\frac{d}{2}+2-2\alpha}u\right\|_{L^{p_0}}\|u\|_{L^{q_0}} \\ &\le C\left\|\Lambda^{\frac{d}{2}+1}u\right\|_{L^2}\left\|\Lambda^{\frac{d}{2}+1}u\right\|_{L^2}\left\|\Lambda^{\frac{d}{2}+1-2\alpha}u\right\|_{L^2},\end{aligned}$$

where $p_0 > 2$ and $q_0 > 2$ are given by

$$\frac{1}{p_0} = \frac{1}{2} - \frac{2\alpha-1}{d}, \quad \frac{1}{q_0} = \frac{2\alpha-1}{d}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{d}{2}+1-\alpha}u\|_{L^2}^2 + \left(\nu - C\left\|\Lambda^{\frac{d}{2}+1-2\alpha}u\right\|_{L^2}\right)\|\Lambda^{\frac{d}{2}+1}u\|_{L^2}^2 \le 0. \tag{6.5}$$

If (6.3) is satisfied with constant $C > 0$ obeying

$$C\left\|\Lambda^{\frac{d}{2}+1-2\alpha}u_0\right\|_{L^2} \le \frac{\nu}{2}, \tag{6.6}$$

then (6.5) implies that the solution u also satisfies (6.6) for all time. We then obtain (6.4). This completes the proof of Proposition 6.1. □

7 Global Existence

This section establishes the global existence part of Theorem 1.1 for the hyperbolic NSE (1.2). As described in the introduction, our idea is to examine the difference $u_\gamma - u$ between the solution u_γ of (1.2) and u of the corresponding NSE (1.3). We make use of the integral representation. We prove via the bootstrapping argument that this difference is bounded globally in time. Since the solution u of the NSE is bounded for all time, we obtain a global bound for u_γ.

The following lemma will be used to bound a product of two functions in Sobolev spaces (see [2]).

Lemma 7.1. *Let $d \geq 2$ be an integer. For any $s_1, s_2 \in \left(-\frac{d}{2}, \frac{d}{2}\right)$ and $s_1+s_2 > 0$, there exists a constant $C > 0$ such that*

$$\|f\, g\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}(\mathbb{R}^d)} \leq C\,\|f\|_{\dot{H}^{s_1}(\mathbb{R}^d)}\,\|g\|_{\dot{H}^{s_2}(\mathbb{R}^d)}, \tag{7.1}$$

where $\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}$ denotes a homogeneous Besov space. Due to the embedding $\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1} \hookrightarrow \dot{H}^{s_1+s_2-\frac{d}{2}}$, (7.1) especially implies

$$\|f\, g\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)} \leq C\,\|f\|_{\dot{H}^{s_1}(\mathbb{R}^d)}\,\|g\|_{\dot{H}^{s_2}(\mathbb{R}^d)}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The solution u to the NSE

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu(-\Delta)^\alpha u = -\nabla p, \quad x \in \mathbb{R}^d,\, t>0,\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x) \end{cases}$$

can be represented as

$$u(t) = e^{-\nu(-\Delta)^\alpha t}u_0 - \int_0^t e^{-\nu(-\Delta)^\alpha(t-s)}\,\mathbb{P}(u\cdot\nabla u)(s)\,ds. \tag{7.2}$$

By Corollary 2.1, the solution u_γ to the fractional hyperbolic NSE (1.2)

$$\begin{cases} \gamma\partial_t^2 u_\gamma + \partial_t u_\gamma + \nu(-\Delta)^\alpha u_\gamma = -\mathbb{P}(u_\gamma\cdot\nabla u_\gamma), \quad x\in\mathbb{R}^d,\, t>0,\\ u_\gamma(x,0) = u_0(x), \quad \partial_t u_\gamma(x,0) = u_1(x) \end{cases}$$

can be written as

$$u_\gamma(t) = \left(K_0 + \frac{1}{2}K_1\right)u_0 + \gamma K_1 u_1 - \int_0^t K_1(t-s)\,\mathbb{P}(u_\gamma\cdot\nabla u_\gamma)(s)\,ds. \tag{7.3}$$

Taking the difference between (7.2) and (7.3) yields

$$\begin{aligned} u_\gamma - u &= \left(K_0 + \frac{1}{2}K_1 - e^{-\nu(-\Delta)^\alpha t}\right)u_0 + \gamma K_1 u_1\\ &\quad - \int_0^t \big(K_1(t-s) - e^{-\nu(-\Delta)^\alpha(t-s)}\big)\mathbb{P}(u\cdot\nabla u)(s)\,ds\\ &\quad - \int_0^t K_1(t-s)\nabla\cdot((u_\gamma-u)\otimes u_\gamma + u\otimes(u_\gamma-u))(s)\,ds\\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{7.4}$$

We now estimate $u_\gamma - u$ in the functional space

$$X := L^4_T\dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha} := L^4\big(0,T;\dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha}(\mathbb{R}^d)\big).$$

Taking the norm of J_1 in X, applying Proposition 5.1 with $q = 4$ and $s =$

$\frac{d}{2}+1-2\alpha$, we have

$$\|J_1\|_X := \left\|\left(K_0+\frac{1}{2}K_1-e^{-\nu(-\Delta)^\alpha t}\right)u_0+\gamma K_1 u_1\right\|_X \\ \le C\,\gamma^{\frac{1}{4}}\|u_0\|_{\dot H^{\frac{d}{2}+1-2\alpha}\cap \dot H^{\frac{d}{2}+1-\alpha}} + C\,\gamma\nu^{-\frac{1}{4}}\|u_1\|_{\dot H^{\frac{d}{2}+1-2\alpha}}. \tag{7.5}$$

The estimate of the second term on the right-hand side of (7.4) is very involved, so we estimate the last term first. By Proposition 4.3 and Lemma 7.1,

$$\begin{aligned}\|J_3\|_X &\le C\,\nu^{-\frac{3}{4}}\|\nabla\cdot((u_\gamma-u)\otimes u_\gamma)\|_{L^2_T\dot H^{\frac{d}{2}+1-3\alpha}} \\ &\quad + C\,\nu^{-\frac{3}{4}}\|\nabla\cdot(u\otimes(u_\gamma-u))\|_{L^2_T\dot H^{\frac{d}{2}+1-3\alpha}} \\ &= C\,\nu^{-\frac{3}{4}}\|(u_\gamma-u)\otimes u_\gamma\|_{L^2_T\dot H^{\frac{d}{2}+2-3\alpha}} + C\,\nu^{-\frac{3}{4}}\|u\otimes(u_\gamma-u)\|_{L^2_T\dot H^{\frac{d}{2}+2-3\alpha}} \\ &\le C\,\nu^{-\frac{3}{4}}\|u_\gamma-u\|_{L^4_T\dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}\Big(\|u\|_{L^4_T\dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}+\|u_\gamma\|_{L^4_T\dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}\Big).\end{aligned} \tag{7.6}$$

We remark that the constraint on the range of α,

$$\frac{2}{3}<\alpha<\frac{1}{3}(2+d) \tag{7.7}$$

is originated from the application of Lemma 7.1. In fact, we applied Lemma 7.1 with

$$s_1=s_2=\frac{d}{2}+1-\frac{3}{2}\alpha$$

and the requirement $s_1,s_2\in\big(-\frac{d}{2},\frac{d}{2}\big)$ and $s_1+s_2>0$ yields (7.7).

We now turn to the second term on the right-hand side of (7.4).

$$\|J_2\|_X := \left\|\int_0^t \big(K_1(t-s)-e^{-\nu(-\Delta)^\alpha(t-s)}\big)\mathbb{P}(u\cdot\nabla u)(s)\,ds\right\|_X.$$

The estimate of this term is very involved. The goal here is to obtain a bound with γ to a positive power so that this term can be made small for small $\gamma>0$. We divide our consideration into the high frequency case and the low frequency case. They are handled differently. We split the spatial integral into two parts,

$$\begin{aligned}\|J_2\|_X &= \left\|\left\||\xi|^{\frac{d}{2}+1-\frac{3}{2}\alpha}\int_0^t\big(\widehat K_1(t-s)-e^{-\nu(t-s)|\xi|^{2\alpha}}\big)|\xi|\widehat{u\otimes u}\,ds\right\|_{L^2}\right\|_{L^4_T} \\ &\le \left\|\left\|\int_0^t\big(\widehat K_1(t-s)-e^{-\eta(t-s)|\xi|^2}\big)|\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha}\widehat{u\otimes u}\,ds\right\|_{L^4_T}\right\|_{L^2} \\ &=: M_1+M_2,\end{aligned}$$

where M_1 and M_2 are given by

$$M_1 = \left\| \left\| \int_0^t \left(\widehat{K}_1(t-s) - e^{-\eta(t-s)|\xi|^2}\right) |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \widehat{u\otimes u}\, ds \right\|_{L^4_T} \right\|_{L^2(S_1)},$$
$$M_2 = \left\| \left\| \int_0^t \left(\widehat{K}_1(t-s) - e^{-\eta(t-s)|\xi|^2}\right) |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \widehat{u\otimes u}\, ds \right\|_{L^4_T} \right\|_{L^2(S_2)}.$$

Recall that

$$|\xi|^\alpha \geq \frac{\sqrt{3}}{4}\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}} \quad \text{for any } \xi \in S_1,$$
$$|\xi|^\alpha < \frac{\sqrt{3}}{4}\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}} \quad \text{for any } \xi \in S_2.$$

According to (3.2) with $\eta = 1$ and $\beta = 0$ in Proposition 3.1, for $\xi \in S_1$,

$$\left|\widehat{K}_1(\xi,t)\right| \leq C\,\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}\,|\xi|^{-\alpha}\,e^{-\frac{1}{8\gamma}t}.$$

By Young's inequality for convolution,

$$\begin{aligned}
M_1 &\leq \left\| \left(\|\widehat{K}_1(t)\|_{L_T^{\frac{4}{3}}} + \|e^{-\nu t|\xi|^{2\alpha}}\|_{L_T^{\frac{4}{3}}}\right) |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \|\widehat{u\otimes u}\|_{L^2_T} \right\|_{L^2(S_1)} \\
&\leq C\, \left\| (\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}\,|\xi|^{-\alpha}\gamma^{\frac{3}{4}} + \nu^{-\frac{3}{4}}|\xi|^{-\frac{3}{2}\alpha})\,|\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha}\|\widehat{u\otimes u}\|_{L^2_T} \right\|_{L^2(S_1)} \\
&\leq C\, \left\| (\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}}\,|\xi|^{-\alpha} + \nu^{-\frac{3}{4}}(\nu^{\frac{1}{4}}\gamma^{\frac{1}{4}})|\xi|^{-\alpha})|\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha}\|\widehat{u\otimes u}\|_{L^2_T} \right\|_{L^2(S_1)} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}} \left\| |\xi|^{\frac{d}{2}+2-\frac{5}{2}\alpha}\|\widehat{u\otimes u}\|_{L^2_T} \right\|_{L^2(S_1)} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}} \left\| |\xi|^{\frac{d}{2}+2-\frac{5}{2}\alpha}\|\widehat{u\otimes u}\|_{L^2_T} \right\|_{L^2(\mathbb{R}^d)} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}} \|\Lambda^{\frac{d}{2}+2-\frac{5}{2}\alpha}(u\otimes u)\|_{L^2_T L^2} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}} \|\Lambda^{\frac{d}{2}+2-\frac{5}{2}\alpha}u\|_{L^2_T L^{p_1}} \|u\|_{L^\infty_T L^{q_1}} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}} \|u\|_{L^2_T \dot{H}^{\frac{d}{2}+1-\alpha}} \|u\|_{L^\infty \dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \\
&\leq C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}}\, \|u\|_{L^2_T \dot{H}^{\frac{d}{2}+1-\alpha}} \|u\|_{L^\infty_T(\dot{H}^{\frac{d}{2}+1-2\alpha}\cap \dot{H}^{\frac{d}{2}+1-\alpha})},
\end{aligned}$$

where p_1 and q_1 are taken to be

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{d}\left(\frac{3}{2}\alpha - 1\right), \quad \frac{1}{q_1} = \frac{1}{d}\left(\frac{3}{2}\alpha - 1\right), \quad \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}. \tag{7.8}$$

By Young's inequality, Lemma 5.1 and the fact that $|\xi|^\alpha < \frac{\sqrt{3}}{4}\gamma^{-\frac{1}{2}}\nu^{-\frac{1}{2}}$ for any $\xi \in S_2$, we have

$$
\begin{aligned}
M_2 &\le \left\| \big\| \widehat{K}_1(t) - e^{-\nu t|\xi|^{2\alpha}} \big\|_{L^2_T} |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \|\widehat{u\otimes u}\|_{L_T^{\frac{4}{3}}} \right\|_{L^2(S_2)} \\
&\le C\,\gamma^{\frac{1}{2}} \left\| |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \|\widehat{u\otimes u}\|_{L_T^{\frac{4}{3}}} \right\|_{L^2(S_2)} \\
&\le C\,\gamma^{\frac{1}{2}} \left\| |\xi|^{\frac{d}{2}+2-\frac{3}{2}\alpha} \|\widehat{u\otimes u}\|_{L^2(\mathbb{R}^d)} \right\|_{L_T^{\frac{4}{3}}} \\
&= C\,\gamma^{\frac{1}{2}} \left\| \Lambda^{\frac{d}{2}+2-\frac{3}{2}\alpha} \|(u\otimes u)\|_{L^2} \right\|_{L_T^{\frac{4}{3}}} \\
&\le C\,\gamma^{\frac{1}{2}} \left\| \|\Lambda^{\frac{d}{2}+2-\frac{3}{2}\alpha} u\|_{L^{p_1}}\, \|u\|_{L^{q_1}} \right\|_{L_T^{\frac{4}{3}}} \\
&\le C\,\gamma^{\frac{1}{2}} \left\| \|u\|_{\dot H^{\frac{d}{2}+1}}\, \|u\|_{\dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \right\|_{L_T^{\frac{4}{3}}} \\
&\le C\,\gamma^{\frac{1}{2}}\, \|u\|_{L^2_T \dot H^{\frac{d}{2}+1}}\, \|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}},
\end{aligned}
$$

where p_1 and q_1 are given by (7.8). In summary, we have obtained

$$
\begin{aligned}
\|J_2\|_X \le\, & C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}}\, \|u\|_{L^2_T \dot H^{\frac{d}{2}+1-\alpha}}\, \|u\|_{L^\infty_T(\dot H^{\frac{d}{2}+1-2\alpha}\cap \dot H^{\frac{d}{2}+1-\alpha})} \\
&+ C\,\gamma^{\frac{1}{2}}\, \|u\|_{L^2_T \dot H^{\frac{d}{2}+1}}\, \|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}. \qquad (7.9)
\end{aligned}
$$

Inserting (7.5), (7.6) and (7.9) in (7.4), we obtain

$$
\begin{aligned}
\|u_\gamma - u\|_X \le\, & C\,\gamma^{\frac{1}{4}} \|u_0\|_{\dot H^{\frac{d}{2}+1-2\alpha}\cap \dot H^{\frac{d}{2}+1-\alpha}} + C\,\gamma\nu^{-\frac{1}{4}} \|u_1\|_{\dot H^{\frac{d}{2}+1-2\alpha}} \\
&+ C\,\nu^{-\frac{3}{4}} \|u_\gamma - u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \Big(\|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} + \|u_\gamma\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \Big) \\
&+ C\,\gamma^{\frac{1}{4}}\nu^{-\frac{1}{2}}\, \|u\|_{L^2_T \dot H^{\frac{d}{2}+1-\alpha}}\, \|u\|_{L^\infty_T(\dot H^{\frac{d}{2}+1-2\alpha}\cap \dot H^{\frac{d}{2}+1-\alpha})} \\
&+ C\,\gamma^{\frac{1}{2}}\, \|u\|_{L^2_T \dot H^{\frac{d}{2}+1}}\, \|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}. \qquad (7.10)
\end{aligned}
$$

Trivially,

$$
\|u_\gamma\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \le \|u_\gamma - u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} + \|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}}.
$$

According to Proposition 6.1, the solution u of the Navier–Stokes equations obeys the following bounds

$$
\begin{aligned}
&\|u\|_{L^\infty \dot H^{\frac{d}{2}+1-2\alpha}} \le \|u_0\|_{\dot H^{\frac{d}{2}+1-2\alpha}}, \quad \|u\|_{L^2_T \dot H^{\frac{d}{2}+1-\alpha}} \le \nu^{-\frac{1}{2}} \|u_0\|_{\dot H^{\frac{d}{2}+1-2\alpha}}, \\
&\|u\|_{L^4_T \dot H^{\frac{d}{2}+1-\frac{3}{2}\alpha}} \le \nu^{-\frac{1}{4}} \|u_0\|_{\dot H^{\frac{d}{2}+1-2\alpha}}, \quad \|u\|_{L^\infty \dot H^{\frac{d}{2}+1-\alpha}} \le \|u_0\|_{\dot H^{\frac{d}{2}+1-\alpha}}, \\
&\|u\|_{L^2_T \dot H^{\frac{d}{2}+1}} \le \nu^{-\frac{1}{2}} \|u_0\|_{\dot H^{\frac{d}{2}+1-\alpha}}.
\end{aligned}
$$

Then (7.10) is reduced to

$$\begin{aligned}\|u_\gamma - u\|_X \le{}& C\,\gamma^{\frac14}\|u_0\|_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}} + C\,\gamma\nu^{-\frac14}\|u_1\|_{\dot H^{\frac d2+1-2\alpha}}\\ &+ C\,\gamma^{\frac14}\big(\nu^{-1}+\gamma^{\frac14}\nu^{-\frac12}\big)\|u_0\|^2_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}}\\ &+ C_1\,\nu^{-\frac34}\,\|u\|_X\,\|u_\gamma-u\|_X + C_1\,\nu^{-\frac34}\,\|u_\gamma-u\|_X^2. \qquad (7.11)\end{aligned}$$

Here C_1 is a constant independent of γ and ν. We apply the bootstrapping argument to establish a uniform global bound for $\|u_\gamma - u\|_X$. Due to the presence of the linear term in (7.11)

$$C_1\,\nu^{-\frac34}\,\|u\|_X\,\|u_\gamma-u\|_X,$$

we need to implement this process on a finite number of sub-intervals of $(0,\infty)$. We recall a basic fact from real analysis.

Lemma 7.2. *Let $(X,\mathcal{B},\mu)$ be a complete measure space. Let f be integrable with respect to the measure μ. Then, for any $\varepsilon>0$, there is a $\delta>0$ such that, if $A\in\mathcal{B}$ and $\mu(A)\le\delta$, then*

$$\int_A |f(x)|\,d\mu(x) < \varepsilon.$$

Since the solution u of the Navier–Stokes equation satisfies

$$\|u\|_X := \|u\|_{L^4_T \dot H^{\frac d2+1-\frac32\alpha}} < \infty,$$

there is $T_1>0$ such that, for any $\rho\ge 0$, $\|u\|_{L^4(\rho,\rho+T_1;\dot H^{\frac d2+1-\frac32\alpha})}$ is small. In particular, we choose $T_1>0$ such that

$$C_1\,\nu^{-\frac34}\,\|u\|_{L^4(\rho,\rho+T_1;\dot H^{\frac d2+1-\frac32\alpha})} \le \frac12. \qquad (7.12)$$

In addition, there is $T_2>0$ such that

$$C_1\,\nu^{-\frac34}\,\|u\|_{L^4(T_2,\infty;\dot H^{\frac d2+1-\frac32\alpha})} \le \frac12.$$

Obviously, there is a positive integer $k_0>0$ such that

$$k_0T_1 \ge T_2.$$

We first apply the bootstrapping argument on $[0,T_1]$ and then repeat this process on the time intervals $[T_1,2T_1]$, $[2T_1,3T_1],\dots,[(k_0-1)T_1,k_0T_1]$ and $[T_2,\infty)$ to obtain a global bound. Inserting (7.12) in (7.11) yields

$$\|u_\gamma-u\|_{L^4(0,T_1;\dot H^{\frac d2+1-\frac32\alpha})} \le C_1\,\gamma^{\frac14}H(u_0,u_1) + C_1\,\nu^{-\frac34}\,\|u_\gamma-u\|^2_{L^4(0,T_1;\dot H^{\frac d2+1-\frac32\alpha})}, \qquad (7.13)$$

where we have written

$$H(u_0,u_1) := \|u_0\|_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}} + \gamma^{\frac34}\nu^{-\frac14}\|u_1\|_{\dot H^{\frac d2+1-2\alpha}} + \big(\nu^{-1}+\gamma^{\frac14}\nu^{-\frac12}\big)\|u_0\|^2_{\dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha}}.$$

We make the ansatz that

$$\|u_\gamma - u\|_{L^4(0,T_1;\dot H^{\frac d2+1-\frac32\alpha})} \le C_2, \tag{7.14}$$

where C_2 satisfies

$$C_1\, C_2\, \nu^{-\frac34} \le \frac12.$$

Inserting (7.14) in the right-hand side of (7.13) yields

$$\|u_\gamma - u\|_{L^4(0,T_1;\dot H^{\frac d2+1-\frac32\alpha})} \le 2C_1\,\gamma^{\frac14} H(u_0,u_1).$$

For γ satisfying (1.15) in Theorem 1.1, namely

$$\gamma^{\frac14}\, H(u_0,u_1) \le C_0$$

for sufficiently small C_0, we have

$$\|u_\gamma - u\|_{L^4(0,T_1;\dot H^{\frac d2+1-\frac32\alpha})} \le 2C_1\,\gamma^{\frac14} H(u_0,u_1) \le \frac{C_2}{2}.$$

The bootstrapping argument then yields the desired bound on $[0,T_1]$. Repeating this process on the time intervals $[T_1,2T_1]$, $[2T_1,3T_1],\dots,[(k_0-1)T_1,k_0T_1]$ and $[T_2,\infty)$ allows us to obtain a global bound on $[0,\infty)$. Combining with the global bound for $\|u\|_{L^4(0,\infty;\dot H^{\frac d2+1-\frac32\alpha})}$ yields the desired global bound for $\|u_\gamma\|_{L^4(0,\infty;\dot H^{\frac d2+1-\frac32\alpha})}$.

The solution u_γ is obtained in $L^4\big(0,\infty;\dot H^{\frac d2+1-\frac32\alpha}\big)$. We can actually show that

$$u_\gamma \in \bigcap_{p=2}^{\infty} L^p\big(0,\infty;\dot H^{\frac d2+1-2\alpha+\frac{2\alpha}{p}}\big)\cap C\big([0,\infty);\dot H^{\frac d2+1-2\alpha}\big).$$

Recall that $u_\gamma \in L^4(0,\infty;\dot H^{\frac d2+1-\frac32\alpha})$ obeys the integral representation

$$u_\gamma(t) = \left(K_0+\frac12 K_1\right)u_0 + \gamma K_1 u_1 - \int_0^t K_1(t-s)\,\mathbb P(u_\gamma\cdot\nabla u_\gamma)(s)\,ds. \tag{7.15}$$

Since

$$u_0 \in \dot H^{\frac d2+1-2\alpha}\cap \dot H^{\frac d2+1-\alpha},\quad u_1\in \dot H^{\frac d2+1-2\alpha},$$

an application of Proposition 4.2 with $s = \frac{d}{2} + 1 - 2\alpha$ implies that

$$\left(K_0 + \frac{1}{2}K_1\right) u_0 + \gamma K_1 u_1 \in \bigcap_{p=2}^{\infty} L^p\big(0,\infty; \dot{H}^{\frac{d}{2}+1-2\alpha+\frac{2\alpha}{p}}\big).$$

Since $u_\gamma \in L^4(0,\infty; \dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha})$, Lemma 7.1 and Hölder's inequality show that

$$u_\gamma \otimes u_\gamma \in L^2\big(0,\infty; \dot{H}^{\frac{d}{2}+2-3\alpha}\big), \quad u_\gamma \cdot \nabla u_\gamma = \nabla \cdot (u_\gamma \otimes u_\gamma) \in L^2\big(0,\infty; \dot{H}^{\frac{d}{2}+1-3\alpha}\big).$$

Because of the boundedness of the Leray projection $\mathbb{P}$ on $\dot{H}^{\frac{d}{2}+1-3\alpha}$, we have

$$\mathbb{P}(u_\gamma \cdot \nabla u_\gamma) \in L^2\big(0,\infty; \dot{H}^{\frac{d}{2}+1-3\alpha}\big).$$

Applying Proposition 4.3 with $s = \frac{d}{2} + 1 - 2\alpha$ implies

$$\int_0^t K_1(t-s)\,\mathbb{P}(u_\gamma \cdot \nabla u_\gamma)(s)\,ds \in L^p\big(0,\infty; \dot{H}^{\frac{d}{2}+1-2\alpha+\frac{2\alpha}{p}}\big).$$

The representation in (7.15) then yields

$$u_\gamma \in \bigcap_{p=2}^{\infty} L^p\big(0,\infty; \dot{H}^{\frac{d}{2}+1-2\alpha+\frac{2\alpha}{p}}\big).$$

We further show that $u_\gamma \in C([0,\infty); \dot{H}^{\frac{d}{2}+1-2\alpha})$. The linear part

$$\left(K_0 + \frac{1}{2}K_1\right) u_0 + \gamma K_1 u_1 \in C\big([0,\infty); \dot{H}^{\frac{d}{2}+1-2\alpha}\big)$$

follows from writing its norm in $\dot{H}^{\frac{d}{2}+1-2\alpha}$ in Fourier space, the continuity of $\widehat{K}_0(t,\xi)$ and $\widehat{K}_1(t,\xi)$ in t and the dominated convergence theorem. To show the time continuity of the nonlinear part

$$N(t) := \int_0^t K_1(t-s)\,\mathbb{P}(u_\gamma \cdot \nabla u_\gamma)(s)\,ds,$$

we take its Fourier transform

$$\widehat{N}(\xi,t) = \int_0^t \widehat{K}_1(\xi,t-s)\,\mathbb{P}(\widehat{u_\gamma \cdot \nabla u_\gamma})(\xi,s)\,ds.$$

For almost every $\xi \in \mathbb{R}^d$, $\widehat{N}(\xi,t)$ is a continuous function of t, and, by Young's inequality for convolution

$$\sup_{0\le\tau\le t} \big|\widehat{N}(\xi,t)\big| \le \big\|\widehat{K}_1(\xi,\cdot)\big\|_{L_t^2} \big\|\mathbb{P}(\widehat{u_\gamma \cdot \nabla u_\gamma})(\xi,\cdot)\big\|_{L_t^2},$$

making use of the upper bounds for $\widehat{K}_1$ in Proposition 3.1, we can easily show that

$$\big\|\widehat{K}_1(\xi,\cdot)\big\|_{L^2_t} \le C\,\nu^{-\frac12}\,|\xi|^{-\alpha},$$

where $C>0$ is a constant independent of ν. Therefore,

$$\begin{aligned}\sup_{0\le\tau\le t}\|N(\cdot,\tau)\|^2_{\dot H^{\frac d2+1-2\alpha}} &\le \int_{\mathbb{R}^d}|\xi|^{d+2-4\alpha}\sup_{0\le\tau\le t}\big|\widehat{N}(\xi,t)\big|^2 d\xi\\ &\le \nu^{-1}\int_{\mathbb{R}^d}|\xi|^{d+2-6\alpha}\big\|P(\widehat{u_\gamma\cdot\nabla u_\gamma})(\xi,\cdot)\big\|^2_{L^2_t}d\xi\\ &\le \nu^{-1}\big\|\|u_\gamma\cdot\nabla u_\gamma\|_{\dot H^{\frac d2+1-3\alpha}}\big\|^2_{L^2_t}\\ &\le \nu^{-1}\big\|\|u_\gamma\otimes u_\gamma\|_{\dot H^{\frac d2+2-3\alpha}}\big\|^2_{L^2_t}\\ &\le \nu^{-1}\|u_\gamma\|^4_{L^4_t\dot H^{\frac d2+1-\frac32\alpha}}.\end{aligned}$$

It then follows from the dominated convergence theorem that $N(t)$ is continuous in $\dot H^{\frac d2+1-2\alpha}$. This completes the proof of the global existence part in Theorem 1.1. □

8 Uniqueness

The section proves the uniqueness part of Theorem 1.1. It is clearly a consequence of the following proposition.

Proposition 8.1. *Let $d\ge 2$, $\nu>0$ and α satisfies*

$$\frac23<\alpha<\frac13(2+d).$$

Consider the generalized hyperbolic Navier–Stokes equations with two different sets of initial data

$$\begin{aligned}(u_0^{(1)},u_1^{(1)}) &\in \dot H^{\frac d2+1-2\alpha}\cap\dot H^{\frac d2+1-\alpha}\times\dot H^{\frac d2+1-2\alpha},\\ (u_0^{(2)},u_1^{(2)}) &\in \dot H^{\frac d2+1-2\alpha}\cap\dot H^{\frac d2+1-\alpha}\times\dot H^{\frac d2+1-2\alpha}.\end{aligned}$$

Let $u_\gamma^{(1)}$ and $u_\gamma^{(2)}$ be the corresponding solutions. Then the difference satisfies, for any $T>0$ and $X_T:=L^4(0,T;\dot H^{\frac d2+1-\frac32\alpha})$,

$$\begin{aligned}\|u_\gamma^{(2)}-u_\gamma^{(1)}\|_{X_T} \le{}& C(\gamma,\nu)\Big(\|u_0^{(2)}-u_0^{(1)}\|_{\dot H^{\frac d2+1-2\alpha}\cap\dot H^{\frac d2+1-\alpha}}+\|u_1^{(2)}-u_1^{(1)}\|_{\dot H^{\frac d2+1-2\alpha}}\Big)\\ &+C\,\nu^{-\frac34}\left(\|u_\gamma^{(1)}\|_{X_T}+\|u_\gamma^{(2)}\|_{X_T}\right)\|u_\gamma^{(2)}-u_\gamma^{(1)}\|_{X_T}. \end{aligned}\tag{8.1}$$

Proof of Proposition 8.1. We use the integral form, for $i=1,2$,

$$u_\gamma^{(i)}(t)=\left(K_0+\frac12K_1\right)u_0^{(i)}+\gamma K_1u_1^{(i)}-\int_0^t K_1(t-s)\,\mathbb{P}(u_\gamma^{(i)}\cdot\nabla u_\gamma^{(i)})(s)ds.$$

Their difference satisfies

$$u_\gamma^{(2)} - u_\gamma^{(1)} = \left(K_0 + \frac{1}{2}K_1\right)(u_0^{(2)} - u_0^{(1)}) + \gamma K_1(u_1^{(2)} - u_1^{(1)})$$
$$+ \int_0^t K_1(t-s)\big(\mathbb{P}(u_\gamma^{(2)} \cdot \nabla u_\gamma^{(2)})(s) - \mathbb{P}(u_\gamma^{(1)} \cdot \nabla u_\gamma^{(1)})(s)\big)ds.$$

We estimate the norm in $X_T := L^4(0,T;\dot{H}^{\frac{d}{2}+1-\frac{3}{2}\alpha})$. By Proposition 4.2,

$$\left\|\left(K_0 + \frac{1}{2}K_1\right)(u_0^{(2)} - u_0^{(1)}) + \gamma K_1(u_1^{(2)} - u_1^{(1)})\right\|_{X_T}$$
$$\leq C(\gamma,\nu)\left(\|u_0^{(2)} - u_0^{(1)}\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}\cap\dot{H}^{\frac{d}{2}+1-\alpha}} + \|u_1^{(2)} - u_1^{(1)}\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}}\right).$$

The nonlinear part can be estimated in (7.6) of Section 7,

$$\left\|\int_0^t K_1(t-s)\left(\mathbb{P}(u_\gamma^{(2)} \cdot \nabla u_\gamma^{(2)})(s) - \mathbb{P}(u_\gamma^{(1)} \cdot \nabla u_\gamma^{(1)})(s)\right)ds\right\|_{X_T}$$
$$\leq C\,\nu^{-\frac{3}{4}}\left(\|u_\gamma^{(1)}\|_{X_T} + \|u_\gamma^{(2)}\|_{X_T}\right)\|u_\gamma^{(2)} - u_\gamma^{(1)}\|_{X_T}.$$

This completes the proof of Proposition 8.1. □

Proof for the Uniqueness Part of Theorem 1.1. According to Lemma 7.2, we can take $T > 0$ to be sufficiently small such that

$$C\,\nu^{-\frac{3}{4}}\left(\|u_\gamma^{(1)}\|_{X_T} + \|u_\gamma^{(2)}\|_{X_T}\right) \leq \frac{1}{2}.$$

It then follows from (8.1) that

$$\|u_\gamma^{(2)} - u_\gamma^{(1)}\|_{X_T}$$
$$\leq 2C(\gamma,\nu)\left(\|u_0^{(2)} - u_0^{(1)}\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}\cap\dot{H}^{\frac{d}{2}+1-\alpha}} + \|u_1^{(2)} - u_1^{(1)}\|_{\dot{H}^{\frac{d}{2}+1-2\alpha}}\right).$$

In particular, if

$$u_0^{(2)} = u_0^{(1)}, \quad u_1^{(2)} = u_1^{(1)},$$

then, on $[0,T]$,

$$u_\gamma^{(2)} = u_\gamma^{(1)}.$$

Repeating this process on the time intervals $[T,2T]$, $[2T,3T]$ and so on yields the desired uniqueness on any time interval. This finishes the proof for the uniqueness part. □

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References

1. Abe S., Thurner S., Anomalous diffusion in view of Einsteins 1905 theory of Brownian motion. Phys. A, 2005, 356(2–4): 403–407
2. Bahouri H., Chemin J.-Y., Danchin R., Fourier Analysis, Nonlinear Partial Differential Equations, Heidelberg: Springer, 2011
3. Brenier Y., Natalini R., Puel M., On a relaxation approximation of the incompressible Navier–Stokes equations. Proc. Amer. Math. Soc., 2004, 132(4): 1021–1028
4. Buckmaster T., Vicol V., Nonuniqueness of weak solutions to the Navier–Stokes equation. Ann. of Math. (2), 2019, 189(1): 101–144
5. Carrassi M., Morro A., A modified Navier–Stokes equation, and its consequences on sound dispersion. Il Nuovo Cimento B, 1972, 9: 321–342
6. Cattaneo C., Sulla conduzione del calore. Atti Sem. Mat. Fis. Univ. Modena, 1949, 3: 83–101
7. Cattaneo C., Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantane. C. R. Acad. Sci. Paris, 1958, 247: 431–433
8. Colombo M., De Lellis C., De Rosa L., Ill-posedness of leray solutions for the hypodissipative Navier–Stokes equations. Comm. Math. Phys., 2018, 362(2): 659–688
9. Coulaud O., Hachicha I., Raugel G., Hyperbolic quasilinear Navier–Stokes equations in $\mathbb{R}^2$. J. Dynam. Differential Equations, 2022, 34(4): 2749–2785
10. De Rosa L., Infinitely many Leray–Hopf solutions for the fractional Navier–Stokes equations. Comm. Partial Differential Equations, 2019, 44(4): 335–365
11. Hachicha I., Global existence for a damped wave equation, convergence towards a solution of the Navier–Stokes problem. Nonlinear Anal., 2014, 96: 68–86
12. Jara M., Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps. Comm. Pure Appl. Math., 2009, 62(2): 198–214
13. Jiu Q., Wang Y., On possible time singular points, eventual regularity of weak solutions to the fractional Navier–Stokes equations. Dyn. Partial Differ. Equ., 2014, 11(4): 321–343
14. Katz N.H., Pavlović N.A., A cheap Caffarelli–Kohn–Nirenberg inequality for the Navier–Stokes equation with hyper-dissipation. Geom. Funct. Anal., 2002, 12(2): 355–379
15. Lions J.-L., Quelques résultats d existence dans des équations aux dérivées partielles non linéaires. Bull. Soc. Math. France, 1959, 87: 245–273
16. Lions J.-L., Quelques Méthodes de Resolution des Problémes aux Limites Non linéaires. Paris: Dunod, 1969
17. Luo T., Titi E.S., Non-uniqueness of weak solutions to hyperviscous Navier–Stokes equations: on sharpness of J.-L. Lions exponent. Calc. Var. Partial Differential Equations, 2020, 59(3): Paper No. 92, 15 pp.
18. Olson E., Titi E.S., Viscosity versus vorticity stretching: Global well-posedness for a family of Navier–Stokes-alpha-like models. Nonlinear Anal., 2007, 66(11): 2427–2458
19. Racke R., Saal J., Hyperbolic Navier–Stokes equations I: Local well-posedness. Evol. Equ. Control Theory, 2012, 1(1): 195–215
20. Racke R., Saal J., Hyperbolic Navier–Stokes equations II: Global existence of small solutions. Evol. Equ. Control Theory, 2012, 1(1): 217–234
21. Tao T., Global regularity for a logarithmically supercritical hyperdissipative Navier–Stokes equation. Anal. PDE, 2009, 3: 361–366
22. Vernotte P., Les paradoxes de la théorie continue de léquation de la chaleur. C. R. Acad. Sci. Paris, 1958, 246: 3154–3155
23. Wu J., Generalized MHD equations. J. Differential Equations, 2003, 195(2): 284–312