The Stable Transformation Path*

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Abstract

Dynamic models of structural transformation are often non-stationary, precluding balanced growth path (BGP) analysis. We develop a generalization of the BGP concept that we call a Stable Transformation Path (STraP). The STraP characterizes the medium-term dynamics of the economy in a turnpike sense; it is the non-stationary path toward which the economy (quickly) converges from an arbitrary initial capital stock. Calibrated simulations demonstrate that these medium-term dynamics have important quantitative implications for structural transformation, investment, and growth, including slow convergence. Medium-term dynamics alone account for the observed 40% secular decline in average growth rates across stages of development.

Keywords: Growth, Investment Dynamics, Non-balanced Growth

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1 Introduction

What is the connection between structural transformation — i.e., the reallocation process of economic activity across sectors — and economic growth and capital accumulation? Empirically, the three phenomena accompany one another as three of the most well-established processes of macroeconomic development.[1] However, theoretically and quantitatively studying possible interactions among them requires analyzing non-stationary models, which are growing in number within the literature.[2] A challenge is that these models do not have balanced growth paths (BGPs) in the medium term, i.e., while structural transformation is occurring. For this class of non-stationary models, how can we then describe their medium-term dynamics, i.e., the capital, investment, and growth dynamics that are independent of the initial level of capital?

In this paper, we introduce a new stable dynamic concept — what we call a Stable Transformation Path (STraP) — which is a generalization of a BGP to non-stationary economies that exhibit medium-term growth dynamics with only asymptotic BGPs. Specifically, we define a STraP as a path from one asymptotic balanced growth path to another. Although models with only asymptotic balanced growth paths have been applied to address many substantive questions, our concrete application focuses on an economy that transitions from being initially agricultural to eventually service-based, moving in and out of industry with dynamic changes in investment and growth along the way. The path is *stable* in the sense that, from arbitrary initial conditions, capital quickly converges to the STraP. The STraP therefore has turnpike-like properties. The dynamics of capital along this convergence reflect standard Neoclassical convergence, whereas the dynamics of capital along the STraP reflect medium-run transformation dynamics toward an asymptotic BGP. Moreover, in addition to defining the STraP, we prove its existence and uniqueness in a general class of growth models with only asymptotic BGPs and provide a simple double-recursive shooting algorithm to solve

[1]See Herrendorf et al. (2014) for a nice review of these empirical facts.

[2]Recent examples include García-Santana et al. (2019), Herrendorf et al. (2018), Kehoe et al. (2018), and Storesletten et al. (2019), but we reference a larger group in our literature review. Stationary models, in which structural transformation is consistent with balanced growth but does not impact that growth, have also been proposed (Ngai and Pissarides, 2007; Kongsamut et al., 2001). The literature on investment-specific technological change provides another prominent example, in the case of general constant elasticity of substitution production functions (Greenwood et al., 1997).

for the STraP.

We then calibrate and simulate a quantitative STraP for a typical model of structural change that starts with simple assumptions, i.e., differential productivity growth, constant elasticity of substitution (CES) sectoral aggregators, common Cobb-Douglas parameters across sectors, and constant intertemporal elasticity of substitution (CIES) preferences. The STraP concept allows us to move the model away from the knife-edge BGP-yielding cases of log intertemporal preferences and investment that (counterfactually) only includes manufacturing value added (Herrendorf et al., 2013). Indeed, we can consider structural transformation within the investment sector, which has been recently shown to be empirically important (Herrendorf et al., 2018). This structural transformation leads to time-varying growth in the effective productivity of the investment sector and the relative price of investment, both of which can preclude BGPs.

The simulations show that the benchmark STraP is able to reproduce the salient features of structural transformation and secular growth patterns. The share of agriculture shows a prolonged decline, while that of services shows a prolonged growth. Interestingly, the simulations yield quantitatively important industrialization and de-industrialization — the hump shape in manufacturing that has eluded previous balanced growth models of structural transformation (Ngai and Pissarides, 2007; Kongsamut et al., 2001). More importantly, the structural transformation in the STraP yields time-varying aggregate productivity growth and a time-varying relative price of investment that affect the aggregate growth process. The model demonstrates a pronounced Baumol's disease slowdown in aggregate growth of chain-weighted Gross Domestic Product (GDP), despite the investment rate increasing over time. We also find that the investment rate increases along the STraP, in spite of the interest rate declining with development.

In addition, we demonstrate that the model's STraP-enabled departures from earlier parameterizations are important for these implications. These departures not only change the quantitative features of the structural transformation, but they can also affect the qualitative growth patterns. For example, the simplifying assumption of log intertemporal preferences implies a declining investment rate rather than an increasing one, while the simplifying assumption of manufacturing-only investment implies an increasing growth rate over time rather than a decreasing growth rate.

Finally, we examine the STraP's predictions for the overall investment and growth process relative to empirical patterns in the data. We show that the model predicts persistent, non-balanced patterns: a rising capital-output ratio, a falling relative price of investment, a falling interest rate, and falling growth rates over the course of development, all of which are clearly observed in the Penn World Tables' cross-country panel, as we document. These patterns contrast starkly with the predictions of the Neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965), endowed with the capital-output ratio observed in poor economies. The Neoclassical growth model implies counterfactually high initial growth rates and interest rates, but rapid convergence of the growth rate, interest rate, and capital-output ratio to the constant BGP, as well as a completely flat relative price of investment.

Thus, the STraP makes progress in addressing the well-known growth convergence puzzle and refocusing it by characterizing the medium-term dynamics of structural transformation. A poor economy, along a shared STraP with advanced economies but with trailing productivity, will grow faster than advanced economies even if each sector's productivity grows at the same rate. Importantly, these higher growth rates are not the product of transitionally low levels of capital. Instead, they are the result of structural transformation shifting resources from high-productivity growth to low-productivity growth sectors despite increasing investment rates. Medium-term dynamics alone account for the observed 40% secular decline in average growth rates as countries develop.

We view the STraP as a natural benchmark for studying sectoral and investment distortions in the macro development process for several reasons. First, from a growth perspective, models that lack BGPs are a starting point, since the very lack of a BGP is precisely what makes structural transformation informative about the overall growth process (see Buera and Kaboski, 2009). Second, from an empirical perspective, the medium-term dynamics of the STraP track the patterns in the data. Third, from an efficiency perspective, the welfare theorems hold in the undistorted STraP model we present. The STraP's medium-term dynamics and stability patterns can be used to normatively evaluate growth trajectories, since departures from the STraP that stem from initial conditions quickly disappear. More persistent departures may reflect underlying distortions. In sum, studying a richer environment of structural transformation without BGPs opens the door to normatively evaluating the sectoral composition of the economy and sectoral distortions — e.g., assessing the empirical evidence in Rodrik (2016) through the lens of dynamic theory.

The remainder of the paper is organized as follows. After reviewing the related literature, we present a benchmark structural transformation model in Section 2. We define the STraP in Section 3 and prove its existence and uniqueness. In Section 4, we simulate various STraPs and show their relevance for understanding structural change and growth patterns in the data. In Section 5, we present our conclusions.

Related Literature The paper builds on and relates to an existing literature on structural transformation. There are some earlier analyses of non-stationary transformation paths from stable equilibria to asymptotic BGPs. Hansen and Prescott (2002) and Gollin et al. (2004) analyze transitions from stagnant or slow-growing agricultural economies to modern growth. While these papers only study transitional dynamics given specific initial conditions, we can show that the dynamic paths in these models are dominated by the more general, medium-term dynamics along the STraP, rather than transitional Neoclassical dynamics. Extending this work, a contribution of our paper is to define the STraP in a general class of environments and use it to distinguish short-run Neoclassical dynamics from medium-term dynamics due to structural change.

Other papers have tried to reconcile structural transformation with perceived Kaldor (1957)'s stylized facts. Kongsamut et al. (2001) used Stone-Geary non-homothetic preferences together with a knife-edge cross-restriction on the preferences and technology to yield a rising service share and declining agricultural share along a BGP. In contrast, Ngai and Pissarides (2007) used biased productivity growth and non-unitary elasticity of substitution across sectors to get structural transformation. They assumed a unitary intertemporal elasticity of substitution and investment being produced exclusively with manufacturing to yield constant growth in terms of the manufacturing good numéraire. As shown by Buera and Kaboski (2009), the assumptions in both studies effectively divorce growth from structural transformation, making the two phenomena orthogonal. Because we only require BGPs *asymptotically*, the STraP we develop allows for a rich, encompassing set of assumptions in models, including non-homotheticities, imperfect substitutability of sectoral output in investment and consumption, and productivity growth in all sectors. It therefore reintegrates the twin macro development phenomena: structural transformation and growth. Moreover, empirically we show that Kaldor's stylized facts do not hold over a wider range of development levels.

The key role of investment in structural transformation has been examined in recent work by García-Santana et al. (2019) and Herrendorf et al. (2018). The former paper argues that the hump shape in the share of value added in the industrial sector can be explained by the combination of a hump shape in the investment rate and the fact that investment is relatively more intensive in manufacturing value added than consumption. They, however, assume a constant sectoral composition of investment. The latter paper shows that structural change occurs within the investment sector, and this is inconsistent with a balanced growth path. Acemoglu and Guerrieri (2008) and Ju et al. (2015) also argue that capital accumulation is central to the transformation process, but do not analyze the medium-term dynamics of the STraP. We solve a model with both the level and composition of investment endogenously time-varying. We find that both are ultimately important in yielding the hump shape of the industrial sector in the STraP.

This paper also relates to the normative literature on structural transformation. Wedge-based normative analyses of structural transformation, including work on the agricultural productivity gap (Gollin et al., 2014) and work on the distortions in command economies (Cheremukhin et al., 2017a,b), have focused exclusively on static distortions or distortions for a given level of capital. The STraP allows for a broadening of these analyses because economies converge to the STraP for different initial conditions. The STraP constitutes a benchmark dynamic model for interpreting the optimality in the aggregate level of capital (given technology levels), and one can therefore infer and interpret distortions away from this benchmark as reflecting dynamic intertemporal distortions. Such an analysis is precisely what our contemporaneous work Buera et al. (2019) undertakes.

In showing that competitive equilibria converge to the STraP for different initial conditions, our work also relates to an early literature studying the turnpike properties of growth models (see McKenzie, 1986, and references therein). While standard turnpike theorems state that the dynamic equilibrium asymptotically approaches a stationary equilibrium, our numerical analyses show that along the transition to the asymptotic stationary equilibrium these trajectories first approach the STraP. Furthermore, when realistic calibrations are used, the convergence to the STraP is fast.[3]

Finally, because our work leads to a declining relative price of capital, it relates to the concept of investment-specific technical change (ISTC). Structural transformation is a distinct, though complementary, explanation for the decline in prices. *Qualitatively*, while a standard ISTC model with constant technical change (e.g., Greenwood et al., 1997) leads to growth in the ratio of real capital to real output, it also yields a BGP and therefore predicts no medium-run growth dynamics. *Quantitatively*, our analysis combines both ISTC and structural transformation. Along the STraP, the forces driving structural transformation, i.e., sectoral productivity growth, explain nearly all of the growth (as noted by Herrendorf et al., 2018) and roughly half of the growth in the capital-output ratio and decline in the relative price of investment.

2 Model

In this section, we present a model of investment and structural transformation based on Ngai and Pissarides (2007) with a general Intertemporal Elasticity of Substitution (IES). We also introduce an investment aggregator as in Herrendorf et al. (2018), which allows for a more generally time-varying relative price of investment. As we show below, a general (non-unitary) IES or the existence of an investment aggregator preclude the existence of a balanced growth path, motivating the need for a more general stable dynamic concept.

2.1 Environment

Consider a standard continuous time intertemporal problem of a representative household with constant intertemporal elasticity preferences over a consumption aggregate $C(t)$. The household exogenously provides labor, which earns a wage, $w(t)$, and owns capital, $K(t)$, which earns a rental rate, $R(t)$. Capital depreciates at a rate $\delta \in [0, 1]$, but can be accumulated through investment, $X(t)$. A bond, $B(t)$, which is priced in units of consumption and pays off in units of consumption, is in zero net supply,

[3] In turn, this result is reminiscent of the fast convergence of the Neoclassical growth model to the balance growth path (King and Rebelo, 1993).

but prices the (consumption-based) interest rate, $r(t)$. The household's problem is therefore:

$$\max_{C(t),X(t),K(t),B(t)} \int_{t=\tau}^{\infty} e^{-\rho(t-\tau)} \frac{C(t)^{1-\theta}}{1-\theta} \tag{1}$$

subject to

$$P_c(t) C(t) + P_x(t) X(t) + P_c(t) \dot{B}(t) = W(t) L + R(t) K(t) + r(t) P_c(t) B(t) \tag{2}$$

and

$$\dot{K}(t) = X(t) - \delta K(t). \tag{3}$$

Note that consumption and investment have distinct, time-varying prices, $P_c(t)$ and $P_x(t)$, respectively. To this investment problem, we add structural transformation, which can impact the price of investment relative to consumption. Specifically, we assume the household also faces an intratemporal problem of choosing consumption of value added from agriculture, $C_a(t)$, manufacturing, $C_m(t)$, and services, $C_s(t)$, to produce the consumption aggregate:

$$C(t) = \left[\sum_{j=a,m,s} \omega_{cj}^{\frac{1}{\sigma_c}} C_j(t)^{\frac{\sigma_c - 1}{\sigma_c}} \right]^{\frac{\sigma_c}{\sigma_c - 1}},$$

where we normalize the CES weights, $\sum_{j=a,m,s} \omega_{cj} = 1$. Consistent with standard structural change patterns, we further assume that sectors are gross complements, i.e., $\sigma_c < 1$.

A competitive firm uses a similar CES aggregator to take value added in agriculture, $X_a(t)$, manufacturing, $X_m(t)$, and services, $X_s(t)$, and produce the final investment aggregate, $X(t)$:

$$X(t) = A_x(t) \left[\sum_{j=a,m,s} \omega_{xj}^{\frac{1}{\sigma_x}} X_j(t)^{\frac{\sigma_x - 1}{\sigma_x}} \right]^{\frac{\sigma_x}{\sigma_x - 1}}. \tag{4}$$

We normalize the weights again to sum to one, $\sum_{j=a,m,s} \omega_{xj} = 1$, but they are specific to the investment sector. Note also that the investment aggregator also differs from the intratemporal utility function in that it experiences sector-neutral technological change through $A_{x,t}$, which we assume occurs at a constant rate:

$$\dot{A}_x(t) = \gamma_x A_x(t), \tag{5}$$

with $\gamma_x > 0$.

Finally, we note that the elasticity of substitution in the investment aggregator, σ_x, can potentially differ from that in the consumption aggregator. However, consistent with standard structural change patterns, we again assume that sectors are gross complements in investment, i.e., $\sigma_x < 1$.

A competitive representative firm in each sector $j \in \{a, m, s\}$ produces value added using common Cobb-Douglas technologies, except for factor-neutral productivity parameters, A_j, which vary by sector:

$$C_j(t) + X_j(t) = A_j(t) K_j(t)^{\alpha} L_j(t)^{1-\alpha}. \tag{6}$$

These productivities grow at constant rates:

$$\dot{A}_j(t) = \gamma_j A_j(t), \tag{7}$$

where the growth rates are also sector-specific and, consistent with standard structural change patterns, ordered as follows: $\gamma_a > \gamma_m > \gamma_s > 0$.

Finally, feasibility requires that the labor and capital used by each sector be less than the aggregate supply:

$$\sum_{j=a,m,s} L_j(t) \leq L, \tag{8}$$

and

$$\sum_{j=a,m,s} K_j(t) \leq K(t). \tag{9}$$

2.2 Equilibrium

Analysis of the equilibrium conditions of the model gives intuition for (i) the important roles for both the effective productivity of the investment sector and the relative price of investment and (ii) how structural change leads to its growth rate varying over time and precludes an *aggregate balanced growth path* (aggregate BGP) for general parameter values.[4] This result is a simple three-sector extension of the results in Herrendorf et al. (2018).

We start with the the Euler equation for the households dynamic problem:

$$\theta \frac{\dot{C}(t)}{C(t)} = r(t) - \rho = \frac{R(t)}{P_x(t)} - \delta - \rho + \left(\frac{\dot{P}_x}{P_x} - \frac{\dot{P}_c}{P_c} \right), \tag{10}$$

which is the standard single-sector Euler equation except for two differences on the right-hand side: the interest rate involves (i) the growth rate of relative price of investment, $\frac{\dot{P}_x}{P_x} - \frac{\dot{P}_c}{P_c}$, and (ii) the rental rate of capital *in terms of investment.* We will highlight the importance of these differences.

The second dynamic equation is the law of motion for capital:

$$\frac{\dot{K}(t)}{K(t)} = \frac{X(t)}{K(t)} - \delta = \frac{P(t)\,Y(t)}{P_x(t)\,K(t)} - \frac{P_c(t)\,C(t)}{P_x(t)\,K(t)} - \delta. \tag{11}$$

In the first equation of (11), one can see that constant growth in capital requires real investment and capital to grow at a constant rate. Using the definition of total output, $P(t)Y(t) = P_c(t)C(t) + P_x(t)X(t)$ and substituting in for $X(t)$, one can see in the second equation of (11) that this implies constant growth in output and consumption expenditures *when translated into units of the investment good.*[5] Moreover, it is not real consumption that grows at a constant rate, but consumption *expenditures* (in units of investment).

Defining $\tilde{C}(t) \equiv P_c(t)\,C(t)\,/P_x(t)$, the Euler equation becomes

$$\theta \frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{R(t)}{P_x(t)} - \delta - \rho + (1-\theta)\left[\frac{\dot{P}_x}{P_x} - \frac{\dot{P}_c}{P_c} \right].$$

[4] Following Ngai and Pissarides (2007), Herrendorf et al. (2018) define an aggregate BGP as an an equilibrium path along which aggregate variables (expressed in a common unit) grow at constant, though potentially different, rates. This latter characteristic allows for structural change.

[5] It is the fact that investment enters the law of motion in terms of real units of capital that makes other variables have constant growth in units of investment. This is the reason that Ngai and Pissarides (2007) and Herrendorf et al. (2018) choose investment as a numeraire. We do not choose a numeraire at this point in order to make the role of the relative price of investment more explicit.

Herrendorf et al. (2018) show that for our preferences, an assumption of log intertemporal preferences (i.e., $\theta = 1$) and a constant productivity growth rate in investment are necessary and sufficient for such a balanced growth path. One can see that $\theta = 1$ eliminates the problematic role of non-constant growth in the relative price of investment, since the household does not respond to it. As we will show, constant productivity growth in investment production leads the rental rate of capital in units of investment to also be constant, but structural change in investment precludes this constant productivity growth in our model.

To study the dynamics of the relative price of investment, we start by solving for the prices of value added. The cost-minimizing competitive price for value added in sector j is

$$P_j(t) = \frac{1}{A_j(t)} \left(\frac{R(t)}{\alpha}\right)^{\alpha} \left(\frac{W(t)}{1-\alpha}\right)^{1-\alpha}. \quad (12)$$

Hence, given the common Cobb-Douglas parameter for all sectors, relative prices become the inverse of relative productivities:

$$\frac{P_j(t)}{P_{j'}(t)} = \frac{A_{j'}(t)}{A_j(t)}. \quad (13)$$

Given the assumption that $\gamma_a > \gamma_m > \gamma_s > 0$, prices move differentially; relative to manufacturing, the price of services rises and the price of agriculture falls. These feed into the price indexes for consumption and investment, which follow from cost-minimization:

$$P_c(t) = \left[\sum_{j=a,m,s} \omega_{cj} P_j(t)^{1-\sigma_c}\right]^{\frac{1}{1-\sigma_c}} \quad (14)$$

and

$$P_x(t) = \frac{1}{A_x(t)} \left[\sum_{j=a,m,s} \omega_{xj} P_j(t)^{1-\sigma_x}\right]^{\frac{1}{1-\sigma_x}}. \quad (15)$$

The relative price of investment is then

$$\frac{P_x(t)}{P_c(t)} = \frac{1}{A_x(t)} \frac{\left[\sum_{j=a,m,s} \omega_{xj} P_j(t)^{1-\sigma_x}\right]^{\frac{1}{1-\sigma_x}}}{\left[\sum_{j=a,m,s} \omega_{cj} P_j(t)^{1-\sigma_c}\right]^{\frac{1}{1-\sigma_c}}}. \quad (16)$$

The structural transformation model provides a theory for an endogenously time-varying relative price of investment. This relative price can trend for three reasons: (i) technical progress in the investment aggregator, $A_x(t)$; (ii) the CES weights differing across consumption and investment; and (iii) the elasticities differing across consumption and investment. The last two will typically lead to different and changing compositions of agriculture, manufacturing, and service value added across investment and consumption, and so differential rates of price changes in value-added will lead to differential rates of change of the relative price. Our model allows for all three of these. Since we allow for the case $\theta \neq 1$, the changing relative price of investment precludes the existence of a balanced growth path. Note that even if the investment aggregator had no structural transformation because it only included manufacturing (i.e., $\omega_{xm} = 1$), a BGP would not exist in this case.[6]

We now turn to the second important feature of the modified Euler equation: the rental rate of capital in units of the investment good, $\frac{R(t)}{P_x(t)}$.

Since all value-added production functions share the same Cobb-Douglas parameter, production in all sectors uses the same capital-labor ratio, $K_j(t)/L_j(t) = \alpha/(1-\alpha)W(t)/R(t) = K(t)/L(t)$. One can solve for an aggregate production function for the investment sector in terms of the total capital and labor embodied in the value-added aggregated into investment:

$$X(t) = \mathcal{A}_x(t) K_x(t)^{\alpha} L_x(t)^{1-\alpha},$$

where

$$\mathcal{A}_x(t) = A_x(t)\left[\sum_{j=a,m,s} \omega_{xj} A_j(t)^{\sigma_x - 1}\right]^{\frac{1}{\sigma_x - 1}}. \tag{17}$$

Herrendorf et al. (2018) refer to $\mathcal{A}_x(t)$ as *effective* productivity because it includes not only the direct productivity of the aggregator, but also the productivities in producing the different sector j value-added components from labor.[7] From this, one can easily calculate the rental rate of capital in units of the investment good:

$$\frac{R(t)}{P_x(t)} = \alpha \mathcal{A}_x(t)\left(\frac{K(t)}{L}\right)^{\alpha - 1}. \tag{18}$$

[6] Ngai and Pissarides (2007) achieve a BGP for the special case where $\omega_{xm} = 1$ and $\theta = 1$.

[7] Herrendorf et al. (2018) also refer to this as a "pseudo" aggregate production function because it holds in equilibrium rather than as a primitive.

Examining (18), since $K(t)$ must grow at a constant rate on a BGP, $\mathcal{A}_x(t)$ must also grow at a constant rate on a BGP. However, equation (17) shows that this effective productivity of investment, like the price of investment, is subject to the changing composition of value added, and so it will not generically grow at a constant rate. Thus, the presence of structural change in the investment aggregator leads to the lack of a BGP, even in the case of log intertemporal preferences.

One can also analogously solve for an aggregate production function for the consumption sector:

$$C(t) = \mathcal{A}_c(t) K_c(t)^{\alpha} L_c(t)^{1-\alpha},$$

where

$$\mathcal{A}_c(t) = \left[\sum_{j=a,m,s} \omega_{cj} A_j(t)^{\sigma_c - 1} \right]^{\frac{1}{\sigma_c - 1}}. \tag{19}$$

Finally, we note that the wage is the value of the marginal product of labor:

$$W(t) = (1-\alpha) P_x(t) \mathcal{A}_x(t) \left(\frac{K(t)}{L} \right)^{\alpha}. \tag{20}$$

Definition 1. *Given an initial state consisting of $K(0)$, $A_x(0)$, and $\{A_j(0)\}_{j=a,m,s}$, a* ***competitive equilibrium*** *for the model is:*

- *an allocation, $C(t)$, $K(t)$, $X(t)$, $\{C_j(t), X_j(t), K_j(t), L_j(t)\}_{j=a,m,s}$, and*
- *prices, $P_c(t)$, $P_x(t)$, $W(t)$, $R(t)$, $r(t)$ and $\{P_j(t)\}_{j=a,m,s}$,*

for $t \geq 0$ that solve:

- $B(t) = 0$;
- *equations (2)-(12), (14), (15), and (17)-(20); and*
- *the transversality condition, $\lim_{t\to\infty} e^{-\rho t} C(t)^{-\theta} K(t) = 0$.*

Structural change in the model can be summarized by the evolution of consumption and investment expenditure shares given by

$$\chi_{jc}(t) \equiv \frac{P_j C_j(t)}{\sum_{j'=a,m,s} P_{j'}(t) C_{j'}(t)} = \frac{\chi_{jc}(0) e^{(\sigma_c - 1)\gamma_j t}}{\sum_{j'=a,m,s} \chi_{j'c}(0) e^{(\sigma_c-1)\gamma_{j'} t}}, \tag{21}$$

$$\chi_{jx}(t) \equiv \frac{P_j X_j(t)}{\sum_{l=a,m,s} P_{j'}(t) X_{j'}(t)} = \frac{\chi_{jx}(0) e^{(\sigma_x - 1)\gamma_j t}}{\sum_{j'=a,m,s} \chi_{j'x}(0) e^{(\sigma_x-1)\gamma_{j'} t}}. \tag{22}$$

From these expressions, along with our assumptions $\sigma_c, \sigma_x < 1$ and $\gamma_a > \gamma_m > \gamma_s$, it follows that consumption and investment shares of the service sector tend to 1 as $t \to \infty$. For this reason, although the model does not have a BGP generally, the model does globally converge to an asymptotic BGP: as $t \to \infty$, the model becomes effectively a single sector model in services. The growth rate of effective investment productivity, $\mathcal{A}_x(t)$, converges to a constant, $\gamma_x + \gamma_s$, and capital and consumption normalized by the factor $\mathcal{A}_x(t)^{1/(1-\alpha)}$ converge to

$$\bar{k}_\infty \equiv \left[\frac{\alpha}{\delta + \rho + \left(1 + \theta \frac{\alpha}{1-\alpha}\right)(\gamma_x + \gamma_s) - (1-\theta)\gamma_s} \right]^{1/(1-\alpha)}, \tag{23}$$

$$\bar{c}_\infty \equiv \bar{k}_\infty^\alpha - \left(\delta + \frac{\gamma_x + \gamma_s}{1-\alpha}\right) \bar{k}_\infty. \tag{24}$$

Formal demonstration of this result is standard. See the proof of Theorem 1.

The absence of $K(t)$ in equations (21) and (22) demonstrates that in this benchmark model, structural change, as measured by the consumption and investment expenditure shares, does not directly depend on the dynamics of capital accumulation.[8] Importantly, the converse is not true. Structural change has a direct impact on the dynamics of capital accumulation and aggregate growth, as we show next.

2.3 Characterization of the Dynamics

Before formally defining the STraP, we motivate it through analysis of the model dynamics in two tractable cases: (i) the local dynamics when the economy is close to the asymptotic BGP; (ii) the global dynamics for the special case where the intertemporal

[8]This is obviously not true generally. For example, this property is not satisfied in models with sector-specific factor elasticities (Acemoglu and Guerrieri, 2008) or non-homothetic preferences (Comin et al., 2021), as we show in the online appendix.

elasticity parameter θ coincides with the capital share, $\alpha = \theta$.

2.3.1 Local Dynamics around the Asymptotic BGP

We characterize the dynamics of normalized variables, where the normalizing factor is $\mathcal{A}_x(t)^{1/(1-\alpha)}$, and we use lowercase to indicate normalized variables. This normalization and notational convention will continue throughout the paper. In terms of the normalized variables, the local dynamics of the economy in the neighborhood of the asymptotic BGP are given by the following system of ordinary differential equations:

$$\theta\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta - \rho + (1-\theta)\left(\bar{\gamma}_c(t) - \gamma_x - \bar{\gamma}_x(t)\right) - \theta\frac{\gamma_x + \bar{\gamma}_x(t)}{1-\alpha}, \tag{25}$$

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - \left(\delta + \frac{\gamma_x + \bar{\gamma}_x(t)}{1-\alpha}\right)k(t), \tag{26}$$

$$\dot{\chi}_{jc}(t) = (\sigma_c - 1)\left(\gamma_j - \bar{\gamma}_c(t)\right)\chi_{jc}(t), \tag{27}$$

$$\dot{\chi}_{jx}(t) = (\sigma_x - 1)\left(\gamma_j - \bar{\gamma}_x(t)\right)\chi_{jx}(t), \tag{28}$$

where $j = a, m, s$. Here $\bar{\gamma}_c(t) = \chi_{ac}(t)\gamma_a + \chi_{mc}(t)\gamma_m + (1 - \chi_{ac}(t) - \chi_{mc}(t))\gamma_s$ denotes the consumption expenditure-weighted average of sectoral productivity growth, and $\gamma_x(t)$ denotes its investment analog.

To a first order, the effect of structural change on growth is summarized by its effect on the weighted averages $\bar{\gamma}_c(t)$ and $\bar{\gamma}_x(t)$. The last two terms in the log-linearized Euler equation, equation (25), capture the two channels through which structural change affects the growth of consumption. The first term corresponds to the effect of structural change on the growth rate of the price of consumption relative to investment, which is given by the difference in the weighted average growth rate of productivity. In the log case ($\theta = 1$) this terms disappears. The second channel is the standard normalizing term that guarantees that the system is (asymptotically) stationary. This term is constant in a Neoclassical growth model, but here it is time-varying for any finite time t, and it also appears in the log-linearized law of motion of capital in equation (26).

Equations (27) and (28) show again that in this benchmark model, structural change, as measured by the evolution of consumption and investment expenditure shares, is independent of the Neoclassical dynamics of capital accumulation.

As discussed earlier, structural change precludes the existence of a BGP, apart from the asymptotic ones. To understand the importance of this departure, it is useful to characterize the speed of convergence of the system to the asymptotic BGPs and to determine whether the dynamics of $\bar{\gamma}_c(t)$ and $\bar{\gamma}_x(t)$ are long-lasting compared with the relatively fast Neoclassical dynamics (King and Rebelo, 1993).

Toward this end, the following proposition characterizes the eigenvalues of the system in (25)-(28), which allows us to compute the speed of convergence of the economy to the asymptotic BGPs.[9]

Proposition 1. *The eigenvalues of the system in (25)-(28) are the solution to the following characteristic polynomial:*

$$\overbrace{\begin{vmatrix} \alpha\left(\bar{k}_\infty\right)^{\alpha-1}-\delta-\frac{\gamma_x+\gamma_s}{1-\alpha}-\lambda & -1 \\ \bar{c}_\infty\frac{\alpha(\alpha-1)}{\theta}\left(\bar{k}_\infty\right)^{\alpha-2} & -\lambda \end{vmatrix}}^{\text{Neoclassical eigenvalues}}$$
$$\times\left[\left(\sigma_c-1\right)\left(\gamma_a-\gamma_s\right)-\lambda\right]\left[\left(\sigma_c-1\right)\left(\gamma_m-\gamma_s\right)-\lambda\right]$$
$$\times\underbrace{\left[\left(\sigma_x-1\right)\left(\gamma_a-\gamma_s\right)-\lambda\right]\left[\left(\sigma_x-1\right)\left(\gamma_m-\gamma_s\right)-\lambda\right]}_{\text{Structural change eigenvalues}}=0 \tag{29}$$

where $\bar{k}_\infty$ and $\bar{c}_\infty$ are the normalized steady-state levels of consumption and capital defined in equations (23) *and* (24)*.*

Each term of the characteristic polynomial (29) defines eigenvalues of the system. There is a sharp separation between two sets of eigenvalues. As labeled, the first term defines a pair of eigenvalues that correspond to those in the standard Neoclassical one-sector growth model. The negative eigenvalue is associated with the stable path, and the positive, with the unstable. The remaining four terms define four eigenvalues, all negative. Among the five negative (stable) eigenvalues, the smallest in absolute value is the dominant eigenvalue because it governs the system dynamics asymptotically.

Importantly, for quantitatively relevant values of the sectoral productivity differentials, the eigenvalues associated with structural change are smaller than those governing Neoclassical dynamics. Equivalently, the half-lives of the structural change dynamics are long relative to those of the Neoclassical transitional dynamics. For

[9] Acemoglu and Guerrieri (2008) provide a related characterization in the proof of their Theorem 2.

instance, if we set $\sigma_x = \sigma_c = 0$ (implying the fastest possible convergence from the structural change eigenvalues), $\gamma_a = 0.05$, $\gamma_m = 0.02$, and $\gamma_s = 0.01$, the dominant eigenvalue governing the local dynamics of structural change implies a half-life of 70 years. This is an order of magnitude larger than the half-life of the Neoclassical transitional dynamics implied by the negative Neoclassical eigenvalue.[10]

2.3.2 An Illustration of the Global Model Dynamics

We now derive the model's dynamics for an easily characterized special case where the intertemporal elasticity parameter θ coincides with the capital share, $\alpha = \theta$. This special case allows us to illustrate in closed form how the medium-term dynamics of the economy are characterized by a path that is independent from the initial conditions of the economy. In the next section, we show that this path is indeed the STraP that exists for more general parameterizations.

We characterize the dynamics of normalized variables, where the normalizing factor is $\mathcal{A}_x(t)^{1/(1-\alpha)}$ and use lowercase to indicate normalized variables as in the previous section. We denote the equilibrium path of capital at time t given an initial condition k_0 as $k(t, k_0)$

Proposition 2. *Suppose that $\alpha = \theta$. The solution to the competitive equilibrium in Definition 1 is*

$$k(t, k_0) = \left\{ \left[k_0^{1-\alpha} - k^*(t_0)^{1-\alpha} \right] \frac{\mu(t_0)}{\mu(t)} + k^*(t)^{1-\alpha} \right\}^{\frac{1}{1-\alpha}} \tag{30}$$

and

$$\tilde{c}(t) = M(t) k(t, k_0), \tag{31}$$

[10]See King and Rebelo (1993) for a thorough discussion on the speed of convergence of the Neoclassical growth model.

where $M(t)$, $\mu(t)$, and $k^*(t)$ are continuous, positively-valued functions satisfying

$$\lim_{t\to\pm\infty} M(t) = \frac{\delta+\rho+(1-\alpha)\gamma_x}{\alpha} - \delta, \tag{32}$$

$$\frac{d\mu(t)}{dt} > 0, \qquad \lim_{t\to\infty} \mu(t) = \infty, \tag{33}$$

$$\lim_{t\to\pm\infty} k^*(t) = \overline{k}_{\pm\infty}. \tag{34}$$

We relegate the details of the derivation to Online Appendix B. Here, we briefly outline the derivation of Proposition 2 and discuss its relevance. The assumption that $\alpha = \theta$ allows us to decouple the dynamic system given by the Euler equation (10) and the law of motion for capital (11) into a differential equation for $\tilde{c}/k$ and another for k. These two equations for $\tilde{c}/k$ and k depend explicitly on time, implying that a balanced growth path cannot be a solution to the competitive equilibrium for any finite value of t. To solve these two differential equations, we need two boundary conditions. One is given by the initial capital level, k_0. The other comes from requiring that, as $t \to \infty$, the competitive equilibrium converges to an economy dominated by the slowest-growing sector (services) featuring a balanced growth path with steady-state capital $\bar{k}_{+\infty}$.[11]

We discuss next the equilibrium time path of capital and rewrite equation (30) as

$$k\left(t,k_0\right)^{1-\alpha} = \underbrace{\left[k_0^{1-\alpha} - k^*\left(t_0\right)^{1-\alpha}\right]}_{\text{Initial Condition}} \underbrace{\frac{\mu\left(t_0\right)}{\mu\left(t\right)}}_{\longrightarrow 0} + \underbrace{k^*\left(t\right)^{1-\alpha}}_{\text{Medium-run}}. \tag{35}$$

Equation (35) shows that the equilibrium trajectory of capital, $k(t,k_0)$, can be decomposed in two parts. The first part is the difference between the initial condition k_0 and the value of capital, $k^*(t_0)$, evaluated along a trajectory that is independent of the initial condition. This difference is constant over time. Moreover, its influence on the level of capital $k(t,k_0)$ diminishes over time, since it is weighted by $\mu(t_0)/\mu(t)$, which tends to 0 as $t \to \infty$.[12] Thus, this first term mostly contributes to short-term

[11] This result comes from the assumption that sectors enter consumption and investment aggregators as gross complements, i.e., $\sigma_c, \sigma_x < 1$. Thus, as services overtake the entire economy, the competitive equilibrium converges to a balanced growth path "as if" we had a one-sector model (services). We compute the normalized steady-state capital level $\bar{k}_\infty$ of this "one-sector" model and require that the economy converges towards this steady state.

[12] We show that the weight $\mu(t_0)/\mu(t)$ features exponential decay. Thus, for practical purposes,

dynamics.

By contrast, the second term, $k^*(t)$, is *independent* of the initial condition k_0 and does not have a dampening factor. Thus, it captures medium-run dynamics of the capital path. Note also that, by construction, it asymptotically converges to the steady-state level of a service-only economy, $\bar{k}_{+\infty}$. Importantly, since $k^*(t)$ is independent from the initial condition k_0, it captures the common component of capital dynamics across economies starting with different initial conditions. The term $k^*(t)$ is constructed by computing the capital path consistent with *both* converging to a one-sector service economy in the future and a one-sector agricultural economy in the past (i.e., as $t \to -\infty$). This second term $k^*(t)$ corresponds to the STraP, which we formalize in the next section.[13]

3 The STraP

In this section, we formally introduce the new concept of a stable growth path, the STraP, define it formally, and show its existence and uniqueness.

Building on the example in the previous section, we utilize the two limiting asymptotic BGPs as key ingredients in the definition of the STraP. In the case of $t \to \infty$ the growth rates of effective investment productivity, $\mathcal{A}_x(t)$, converges to a constant $\gamma_x + \gamma_s$, while in the case of $t \to -\infty$ it converges to an analogous $\gamma_x + \gamma_a$. In both cases, the composition of investment and consumption are identical to each other, so the growth rate of the relative price of investment converges to $-\gamma_x$, and, similarly, in both cases the normalizing factor is $\mathcal{A}_x(t)^{1/(1-\alpha)}$. Again, we use lowercase to indicate normalized variables and, therefore, define the limiting normalized capital stock for the asymptotic economy as $t \to -\infty$ as:

this decay is "fast" and the contribution of the first term is negligible at "short" horizons. This is exemplified in the model simulations of Section 4.2.1.

[13]Note also that the dynamics for consumption inherit analogous properties in terms of the decomposition to the ones we have discussed for capital. Equation (31) shows that the optimal consumption path is proportional to the optimal capital path. This factor of proportion, $M(t)$, is time dependent but does not depend on the initial capital condition.

$$\bar{k}_{-\infty} \equiv \left[\frac{\alpha}{\delta + \rho + \left(1 + \theta \frac{\alpha}{1-\alpha}\right)(\gamma_x + \gamma_a) - (1-\theta)\gamma_a} \right]^{1/(1-\alpha)},$$

while the limiting normalized capital stock for the asymptotic economy as $t \to \infty$, $\bar{k}_{\infty}$, is given by equation (23).

3.1 Defining the STraP

We can now define the STraP as the time path of objects connecting the asymptotic agricultural BGP to the asymptotic service BGP.

Definition 2. *Given a sequence of productivities, $A_x(t)$, and $\{A_j(t)\}_{j=a,m,s}$, the* ***Stable Transformation Path (STraP)*** *is:*

- *an allocation, $C(t)$, $K(t)$, $X(t)$, $\{C_j(t), X_j(t), K_j(t), L_j(t)\}_{j=a,m,s}$, and*
- *prices, $P_c(t)$, $P_x(t)$, $W(t)$, $R(t)$, $r(t)$ and $\{P_j(t)\}_{j=a,m,s}$,*

defined $\forall t \in \mathbb{R}$ that solve:

- *$B(t) = 0$;*
- *equations (2)-(12),(14), (15), and (17)-(20); and*
- *asymptotic conditions, $\lim_{t\to\infty} \frac{K(t)}{\mathcal{A}_x(t)^{1/(1-\alpha)}} = \bar{k}_{\infty}$ and $\lim_{t\to-\infty} \frac{K(t)}{\mathcal{A}_x(t)^{1/(1-\alpha)}} = \bar{k}_{-\infty}$.*

Comparing the STraP with the definition of an equilibrium in the previous section, we see two main differences. First, whereas an equilibrium is only defined forward from an initial value of $t = 0$, the STraP is defined for all real numbers. Second, an equilibrium is solved for a specific initial value and asymptotic boundary condition (the transversality condition) of the capital stock, while the STraP uses two asymptotic boundary conditions.

The fact that the initial value of capital, $K(0)$, is not an arbitrary initial condition in the STraP implies that, given the productivity process, the STraP passes through *a particular value of* $K(0)$, which we call $K(0)_{STraP}$. That is, the STraP gives a particular time path of capital that is stable for the productivity process, whereas an equilibrium can be defined for any positive value of $K(0)$. The STraP path for capital is stable in that for $K(0) \neq K(0)_{STraP}$, the dynamic equations in the equilibrium will lead capital to converge to the STraP level of capital over time via standard Neoclassical convergence dynamics as illustrated in equation (35). As anticipated by Proposition 1, in simulation, these Neoclassical convergence dynamics are quick, as they are in the single-sector model. In the next subsection, we formally define a more general version of the STraP and prove its existence and uniqueness.

3.2 Existence and Uniqueness of the STraP

We start by presenting a more general class of growth models for which we will prove existence and uniqueness of the STraP. We consider models in which the First and Second Welfare Theorems hold, so that competitive equilibria coincide with solutions to the planner's problem. In particular, we suppose that starting at an arbitrary time, $-\tau$, the planner's intertemporal problem can be written as follows:

$$\max_{c(t),k(t)} \int_{t=-\tau}^{\infty} e^{-\rho t} \mathcal{A}_u(t) u(c(t), t) dt, \tag{36}$$

where

$$\dot{k}(t) = f(k(t), t) - (\delta + \gamma_k(t))\, k(t) - c(t). \tag{37}$$

To ensure everything is well behaved, we assume that $\gamma_u(t) \equiv \frac{\dot{\mathcal{A}}_u(t)}{\mathcal{A}_u(t)}$, $\gamma_k(t)$ are continuously differentiable; u is three times continuously differentiable, strictly concave for each t, and $\lim_{c \to 0} u'(c,t) = \infty$ for all t; the function $f(k,t)$ is twice continuously differentiable and satisfies the Inada conditions for each t; and $\gamma_k(t) > 0$ for all t.

For the concept of the STraP to make sense, we need that this problem converges to a standard optimal growth problem in the limits as $t \to \pm\infty$. This means the growth rates, the production function, and the utility function all need to converge. We assume

$$\lim_{t\to\infty} \gamma_u(t) = \gamma_{+u} > 0, \lim_{t\to-\infty} \gamma_u(t) = \gamma_{-u} > 0,$$

$$\lim_{t\to\infty} \gamma_k(t) = \gamma_{+k} > 0, \lim_{t\to-\infty} \gamma_k(t) = \gamma_{-k} > 0.$$

For the production functions, we assume that there exist functions $f_+(k), f_-(k)$ such that

$$\lim_{t\to\infty} f(k,t) = f_+(k), \lim_{t\to-\infty} f(k,t) = f_-(k),$$

$$\lim_{t\to\infty} \frac{\partial f(k,t)}{\partial k} = f'_+(k), \lim_{t\to-\infty} \frac{\partial f(k,t)}{\partial k} = f'_-(k)$$

uniformly on $k \in [\varepsilon, \bar{k}]$ for all $\varepsilon, \bar{k} > 0$. Similarly, we assume there exist functions $u_+(c)$ and $u_-(c)$ such that

$$\lim_{t\to\infty} u(c,t) = u_+(c), \lim_{t\to-\infty} u(c,t) = u_-(c),$$

$$\lim_{t\to\infty} \theta(c,t) = \theta_+(c), \lim_{t\to-\infty} \theta(c,t) = \theta_-(c)$$

uniformly on $c \in [\varepsilon, \bar{C}]$ for all $\varepsilon, \bar{C} > 0$, where $\theta(c,t) \equiv -\frac{\frac{\partial^2 u(c,t)}{\partial c^2} c}{\frac{\partial u(c,t)}{\partial c}}$. To these assumptions, we add the assumption that $\gamma_{+u}, \gamma_{-u} < \rho$ so that utility is well defined moving forward and in both asymptotic balanced growth paths.

Our model in Section 2 is a special case of this more general setting. Once normalized, consumption and investment correspond to those in the model. Similarly, $u(c,t) = \frac{c^{1-\theta}}{1-\theta}$, $f(k,t) = k^\alpha$, $\gamma_k(t) = \frac{1}{1-\alpha}\frac{\dot{\mathcal{A}}_x(t)}{\mathcal{A}_x(t)}$, and $\mathcal{A}_u(t) = \left(\mathcal{A}_c(t)\mathcal{A}_x(t)^{\frac{\alpha}{1-\alpha}}\right)^{1-\theta}$, which all fit the model assumptions. It is also straightforward to show that the Welfare Theorems hold in the model of Section 2 using standard techniques as in Acemoglu (2008), for example.[14]

Similarly, we can define the asymptotic conditions more generally. Denote k_∞, the asymptotic steady-state level of capital, as the steady-state level of capital for the problem

$$\max_{c(t),k(t)} \int_0^\infty e^{(-\rho+\gamma_{+u})t} u_+(c(t))dt,$$

where

$$\dot{k}(t) = f_+(k(t)) - (\delta + \gamma_{+k})\, k(t) - c(t),$$

[14]In the online appendix, we formally show the above mapping, as well as the mapping for two extensions: a version of the benchmark model with non-homothetic CES preferences as in Comin et al. (2021) and one with sector-specific factor intensities following Acemoglu and Guerrieri (2008).

and $k_{-\infty}$ as the steady-state levels of capital for the problem

$$\max_{c(t),k(t)} \int_0^\infty e^{(-\rho+\gamma_{-u})t} u_-(c(t))dt,$$

where

$$\dot{k}(t) = f_-(k(t)) - (\delta + \gamma_{-k})\, k(t) - c(t).$$

Naturally, there are also corresponding steady-state levels of consumption, c_∞ and $c_{-\infty}$.

For the sake of completeness and ease of stating the theorem, we now restate the definition of the STraP in this more general setup.

Definition 3. *Given time-τ productivities, $\mathcal{A}_u(\tau)$ and $\gamma_k(\tau)$, the* ***Stable Transformation Path (STraP)*** *is an allocation, $c\,(t)$ and $k\,(t)$ defined $\forall t \in \mathbb{R}$, that solves the maximization problem in equations (36) and (37) and satisfies the boundary conditions:*

- $\lim_{t\to\infty} k(t) = k_\infty$ *and*
- $\lim_{t\to-\infty} k(t) = k_{-\infty}$.

Given the model and definitions, we make an additional assumption.

Assumption 1. *There exists a function $h : \mathbb{R} \to (0, 1)$ such that:*

- *h is strictly increasing and invertible,*
- *both h and h^{-1} are twice continously differentiable,*
- $\lim_{t\to\pm\infty} h'(t)$ *exists,*
- $\lim_{t\to\pm\infty} \frac{\dot{\gamma}_k(t)}{h'(t)} = \lim_{t\to\pm\infty} \frac{\dot{\gamma}_u(t)}{h'(t)} = 0,$
- $\lim_{t\to\pm\infty} \frac{\frac{\partial f(k,t)}{\partial t}}{h'(t)} = \lim_{t\to\pm\infty} \frac{\frac{\partial^2 f(k,t)}{\partial k \partial t}}{h'(t)} = 0$ *uniformly on* $k \in [\underline{\varepsilon}, \bar{k}]$,
- $\lim_{t\to\pm\infty} \frac{\frac{\partial \theta(c,t)}{\partial t}}{h'(t)} = 0$ *uniformly on* $c \in [\varepsilon, \bar{C}]$ *for all* $\varepsilon > 0$, *and*
- $\lim_{t\to\infty} \frac{h''(t)}{h'(t)} = a_+ \in (-\infty, 0), \lim_{t\to-\infty} \frac{h''(t)}{h'(t)} = a_- \in (0, \infty).$

We construct an $h(t)$ in the online appendix that satisfies Assumption 1 for any monotonic, twice-differentiable $\gamma_u(t)$ and $\gamma_k(t)$, including the model in Section 2. The theorem can now be stated quite simply.

Theorem 1. *If Assumption 1 holds, then the STraP exists and is unique.*

We provide a brief overview of the proof here and leave the formal proof to Online Appendix A. Given the setup, the Hamiltonian conditions and transversality condition, which k_∞ satisfies, are sufficient to yield a unique path forward from any $k(t) > 0$. We denote the unique optimal consumption level by $c(k, t)$. This simplifies the system to a one-dimensional non-autonomous system in $k(t)$. Proving the existence and uniqueness of the STraP therefore comes down to proving that from time τ there exists one unique path that has $k(t) \to k_{-\infty}$ as $t \to -\infty$.

We apply an existing theorem (Theorem 4.7.5 in Hubbard and West, 1991) for the existence and uniqueness of time paths in an antifunnel, which requires several conditions: 1) a single differential equation, 2) a Lipschitz condition within the antifunnel, 3) narrowing upper and lower fences that define the antifunnel; and 4) a condition on the derivative of the right-hand side of the differential equation that bounds it away from $-\infty$ in a particular sense.[15] Verifying these conditions requires characterizing $c(k, t)$ to some extent. The function $h(t)$ in Assumption 1 is used to transform the original non-autonomous two-dimensional system into a more easily analyzed three-dimensional autonomous system by including time as a variable and reparameterizing it onto the compact interval $[0, 1]$. This requires that the system is well behaved in the limit as $t \to \pm\infty$, which the conditions in Assumption 1 ensure.

That reparametrization allows us to show that $c(k, t)$ gets arbitrarily close to the consumption function in the negative asymptotic growth problem. With that, we can construct the upper and lower narrowing fences that define the antifunnel and verify that the other conditions hold.

[15] Heuristically, a fence is a one-way gate for a dynamic path. A lower fence pushes solutions up; an upper fence pushes them down. An antifunnel is a region defined by upper and lower fences, as lower and upper limits, respectively, which a path, starting from the outside, cannot enter. Narrowing is the property that the size of this region shrinks to 0. For a deeper explanation of antifunnels and fences, see Hubbard and West (1991). In particular, chapter 1 gives a nice intuitive introduction.

4 STraP Implications for Medium-Term Structural Change and Growth

In this section, we (i) calibrate the model and show the relevance of the relaxed parameterizations it enables for structural change and growth dynamics, and (ii) show the STraP's implications for aggregate development patterns. The medium-term STraP dynamics allow us to better understand important development patterns in the data, including slow convergence, which constitutes a well-known puzzle for the Neoclassical growth model.

4.1 Calibration

To illustrate and compute the STraP, we start with a benchmark model that sticks closely with the existing literature but adds in the new variations that the STraP allows for. For its computation we move to discrete time. We maintain the same growth notation, but we use the discrete analogs, e.g., $A_{x,t+1}/A_{x,t} = 1 + \gamma_{x,t}$, and the discount rate, ρ, is replaced by the discount factor, β.[16]

We assign standard values for depreciation and the capital elasticity parameter, $\delta = 0.05$ and $\alpha = 0.33$. We choose an intertemporal elasticity parameter of $\theta = 2$, which diverges from the log utility in the Ngai and Pissarides (2007) and Herrendorf et al. (2018) but is a more common value in the broader macro literature.[17] We calibrate the rest of the parameters to match the available time series for the United States.[18] The relative sectoral productivity growth rates are calibrated using time series data on relative prices according to equations (13) and (16). The absolute growth rates are scaled to match growth in income per capita. This yields values of

[16]Although Theorem 1 ensures that a unique STraP exists, an issue of practical relevance is how to solve for the STraP. In the online appendix we provide an algorithm that addresses less straightforward features encountered when solving models with non-trivial medium-term dynamics. We also present a calibrated extension with non-homothetic preferences.

[17]Havránek (2015)'s meta-analysis of the 2,735 estimates of the elasticity of intertemporal substitution in consumption from 169 published studies covering 104 countries produces a mean of 0.5, which implies $\theta = 2$.

[18]We combine the historical GDP by Industry data over the period 1947–1997 together with such data over the period 1997–2018 to yield sectoral prices and value added. We use aggregate real (chain-weighted) GDP, real personal consumption expenditures (PCE), and real private investment from BEA Table 1.1.6, and the prices of PCE and real private investment from BEA Table 1.1.4.

$\gamma_a = 0.050$, $\gamma_m = 0.021$, $\gamma_s = 0.012$, and $\gamma_x = 0.0026$. Absent non-homotheticities, Leontieff substitution between sectors provides a best fit to long-run data (Buera and Kaboski, 2009). We therefore assign a common elasticity of substitution for consumption and investment that approaches Leontieff, $\sigma_c = \sigma_x = 0.01$. We then calibrate the aggregator weights to match the average shares over the time series, where we use the input-output tables to yield the sectoral composition of investment. These values are $\omega_{c,a} = 0.013$, $\omega_{x,a} = 0.015$, $\omega_{c,m} = 0.231$, $\omega_{x,m} = 0.502$, $\omega_{c,s} = 0.756$, and $\omega_{x,s} = 0.483$.[19] Finally, we choose $\beta = 0.99$, to match the average interest rate, which we take from Gomme et al. (2011). They calculate the after-tax return to business capital to average 6.1% over the period 1950-2000.

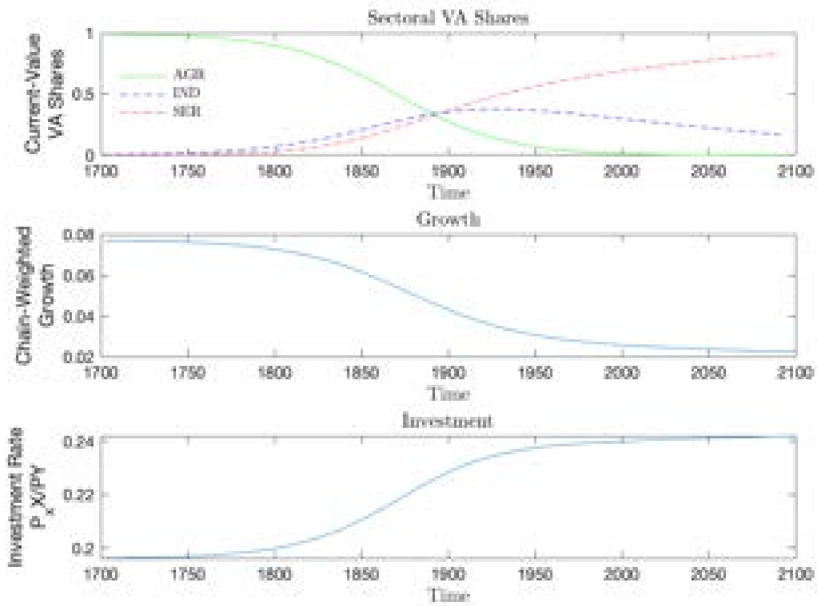
4.2 Implications for structural transformation, growth, and investment

In this section, we show that the calibrated benchmark STraP combines interesting implications for structural transformation, growth, and investment patterns. Simulations of the key growth variables for the benchmark economy are displayed in Figure 1.

The top panel of Figure 1 shows the structural transformation patterns over time by tracking the current-value, value-added share of each sector over time. The patterns of the benchmark economy replicate the qualitative empirical patterns of most countries, including a sharp decline of the share of agriculture that asymptotes toward zero, a hump shape in manufacturing's share (with a peak under 50% of the economy), and an increase and late acceleration of the share of services that eventually constitutes the majority of the economy. Using industry as a gauge, one can see that the dynamics of structural transformation are of comparable orders of magnitude to the historical data (see Buera and Kaboski, 2012, and the original sources therein). In 1870, when data are first available, the share is roughly 0.28 in the model, somewhat higher than the 0.24 in the data. By the middle of the twentieth century, that share has risen considerably. The peak in the model is 0.38 in 1927, and the peak in the data is 0.40 in 1941. The growth to the peak is therefore 10 percentage points

[19] We solve for the relative weights using the relationship $\left(\frac{P_{j,t}}{P_{m,t}}\right)^{\sigma_c} = \frac{\omega_{c,j}}{\omega_{c,m}} \frac{C_{m,t}}{C_{j,t}}$, with $j = a, m$. We then use the normalization that weights sum to one to obtain values for $\omega_{c,j}$, with $j = a, m, s$.

Figure 1: Structural Transformation, Growth, and Investment



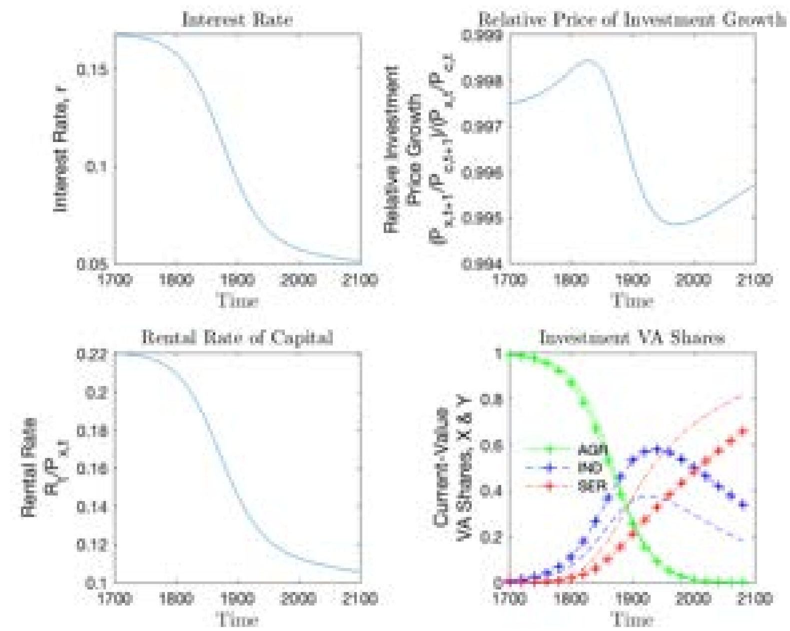
in the data, and 16 percentage points in the model. By 2000, however, the share has fallen to 0.30 in the model and to 0.19 in the data. The decline of 8 percentage points in the model is therefore also smaller than the observed decline of 18 percentage points over this period. Nevertheless, the predicted hump shape in industry is quantitatively important and not something that the Ngai and Pissarides (2007) model accomplished.

The middle panel of Figure 1 presents the benchmark economy's growth rate of real GDP, where real aggregates are constructed using a chain-weighted index, as done in national income data. We see that the real growth rate varies along the STraP, and indeed structural transformation has implications for growth. As the economy moves from the fastest total factor productivity (TFP) growth sector (agriculture) to the slowest (services), Baumol (1967)'s disease is at work, and the real growth rate slows from 4.3% to 2.6% over the twentieth century.[20]

The bottom panel of Figure 1 shows that the growth rate is determined not only by the sectoral TFP patterns that drive Baumol's disease, but also by the dynamics of capital

[20] If we use investment as the numeraire in constructing real growth, as in Ngai and Pissarides (2007), the growth rate shows a similar fall, but from about 4.7% to 3.0%.

Figure 2: Understanding Growth Dynamics



that motivate the dynamic model. The investment rate increases with structural transformation — about 22% to 24% over the twentieth century. This acts to partially counteract Baumol's disease. Interestingly, the growth of investment coincides with an increase in the value-added share of manufacturing, consistent with García-Santana et al. (2019). However, the subsequent decline in the share of manufacturing occurs despite maintaining the high rate of investment.

Figure 2 explores the factors behind the investment dynamics in more depth. The top left panel plots the interest rate over time, which is crucial to growth in the Euler equation, i.e. equation (10). It falls over time, with a substantial drop from 17% to 5%. This interest rate drop can be decomposed into the (gross) growth rate of the relative price of investment and the rental rate in terms of the investment good, per equation (10). The top right panel shows that the growth rate of the relative price of investment is below one, but it is relatively stable with a slight hump-shaped pattern. The lower left panel shows that the drop in the interest rate is driven primarily by the decline in the rental rate of capital in terms of investment good. Recall that the price of the investment good is a function of the productivity growth rate in the investment aggregator, as well as the changing composition of investment caused by structural transformation within the investment. The bottom right panel shows this sectoral composition of investment (the lines with the plus symbols) compared with the overall sectoral composition of value added (the lines without the plus symbols). Investment undergoes the same qualitative transformation as the economy overall, but is always more intensive in manufacturing — and eventually services. This changing composition gives insight into the de-industrialization process: the rate of investment stays high during the period when the share of manufacturing is declining, but given the high productivity growth in manufacturing, the composition of investment shifts away from manufacturing. The composition of investment also contributes to changes in the growth rate of the relative price of investment. Given the productivity growth in the investment aggregator, the relative price of investment is always falling, but it falls more slowly in the middle period when agriculture — the sector with the fastest TFP growth — is relatively unimportant in investment, but still vital to consumption.

4.2.1 Capital Dynamics and Stability

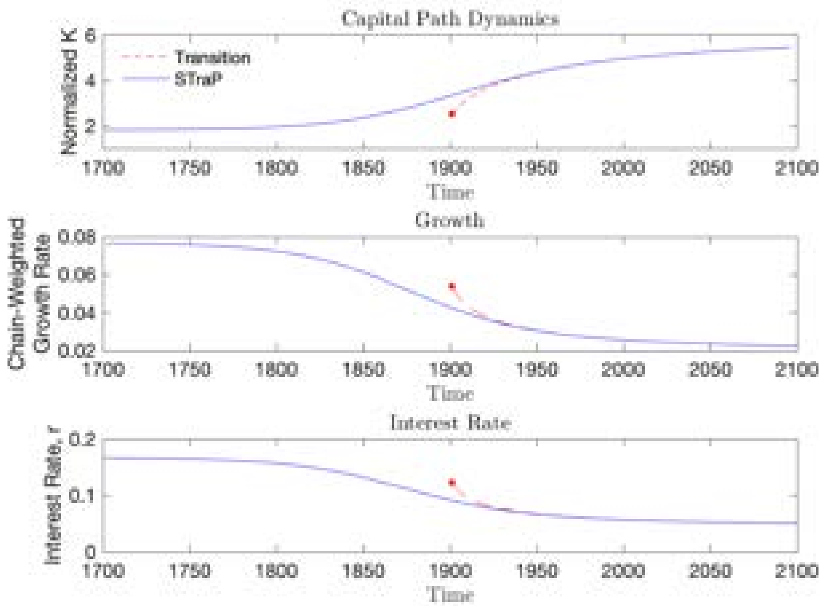
The STraP leads to time-varying dynamics in capital, even appropriately normalized capital (i.e., $K_t/\mathcal{A}_{x,t}^{1/(1-\alpha)}$), and we have emphasized that this time path is stable, as illustrated by the example in equation (35). Figure 3 further shows both of these features of the STraP. The solid line in the top panel shows the time path for the normalized capital stock along the STraP. Over time, capital increases relative to normalized investment productivity. Hence, the STraP predicts a lengthy capital deepening phenomenon over the development process.

The dashed line in the top panel of Figure 3 shows a transition path from a low level of normalized capital starting in 1900. The initial level of capital is low relative to the value of capital of the STraP in that year, given the level of the profile of productivities.[21]

We describe the STraP as stable because capital quickly converges from the relatively lower initial level to the STraP. Initially, there is also a relatively large growth rate (see

[21] The level of capital could be initially low because there was a negative shock, e.g., a war, destroying part of the capital stock. Alternatively, the initial level of capital could correspond to the value of a capital in an alternative STraP with a lower level of productivity. In this second interpretation, the transition is triggered by an unanticipated productivity shock, and the new STraP is a scaled-up version of the original one.

Figure 3: Transitional Dynamics



the middle panel of Figure 3), as the marginal product of capital is relatively large and capital is accumulated at a higher rate. Thus, the mechanics are akin to Neoclassical dynamics toward a BGP, except that the path for stable (normalized) capital is itself time-varying. This example illustrates not only the stability of the STraP but also the speed of convergence. The half-life in this simulated case is just eight years. This stands in contrast to the relatively slow capital-deepening dynamics along the STraP. Thus, Neoclassical forces lead to rapid convergence of capital, whereas structural transformation itself leads to slower time-varying capital deepening. This is in turn a global numerical counterpart to the local analytical result in Proposition 1 (and the particular case of Proposition 2).

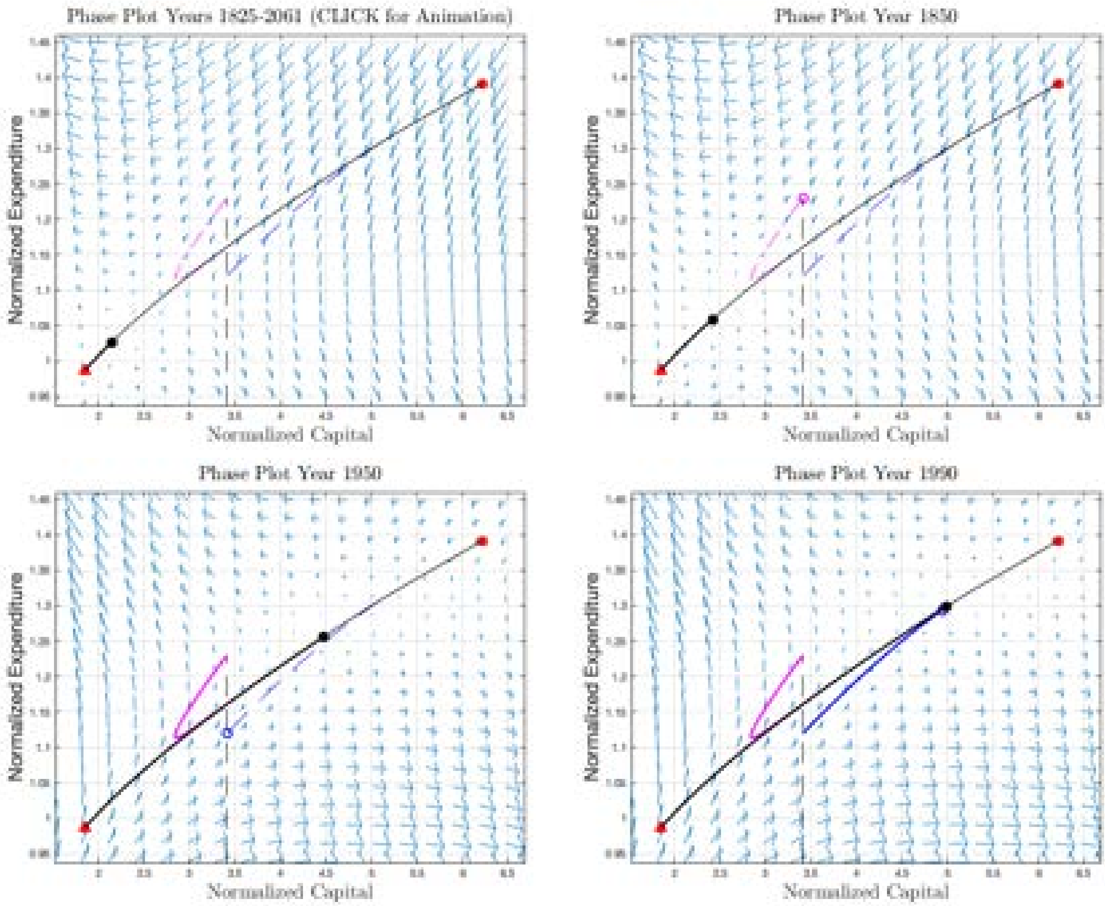
Figure 4 illustrates the multidimensional aspect of convergence, by plotting the dynamic paths in the normalized capital–normalized consumption expenditure space. The top left panel of the figure adds the remaining dimension of time, which controls productivity — this is shown with an animation of the movement of the economy (the black dot) along the STraP (black line), from the agricultural BGP (the red triangle) through the transformation to the services BGP (the red square). (The animation starts when clicked on.) The vector field of time-varying arrows is a phase diagram showing the systems instantaneous trajectories for arbitrary expenditure-capital combinations. To further illustrate convergence properties, we start two economies from the identical normalized capital stock, but at two different points in time. The upper pink trajectory is an economy that starts at an earlier time with a higher-than-STraP level of capital given its productivity at that time, while the lower blue trajectory starts at a later date with a lower-than-STraP level of capital. The animation shows the rapid convergence to the STraP over time for both initial values. Note that the distances of the convergence paths in normalized expenditure-capital space are *not* reflective of the time required to converge. (Indeed, for more extreme levels of productivity — either extremely early or extremely late time periods — the convergence paths could go directly toward the BGPs.)

The other panels of Figure 4 illustrate snapshots from this animation. The different panels are discrete jumps in time, which serve to emphasize how the vector field — and the point toward which an economy moves — varies with time. The top right panel illustrates this starting point of the upper pink trajectory in the year 1850. Given high levels of capital, initial expenditures also exceed those of the STraP economy as the open pink circle indicates. The higher expenditures of the pink economy drive down the (normalized) capital stock over time. The bottom left panel brings the economy forward to the year 1950, where the full, completed convergence path of the pink economy can be seen. Interestingly, although convergence of the pink economy is from above, it involves a period of both decreasing and increasing normalized capital stocks. The same panel also shows the starting point of the blue dashed trajectory. Given the later date, the same capital stock is now to the left of the STraP and the vector field has rotated. Hence, the economy chooses a lower-than-STraP level of expenditures and will accumulate capital. The bottom right panel shows the economy in the year 1990, where both economies have converged to the STraP (here we show the instance where the blue hollow dot has almost converged so that it can be compared with the black dot).

4.2.2 Comparative Statics and Variant Models

Our benchmark model relaxed assumptions that were necessary for a BGP but no longer necessary in the STraP. In this section, we examine the importance of these

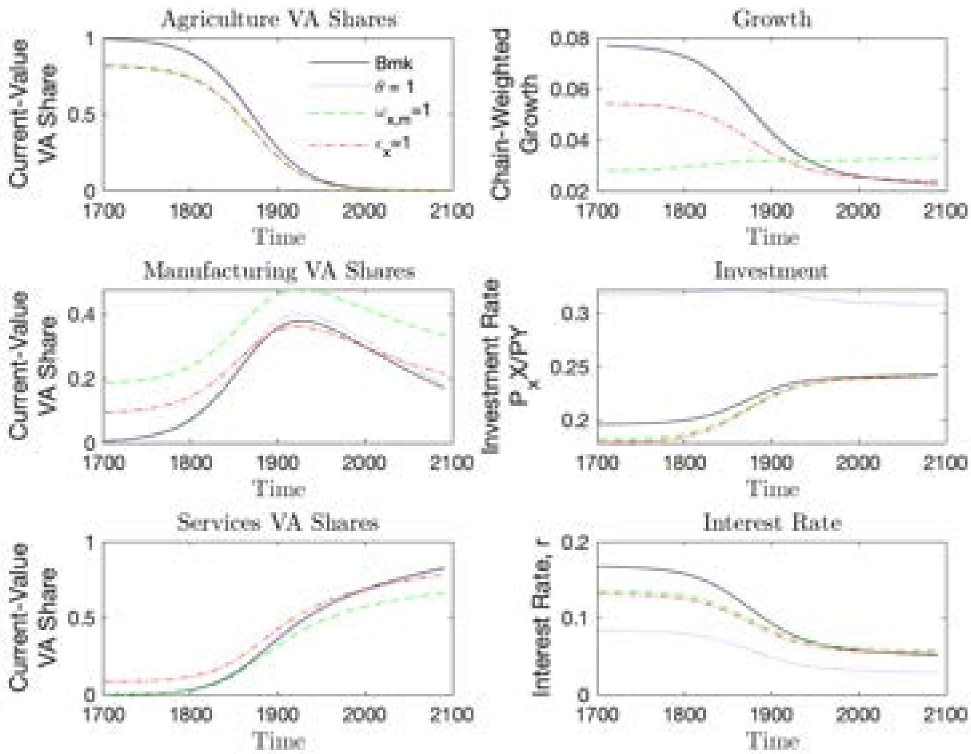
Figure 4: STraP Convergence in the Capital-Expenditure Space



assumptions by simulating alternative models with different parameter choices. That is, we perform simulated comparative statics changing one assumption at a time. Ultimately, not only do these parameters matter quantitatively for structural transformation and interest rates, but they also influence the qualitative patterns of growth and investment.

The alternative models we present are as follows. The first two adopt the assumptions in Ngai and Pissarides (2007), one at a time. The first alternative is their assumption of $\theta = 1$, i.e., log intertemporal preferences (relative to our benchmark of $\theta = 2$). The second alternative is their assumption that investment consists only of manufacturing value added, i.e., $\omega_{x,m} = 1$, (relative to the benchmark where it is heavy in manufacturing, but still a mix that undergoes structural transformation). Finally, we consider an alternative in which the elasticity of substitution in the investment sector

Figure 5: Alternative Models



is unitary. This captures the idea of an investment sector with a mixed but stable composition, as in García-Santana et al. (2019), and contrasts it with an investment sector that undergoes structural transformation itself.

Figure 5 plots these alternatives versus the benchmark to demonstrate their impact on sectoral shares (left panels), growth (top right), investment (middle right), and interest rates (bottom right).

The impact of the first alternative, log preferences (the dotted blue lines), shows up most strikingly in the investment rate and interest rate. On the one hand, the level of the investment rate starts out much higher than in the benchmark (roughly, 0.31 versus 0.20) and falls slightly over time rather than rising as in the benchmark (and other alternatives). On the other hand, the interest rate starts substantially lower (8% versus 17%) and declines over time as in the benchmark, though by substantially less (5 percentage points relative to 11 percentage points). Focusing on the structural transformation patterns, one can see the impact of the higher investment rate; the peak in the manufacturing hump is slightly higher, since investment is relatively intensive in manufacturing value added.

The second alternative, only manufacturing in investment (the dashed green line), leads to different sectoral distributions, growth rates, and interest rates. The fact that all investment comes from the manufacturing sector changes the sectoral distributions. Manufacturing naturally constitutes a higher share of output, but the impact can be most easily seen in the asymptotic sectoral compositions; agriculture is less than one initially, while the economy never fully becomes exclusively services as it grows. Compared with the growth rate in the benchmark economy, the growth rate is lower and varies less over time.[22] Moreover, it rises with structural transformation rather than declining steeply as the economy leaves agriculture. Finally, we see that the interest rate starts lower (roughly 14.5% versus 17%), but it is relatively stable and indeed 1 percentage point higher by the end of the sample than the benchmark rate, which falls much more. (The impact on the investment rate is more subtle. It starts half a percentage point lower than the benchmark rate, but rises a bit faster and is comparable to the benchmark rate by the end of the sample.)

The third alternative, a unitary elasticity of substitution in investment (the dash-dotted red line), tends to dampen the patterns relative to our benchmark. The decline in agriculture, hump shape in manufacturing, and increase in services are all less pronounced that in the benchmark simulations. Looking at the panels on the right, we see that the decline in growth is also less pronounced, as is the decline in the interest rate. The intuition is clear: structural transformation is weaker because it is only occurring within the consumption sector. However, the investment rate rises somewhat more without structural transformation as agriculture is not as important to investment early on, so the relative price of investment does not fall as rapidly.

4.3 STraP and the Development Path

We now shift our focus away from the US to the STraP in relation to empirical growth and development patterns. We use Penn World Tables (PWT) 9.1 cross-country panel

[22] It is not absolutely constant as in Ngai and Pissarides (2007) because we report a chain-weighted growth rate rather than using manufacturing as the numeraire.

(Feenstra et al., 2015) to establish these empirical patterns and compare the model's predictions with respect to real (purchasing power parity, PPP) expenditure income per capita for real capital-output ratios, current-value investment rates, relative price of investment, (consumption-based) interest rates, and real (within-country) growth rates per capita.[23]

Relative to the US data, the PWT data differ in three important ways that are relevant to the STraP. First, capital's share is somewhat higher. Second, growth rates in real income per capita are somewhat slower than in the US BEA data. Third, the capital-output ratio is substantially higher in the PWT data. For calibration, we focus on comparable countries, and include all country-year observations within the US real income per capita range for the period 1950–2000, which is approximately 14,600-46,500 US dollars.[24] To account for the differences, we change three parameter values to ensure that our simulations for the years 1950–2000 match the corresponding moments in the PWT data (an average annual growth of 1.55%, a capital's share of 0.40, and a capital-output ratio of 3.99).[25] First, we raise α from 0.33 to 0.4. Second, to account for lower growth, we scale all primitive productivity growth rates down by a common factor, 0.565. To account for the higher capital-output ratio, we adjust the discount factor up (and no longer target the interest rate) to $\beta = 0.9782$.

We present the results over a wide range of development that spans well beyond the calibrated range, from a log real income per capita of 7 (roughly \$1100) up to the US income per capita in 2000, 10.75 (roughly \$46,500).[26] With 126 countries and over 4,000 observations, the data themselves are dense and have wide variance. To make the data clearer, we use three lines to characterize them: nonparametric fits (using 100 income bins) of the 25th and 75th percentile of the data at each income level and a linear fit.[27] For comparison's sake, we include a simulated, calibrated Cass-

[23]Given the importance of capital dynamics, we include only country-year observations for which arbitrarily initialized capital stocks are no longer relevant, as discussed in Inklaar et al. (2019). Such indicators are not in the publicly available PWT 9.1 data series, but were provided upon request. See Online Appendix E for details on the construction of the sample and all data variables.

[24]All dollars are in real international PPP dollars, which equals 2011 US dollars.

[25]Another difference between the model and the data is the presence of cyclical fluctuations in the data. Given these fluctuations and the fact that we are more interested in medium-term growth patterns, we construct annual growth rates using ten-year growth averages.

[26]Although limited data are available above and below these ranges, the set of countries thins out quickly, and patterns can be easily driven by country-specific effects and the changing sample.

[27]In all the figures that we present, the coefficient in the linear fit is significant at the 5% level.

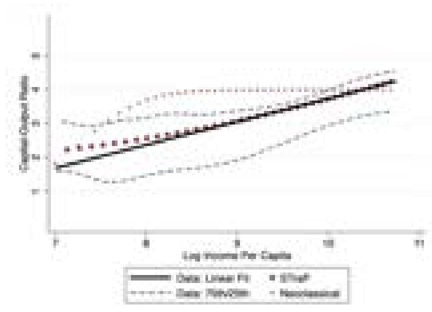
Koopmans-Ramsey growth model — the benchmark one-sector Neoclassical growth model. The comparison is insightful, since this benchmark exhibits no medium-term dynamics and therefore isolates the standard short-term Neoclassical dynamics, while the STraP exhibits no short-term dynamics and isolates the medium-term dynamics. We start the Neoclassical simulation off at log real income per capita of 7 with the average capital-output ratio in the data bin at that level of income.[28]

In the figures that follow, we stress that everything is out-of-sample except for the *average* capital-output ratios and annual growth rates at very high incomes (log incomes above 9.59) and the initial (i.e., log income of 7) capital-output ratio in the Neoclassical model. Looking back on lower incomes is a test of whether the stability of the productivity process and the structural transformation forces modeled can add insight into broader development patterns.

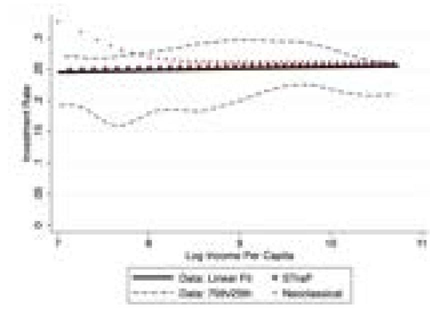
We start by examining the patterns and determinants of the capital-output ratio, since, at the aggregate level, structural transformation provides a theory for a time-varying capital-output ratio over the medium term, and the STraP characterizes these dynamics. Figure 6 presents these results. The top left panel plots the capital-output ratio over development. The black lines show the data, which have wide variance but clearly trend up and in a economically important way. In the linear fit, the capital-output ratio rises from roughly 1.8 to 4.2. The capital-output ratios in the STraP (diamonds) structural transformation model mirrors this increase fairly well though not fully: the STraP can explain an increase in the capital-output ratio from roughly 2.2 to 4. By construction, the Neoclassical growth model ('+' symbols) explains the full increase in K/Y, but the path displays its well-known short-lived dynamics, rising quickly and then remaining flat. The key point here is over the broad range of development, the assumption of a constant, balanced growth capital-output ratio does not hold in the data. The structural transformation models yield persistent medium-run capital accumulation dynamics that (i) go beyond the rapid convergence dynamics in the Neoclassical growth model that stem from initial levels of capital, and (ii) can partially explain the overall idea of a rising capital-output ratio over development.

[28]The calibrated Neoclassical growth model uses the same Cobb-Douglas parameter and depreciation rate, and productivity growth and the discount factor are again chosen to match the identical calibration targets.

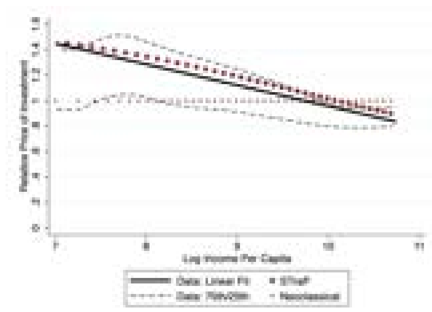
Figure 6: Capital Accumulation and its Determinants over Development



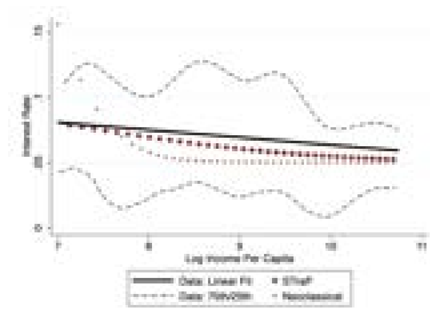
(a) Capital-Output Ratio



(b) Investment Rate



(c) Relative Price of Investment

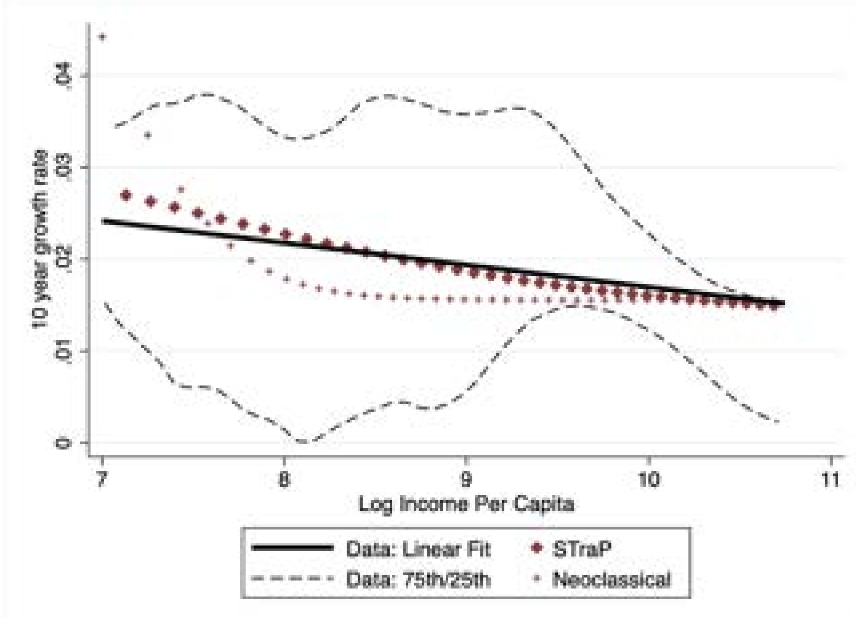


(d) Interest Rate

The remaining panels of Figure 6 explore the determinants of these capital accumulation dynamics. The top right panel shows the (current value) investment rate. The linear fit of the data shows a very mild, though statistically significant, increase. The relative flatness of this pattern was emphasized by Hsieh and Klenow (2007), and the statistical significance of the increase was noted by Inklaar et al. (2019). The STraP shows an increase in the investment rate, roughly identical to that of the linear fit of the data. Again, the Neoclassical growth model, in contrast, has a very high initial investment rate and a rapid stabilization consistent with the well-known short-run dynamics.

The bottom left panel shows the relative price of investment over development. The marked decline in the data, from over 1.4 to less than 0.9, is another pattern emphasized by Hsieh and Klenow (2007) as important for understanding development: rich countries get more real investment bang for their current-value investment rate buck. Here the structural transformation in the STraP leads to variation over development that is consistent with a declining relative price of investment that is again remark-

Figure 7: Ten-Year Growth Over Development



ably similar to the declining relative price given by the linear fit in the data. Again, in contrast, the Neoclassical growth model delivers a flat prediction for relative prices.

The bottom right panel plots interest rates over development.[29] The data show a mild decline, with the linear fit dropping from roughly 8% to 6%. The STraP interest rate is only slightly lower, droppomg from 7.5% to 5%. In contrast, the initial interest rate is exceedingly high in the Neoclassical growth model, with the decline in the model transition immediate, intense, and short-lived.

Figure 7 shows what is really the key finding of this analysis. It focuses on the models' implications for (ten-year) average annual per capita income growth (and implicitly convergence) over development. Again, the ten-year growth rate is used because our focus is on medium-term growth dynamics. In the data, ten-year annual growth rates, on average, fall with development from 2.5% to under 1.5%, or a decline of 40%. The Neoclassical growth model displays a version of the well-known convergence puzzle.[30]

[29]We measure interest rates in the PWT data to be consistent with our consumption-based definition of interest rates.

[30]The more extreme version of the convergence puzzle in Barro and Sala-i Martin (1992) assumes that all countries have the same technology (at a point in time). We combine time and cross-sectional variation and allow poorer countries at any point in time to be further behind in the productivity process.

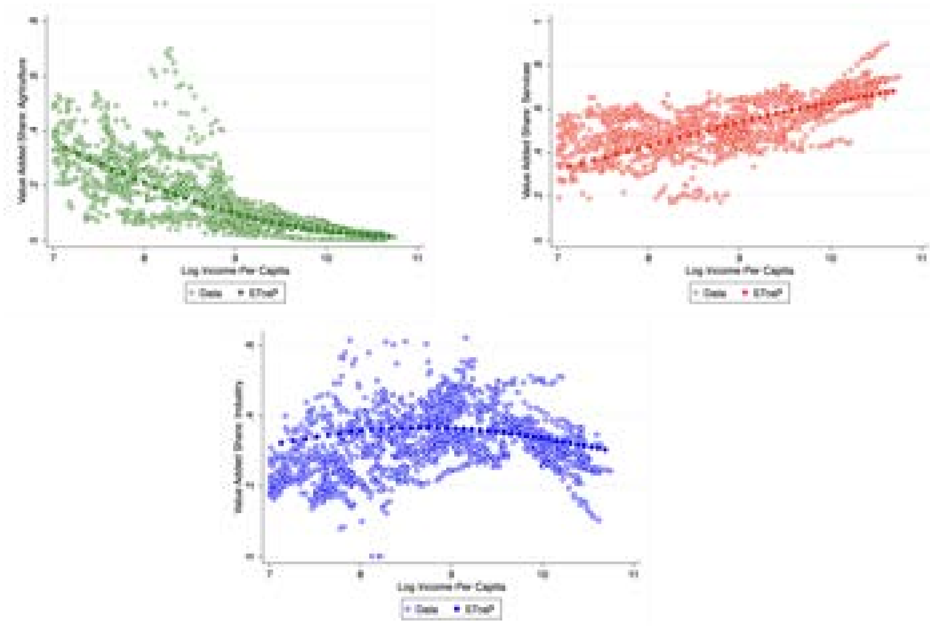
Growth declines very quickly over development, as the economy rapidly converges to its BGP.

In contrast, the medium-term dynamics of the STraP exhibit slowly and persistently declining growth rates that are quantitatively important. This is not simply Baumol's disease. While the movement of resources from faster to slower productivity growth sectors dominates the dynamics, the increasing capital accumulation over development in response to a falling relative price of investment and the productivity growth slowdown partially counteracts this. Moreover, one can see that the growth slowdown is much stronger at incomes below \$13,000 PPP (log of 9.5), a range of development well below the range emphasized by Baumol. The point here is that even without an initially low capital stock and despite increasing investment rates over development, a country with an income per capita of \$1,100 (i.e., log of 7) but following the productivity growth rates of the richer countries could expect its growth rate to fall by more than one-third (from 2.8% to 1.8%) as it grows to an income per capita of \$13,000.

Our analysis combines structural transformation dynamics with the better understood forces of investment-specific technical change (ISTC) — a potential alternative to explain the declining relative price of investment shown in Figure 6. To distinguish these forces, we compare our results with a model in which we turn off the other sources of exogenous technical change (i.e., $\gamma_a = \gamma_m = \gamma_s = 0$), leaving only investment-specific technical change (γ_x). We note four things. First, ISTC explains almost no GDP growth in the model (an annual growth rate of less than 0.1%). Second, ISTC explains only 58% of the decline in relative prices. Although this pattern is often solely attributed to ISTC, structural transformation is responsible for 42% of the decline. Third, ISTC explains only 42% of the real increase in (log) K/Y. (These two numbers matching is merely incidental.) Fourth, ISTC alone delivers a balanced growth path (Greenwood et al., 1997), and so, unsurprisingly, leads to no change in the growth rate over development. The remainder of these two last patterns are instead the result of the time-varying nature of the rate of change in relative prices and Baumol's disease — both endogenous processes.

The importance of the structural transformation mechanism for aggregate dynamics leads to the natural question of whether the structural transformations that underlie the aggregate behavior in the model also align with the data. Sectoral data are not

Figure 8: Structural Transformation over Development



available in the PWT data, but we turn to the Groningen Growth and Development Centre (GGDC) 10-Sector Database (Timmer et al., 2015), which provides data on sectoral shares for 39 countries of varying levels of development over the years 1950–2010. Given the smaller sample, we can plot the STraP against the actual data. (We omit the Neoclassical model because it has no sectoral implications.)

Figure 8 shows the fit of the STraP simulation relative to the GGDC data (empty circles) for agriculture (top left panel), industry (lower panel), and services (top right panel). Despite the fact that, viewed together, the figures make a sad face, the model actually follows the overall patterns of structural change in the data quite well. Again, we emphasize that these patterns are out of sample: none of these sectoral patterns have been used to calibrate the model. In sum, the structural transformation mechanisms driving the growth dynamics in the model have supporting evidence in the data as well.

Lastly, the STraP can be used to decompose growth into transitional, medium-term, and residual elements. Specifically, by comparing (i) data, (ii) STraP dynamics, and (iii) simulations of the structural transformation model from observed initial capital levels, we can decompose how much observed additional growth came from medium-term STraP dynamics and how much came from initial conditions, and determine the residual growth that came from faster-than-benchmark technical change. A comparison of the United States and Thailand is a nice illustrative example. Empirically, post-1950, real income per capita in the United States grew at 1.6% annually, two percentage points lower than Thailand's annual growth of 3.6%.[31] For the US, the data, transition path from an initial condition, and STraP each yield the same 1.6% contribution, indicating that initial conditions played essentially no role. The US was effectively on its STraP and is well approximated by the benchmark productivity process. For Thailand, however, the STraP yields a growth rate of 2.1%, indicating that 0.5 (i.e., 2.1% - 1.6%) percentage points of its additional growth (one-quarter of the 2 percentage point difference) came from STraP dynamics. The transition path to the STraP from initial capital yielded an average annual growth of 2.3%, indicating that initial conditions accounted for an additional 0.2 (i.e., 2.3% - 2.1%) percentage points, somewhat smaller than the contribution of medium-term dynamics.[32] The remaining 1.3 percentage points come from Thailand's higher productivity growth.

Much of Thailand's additional growth along the STraP is driven by structural transformation, where the share of agriculture declines by 12 percentage points in Thailand and only 2 percentage points in the US. (Although this difference is substantial, it is much smaller than the 38 and 6 percentage point declines implied, respectively, in the data.) Again, structural transformation leads to both Baumol's disease and capital accumulation. Capital per capita increases 1.8% in the US but 3.0% in Thailand annually in the simulations from observed capital levels. Of this 1.2 (i.e., 3.0% - 1.8%) percentage point difference in capital accumulation in the model, roughly half (i.e., 2.5%-1.8%, where the difference is rounding) comes along the STraP, and the other half is due to a low initial stock of capital. (Empirically, capital grew at 4.4% annually in Thailand, which is 1.2 percentage points higher than in the Thai simulation and again reflects faster-than-benchmark technical change.)

[31] We use the periods available in the Groningen 10-sector database. Thai data starts in 1951.

[32] When we compute a country-specific STraP using country-specific technological progress, we can account for the full growth, but the contribution of initial conditions remains nearly the same, at 0.4 percentage points. The country-specific model can fully account for the difference in growth rates.

5 Conclusion

We have developed a new dynamic concept to characterize growth models with asymptotic BGPs, but non-trivial medium-run dynamics like structural change models. We have proven its existence and uniqueness for a general class of growth models, and we have presented an algorithm for computing it.

The STraP allows us to study a broader class of structural change models and enables us to link two important sets of development patterns — namely, those of structural transformation with those of aggregate growth. The STraP is therefore valuable in that it allows us to characterize the medium-term dynamics of capital accumulation and growth. The model and its predictions, even out-of-sample predictions, make progress in matching and understanding important and well-known empirical patterns of development. Indeed, much of the structural change studies previously done in static settings can be extended to dynamic frameworks.

We believe that the STraP concept can be useful for future research in development economics, macroeconomics, and international economics. In development economics, for instance, our concurrent research, Buera et al. (2019), is using the STraP as a benchmark for normative wedge analysis of structural transformation. In macroeconomics, the STraP can be helpful for disentangling business cycle frequency from medium-term transformation dynamics. In international economics, the STraP can be useful for work analyzing premature de-industrialization in the global economy. These are examples of the potentially wide applicability of the concept.

Link to Online Appendix (Click Here)

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