AN INTRODUCTION TO QUASI-ISOMETRY AND HYPERBOLIC GROUPS

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ABSTRACT

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A Quasi-Isometry is map between metric spaces that allows us to distort distances but preserves the large scale geometry of the spaces. By relaxing the distance preservation of an isometry and allowing distances to dilate and contract, we arrive at a bi-Lipschitz map between metric spaces. However, bi-Lipschitz maps still preserve local information. If we relax these conditions even further and allow an additive error, we can 'zoom out' and investigate the global structure of spaces. This leads to the notion of quasi-isometry.

A common introduction to hyperbolic space is through the fact that the sum of the angles of a triangle is less than 180 degrees. Another feature of these triangles is that they are in some sense 'thin'. The notion of thin triangles extends to other metric spaces as well. This leads to the study of hyperbolic metric spaces, which are spaces characterized by the uniform thinness of geodesic triangles.

Contents

1. Introduction

1.1 Motivation

Geometric group theory is study of groups as geometric objects as well as the study of nice group actions on geometric objects. While we usually start with a space and associate a group to it, such as homotopy and homology groups in topology, geometric group theory is somewhat the reverse of this. Instead we start with a group and associate a natural space to it, its Cayley graph and define a natural metric to associate a metric space to the group we started with. Armed with a metric, we can explore groups as geometric objects and study geometric properties such as geodesics, curvature, and boundary.

1.2 Main Results

Our main results are primarily concerned with the characterization of metric groups and the invariance of certain qualities under quasi-isometry.

Theorem 1.2.1. *(Bi-Lipschitz Equivalence of Groups) Let G be a finitely generated group and let S and S' be two finite generating sets. The identity map* $\mathrm{Id} : (G, d_S) \to (G, d_{S'})$ *is a bi-Lipschitz equivalence.*

Theorem 1.2.2. *(*Švarc-Milnor Lemma*) Let G be a group and let (X, d) be a proper geodesic metric space. If G acts geometrically on X, then G is finitely generated and G is quasi-isometric to X.*

Theorem 1.2.3. *(Quasi-Isometry Invariance of Hyperbolicity) Let X and Y be geodesic metric spaces. Suppose $f : X \to Y$ is a quasi-isometry. Then X is hyperbolic if and only if Y is hyperbolic.*

Theorem 1.2.4. *(Quasi-Isometry Invariance of the Gromov Boundary) Let X and Y be hyperbolic metric spaces. If $f : X \to Y$ is a quasi-isometry, then the induced map $\partial f : \partial X \to \partial Y$ is a homeomorphism.*

2. Finitely Generated Groups as Metric Spaces

2.1 Cayley Graphs

When first learning about groups, we usually learn them from a purely symbolic algebraic standpoint. We get a glimpse of what groups are about when learning about dihedral groups. We see that they encode 'symmetries' of an n-gon when we draw some n-gon and start rotating and reflecting it. More precisely, dihedral groups are the isometries on the vertices of an n-gon. However, given any finitely generated group, it is possible to draw pictures of what these groups "look" like.

Definition 2.1.1. Let G be a group with finite generating set $S \subset G$. The ***Cayley graph*** of G with respect to S, denoted $\mathrm{Cay}(G, S)$ is the directed labeled graph where

1. the vertex set $\mathcal{V}(\mathrm{Cay}(G, S)) = G$,

2. the edge set is $\mathcal{E}(\mathrm{Cay}(G, S)) = \{(g, gs) \mid g \in G,\ s \in S \setminus \{e\}\}$ and the edge (g, gs) is labeled by s.

The Cayley graph of a group allows us to draw pictures of groups and to see connections between elements in a different light.

Example 2.1.2. The Cayley graph of the additive group $\mathbb{Z}$ with $S = \{1\}$, $\mathrm{Cay}(\mathbb{Z}, \{1\})$, is a simple line with directed edges $(n, n+1)$.

Example 2.1.3. The Cayley graph of the additive group $\mathbb{Z}$ but with $S = \{2, 3\}$, $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$ looks very different from $\mathrm{Cay}(\mathbb{Z}, \{1\})$. It is no longer a simple line, but instead a graph with two different types of labeled edges: there are the directed edges labeled with a 2, $(n, n+2)$, and the directed edges labeled with a 3, $(n, n+3)$. Although at close range, $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$ looks nothing like a line, if we step back far enough, the edges will start to blur and in fact the graph will actually start to look like a line. This sort of "zooming out" is a main theme in geometric group theory.

Example 2.1.4. The Cayley graph $\mathrm{Cay}(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$ is the integer lattice with directed edges labeled by $(1, 0)$ and $(0, 1)$. Similar to the previous example, if we draw this graph and step back far enough, our vertices and edges will blur and the graph will in fact start to look like the Euclidean plane $\mathbb{R}^2$.

Example 2.1.5. Consider the dihedral group $\mathcal{D}_4 = \langle r, s \mid r^4,\ s^2,\ rsrs\rangle$. The Cayley graph $\mathrm{Cay}(\mathcal{D}_4, \{r, s\})$ can be drawn to look like a cube. Unlike the other cases, if we step back far enough, our graph will just blur to a point. In fact, any finite group will have this property.

Example 2.1.6. $\mathrm{Cay}(F_2, \{a, b\})$, the Cayley graph of the free group on two generators, is seen to be an infinite tree. In fact, the Cayley graph of any free group on a finite set of generators is a tree. The converse of this is not true however. [2]

Example 2.1.7. Consider the group $\mathbb{Z} \times \mathbb{Z}_2$ with generating set $\{(1,[0]),(0,[1])\}$. Then $\text{Cay}(\mathbb{Z} \times \mathbb{Z}_2, \{(1,[0]),(0,[1])\})$ is a graph which resembles a ladder. Think of $(\mathbb{R} \times \{0\}) \coprod (\mathbb{R} \times \{1\})$ where there is an edge connecting $(n,0)$ and $(n,1)$ for $n \in \mathbb{Z}$.

2.2 The Word Metric

Now that we can "see" our group concretely via its Cayley graph, it is natural to try to measure distance between elements: choose any two vertices and see how many vertices are in between the two chosen ones. In general, there may not be a unique path between any two vertices and thus taking different routes between any two vertices may give us different measurements. We can begin defining a well-defined metric on any finitely generated group G by considering the shortest path between two elements. We begin making G into a metric space by defining a length function. We already have a Cayley graph to see our groups and by defining a metric, we can study the geometry of groups.

Definition 2.2.1. Let G be a finitely generated group with generating set S. For any $g \in G$, the ***length*** $\ell_S(g)$ of g with respect to S is the smallest integer n such that there exists a sequence $(s_1, \ldots, s_n)$ of generators for which $g = s_1 \cdots s_n$.

So, if $g = s_1 \cdots s_n$ is the shortest way to write g, then $\ell_S(g) = \ell_S(s_1 \cdots s_n) = n$.

Proposition 2.2.2. *Let G be a finitely generated group with generating S. Then ℓ_S induces the metric $d_S : G \times G \to \mathbb{N}$ defined by $d_S(g,h) = \ell_S(g^{-1}h)$. Hence (G, d_S) is a metric space.*

Proof. For $g \in G$, we have $d(g,g) = \ell_S(g^{-1}g) = \ell_S(e) = 0$. If $d_S(g,h) = 0$, then $\ell_S(g^{-1}h) = 0$. So, $g^{-1}h = e$ and thus $g = h$.

For $g, h \in G$ with $g \neq h$, we have $g^{-1}h \neq e$. So, $d_S(g,h) = \ell_S(g^{-1}h) > 0$. So, d_S is positive-definite.

For $g, h \in G$, suppose $g^{-1}h = s_1 \cdots s_n$. Then $h^{-1}g = (g^{-1}h)^{-1} = s_n^{-1} \cdots s_1^{-1}$, and we have $d_S(g,h) = \ell_S(g^{-1}h) = \ell_S(h^{-1}g) = d_S(h,g)$. So, d_S is symmetric.

For any $g_1, g_2, g_3 \in G$, let $g_1^{-1}g_2 = r_1 \cdots r_m$ for $r_i \in S$ and let $g_2^{-1}g_3 = s_1 \cdots s_n$ for $s_i \in S$. Then $g_1^{-1}g_3 = g_1^{-1}g_2g_2^{-1}g_3 = r_1 \cdots r_m s_1 \cdots s_n$. So, $\ell_S(g_1^{-1}g_3) \leq m + n$. Thus, $d_S(g_1, g_3) \leq m + n = d_S(g_1, g_2) + d_S(g_2, g_3)$. So, d_S satisfies the triangle inequality. □

We call d_S the ***word metric*** on G associated to S. So, every finitely generated group we know can also be seen as a metric space! Note that our metric is dependent on the chosen generating set. For example consider the metric spaces $(\mathbb{Z}, d_{\{1\}})$ and $(\mathbb{Z}, d_{\{2,3\}})$. We have $d_{\{1\}}(0,1) = \ell_{\{1\}}(1) = 1$, but with the generating set $\{2,3\}$, we have $1 = 3 - 2$, and so $d_{\{2,3\}}(0,1) = \ell_{\{2,3\}}(1) = \ell_{\{2,3\}}(3-2) = 2$. Therefore, these are not equivalent metrics. We will soon see, however, that we can get around this limitation by studying maps in which the choice of generating set does not matter.

Given any connected graph Γ, we can make $\mathcal{V}(\Gamma)$ into a metric space in a similar way as the word metric: the distance between two vertices is the length of the shortest path between the two vertices. In particular, for any finitely generated group G with finite generating set S, the word metric d_S is equivalent to the usual metric on the Cayley graph. We call the metric on the Cayley graph the ***path metric***.

However, we would also like to take into account the edges of Γ as well. We define the ***geometric realization***, X, of a graph Γ as follows: Take the disjoint union consisting of one point for each vertex, $\{p_{v_i}\}$, in (Γ) and one copy of the unit interval for each edge, $[0,1]_{(v_i,v_j)}$, in $\mathcal{E}(\Gamma)$. Next we form the quotient by identifying the endpoints of $[0,1]_{(v_i,v_j)}$ to the points representing the vertices, i.e., $0 \sim p_{v_i}$ and $1 \sim p_{v_j}$. We can now define a metric, d_X, in the following way:

- the distance between two vertices is inherited from the path metric, i.e., it is still the length of the shortest path between them
- if two points x, y are on the same edge then $x, y \in [0,1]_{(v_i,v_j)}$ and the distance between them is $d_X(x,y) = |x - y|$
- if two points x, y lie are on different edges, $x \in [0,1]_{(v_i,v_j)}, y \in [0,1]_{(v_k,v_\ell)}$, then we define their distance to be

$$\min\Big\{d_X(x,v_i) + d_X(v_i,v_k) + d_X(v_k,v_y), d_X(x,v_i) + d_X(v_i,v_\ell) + d_X(v_\ell,v_y),$$
$$d_X(x,v_j) + d_X(v_j,v_k) + d_X(v_k,v_y)\}, d_X(x,v_j) + d_X(v_j,v_\ell) + d_X(v_\ell,v_y)\Big\}.$$

With this way of looking at graphs, we have another way of looking at groups as metric spaces, by looking at their Cayley graphs as metric spaces.

3. Bi-Lipschitz Equivalence with Groups

3.1 Isometry

We have seen that any finitely generated group can be made into a metric space. Therefore, in addition to studying groups by studying group homomorphisms, we now have another class of maps we can use to study groups: distance preserving maps.

Definition 3.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. An ***isometric embedding*** is map $f : X \to Y$ that preserves distances, i.e., for all $x, x' \in X$, we have $d_X(x, x') = d_Y(f(x), f(x'))$. An isometric embedding f is an ***isometry*** if f is also surjective.

Since $f(x) = f(x')$ implies $0 = d_Y(f(x), f(x')) = d_X(x, x')$, we have that $x = x'$. So, every isometric embedding is injective. Furthermore, every isometric embedding is Lipschitz continuous. Thus, if $f : X \to Y$ is an isometric embedding, then X is homeomorphic to $\text{Im}(f)$.

Example 3.1.2. Consider $\mathbb{Z}^n$ and $\mathbb{R}^n$ both with the taxicab metric. The inclusion map $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is an isometric embedding.

Example 3.1.3. Consider $\mathbb{R}^n$ with the Euclidean metric. For any $v \in \mathbb{R}^n$, translation by v is an isometry. For any $p \in \mathbb{R}^n$, rotation about p is also an isometry.

Example 3.1.4. The geometric realization of $\mathrm{Cay}(\mathbb{Z}, \{1\})$ is isometric to $\mathbb{R}$ with the Euclidean metric.

Example 3.1.5. Considering $\mathbb{R}$ and $\mathbb{R}^2$ with their Euclidean metrics, the set of functions

$$\left\{ f : \mathbb{R} \to \mathbb{R}^2 \;\middle|\; f(t) = \left(\frac{kt}{\sqrt{k^2+1}}, \frac{t}{\sqrt{k^2+1}} \right) \right\}_{k \in \mathbb{Z}}$$
$$\left\{ f : \mathbb{R} \to \mathbb{R}^2 \;\middle|\; f(t) = \left(\frac{t}{\sqrt{k^2+1}}, \frac{kt}{\sqrt{k^2+1}} \right) \right\}_{k \in \mathbb{Z}}$$

are isometric embeddings.

Example 3.1.6. Considering $\mathbb{R}^n$ with the taxicab metric, translation by some vector is an isometry. Rotations, however, in general are not isometries: e.g., the rotation about $(0, 0)$ given by $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

Example 3.1.7. Considering the metric space (G, d_S), left multiplication by a fixed $g \in G$ is an isometry of the group as a metric space. In fact, every group acts by isometries on its itself by left multiplication. Similarly, every group acts by isometries on any of its Cayley graphs by left multiplication.

The next theorem shows the power of studying groups acting on metric spaces to extract extra information about the group. Recall that a group action of G on X is free, or fixed-point free, if the stabilizer of every point in X is trivial. That is, if $g \cdot x = x$, then $g = e$. Also, recall that the torsion elements of a group are the non-identity elements of finite order. So, if a group is torsion-free, then only the identity element has finite order.

Theorem 3.1.8. *Let G be a group acting by isometries on $\mathbb{R}^n$ with the Euclidean metric. If the action is free, then G is torsion-free.*

Proof. Let $g \in G$ have finite order m and consider the cyclic subgroup generated by g. Choose any $x \in \mathbb{R}^n$ and consider the orbit of x of the action restricted to $\langle g \rangle$, $\mathcal{O}_x^{\langle g \rangle} = \{x, gx, \ldots, g^{m-1}x\}$. Consider the barycenter, $b \in \mathbb{R}^n$, of the convex hull of $\mathcal{O}_x^{\langle g \rangle}$. Since $g \cdot \mathcal{O}_x^{\langle g \rangle} = \{gx, g^2x, \ldots, x\} = \mathcal{O}_x^{\langle g \rangle}$, and the action of G is by isometries, the barycenter of the convex hull of $g \cdot \mathcal{O}_x^{\langle g \rangle}$ is $g \cdot b = b$. Since the action is free, then $g = 1$ and $m = 1$. Thus, G is torsion-free. □

3.2 Bi-Lipschitz Equivalence

In general, choosing different generating sets for our groups will result in different word metrics. In particular, we have seen $(\mathbb{Z}, d_{\{1\}})$ and $(\mathbb{Z}, d_{\{2,3\}})$ are not isometric. We mentioned earlier that for our purposes, the choice of generating set would not matter. We make this precise with bi-Lipschitz maps. These maps weaken the rigidness of an isometric embedding by allowing distances to be contracted and dilated by a fixed amount.

Definition 3.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be a ***bi-Lipschitz embedding*** if there exists a constant $K \geq 1$ so that for all $x, x' \in X$ we have

$$\frac{1}{K} d_X(x, x') \leq d_Y(f(x), f(x')) \leq K d_X(x, x').$$

If f is also surjective, then f is said to be a ***bi-Lipschitz equivalence*** and X and Y are said to be bi-Lipschitz equivalent.

As the name suggests, a bi-Lipschitz embedding is injective since if $f(x) = f(x')$, then $d_Y(f(x), f(x')) = 0$ and the first inequality implies $d_X(x, x') = 0$. Thus, $x = x'$ and so f is injective. The second inequality implies that f is Lipschitz continuous and the first inequality then also implies that X is homeomorphic to $\mathrm{Im}(f)$. Note also that if $K = 1$, then f is an isometric embedding. Also, as the name suggests, a bi-Lipschitz equivalence is an equivalence relation.

Proposition 3.2.2. *Bi-Lipschitz equivalence is an equivalence relation on the class of metric spaces.*

Proof. Reflexivity: The identity map $\mathrm{Id} : X \to X$ is a bi-Lipschitz equivalence since it is an isometry. Thus, $X \sim X$.

Symmetry: If $X \sim Y$ and $f : X \to Y$ is a bi-Lipschitz equivalence, then there exists a constant $K \geq 1$ so that for all $x, x' \in X$ we have $(1/K)d_X(x, x') \leq d_Y(f(x), f(x')) \leq K d_X(x, x')$. Since $f^{-1}f(x) = x$, from the first inequality we get $d_X(f^{-1}f(x), f^{-1}f(x')) \leq K d_Y(f(x), f(x'))$, and from the second inequality we get

$(1/K)d_Y(f(x), f(x')) \leq d_X(f^{-1}f(x), f^{-1}f(x'))$. So, we have

$$\frac{1}{K}d_Y(f(x), f(x')) \leq d_X(f^{-1}f(x), f^{-1}f(x')) \leq Kd_Y(f(x), f(x'))$$

and so f^{-1} is a bi-Lipschitz equivalence. So, $Y \sim X$.

Transitivity: Suppose $X \sim Y$, $Y \sim Z$, and that $f : X \to Y$ and $g : Y \to Z$ are bi-Lipschitz equivalences. Let K_1 be the constant associated to f and K_2 the constant associated to g. Then

$$\frac{1}{K_1}\frac{1}{K_2}d_X(x, x') \leq \frac{1}{K_2}d_Y(f(x), f(x')) \leq d_Z(gf(x), gf(x')$$
$$\leq K_2 d_Y(f(x), f(x')) \leq K_1K_2d_X(x, x').$$

So, gf is a bi-Lipschitz equivalence and thus $X \sim Z$. □

We come to our first main theorem. When studying groups as metric spaces, the following theorem allows us the freedom to choose any finite generating set we please. The idea of the proof is to show that the identity map of a group is a bi-Lipschitz equivalence by comparing the maximum length of the elements in one generating set with respect to the other generating set and vice versa. Then we can use these bounds to bound the length of any two elements from our group.

Theorem 3.2.3. *(Bi-Lipschitz Equivalence of Groups) Let G be a finitely generated group and let S and S' be two finite generating sets. Then the identity map of G, $\mathrm{Id} : (G, d_S) \to (G, d_{S'})$, is a bi-Lipschitz equivalence.*

Proof. First, we note that with respect to any word metric, $d_\bullet$, for any $g, h \in G$ we have $d_\bullet(g,h) = \ell_\bullet(g^{-1}h) = \ell_\bullet(1g^{-1}h) = d_\bullet(1, g^{-1}h)$. So, to show that Id is a bi-Lipschitz equivalence, it suffices to show there exists $K \geq 1$ so that for all $g \in G$

$$\frac{1}{K} d_S(1,g) \leq d_{S'}(1,g) \leq K d_S(1,g).$$

Let $M = \max\{d_S(1,s') \mid s' \in S'\}$. Then $M \geq 1$. Let $g \in G$ and suppose $d_{S'}(1,g) = n$. Then $g = s'_1 \cdots s'_n$ where $s'_i \in S'$. By the triangle inequality and the fact that $d_\bullet(g,h) = d_\bullet(1, g^{-1}h)$, we have

$$\begin{aligned}
d_S(1,g) &= d_S(1, s'_1 \cdots s'_n) \\
&\leq d_S(1, s'_1) + d_S(s'_1, s'_1 s'_2) + \cdots + d_S(s'_1 \cdots s'_{r-1}, s'_1 \cdots s'_n) \\
&\leq d_S(1, s'_1) + d_S(1, s'_2) + \cdots + d_S(1, s'_n) \\
&\leq Mn \\
&= M d_{S'}(1,g).
\end{aligned}$$

So, we have $\frac{1}{M} d_S(1,g) \leq d_{S'}(1,g)$.

Similarly, let $N = \max\{d_{S'}(1,s) \mid s \in S\}$. Then $N \geq 1$. Suppose $d_S(1,g) = m$.

Then $g = s_1 \cdots s_m$ where $s_i \in S$. As above, we have

$$\begin{aligned} d_{S'}(1,g) &= d_{S'}(1, s_1 \cdots s_m) \\ &\le d_{S'}(1,s_1) + d_{S'}(s_1, s_1 s_2) + \cdots + d_{S'}(s_1 \cdots s_{s-1}, s_1 \cdots s_m) \\ &\le d_{S'}(1,s_1) + d_{S'}(1,s_2) + \cdots + d_{S'}(1,s_m) \\ &\le Nm \\ &= N d_S(1,g). \end{aligned}$$

So, $d_{S'}(1,g) \le N d_S(1,g)$.

Letting $K = \max\{M, N\}$ we have

$$\frac{1}{K} d_S(1,g) \le d_{S'}(1,g) \le K d_S(1,g). \qquad \square$$

So, although $(\mathbb{Z}, d_{\{1\}})$ and $(\mathbb{Z}, d_{\{2,3\}})$ are not isometric, they are bi-Lipschitz equivalent. The same can be said about their Cayley graphs: $\mathrm{Cay}(\mathbb{Z}, \{1\})$ and $\mathrm{Cay}(\mathbb{Z}, \{2,3\})$ with their path metrics are also bi-Lipschitz equivalent. Unfortunately, we can not extend this to the geometric realizations of the Cayley graphs. So, we look for another class of maps that will allow us more freedom.

4. Quasi-Isometry and the Švarc-Milnor Lemma

4.1 Quasi-Isometry

As we have seen, bi-Lipschitz maps allow us the flexibility to work with any finite generating set we want, but it is still too rigid for our purposes. Bi-Lipschitz maps already seem flexible, we can dilate and contract distances. How can we allow more flexibility? We can allow an additive error on our bounds.

Definition 4.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is a (K, C)-***quasi-isometric embedding*** if there exists constants $K \geq 1$ and $C \geq 0$ so that for all $x, x' \in X$ we have

$$\frac{1}{K} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + C.$$

Furthermore, f is called a (K, C)-***quasi-isometry*** if it is ***coarsely surjective***, i.e., there exists $r \geq 0$ so that for all $y \in Y$ there exists $x \in X$ such that $y \in \overline{B_r(f(x))}$. In this case, we say X and Y are quasi-isometric.

Remark: Although we have used the term 'embedding', a quasi-isometric embedding need not be injective. A quasi-isometric embedding need not be continuous either.

Notice that when $C = 0$ we have that f is a bi-Lipschitz embedding. Note also that if a quasi-isometric embedding $f : X \to Y$ is surjective, then it is trivially coarsely surjective.

Example 4.1.2. We saw in Theorem 3.2.3, for a finitely generated group G, the metric spaces (G, d_S) and $(G, d_{S'})$ are bi-Lipschitz equivalent. Hence they are quasi-isometric.

Example 4.1.3. Considering $\mathbb{Z}^n$ and $\mathbb{R}^n$ with the taxicab metric, we saw the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ was an isometric embedding. This map is also coarsely surjective since for any $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x} \in \overline{B_1(\lfloor \mathbf{x} \rfloor)}$ where $\lfloor \mathbf{x} \rfloor$ takes the floor of each component of $\mathbf{x}$. So, $\mathbb{Z}^n$ and $\mathbb{R}^n$ are quasi-isometric.

Example 4.1.4. Consider $\mathbb{R}$ with the Euclidean metric. The floor function $\lfloor \bullet \rfloor : \mathbb{R} \to \mathbb{R}$ is a quasi-isometry. Given $x \in \mathbb{R}$, we have $\Big| x - \lfloor x \rfloor \Big| < 1$. So, f is coarsely surjective. Next, write $x = \lfloor x \rfloor + r$ and $y = \lfloor y \rfloor + s$ where $0 \leq r, s < 1$. Then

$$|x - y| = \Big| \lfloor x \rfloor + r - (\lfloor y \rfloor + s) \Big| = \Big| \lfloor x \rfloor - \lfloor y \rfloor + r - s \Big| \leq \Big| \lfloor x \rfloor - \lfloor y \rfloor \Big| + |r - s|.$$

So, $|x - y| - |r - s| \leq \Big| \lfloor x \rfloor - \lfloor y \rfloor \Big|$. Also,

$$\Big| \lfloor x \rfloor - \lfloor y \rfloor \Big| = |x - r - (y - s)| = |x - y + s - r| \leq |x - y| + |r - s|.$$

Since $|r - s| \leq 1$, choose $K = 1 = C$. Then we have

$$|x - y| - 1 \leq \Big| \lfloor x \rfloor - \lfloor y \rfloor \Big| \leq |x - y| + 1$$

and thus $\lfloor \bullet \rfloor : \mathbb{R} \to \mathbb{R}$ is a quasi-isometry.

Note that the relevance of this example is not to show that $\mathbb{R}$ is quasi-isometric to itself, as the identity map can show this, but instead to show that since the floor function surjects onto $\mathbb{Z}$, then the floor function is trivially coarsely surjective and thus we also have a quasi-isometry between $\mathbb{R}$ and $\mathbb{Z}$.

Example 4.1.5. The inclusion map $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ is a quasi-isometry. Therefore, for any $n \in \mathbb{Z}$, we have $n\mathbb{Z}$ is quasi-isometric to $\mathbb{R}$.

Definition 4.1.6. Let $f, g : X \to Y$ be maps between metric spaces. We say f and g have ***finite distance*** if there exists a real number $c \geq 0$ such that $d(f(x), g(x)) \leq c$ for all $x \in X$.

Definition 4.1.7. Let X and Y be metric spaces. A map $g : Y \to X$ is a ***quasi-inverse*** for $f : X \to Y$ if gf has finite distance from Id_X and fg has finite distance from Id_Y. That is, there exists a constant $r \geq 0$ so that for all $x \in X$, we have $d_X(gf(x), x) \leq r$, and there exists a constant $s \geq 0$ so that for all $y \in Y$ we have $d_Y(fg(y), y) \leq s$.

Note that in the definition of quasi-inverse, we could take one constant $D = \max\{r, s\}$ which would work for both inequalities. In fact, in the definition of quasi-isometric embedding, we could have also let $R = \max\{K, C\}$ and the condition for quasi-isometric embedding is still satisfied. Although we are expanding our bounds, this allows us a bit more flexibility in proofs. Going one step further, if we have a quasi-isometry we could take $R = \max\{K, C, D\}$ where D is the constant of the coarse surjectivity.

Theorem 4.1.8. *(a) A quasi-isometric embedding $f : X \to Y$ is a quasi-isometry if and only if f has a quasi-inverse $g : Y \to X$.*

(b) Any quasi-inverse for f is a quasi-isometry.

(c) A composition of quasi-isometries is a quasi-isometry.

Proof. (a) If a quasi-isometric embedding $f : X \to Y$ has a quasi-inverse $g : Y \to X$, then there exists $r \geq 0$ so that for all $y \in Y$ we have $d_Y(fg(y), y) \leq r$. Thus, f is also coarsely surjective. So, f is a quasi-isometry.

Conversely, suppose $f : X \to Y$ is a quasi-isometry. Then there exists $R \geq 1$ such that

$$\frac{1}{R} d_X(x, x') - R \leq d_Y(f(x), f(x')) \leq R d_X(x, x') + R$$

and $d_Y(f(x), y) \leq R$ for all $y \in Y$. Consider $\left\{f^{-1}\left(\overline{B_R(y)}\right)\right\}_{y \in Y}$. Since every $y \in Y$ is in a closed R-neighborhood of $f(x)$ for some x, then the family $\left\{f^{-1}\left(\overline{B_R(y)}\right)\right\}_{y \in Y}$ covers X. Thus, we use the axiom of choice to construct $g : Y \to X$ where we choose $g(y) \in f^{-1}\left(\overline{B_R(y)}\right)$. Since $f\left(f^{-1}\left(\overline{B_R(y)}\right)\right) \subset \overline{B_R(y)}$, then $d_Y(fg(y), y) \leq R$ for all $y \in Y$.

Since f is a quasi-isometry, then for the points $x, gf(x) \in X$ we have that $(1/R) d_X(gf(x), x) - R \leq d_Y(f(gf(x)), f(x)) = d_Y(fg(f(x)), f(x))$. Since $fg(f(x)) \in$

$\overline{B_R(f(x))}$ by construction, we have

$$d_X(gf(x), x) \leq R d_Y(f(gf(x)), f(x)) + R^2 \leq R^2 + R^2 = 2R^2.$$

So, g is a quasi-inverse for f.

(b) To show that g is a quasi-isometry, by part (a) it suffices to show that g is a quasi-isometric embedding. Let $y, y' \in Y$. Since f is a quasi-isometry, then

$$\begin{aligned} d_X(g(y), g(y')) &\leq R d_Y(fg(y), fg(y')) + R^2 \\ &\leq R(d_Y(fg(y), y) + d_Y(y, y') + d_Y(y', fg(y'))) + R^2 \\ &\leq R(R + d_Y(y, y') + R) + R^2 \\ &= R d_Y(y, y') + 3R^2. \end{aligned}$$

Also, since $d_Y(y, y') \leq d_Y(y, fg(y)) + d_Y(fg(y), fg(y')) + d_Y(fg(y'), y')$, then we have $d_Y(fg(y), fg(y')) \geq d_Y(y, y') - d_Y(y, fg(y)) - d_Y(fg(y'), y')$. So,

$$\begin{aligned} d_X(g(y), g(y')) &\geq \frac{1}{R} d_Y(fg(y), fg(y')) - 1 \\ &\geq \frac{1}{R}(d_Y(y, y') - d_Y(y, fg(y)) - d_Y(fg(y'), y')) - 1 \\ &\geq \frac{1}{R}(d_Y(y, y') - R - R) - 1 \\ &= \frac{1}{R} d_Y(y, y') - 3. \end{aligned}$$

Since $R \geq 1$, letting $K = R$ and $C = 3R^2$ gives us our constants for g to be a quasi-isometric embedding.

(c) Suppose $f : X \to Y$ and $g : Y \to Z$ are quasi-isometries. Then we can choose $R \geq 1$ so that

$$\frac{1}{R} d_X(x, x') - R \leq d_Y(f(x), f(x')) \leq R d_X(x, x') + R,$$

$$\frac{1}{R} d_Y(y, y') - R \leq d_Z(g(y), g(y')) \leq R d_Y(y, y') + R.$$

So, we have

$$\begin{aligned} \frac{1}{R^2} d_X(x, x') - 1 - R &\leq \frac{1}{R} d_Y(f(x), f(x')) - R \\ &\leq d_Z(gf(x), gf(x')) \\ &\leq R d_Y(f(x), f(x')) + R \\ &\leq R(R d_X(x, x') + R) + R \\ &= R^2 d_X(x, x') + R^2 + R. \end{aligned}$$

Since $R \geq 1$, we have

$$\frac{1}{R^2} d_X(x, x') - (R^2 + R) \leq d_Z(gf(x), gf(x')) \leq R^2 d_X(x, x') + (R^2 + R).$$

So, $gf : X \to Z$ is a quasi-isometric embedding.

Now, let $z \in Z$. Then there exists $y \in Y$ with $d_Z(g(y), z) \leq R$ and there exists $x \in X$ with $d_Y(f(x), y) \leq R$. So,

$$d_Z(gf(x), z) \leq d_Z(gf(x), g(y)) + d_Z(g(y), z) \leq R d_Y(f(x), y) + R + R \leq R^2 + 2R.$$

So, gf is coarsely surjective and thus a quasi-isometry. □

Corollary 4.1.9. *Quasi-isometry is an equivalence relation on the class of metric spaces.*

With the notion of quasi-isometry, we can begin to consolidate the study of group metric spaces, their Cayley graph, and the geometric realization of the Cayley graph.

Theorem 4.1.10. *Let G be a finitely generated group and let S and S' be two finite generating sets. Then the geometric realization of* $\mathrm{Cay}(G, S)$ *is quasi-isometric to the geometric realization of* $\mathrm{Cay}(G, S')$.

Proof. The map $\varphi : \mathrm{Cay}(G, S) \to (G, d_S)$ where x maps to its nearest vertex, making a choice for when x is the midpoint of an edge, is a $(1, 0)$-quasi-isometry and the map $\psi : (G, d_{S'}) \hookrightarrow \mathrm{Cay}(G, S')$ is a $(1, 1)$-quasi-isometry. We saw the identity map $\mathrm{Id}_G : (G, d_S) \to (G, d_{S'})$ is a quasi-isometry and since a composition of quasi-isometries is a quasi-isometry then $\psi \, \mathrm{Id}_G \, \varphi : \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$ is a quasi-isometry. □

Example 4.1.11. We know the metric spaces $(\mathbb{Z}, d_{\{1\}})$ and $(\mathbb{Z}, d_{\{2,3\}})$ are bi-Lipschitz equivalent. Now, we can say more: the geometric realization of their Cayley graphs $\mathrm{Cay}(\mathbb{Z}, \{1\})$ and $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$ are quasi-isometric.

Example 4.1.12. Notice that the geometric realization of $\mathrm{Cay}(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$ is exactly $\mathbb{Z}^2$ with the taxicab metric. Since $\mathbb{Z}^2$ and $\mathbb{R}^2$ with the taxicab metric are quasi-isometric, then in fact $\mathrm{Cay}(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$ is quasi-isometric to $\mathbb{R}^2$ with the taxicab metric. By the theorem, any Cayley graph for $\mathbb{Z}^2$ is quasi-isometric to $\text{Cay}(\mathbb{Z}^2, \{(1,0),(0,1)\})$ and hence quasi-isometric to $\mathbb{R}^2$ with the taxicab metric and thus quasi-isometric to the plane.

4.2 Švarc-Milnor Lemma

Our second main result will concern finitely generated groups acting by isometries in some nice way on nice metric spaces from which we can conclude that the group is finitely generated and quasi-isometric to the metric space. Before we prove this, we first set up the machinery.

Definition 4.2.1. Let (X, d) be a metric space. A ***geodesic*** is an isometric embedding $\gamma : \mathbb{R} \to X$. A ***geodesic segment*** is an isometric embedding $\gamma : [a, b] \to X$. The metric space X is said to be a ***geodesic metric space*** if for any two points $x, y \in X$, there exists a geodesic segment $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. We sometimes identify γ with its image and write $[x, y]$ for the geodesic segment.

Example 4.2.2. The classic example of a geodesic space is $\mathbb{R}^n$ with the Euclidean metric. Geodesics are straight lines and thus a geodesic segment between any two points is the unique line segment connecting the points. The length of a geodesic segment is the usual Euclidean length between two points.

Example 4.2.3. For any finitely generated group, the Cayley graph $\text{Cay}(G, S)$ with the path metric is a geodesic space. Unlike $\mathbb{R}^n$ where there exists a unique geodesic segment between any two points, in $\text{Cay}(G, S)$ there may be many geodesic segments between two points. The geodesic segments between any two vertices $g, h \in G$ is the set of shortest paths between these two points. For example, in $\text{Cay}(F_2, \{a, b\})$ there is a unique geodesic segment between any two vertices. However, for the Cayley graph $\text{Cay}(\mathbb{Z} \times \mathbb{Z}_2, \{(1, [0]), (0, [1])\})$, choosing a vertex $v_0 \in (n, [0]) \in \mathbb{Z} \times \{[0]\}$ and a vertex $v_1 = (n + m, [1]) \in \mathbb{Z} \times \{[1]\}$ where $m \in \mathbb{Z}$ and $|m| \geq 1$, there exists at least two geodesics between v_0 and v_1.

Example 4.2.4. The hyperbolic plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the usual hyperbolic metric is a geodesic space. Geodesics are vertical lines and semi-circles meeting the x-axis at a right angle. Like $\mathbb{R}^n$, for any two points z, z' in the hyperbolic plane, there exists a unique geodesic segment connecting them.

Example 4.2.5. An example of a space which is not geodesic is $\mathbb{R}^2 \setminus \{(0, 0\}$ with the Euclidean metric. There is no geodesic segment between $(1, 0)$ and $(-1, 0)$ since the unique geodesic segment between these points must pass through $(0, 0)$.

Definition 4.2.6. A metric space X is ***proper*** is every closed ball is compact, i.e., for every $x \in X$ and for every $r > 0$, the closed ball $\overline{B_r(x)}$ is compact.

Definition 4.2.7. An action of a group G on a metric space X is said to be ***properly discontinuous*** if for every compact subset $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite.

Definition 4.2.8. An action of a group G on a metric space X is said to be ***cocompact*** if for any point $x \in X$, there exists $r > 0$ such that $X = \bigcup_{g \in G} \overline{B_r(gx)}$. Equivalently, for any point $x \in X$, there exists $r > 0$ such that for any $y \in X$ there exists $g \in G$ so that $y \in \overline{B_r(gx)}$.

Definition 4.2.9. An action of a group G on a metric space X is called ***geometric*** if it is a properly discontinuous and cocompact action by isometries.

Example 4.2.10. The action of the group $\mathbb{Z}^n$ on $(\mathbb{R}^n, d_{Euc})$ by translation is a geometric action.

Example 4.2.11. Any finitely generated group G acts geometrically on its Cayley graph $\text{Cay}(G, S)$ by left multiplication.

As can see, there is a lot involved in a geometric action. The reason for this is because we want our action to provide us concrete information. We want the action to be by isometries so that the metric structure of X is preserved; we want the action to be properly discontinuous so that the points of X are actually moved around the space and not fixed by all elements of the group (note that singletons are compact and thus if the action is properly discontinuous then for each $x \in X$, $\{g \in G \mid g\{x\} \cap \{x\} \neq \varnothing\} = \{g \in G \mid gx = x\}$ is finite); and finally we want the action to be cocompact so that for some fixed radius, R, the closed R neighborhoods of the orbit of any point in X is all of X. We also have the requirement that X be proper. This is so that for each $x \in X$ and each $r > 0$, only finitely many $g \in G$

will keep $\overline{B_r(gx)}$ overlapped with $\overline{B_r(x)}$ by proper discontinuity.

Theorem 4.2.12. *(*Švarc-Milnor Lemma*) Let G be a group and let (X,d) be a proper geodesic metric space. If G acts geometrically on X, then G is finitely generated and G is quasi-isometric to X.*

Proof. Fix some $x_0 \in X$. Since the action of G on X is cocompact, choose $R > 0$ so that $X = \bigcup_{g \in G} \overline{B_R(gx_0)}$. Since the action is by isometries then for each $g \in G$ we have $\overline{B_R(gx_0)} = g\overline{B_R(x_0)}$ and thus $X = \bigcup_{g \in G} g\overline{B_R(gx_0)}$. Let $S = \left\{g \in G \mid g\overline{B_R(x_0)} \cap \overline{B_R(x_0)} \neq \varnothing\right\}$. Then $S = S^{-1}$ and since X is proper and the action is properly discontinuous, S is finite.

Claim: S generates G. First let $c = \inf\left\{d\left(\overline{B_R(x_0)}, g\overline{B_R(x_0)}\right) \mid g \notin S, g \neq 1\right\}$. We show that $c > 0$. For any $g \in G$ where $g \notin S$ and $g \neq 1$, we have that $g\overline{B_R(x_0)} \cap \overline{B_R(x_0)} = \varnothing$. Suppose $d\left(\overline{B_R(x_0)}, g\overline{B_R(x_0)}\right) = D > 0$. Now consider $\overline{B_D(x_0)}$. Since the action is properly discontinuous, only finitely many translates of $\overline{B_R(x_0)}$ intersect $\overline{B_D(x_0)}$. Therefore, $\left\{d\left(\overline{B_R(x_0)}, g\overline{B_R(x_0)}\right) \leq D \mid g \notin S, g \neq 1\right\}$ is finite. Thus, c is a minimum of finitely many positive values and so $c > 0$.

Let $g \in G \setminus S$ where $g \neq 1$. Then $d(x_0, gx_0) \geq 2R + c > R + c$. If we keep adding c to the right side of this inequality, eventually we will have $d(x_0, gx_0) < R + mc$ for some $m \geq 2$. So, let $k \geq 2$ be the smallest positive integer such that $R + (k-1)c \leq d(x_0, gx_0) < R + kc$. Now choose points $x_1, \ldots, x_{k+1} = gx_0$ on a geodesic segment between x_0 and gx_0 with $d(x_0, x_1) \leq R$ and $d(x_i, x_{i+1}) < c$ for $1 \leq i \leq k$. For these points, there exists elements $1 = g_0, g_1, \ldots, g_k = g$ in G such that $x_{i+1} \in g_i\overline{B_R(x_0)}$

for $0 \le i \le k$.

Now we define $s_i = g_{i-1}^{-1} g_i$ for $1 \le i \le k$ and we have

$$\begin{aligned} d\left(\overline{B_R(x_0)}, s_i \overline{B_R(x_0)}\right) &= d\left(\overline{B_R(x_0)}, g_{i-1}^{-1} g_i \overline{B_R(x_0)}\right) \\ &= d\left(g_{i-1}\overline{B_R(x_0)}, g_i \overline{B_R(x_0)}\right) \\ &\le d(x_{i-1}, x_i) \\ &< c. \end{aligned}$$

So, $s_i \in S$. Finally, $s_1 s_2 \cdots s_k = (1g_1)(g_1^{-1}g_2)\cdots(g_{k-1}^{-1}g) = g$. So, S generates G.

To show that (G, d_S) is quasi-isometric to (X, d), consider the map $f : G \to X$ defined by $g \mapsto gx_0$. Since the action of G on X is cocompact, every point $y \in X$ is contained in $\overline{B_R(gx_0)}$ for some $g \in G$. So, the map is coarsely surjective. Since the action is by isometries we know that $d(gx_0, hx_0) = d(x_0, g^{-1}hx_0)$ and since $d_S(g, h) = d_S(1, g^{-1}h)$, it suffices to find constants $K \ge 1$ and $C \ge 0$ so that for all $g \in G$

$$\frac{1}{K} d_S(1, g) - C \le d(x_0, gx_0) \le K d_S(1, g) + C.$$

If $g = 1$, then $d_S(1, g) = 0 = d(x_0, gx_0)$. So, the inequalities are trivially satisfied.

If $g \in S$, $g \neq 1$, then $d(1, g) = 1$ and since $\overline{B_R(x_0)} \cap \overline{B_R(gx_0)} \neq \varnothing$ we have $d(x_0, gx_0) \le 2R$. So, letting $K = R + 1 = C$, we have $1/K < 1$, $C + 1 > 1$ and so the inequalities are satisfied.

Suppose $g \notin S$, $g \neq 1$. From above, we know that if k is the smallest integer such that $R + (k-1)c \leq d(x_0, gx_0) < R + kc$ then $d_S(1, g) \leq k$ since $s_1 \cdots s_k = g$ where $s_i \in S$. So, $R + (d_S(1,g) - 1)c \leq R + (k-1)c \leq d(x_0, gx_0)$, and thus $R + (d_S(1,g) - 1)c \leq d(x_0, gx_0)$. So, $cd_S(1,g) - c \leq d(x_0, gx_0) - R \leq d(x_0, gx_0)$. Let $M = \max\{d(x_0, sx_0) \mid s \in S\}$ and let $g = s_1 \cdots s_j$ where $s_i \in S$ and $j \leq k$. Then

$$\begin{aligned} d(x_0, gx_0) &\leq d(x_0, s_1x_0) + d(s_1x_0, s_1s_2x_0) + \cdots + d(s_1 \cdots s_{j-1}x_0, gx_0) \\ &= d(x_0, s_1x_0) + d(x_0, s_2x_0) + \cdots + d(x_0, s_jx_0) \\ &\leq Md_S(1, g). \end{aligned}$$

So, we have $cd_S(1,g) - c \leq d(x_0, gx_0) - R \leq d(x_0, gx_0) \leq Md_S(1,g)$.

Letting $K = \max\{R+1, M, 1/c\}$ and $C = \max\{R+1, c\}$ gives us f is a (K, C)-quasi-isometry. $\square$

5. Hyperbolic Metric Spaces, Hyperbolic Groups, and the Geodesic Boundary

5.1 Hyperbolic Metric Spaces

Hyperbolic space is famous for its curved geodesics and the fact there exists triangles whose interior angles sum to less than 180 degrees. One qualitative feature of this is that there exists a fixed real constant $\delta > 0$ so that for any 'nice' triangle and any point on any side of the triangle, there exists a point on one of the other two sides such that the distance, with the respective metric, between the two points is less than δ. Amazingly we can even extend this notion to groups by seeing if some Cayley graph of the group has this 'thin' triangle property.

Definition 5.1.1. Let (X, d) be a geodesic metric space. A ***geodesic triangle*** in X is an ordered triple of geodesic segments $(\gamma_1, \gamma_2, \gamma_2)$ where $\gamma_i : [0, L_i] \to X$ such that

$$\gamma_1(L_1) = \gamma_2(0), \quad \gamma_2(L_2) = \gamma_3(0), \quad \gamma_3(L_3) = \gamma_1(0).$$

We sometimes identify $(\gamma_1, \gamma_2, \gamma_3)$ with its image and write the geodesic triangle as $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ where $\gamma_i(0) = x_i$.

Example 5.1.2. The usual triangles in $\mathbb{R}^n$ are examples of geodesic triangles.

Example 5.1.3. Consider the geometric realization of a tree. Geodesic triangles here look very differently than in $\mathbb{R}^n$. Instead, geodesic triangles here look like a tripod and they have the property that every point on the triangle is on at least two sides of the triangle.

Example 5.1.4. In $\mathbb{H}^2$, since geodesics are semi-circles and vertical lines, if we choose three points in $\mathbb{H}^2$, where at most two are on a vertical line, we can see that geodesic triangles look like curved Euclidean triangles.

Definition 5.1.5. We say that the geodesic triangle is ***δ-thin*** if there exist a real number $\delta \geq 0$ such that $\text{Im}(\gamma_i) \subset B_\delta(\text{Im}(\gamma_j) \cup \text{Im}(\gamma_k))$ where $B_\delta(A) = \{x \in X \mid d(x,y) \leq \delta,\ y \in A\}$ for $A \subset X$.

Definition 5.1.6. Let (X, d) be a geodesic metric space. We say X is *(Gromov)-**hyperbolic*** if there exsits $\delta \geq 0$ so that all geodesic triangles are δ-thin. We call δ the constant of hyperbolicity and for convenience we say X is δ-hyperbolic.

An equivalent way to define X to be hyperbolic is if there exists a real number $\delta \geq 0$ so that for any geodesic triangle $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ and $p \in [x_i, x_j]$, there exists $q \in [x_i, x_k] \cup [x_j, x_k]$ such that $d(p, q) \leq \delta$.

Example 5.1.7. The defining example of a hyperbolic metric space is the hyperbolic plane $\mathbb{H}^2$. Refer to Theorem A.3.27 in [2] for a proof showing geodesic triangles are slim.

Example 5.1.8. For any tree, since any point on a geodesic triangle lies on at least two sides of the triangle, we can conclude that trees are 0-thin and hence trees are 0-hyperbolic.

Example 5.1.9. $\mathbb{R}$ is hyperbolic since geodesic triangles are 0-thin like in trees. Similarly, $\mathbb{Z}$ is hyperbolic.

Example 5.1.10. $(\mathbb{R}^n, d_{Euc})$ does not have thin triangles for any fixed δ. To see this, fix $\delta > 0$. Choose any point in $x_0 \in \mathbb{R}^n$ and consider the $B_\delta(x_0)$. Now, we can form a geodesic triangle large enough with x_0 a midpoint of one of the sides so that for any point y on the other two sides $d(x_0, y) > \delta$.

Definition 5.1.11. Let G be a finitely generated group. We say G is a ***hyperbolic group*** if there exists a finite generating set S such that $\text{Cay}(G, S)$ is hyperbolic.

Since any two Cayley graphs for G are quasi-isometric, we will see shortly that if G has one hyperbolic Cayley graph, then all of its Cayley graphs are hyperbolic.

Example 5.1.12. Any finite group is hyperbolic since its Cayley graph is bounded. So, just choose δ large enough.

Example 5.1.13. Since the Cayley graph of any free group is a tree, we have that free groups are hyperbolic groups.

Example 5.1.14. $\mathbb{R}$ as a group is hyperbolic. Similarly, so is $\mathbb{Z}$ as a group.

Example 5.1.15. Since $\mathbb{Z}^2$ is quasi-isometric to the plane, it is not hyperbolic.

In order to prove the third main theorem, we need to concept of quasi-geodesic. We saw that a geodesic in a metric space X is an isometric embedding $\mathbb{R} \to X$. Naturally, we can ask: what if we consider quasi-isometric embeddings of $\mathbb{R} \to X$ instead?

Definition 5.1.16. Let (X, d) be a metric space. A ***quasi-geodesic*** is a quasi-isometric embedding $\gamma : \mathbb{R} \to X$. A ***quasi-geodesic segment*** is an quasi-isometric embedding $\gamma : [a, b] \to X$.

Definition 5.1.17. Let (M, d) be a metric space and let X and Y be subsets of M. The ***Hausdorff distance*** between X and Y as

$$d_{Haus}(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y}\{d(x, y)\}, \sup_{y\in Y} \inf_{x\in X}\{d(x, y)\}\}.$$

In general, the Hausdorff distance is not a well-defined metric. However, it does define a metric on the set compact subsets of a space. We will not need this fact. We use the Hausdorff distance in the following lemma; a proof can be found in [1].

Lemma 5.1.18. *Let X be hyperbolic. Let $\varphi : [0, L_1] \to X$ be any geodesic segment, and let $\gamma : [0, L_2] \to X$ be any (K, C)-quasi-geodesic segment with the same endpoints as φ. Then there exists a real number $D \geq 0$ such that the $d_{Haus}(\mathrm{Im}(\varphi), \mathrm{Im}(\gamma)) \leq D$.*

The third main theorem comes as a corollary to the following theorem which tells us if we have a quasi-isometric embedding from a geodesic space into a hyperbolic space, the domain is also hyperbolic. The idea is $f : X \to Y$ is a quasi-isometric embedding between geodesic spaces where Y is hyperbolic, we push forward a geodesic triangle in X which becomes a quasi-geodesic triangle in Y, use the lemma to get a geodesic triangle at finite distance from this quasi-geodesic triangle, then use the hyperbolicity of Y and the fact the f is a quasi-isometric embedding to pull back points and get a hyperbolicity constant in X.

Theorem 5.1.19. *Let X and Y be geodesic metric spaces. Suppose $f : X \to Y$ is a quasi-isometric embedding. If Y is hyperbolic, then X is hyperbolic.*

Proof. Suppose Y is δ-hyperbolic and $f : X \to Y$ is (K, C)-quasi-isometric embedding. Let $\Delta = (\varphi_1, \varphi_2, \varphi_3)$ be a geodesic triangle in X. We want to show that Δ is a thin geodesic triangle. Without loss of generality, it suffices to show that if $p \in \mathrm{Im}(\varphi_1)$, then there exists $q \in \mathrm{Im}(\varphi_2) \cup \mathrm{Im}(\varphi_3)$ such that $d_X(p, q) < \infty$.

Since φ_i is an isometric embedding, then it is a $(1, 0)$-quasi-isometric embedding. Since a composition of quasi-isometric embeddings is again a quasi-isometric embedding, then $f\varphi_i$ is a quasi-geodesic segment in Y. This gives us a quasi-geodesic triangle in Y. Let $\varphi_i(0) = x_i$. Then $f\varphi_1$ is a quasi-geodesic segment from $f(x_1)$ to $f(x_2)$.

Consider the geodesic $[f(x_1), f(x_2)]$ in Y. By Lemma 5.1.17, there exists a constant $D_1 \geq 0$ so that the Hausdorff distance $d_{Haus}([f(x_1), f(x_2)], \mathrm{Im}(f\varphi_1)) \leq D_1$. Since $p \in \mathrm{Im}(\varphi_1)$, then $f(p) \in \mathrm{Im}(f\varphi)$ and by the lemma, there exists $p' \in [f(x_1), f(x_2)]$ such that $d_Y(p', f(p)) \leq D_1$. Since Y is δ-hyperbolic, there exists $q' \in [f(x_2), f(x_3)] \cup [f(x_3), f(x_1)]$ such that $d_Y(p', q') \leq \delta$. Without loss of generality suppose $q' \in [f(x_2), f(x_3)]$.

Consider the quasi-geodesic segment $f\varphi_2$ between $f(x_2)$ and $f(x_3)$. Again, by the lemma, there exists $D_2 \geq 0$ such that $d_{Haus}([f(x_2, f(x_3)], \mathrm{Im}(f\varphi_2)) \leq D_2$. So, there exists $f(q) \in \mathrm{Im}(f\varphi_2)$ such that $d_Y(q', f(q)) \leq D_2$.

Putting this all together gives us $d_Y(f(p), f(q)) \leq D_1 + \delta + D_2$. Since f is a (K, C)-quasi-isometric embedding we have

$$d_X(p, q) \leq K d_Y(f(p), f(q)) + KC \leq K(D_1 + \delta + D_2) + KC < \infty.$$

So, Δ is thin. $\square$

Corollary 5.1.20. *(Quasi-Isometry Invariance of Hyperbolicity) Let X and Y be geodesic metric spaces. Suppose $f : X \to Y$ is a quasi-isometry. Then X is hyperbolic if and only if Y is hyperbolic.*

Proof. If $f : X \to Y$ is a quasi-isometry, and Y is hyperbolic, then X is hyperbolic.

Since f is a quasi-isometry, it has a quasi-inverse quasi-isometry $g : Y \to X$. So, if X is hyperbolic, then Y is hyperbolic. $\square$

Putting together this corollary and the Švarc-Milnor Lemma we obtain the result:

Corollary 5.1.21. *Let G be a group and let X be proper and hyperbolic. If G acts geometrically on X, then G is a hyperbolic group.*

Applying the theorem to groups we have:

Corollary 5.1.22.

- *Let G and G' be finitely generated groups with generating sets S and S', respectively, and suppose G' is hyperbolic. If $f : \mathrm{Cay}(G,S) \to \mathrm{Cay}(G',S')$ is a quasi-isometric embedding, then G is hyperbolic.*

- *If $f : \mathrm{Cay}(G,S) \to \mathrm{Cay}(G',S')$ is a quasi-isometry, then G is hyperbolic if and only if G' is hyperbolic.*

5.2 Gromov Boundary

Fix a base-point x_0 in some geodesic space X and consider the geodesic rays $\gamma : [0,\infty)$ starting at x_0. In general, there may be many geodesics that look similar as we go out to infinity. If we quotient out by setting two geodesic rays equivalent if they have finite Hausdorff distance then what we have is a way of visualizing boundary points in our space. We can even topologize this boundary and one can then ask if quasi-isometric spaces have homeomorphic boundary. In general this is not the case for geodesic spaces. However, it is true for hyperbolic spaces.

Definition 5.2.1. Let (X,d) be a proper hyperbolic metric space. The *(Gromov)-**boundary*** of X is defined as

$$\partial X = \{\gamma : [0,\infty) \to X \mid \gamma \text{ is a geodesic ray}\}/\sim$$

where $\gamma_1 \sim \gamma_2$ if and only if $d_{Haus}(\operatorname{Im}(\gamma_1), \operatorname{Im}(\gamma_2)) < \infty$.

We put a topology on ∂X through convergence of sequences: Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in ∂X and $x \in \partial X$. We define convergence in ∂X by saying that $(x_n)_{n=1}^{\infty}$ *converges* to x if there exists a sequence of geodesic rays $(\gamma_n)_{n=1}^{\infty}$ representing $(x_n)_{n=1}^{\infty}$, i.e., $[\gamma_k] = x_k$, and if there exists a geodesic ray γ representing x, i.e., $[\gamma] = x$ so that every subsequence of $(\gamma_n)_{n=1}^{\infty}$ contains a further subsequence which converges uniformly on compact subsets of $[0, \infty)$ (in the sense of functions) to γ.

We will see that this boundary is preserved under quasi-isometry. We can extend the definition of the boundary to hyperbolic groups and by quasi-isometry invariance of the boundary, this is independent of the choice of generating set.

Definition 5.2.2. Let G be a hyperbolic group with finite generating set S. We define the *(Gromov)-**boundary*** of G to be $\partial G = \partial \operatorname{Cay}(G, S)$.

Example 5.2.3. If X is a hyperbolic space of finite diameter, then there cannot exist a geodesic ray in X and thus $\partial X = \varnothing$.

Example 5.2.4. In $\mathbb{R}^n$, if we consider the geodesic rays starting at $\mathbf{0}$, then there is a unique geodesic ray for each direction in $\mathbb{R}^n$. Thus, $\partial \mathbb{R}^n$ is just the space of directions, i.e., $\partial \mathbb{R}^n \cong S^{n-1}$.

Example 5.2.5. A similar argument as for $\mathbb{R}^n$ works for $\mathbb{H}^n$. For all $n \in \mathbb{N}$ such that $n \geq 2$, $\partial \mathbb{H}^n \cong S^{n-1}$.

Theorem 5.2.6. *(Quasi-Isometry Invariance of the Gromov Boundary) Let X and Y be hyperbolic metric spaces. If $f : X \to Y$ is a quasi-isometry, then the induced map $\partial f : \partial X \to \partial Y$ is a homeomorphism.*

Corollary 5.2.7. *Let $n, m \in \mathbb{N}$ where $n, m \geq 2$. Then $\mathbb{H}^n$ is quasi-isometric to $\mathbb{H}^m$ if and only if $n = m$.*

Proof. If $\mathbb{H}^n$ is quasi-isometric to $\mathbb{H}^m$, then by quasi-isometric invariance of the boundary $S^{n-1} = \partial\mathbb{H}^n \cong \partial\mathbb{H}^m = S^{m-1}$. By the homology of spheres, $S^{n-1} \cong S^{m-1}$ if and only if $n = m$. □

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