Beginning Logic
MATH 10130
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Contents

0.1 What is logic? What will we do in this course?

The main goal of this course is to develop some formal tools for analyzing arguments. An argument has a set of *premises*, or *assumptions*, and a *conclusion.* The argument is “valid” if, whenever the premises are all true, the conclusion is also true. In logic, we look at the form of the argument. We will develop tools that allow us to demonstrate that an argument is valid based solely on its form, so long as one agrees with the basic principles that our tools rely on. These premises will be such that any ”reasonable” person will accept them.

Example 0.1.1.

> All men are mortal.
> Socrates is a man.
> Therefore, Socrates is mortal.

This argument may be familiar. The first two statements are the premises. The final statement is the conclusion.

Example 0.1.2.

> All Notre Dame women are smart.
> Susan is a Notre Dame woman.
> Therefore, Susan is smart.

The second argument has the same form as the first. We can replace Socrates with Susan, men with Notre Dame women, and mortal with smart to see this. If we agree that the first argument is valid, then we should believe the second as well. (Note that an argument being valid is different than the conclusion being valid: You might believe an argument to be valid but disagree with one of the premises.)

We will develop formal languages, a formal proof system, and a formal definition of truth to help us analyze arguments.

I. Formal languages

Some arguments, in English, are difficult to analyze because of the length and complexity of the sentences. It helps to break things into simpler parts, and focus on the way that the parts are put together. We will translate arguments from English into some formal languages. This process makes the form of an argument transparent—we can see how the simple parts appear in different statements. For an argument stated in English, there may be some ambiguous statements—capable of being interpreted in more than one way. The process of translating forces us to resolve ambiguities.

We will consider first *propositional* languages. These are useful for arguments in which what is important is the way statements are combined by *connectives* "and", "or", "implies", etc. We will then consider *predicate languages*, with *predicate* symbols for properties of objects (such as a man being mortal or a Notre Dame woman being smart), and *quantifiers* "for all" and "for some". The sample arguments above require predicate logic for a successful translation.

II. Proofs and Truth

In both propositional logic and predicate logic, we will consider two formal ways in which a *conclusion* can be argued from a set of *premises* (or *assumptions*).

1. The first method is to write a *proof*, which is a finite sequence of steps leading from the set of assumptions to the desired conclusion. Each step will be an instance of a *rule of inference*, which is a basic axiom that we assume is valid within our proof system. If someone agrees with the rules of inference and believes all of the assumptions, then they should accept the conclusion of the proof.

2. The second method is via a formal definition of *truth* for a statement in propositional logic or predicate logic. We will then say that a conclusion is a *logical consequence* of a set of assumptions if the conclusion is true whenever the assumptions are true. To show that a conclusion is *not* a logical consequence of a certain set of assumptions, we will look for a "counter-example", i.e. a setting in which the assumptions are all true and the conclusion is false.

Let Γ (capital Gamma) denote a list of assumptions, and let C be a conclusion. For example, in Example 0.1.1, Γ would consist of "All men are mortal" and "Socrates is a man", while C would be "Socrates is mortal".

1. *C follows from* Γ if there is a proof of C from the assumptions in Γ, i.e. we have a finite sequence of steps as described in 1. above which takes us from the assumptions to the conclusion. We will write this as $\Gamma \vdash C$, which we read as "Γ proves C". (Example: "All men are mortal" and "Socrates is a man" proves 'Socrates is mortal.")

2. *C follows from* Γ if C is true whenever the assumptions in Γ are true. Here we mean under our defined notion of truth from 2., not necessarily "real world" truth. We will write this as $\Gamma \models C$, which we read as "Γ logically implies C."

These two methods appear different on the surface, and each has its own strengths and weaknesses, which we will discuss. However, a remarkable fact is that the two methods are the same: if we can prove something, then it is true, and vice versa. Using the above notation, this means that given a set Γ of assumptions and a conclusion C, we have:

Soundness: If $\Gamma \vdash C$ then $\Gamma \models C$.
Completeness: If $\Gamma \models C$ then $\Gamma \vdash C$.

The words "Soundness" and "Completeness" are adjectives describing the formal proof system. Soundness says that the proof system is *sound* in the sense that we cannot prove an implication that isn't true: If we can prove it, then it must be true. The second says that the proof system is *complete* in the sense that if C is true whenever the assumptions in Γ are true, then there is a formal proof of C from Γ: If something is true, then there is a proof of it. Both propositional and predicate logic have both of these properties, but there are some useful proof systems which are sound but not complete.

III. Analysis of arguments

We will apply all of the tools above to analyze arguments. We will translate our argument into an appropriate language and then use proof and/or truth analysis to determine whether the argument is valid. For an argument that translates successfully into propositional logic, we can carry out the truth analysis in a definite way to determine whether the argument is valid. For an argument that requires predicate logic, we will search simultaneously for a proof and for a counterexample. If the argument is valid, then there is a proof, and there is no counterexample. If the argument is not valid, then there is a counterexample, and there is no proof. It is a deep result that there is no definite procedure for deciding whether an argument is valid. Therefore, finding a proof in predicate logic is often quite difficult and requires a certain level of creativity. In most cases, a proof can only be found after a significant number of failed attempts, which ultimately inform the correct proof (if one exists).

IV. Further topics in logic

1. **Computability**. What problems can a machine solve? To show that there is a mechanical procedure for solving some kind of problem, we simply need to describe the procedure. To show that there is *no* mechanical procedure for solving some kind of problem, we need to know exactly what qualifies as a mechanical procedure. There are many seemingly different definitions that turn out to all be equivalent. The *Church-Turing Thesis* says that the notion given by these definitions is "correct"—that they capture the idea of a mechanical procedure.

2. **Computer-aided proofs**. There are some important mathematical results that have been proved with the aid of a computer. In particular, the "four-color theorem", which says given a map of "countries" or regions, there is a way to color all of them so that no neighboring regions share the same color, was proven using a computer. Do "proofs" like these really count as proofs in the sense above if a human can't check them?

3. **Logical paradoxes**. One example of a logical paradox is the statement "I am lying". Is this statement true or false?

4. **Incompleteness**. Gödel showed that for the natural numbers with the usual addition and multiplication, no "nice" set of axioms (true and recognizable) is enough to prove all true statements. The idea is to write a sentence that refers to itself and says, in a coded way, "I am unprovable". This sentence turns out to be true and unprovable. (The statement was inspired by the *Liar Paradox* stated above.) The requirement that the set of axioms be "nice" is why this is not contradicting our claim above that predicate logic is complete: if a statement is true then a proof exists, but there isn't a "nice" set of axioms that lets us prove EVERY true statement.

V. Final Remarks

Students who have taken Beginning Logic in the past have said that the course improved their ability to analyze complex material, and their ability to write so as to make a point. Although logical arguments and formal proof systems are most widely applied in branches of mathematics, an understanding of these ideas will help you to discern the validity of arguments found in everyday life. You should view this logical system as the foundation on which you can build an understanding of the world.

Part I

Propositional Logic

Chapter 1

Beginning propositional logic

We shall consider two different kinds of formal languages. The first languages are *propositional*. In propositional logic, we let single letters P, Q, etc., stand for basic statements, and we combine them using symbols for "and", "or", "implies", "if and only if", and "not".

1.1 Symbols

Here are the symbols of propositional logic.

propositional variables: P, Q, R, etc.

logical connectives (and their intended meanings):

$\wedge$ (and)
$\vee$ (or)
$\neg$ (not)
$\rightarrow$ (implies, or if $\cdots$ then $\cdots$)
$\leftrightarrow$ (if and only if)

parentheses: (,).

Remark 1.1.1. The symbol $\leftrightarrow$ is included only temporarily among the formal symbols. Later, we will think of it as an abbreviation.

1.2 Well-formed formulas

All of the statements we will use are called *well-formed formulas*. (Or *wff* for short.) Well-formed formulas are exactly the statements described by the following rules:

Definition 1.2.1 (Well-formed formulas)**.**

(1) A propositional variable P, Q, R, etc., is a *wff*, just by itself.

(2) If F is a *wff*, then so is $\neg F$.

(3) If F and G are *wff* s, so are $(F \wedge G)$, $(F \vee G)$, $(F \to G)$, and $(F \leftrightarrow G)$.

(4) A string of symbols is a *wff* if and only if it can be obtained by finitely many applications of conditions 1, 2, and 3.

The definition tells us how more complicated formulas are built up from simpler ones. Statements which are not *wff* s, such as "$(Q \to)(($" are completely meaningless in our setting, and as such we will ignore them. To show that something is a well-formed formula, we give a finite sequence of steps, where each step is obtained by an application of Rule 1, or the result of applying Rule 2 or Rule 3 to earlier steps. Such a sequence is called a *formation sequence*.

Example 1.2.2. The following is a *wff*: $(((P \wedge \neg Q) \vee R) \to (Q \wedge R))$.

We give a formation sequence.

1. P is a *wff* by condition (1).
2. Q is a *wff* by condition (1).
3. R is a *wff* by condition (1).
4. $\neg Q$ is a *wff* by condition (2) applied to step 2.
5. $(P \wedge \neg Q)$ is a *wff* by condition (3) applied to steps 1 and 4.
6. $((P \wedge \neg Q) \vee R)$ is a *wff* by condition (1) applied to steps 5 and 3.
7. $(Q \wedge R)$ is a *wff* by condition (3) applied to steps 3 and 2.
8. $(((P \wedge \neg Q) \vee R) \to (R \wedge Q))$ is a *wff* by condition (3) applied to steps 6 and 7.

Note that the same formula could have different looking formation sequences, depending on the order of the steps. In general a *wff* might have many different formation sequences: we really only care about finding one: no formation sequence is the "correct" one.

Example 1.2.3. The following is not a *wff*: $P\neg Q \rightarrow ($

Notice that this statement ends in a left parenthesis. We can see, though, that no *wff* ends in a left parenthesis. If it did, then there would have to be some rule that gave us the first appearance of the final parenthesis:

- Anything obtained from Rule 1 contains no parentheses, so it certainly doesn't end in a left one.
- If F is a *wff* not ending in a left parenthesis, then $\neg F$ does not either.
- Everything obtained from Rule 3 ends in a right parenthesis.

Since there is no rule that creates a *wff* ending in a left parenthesis, $P\neg Q \rightarrow ($ cannot be a *wff* because there cannot be a formation sequence for it.

1.3 Basic translation

We are ready to do some translation. In the process, we will firm up the meaning of our connectives, removing some ambiguities that are present in English.

Translate the following into English, letting P stand for "Peter walks the dog", letting D stand for "the dog barks", and letting C stand for "the cat makes trouble".

Propositional Formulas

1. $(P \vee D)$
2. $(C \rightarrow D)$
3. $(C \leftrightarrow D)$
4. $\neg D$
5. $\neg(C \wedge D)$
6. $(\neg C \vee D)$

Translations

1. We may write "Peter walks the dog or the dog barks."
2. We may write, "If the cat makes trouble, then the dog barks", or "When the cat makes trouble, the dog barks". Note that our "if...then" does not tell us that the second clause actually holds. It means that if the first clause is true, then the second is also true—if the first clause is false, then the second may or may not be true.

3. We may write "The cat makes trouble if and only if the dog barks", or "The cat makes trouble exactly when the dog barks". Note that "if and only if" means that both implications hold. Either both things are true or both are false.

4. We may write "The dog does not bark", or "The dog is not barking". There is a subtle difference in English, which cannot be expressed in propositional logic.

5. We may write "It is not the case that the cat makes trouble and the dog barks", which has the same logical meaning as "Either the cat does not make trouble or the dog does not bark". So the meaning of $\neg(C \wedge D)$ appears to be the same as the meaning of $(\neg C \vee \neg D)$. We will eventually rigorously show this is always the case. On the other hand, the meaning of $\neg(C \wedge D)$ is quite different from that of $(\neg C \wedge \neg D)$. In the first statement, we mean that at least one of C or D are false, while in the second, we mean that both are false.

6. We may write "Either the cat does not make trouble or the dog barks." As an exercise, you should compare the meaning of this statement to that of statement (2).

Remark 1.3.1. IMPORTANT! In propositional logic, the symbol $\vee$ is used for *inclusive* or. (That is, it includes the case when both are true.) In other words, when we say "A or B", we allow the possibility that both A and B are true.

If we want to express *exclusive or* (which excludes the case when bother are true), we could do so with a *wff* of the following form

$$((A \vee B) \wedge \neg(A \wedge B)).$$

Now let us consider a sequence of assertions, which together forms something we might identify as an *argument*.

> If the river floods, then our entire wheat crop will be destroyed. If the wheat crop is destroyed, then our community will be bankrupt. The river will flood if there is an early thaw. In any case, there will be heavy rains later in the summer. Therefore, if there is an early thaw, then our entire community will be bankrupt or there will be heavy rains later in the summer.

For now, we will not worry about the validity of this argument, or the possible ambiguities in the choice of English words. Our task is to identify the assumptions and the conclusion, and then translate these statements to propositional formulas.

Assumptions

1. If the river floods, then our entire wheat crop will be destroyed.
2. If the wheat crop is destroyed, then our community will be bankrupt.
3. The river will flood if there is an early thaw.
4. In any case, there will be heavy rains later in the summer.

Conclusion: If there is an early thaw, then our entire community will be bankrupt or there will be heavy rains later in the summer.

We choose some basic propositional variables—symbols to stand for the important smaller pieces that occur in the statements.

R: The river floods.
D: The wheat crop will be destroyed.
E: There is an early thaw.
B: The community will be bankrupt.
H: There are heavy rains later in the summer.
The assumptions are:

1. $(F \to D)$
2. $(D \to B)$
3. $(E \to F)$
4. H

The conclusion is:

$$(E \to (B \vee H)).$$

If we moved the comma from after "thaw" to after "bankrupt", we would probably change the translation to

$$((E \to B) \vee H).$$

You have seen that propositional logic lacks the nuances of English. When you translate, try to check that the statement you are translating would be true in the same cases as your translation. When you translate from English into propositional logic, you may find it helpful to first re-write the statement, still in English, but closer to the idiom of propositional logic.

Here is an example of such a nuance. The word "but" means roughly the same as "and". For example, the following are approximately the same:

(1) It is rainy but warm.

(2) It is rainy and warm.

1.4 Implications

The translation of the implication sign $\to$ can be tricky. When we write $(P \to Q)$, we mean that *if P is true, then so is Q*. Keeping this in mind, the only possible way for the statment $(P \to Q)$ to be false is if P is true and Q is false. In symbols, $\neg(P \to Q)$ is logically the same as $(P \wedge \neg Q)$.

Remark 1.4.1. In English, there is some ambiguity associated with implications. When we say "if P, then Q", or "P implies Q", we might simply mean that if P is true, then Q is true, or we might mean that P is the *cause* of Q. We do not want to deal with causation, so our meaning for the formal $\to$ symbol is always the former.

"If" and "only if"

The placement of the word "if" in an implication can change. In particular, the following statements all mean the same thing:

1. If P is true then Q is true. (More briefly: If P then Q.)
2. Q is true if P is true. (More briefly: Q if P.)
3. P is true only if Q is true. (More briefly: P only if Q.)

You should spend as much time as you need to convince yourself these are all the same. Here is an example:

1. If there is a tornado warning then classes are canceled.
2. Classes are canceled if there is a tornado warning.
3. There is a tornado warning only if classes are canceled.

These three statements are all saying the same thing. *It is helpful to think of the phrase "only if" has having the same meaning as "implies".*

"If and only if"

We have not yet discussed the symbol $\leftrightarrow$. Recall that our original translation of this symbol was "if and only if". This symbol is mean to represent that two things are equivalent, i.e. that one is true exactly when the other is true. In other words, $(P \leftrightarrow Q)$ says that if P is true then Q is true *and* if Q is true then P is true. In symbols:

$$(P \leftrightarrow Q) \text{ is logically equivalent to } ((P \to Q) \wedge (Q \to P)).$$

Recall that $(Q \to P)$ can be translated as "P if Q", and $(P \to Q)$ can be translated as "P only if Q. Altogether, this justifies our translating $(P \leftrightarrow Q)$ as "P if and only if Q".

Example 1.4.2. Let C stand for "there are clouds in the sky". Let R stand for "It is raining outside".

1. $(R \to C)$ translates to "there are clouds in the sky if it is raining outside".

2. $(C \to R)$ translates to "there are clouds in the sky only if it is raining outside".

3. $(C \leftrightarrow R)$ translates to "there are clouds in the sky if and only if it is raining outside".

1.5 Nuances with negations

To translate $\neg(F \wedge G)$, the safest thing is to say "it is not the case that both F and G". Similarly, to translate $\neg(F \vee G)$, the safest thing is to say "it is not the case that either F or G". It is awfully tempting to believe that negations can just be brought "inside" the other connectives. However, we must be careful, or we will lose our intended meaning.

1. $\neg(F \wedge G)$ is not the same as $(\neg F \wedge \neg G)$. The latter says that *both* F and G are false. The former says that $(F \wedge G)$ is false. Now, $(F \wedge G)$ is true exactly when F and G are both true, so it is false whenever at least one of F or G is false. We conclude that $\neg(F \wedge G)$ has the same meaning as $(\neg F \vee \neg G)$.

2. $\neg(F \vee G)$ is not the same as $(\neg F \vee \neg G)$. The latter says that *at least one* of F and G is false. The former says that $(F \vee G)$ is false. Now, $(F \vee G)$ is true whenver at least one of F and G is true, so it is false exactly when both are false. We conclude that $\neg(F \vee G)$ has the same meaning as $(\neg F \wedge \neg G)$.

3. $\neg(F \to G)$ is not the same as $(\neg F \to \neg G)$. The first is true only when $(F \to G)$ is false. Recall that $(F \to G)$ means that if F is true, then so is G. The implication is false only when F is true and G is false. We conclude that $\neg(F \to G)$ has the same meaning as $(F \wedge \neg G)$. The statement $(\neg F \to \neg G)$ is true if whenever $\neg F$ is true, then $\neg G$ is true. In other words, $(\neg F \to \neg G)$ is true if $\neg G$ is true or $\neg F$ is false. To test your understanding, right down a *wff* that captures when $(\neg F \to \neg G)$ is true using a different connective as we did in the previous examples.

Chapter 2

Proofs

2.1 Introduction to proofs

We shall now begin discussing formal proofs. While finding a proof of a conclusion from some assumptions can often involve some creative thinking, you will see that proofs written in a formal system can be checked by a human with no special ingenuity—they can even be checked by a machine.

Definition 2.1.1. A *proof* of a *conclusion* C, from a set Γ of *assumptions*, is a finite sequence of propositional formulas ending with C, such that each *wff* in the list is obtained by applying *rule of inference* to the assumptions or earlier steps in the proof. We write $\Gamma \vdash C$ if there is a proof of C from assumptions in Γ.

This definition is fairly technical, and there are some things we still need to elaborate on, most importantly the *rules of inference*. When you present your proofs, you will be required to organize the steps in a table, clearly stating in each step the assumptions required, the rule of inference used, and which steps you are applying the rule to. We will give many examples to illustrate the format.

Presentation of a proof:

Suppose you want to prove an assertion of the form $\Gamma \vdash C$, where Γ is a set of assumptions and C is a conclusion. Your proof will be presented as a table with three columns and numbered rows representing each logical step necessary to complete the proof. The three columns will be labeled "Conclusion", "Rule", and "Assumptions", respectively, and each row will satisfy the following properties:

- The first entry (under "Conclusion") will be a propositional *wff* which we conclude in the present step of the proof. If we are translating a proof from English, then the propositional formulas that we get from translating each step of the argument go in this column.

- The second entry (under "Rule") will be the abbreviation for the rule of inference used in the present step, possibly with additional numbers signifying the previous steps in the proof this rule is being applied to. Specific directions for this will depend on the rule of inference, and will be given after the definition of each rule.
- The third entry (under "Assumptions") will list the assumptions from Γ that we used in any step required to reach the current step. More precisely, we will list the row number corresponding to the line number where the given assumption from Γ first entered the proof. Again, specific directions for this will depend on the rule of inference, and will be given after the definition of each rule.

Finally, in order to be a complete and acceptable proof, the final row in the table must contain the conclusion C as the "Conclusion" entry and the "Assumptions" entry must contain only statements in Γ.

Here is an example of what a proof will *look like*. At the moment, we cannot read the proof because we have not learned any rules of inference, nor the specific directions for how to complete rows of a proof.

Example proof: $\{P, \neg R\} \vdash \neg(P \to R)$

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $\neg R$	**A**	2
3. $(P \to R)$	**A***	3^*
4. R	**MP** $3, 1$	$1, 3^*$
5. $(R \wedge \neg R)$	$\wedge$**I** $2, 4$	$1, 2, 3^*$
6. $((P \to R) \to (R \wedge \neg R))$	**CP** $3, 5$	$1, 2$
7. $\neg(P \to R)$	**RAA** 6	$1, 2$

2.2 The first rules of inference

Before doing some proofs ourselves, we must learn a few rules of inference. In and of itself, a rule of inference is just a specific instance of an argument. In other words, each rule of inference will have the form: "if certain requirements are met, then *we can conclude* a statement of a given form." We will denote this by $\Gamma \rhd C$ (read: "from Γ, one may conclude C"), where Γ is the collection of requirements we need to meet to apply the rule, and C is the result we get out. In each rule, the conclusion C will be a propositional *wff*. In most rules, the same will be true about the assumptions in Γ. However, in one particular rule (specifically, Rule 5 below), the assumption in Γ will itself be a proof! How this works will be clearer once we see examples.

The best way to think about a rule of inference $\Gamma \rhd C$ is that it is an argument which we take to be valid without any further justification, something that should be "obvious." In other words, the rules of inference are *axioms of our proof system*. The arguments given by the rules of inference will be treated as the fundamental ingredients with which we can build more complicated proofs. For this reason, our rules of inference should be extremely basic and reasonably sound, in the sense that it is uncontroversial to accept the rule without any justification. A test for soundness of $\Gamma \triangleright C$ can be achieved by asking, "If the assumptions in Γ are true, then do I believe that C is true?" If the answer is a resounding "no", then $\Gamma \triangleright C$ is not a reasonable candidate for a rule of inference.

2.2.1 Rule of Inference: Assumption

Rule 1: **Assumption**, abbreviated **A**.
This rule says that from an assumption F we can conclude F, i.e. $$F \triangleright F.$$

We can test this rule for soundness: If F is true then do we believe F is true? The answer is surely yes.

Directions for using **A** *in proofs*:
Suppose that, in step number n of our proof, we want to use **A** to invoke the assumption F from Γ. Then we add the following row:

Conclusion	Rule	Assumptions
n. F	**A**	n

We're now ready to do our first proof:

Example 2.2.1. $\{F\} \vdash F$

Conclusion	Rule	Assumptions
1. F	**A**	1

Obviously, this is not a very interesting proof, but it is a good exercise to check that it is indeed a proof by checking all of the properties we listed above. As you will see, rule **A** is extremely important, in the sense that it is used in most proofs. However, it can ONLY be applied to assumptions: you cannot apply rule **A** to obtain anything outside of our list of assumptions Γ. It is not very powerful by itself, so we need at least one more rule before we can execute any nontrivial proofs.

2.2.2 Rule of Inference: Modus Ponens

Rule 2: **Modus Ponens**[1], abbreviated **MP**.
This rule says that from $(F \to G)$ and F, we can conclude G, i.e. $$(F \to G), F \rhd G.$$

Note that this rule can be applied to anything of this form: F and G could themselves be complicated wffs. For example, **MP** gives us

$$((F \vee G) \to (P \to Q)), (F \vee G) \rhd (P \to Q)$$

The same principle will apply to later rules of inference as well.

Once again, we test the rule for soundness: Suppose F is true and "F implies G" is true, do we believe that G is true? Recall that one way of reading $F \to G$ is "If F is true, then G is true." From this interpretation, we should believe that if both F and $F \to G$ are true, G should be true.

Directions for using **MP** *in proofs*:

Suppose that, at step number n of our proof, we want to conclude G by applying rule **MP** to $(F \to G)$ and F, which are conclusions previously obtained at steps i and j, respectively, of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
n. G	**MP** i, j	(copy assumptions from rows i and j)

We can now give a much more interesting example of a proof.

Example 2.2.2. We give a formal proof of S from the assumptions P, $(P \to Q)$, $(Q \to R)$, and $(R \to S)$, i.e. we show

$$\{P, (P \to Q), (Q \to R), (R \to S)\} \vdash S\ .$$

Conclusion	Rule	Assumptions
1. $(P \to Q)$	**A**	1
2. P	**A**	2
3. Q	**MP** $1, 2$	$1, 2$
4. $(Q \to R)$	**A**	4
5. R	**MP** $4, 3$	$1, 2, 4$
6. $(R \to S)$	**A**	6
7. S	**MP** $6, 5$	$1, 2, 4, 6$

Look carefully at the proof to understand how each step is generated, and see that it satisfies the specifications given above.

[1] Latin for "mode that affirms".

Remark 2.2.3.

1. The order of the numbers in the "Assumptions" column does not matter, but you should try to be consistent (for example, listing numbers in increasing order, as we do here). However the order of the numbers in the "Rule" column is important. For example, if you apply rule **MP** to $(F \to G)$ and G, which were conclusions previously obtained in steps i and j, respectively, then you should enter "**MP** i, j" and not "**MP** j, i".

2. The inclusion of third column may feel pedantic. However, keeping track of assumptions will be useful when we learn more complicated rules of inference (especially rule **CP** below). This can also help us trim down a large list of assumptions by seeing which ones we actually used in the proof. For now, we can use the third column to see clearly which of our assumptions are actually used in the proof.

Another observation is that proofs are not unique. In other words, the order of the steps in your proof might differ from mine, while both proofs are still correct. For example, in the previous proof, the order the first two steps could be switched.

2.2.3 Rule of Inference: Modus Tollens

Rule 3: **Modus Tollens**[2], abbreviated **MT**.
This rule says that from $(F \to G)$ and $\neg G$, we can conclude, $\neg F$, i.e. $$(F \to G), \neg G \rhd \neg F \ .$$

Once again, you should test this rule for soundness. (If G is true whenever F is, and G is not true, is F true?)

Directions for using **MT** *in proofs*:

Suppose that, at step number n of our proof, we want to conclude $\neg F$ by applying rule **MT** to $(F \to G)$ and $\neg G$, which are conclusions previously obtained at steps i and j of the proof, respectively. Then we add the row:

Conclusion	Rule	Assumptions
n. $\neg F$	**MT** i, j	(copy assumptions from rows i and j)

Let us see how **MT** is used in an example.

[2]Latin for "mode that denies".

Example 2.2.4. We show $\{\neg R, (P \to Q), (Q \to R)\} \vdash \neg P$.

Conclusion	Rule	Assumptions
1. $(Q \to R)$	**A**	1
2. $\neg R$	**A**	2
3. $\neg Q$	**MT** $1, 2$	$1, 2$
4. $(P \to Q)$	**A**	4
5. $\neg P$	**MT** $4, 3$	$1, 2, 4$

Here is a question you can ask yourself to test your understanding of the proof format: *In the previous proof, why does the number* 3 *never appear in any entry of the "Assumptions" column?*

2.2.4 Rule of Inference: Double Negation

Rule 4: **Double Negation**, abbreviated **DN**.

This rule says that F and $\neg\neg F$ are logically equivalent. Explicitly:

(a) From $\neg\neg F$, we may conclude F, i.e. $\neg\neg F \rhd F$.

(b) From F, we may conclude $\neg\neg F$, i.e. $F \rhd \neg\neg F$.

The test for soundness of this rule is worth emphasizing. In particular, suppose you believe $\neg\neg F$ is true. Should you believe F is true? In this case, your gut reaction may or may not be "yes". In our proof system, we are working with a notion of "truth" in which, given any statement F, exactly one of F or $\neg F$ is true. This is sometimes called the *law of excluded middle*; once we have defined all of our rules of inference, we will give a proof of this law (see Example 2.6.6). With this caveat, the soundness of rule **DN** is uncontroversial. However, there are other proof systems in which the notion of truth does not come with this assumption.[3] If you are skeptical of the law of the excluded middle, then you may think of statements as being "not true" instead of being "false."

[3]The interested reader should research *intuitionistic logic*. Like our proof system, intuitionistic logic comes with the assumption that F and $\neg F$ are never true simultaneously. However, unlike our proof system, in intuitionistic logic, not all statements have a determined truth value ("true" or "false"), and so it is possible that neither F nor $\neg F$ is assigned the value "true". While this might seem strange, consider the following situation. Let F be the statement "Socrates' favorite number was 6". From one perspective, we might say that F is either true or false. From another perspective, we might say that no one currently knows whether F is true or false, and so we cannot assign it a known truth value. You should not view these as conflicting perspectives, but simply two different logical systems in which a person can make arguments and proofs.

Directions for using **DN** *in proofs*:

Suppose that, at step number n of our proof, we want to either:

(a) deduce F by applying rule **DN** to $\neg\neg F$, which was previously obtained in step i, or

(b) deduce $\neg\neg F$ by applying rule **DN** to F, which was previously obtained in step i.

Then we add the row:

	Conclusion	Rule	Assumptions
if case (a)	n. F	**DN** i	(assumptions from row i)
or			
if case (b)	n. $\neg\neg F$	**DN** i	(assumptions from row i)

Example 2.2.5. We show $\{(P \to \neg Q), Q\} \vdash \neg P$.

Conclusion	Rule	Assumptions
1. $(P \to \neg Q)$	**A**	1
2. Q	**A**	2
3. $\neg\neg Q$	**DN** 2	2
4. $\neg P$	**MT** $1,3$	$1,2$

It is tempting to skip step 3 here, but you need to resist the temptation. To apply **MT**, we must have something of the form $(F \to G)$ and $\neg G$, and if G is itself a negation (e.g. G is $\neg Q$ in the above example), then $\neg G$ will be the result of adding another negation (e.g. $\neg G$ is $\neg\neg Q$ in the above example).

2.2.5 More Examples of Proofs

Example 2.2.6. $\{P, (P \to \neg R), (Q \to R)\} \vdash \neg Q$.

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $(P \to \neg R)$	**A**	2
3. $(Q \to R)$	**A**	3
4. $\neg R$	**MP** $2,1$	$1,2$
5. $\neg Q$	**MT** $3,4$	$1,2,3$

Example 2.2.7. $\{(\neg P \to Q), \neg Q\} \vdash P$.

Conclusion	Rule	Assumptions
1. $(\neg P \to Q)$	**A**	1
2. $\neg Q$	**A**	2
3. $\neg\neg P$	**MT** $1,2$	$1,2$
4. P	**DN** 3	$1,2$

Example 2.2.8. $\{(\neg P \to Q), (Q \to R), \neg R\} \vdash P$.

Conclusion	Rule	Assumptions
1. $(\neg P \to Q)$	**A**	1
2. $(Q \to R)$	**A**	2
3. $\neg R$	**A**	3
4. $\neg Q$	**MT** 2, 3	2, 3
5. $\neg\neg P$	**MT** 1, 4	1, 2, 3
6. P	**DN** 5	1, 2, 3

Example 2.2.9. Consider the following "proof" for $\{(P \to \neg Q), Q\} \vdash \neg P$

Conclusion	Rule	Assumptions
1. $(P \to \neg Q)$	**A**	1
2. Q	**A**	2
3. $\neg P$	**MT** 1, 2	1, 2

This argument is incorrect because we have not accurately applied **MT**. The conclusions in steps 1 and 2 are not of the form $(F \to G)$ and $\neg G$ for some choice of F and G. We can fix the proof as follows.

Conclusion	Rule	Assumptions
1. $(P \to \neg Q)$	**A**	1
2. Q	**A**	2
3. $\neg\neg Q$	**DN** 2	2
4. $\neg P$	**MT** 1, 3	1, 2

2.3 Rule for implications

The next rule is for proving *conditional* statements, or implications. Recall that when we write $(F \to G)$, we mean that if F is true, then G is also true. We are not asserting that F is true. Moreover, if F is false, we make no claims about the truth of G. When we argue that a conditional statement $(F \to G)$ is true, it is natural to *suppose* that F is true and argue that G must then also be true. The next rule of inference formalizes this.

2.3.1 Rule of Inference: Conditional Proof

Rule 5: **Conditional Proof**, abbreviated **CP**.
This rule says that from $F \vdash G$ we can conclude $(F \to G)$, i.e., $$(F \vdash G) \rhd (F \to G)$$

Earlier, we used in our proofs only assumptions that were actually given—assumptions that we would allow on the last step of a proof. The strategy associated with rule **CP** calls for making an extra assumption, temporarily. We get rid of the extra assumption when we actually apply the rule.

Strategy for using **CP**:

To prove $(F \to G)$, using **CP**, we must do the following.

1. Temporarily assume F using rule **A**. When we assume something outside of our regular assumptions in order to use **CP**, we will use **A*** to denote that it is an extra assumption, and use a * next to the number as well, e.g. n*.

2. Prove G. This may take several steps. We may use the assumption F, although if we don't need it, that is fine as well.

3. Apply **CP** to conclude $(F \to G)$. At this point, we discharge the temporary assumption F. In particular, if we should no longer be able to use F or the line n* in any later steps of the proof.

Directions for using **CP** *in proofs*:

Suppose that, at step n, we decide we want to add the temporary assumption F, prove G with this extra assumption, and then use **CP** to conclude $(F \to G)$. Then we must do several things:

1. We add the row

Conclusion	Rule	Assumptions
n. F	$\mathbf{A}^*$	n^*

In other words, we are using rule **A** to temporarily add the new assumption F, which we will discharge later. As mentioned above, the asterisks in the "Rule" and "Assumptions" columns are there to remind us that F is an extra assumption which we must eventually remove using rule **CP** in order to obtain a valid proof.

2. Work toward the proof of G, using any rules of inference we wish along with the extra assumption F where necessary. Add rows to the proof as directed by these rules of inference. Whenever the extra assumption F is used, we continue to include the asterisk in the "Assumptions" column to remind ourselves that this extra assumption must eventually be removed using rule **CP**.

3. The previous proof of G may take any number of steps. Say we obtain G at step m of the proof (where necessarily $m \geq n$). Then, in the next step of the proof (step $m + 1$) we may use rule **CP** to conclude $(F \to G)$, and we remove the extra assumption F. We signify this by adding the row:

Conclusion	Rule	Assumptions
$m + 1$. $(F \to G)$	**CP** n, m	(assumptions from row m, *except for* n^*)

> Once again in the second column, the numbers n, m correspond, respectively, the step where we added the extra assumption F, and the step where we eventually obtained G (possibly using this extra assumption). In the third column, we are listing all of the assumptions used to obtain G *other than* F.

Clearly, rule **CP** is more complicated than the previous rules in both its formal execution and its underlying logical soundness. So we will do several examples.

Example 2.3.1. $\{(P \to Q)\} \vdash (\neg Q \to \neg P)$.

Conclusion	Rule	Assumptions
1. $(P \to Q)$	**A**	1
2. $\neg Q$	**A***	2^*
3. $\neg P$	**MT** $1, 2$	$1, 2^*$
4. $(\neg Q \to \neg P)$	**CP** $2, 3$	1

To summarize, we wanted to prove the implication $(\neg Q \to \neg P)$. So we added the extra assumption $\neg Q$ in step 2, and then used this extra assumption along with our original assumption $(P \to Q)$ to conclude $\neg P$. This took one more step, bringing us to step 3. Therefore, at step 4, we apply **CP** to obtain the desired implication $(\neg Q \to \neg P)$. In the "Assumptions" column, we list all assumptions underlying the deduction of $\neg P$ (i.e. all assumptions in step 3) *except for* the extra assumption $\neg Q$.

Example 2.3.2. $\{P\} \vdash (Q \to P)$

Conclusion	Rule	Assumptions
1. Q	**A***	1^*
2. P	**A**	2
3. $(Q \to P)$	**CP** $1, 2$	2

This example has the feature that the extra assumption Q is not actually used in obtaining P. It does not appear as an assumption in step 2.

The next example will be different from anything we have seen thus far, because we will prove a statement using *no initial assumptions.* Statements like these are special; we think of them as being always true, regardless of the assumed truth of any constituent parts of the formula.

Example 2.3.3. $\vdash ((P \to Q) \to ((Q \to R) \to (P \to R)))$

The proof is a bit more complicated than previous examples, and so we will first outline the rough strategy.

1. Add $(P \to Q)$ as an extra assumption. Then prove $((Q \to R) \to (P \to R))$ and use rule **CP** to obtain $((P \to Q) \to ((Q \to R) \to (P \to R)))$

2. In order to prove $((Q \to R) \to (P \to R))$, we add $(Q \to R)$ as an extra assumption and then prove $(P \to R)$. Then use rule **CP** to obtain $((Q \to R) \to (P \to R))$.

3. In order to prove $(P \to R)$, we add P as an extra assumption and then prove R (using the extra assumptions from the first two steps). Then we use rule **CP** to obtain $(P \to R)$.

Here is the proof.

Conclusion	Rule	Assumptions
1. $(P \to Q)$	$\mathbf{A}^*$	1^*
2. $(Q \to R)$	$\mathbf{A}^*$	2^*
3. P	$\mathbf{A}^*$	3^*
4. Q	**MP** $1, 3$	$1^*, 3^*$
5. R	**MP** $2, 4$	$1^*, 2^*, 3^*$
6. $(P \to R)$	**CP** $3, 5$	$1^*, 2^*$
7. $((Q \to R) \to (P \to R))$	**CP** $2, 6$	1^*
8. $((P \to Q) \to ((Q \to R) \to (P \to R)))$	**CP** $1, 7$	

We discharged all three temporary assumptions, and we have proved the conclusion from nothing, as required.

Example 2.3.4. $\vdash (P \to P)$

Conclusion	Rule	Assumptions
1. P	$\mathbf{A}^*$	1^*
2. $(P \to P)$	**CP** $1, 1$	

The proof has the unusual feature that the two lines we refer to when we apply rule **CP** are the same. The rule doesn't say they have to be different.

You have seen how rule **CP** works formally. It occurs often in informal everyday arguments.

Example 2.3.5. Suppose we want to argue for the following statement:

If the only thing I eat every day is ice cream, I will get sick.

We might argue as follows:

If one eats only ice cream every day, then they will not get enough nutrients. If one does not get enough nutrients, they will get sick. If I only eat ice cream every day, then I will not get enough nutrients. If I do not get enough nutrients, then I will get sick. Therefore if I only eat ice cream every day, I will get sick.

The use of rule **CP** in this argument is correct, however we claim nothing about the accuracy of the assertions about ice cream and nutrients.

Remark 2.3.6. As a reminder, we should *never* make extra assumptions without a plan for getting rid of them. If we are proving conclusion C from a set Γ of assumptions then, in the last step of the proof, where we obtain C, we can only use assumptions occurring in Γ.

2.3.2 More examples of rule CP

Example 2.3.7. $\{(P \to \neg Q)\} \vdash (Q \to \neg P)$

Conclusion	Rule	Assumptions
1. Q	$\mathbf{A}^*$	1^*
2. $\neg\neg Q$	**DN** 1	1^*
3. $(P \to \neg Q)$	**A**	3
4. $\neg P$	**MT** $3, 2$	$1^*, 3$
5. $(Q \to \neg P)$	**CP** $1, 4$	3

Example 2.3.8. $\{(P \to \neg Q), (R \to Q)\} \vdash (P \to \neg R)$

Conclusion	Rule	Assumptions
1. P	$\mathbf{A}^*$	1^*
2. $(P \to \neg Q)$	**A**	2
3. $\neg Q$	**MP** $2, 1$	$1^*, 2$
4. $(R \to Q)$	**A**	4
5. $\neg R$	**MT** $4, 3$	$1^*, 2, 4$
6.$(P \to \neg R)$	**CP** $1, 5$	$2, 4$

2.4 Rules for conjunctions

The next two rules of inference involve $\wedge$. The first rule lets us prove a conjunction, and the second lets us use a conjunction.

2.4.1 Rule of Inference: Conjunction Introduction

Rule 6: **Conjunction Introduction**, abbreviated $\wedge$**I**.
This rule says that from F and G we can conclude $F \wedge G$, i.e. $$F, G \rhd (F \wedge G)$$

Directions for using $\wedge$**I** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $F \wedge G$ by applying rule $\wedge$**I** to F and G, which are conclusions previously obtained at steps i and j, respectively, of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ F \wedge G$	$\wedge$**I** i, j	(copy assumptions from rows i and j)

2.4.2 Rule of Inference: Conjunction Elimination

Rule 7: **Conjunction Elimination**, abbreviated $\wedge$**E**.
This rule says that from $F \wedge G$, we can conclude F and we can conclude G. This gives two statements: (a) $(F \wedge G) \rhd F$ (b) $(F \wedge G) \rhd G$

Directions for using $\wedge$**E** *in proofs*:

Suppose that, at step n of our proof, we want to conclude F by applying rule $\wedge$**E** to $(F \wedge G)$, which is a conclusion previously obtained at step i of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ F$	$\wedge$**E** i	(copy assumptions from row i)

The directions for concluding G from $F \wedge G$ are analogous.

2.4.3 Examples of rules $\wedge$E and $\wedge$I

Example 2.4.1. $\{(P \wedge Q)\} \vdash (Q \wedge P)$

Conclusion	Rule	Assumptions
1. $(P \wedge Q)$	**A**	1
2. P	$\wedge$**E** 1	1
3. Q	$\wedge$**E** 1	1
4. $(Q \wedge P)$	$\wedge$**I** $3, 2$	1

Example 2.4.2. $\{(P \wedge Q), (Q \to R)\} \vdash (P \wedge R)$

Conclusion	Rule	Assumptions
1. $(P \wedge Q)$	**A**	1
2. $(Q \to R)$	**A**	2
3. P	$\wedge$**E** 1	1
4. Q	$\wedge$**E** 1	1
5. R	**MP** 2, 4	1, 2
6. $(P \wedge R)$	$\wedge$**I** 3, 5	1, 2

Example 2.4.3. $\{P, (P \to Q), (P \to R)\} \vdash (Q \wedge R)$

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $(P \to Q)$	**A**	2
3. $(P \to R)$	**A**	3
4. Q	**MP** 2, 1	1, 2
5. R	**MP** 3, 1	1, 3
6. $(Q \wedge R)$	$\wedge$**I** 4, 5	1, 2, 3

Example 2.4.4. $\vdash (P \to (Q \to (P \wedge Q)))$

Conclusion	Rule	Assumptions
1. P	**A***	1^*
2. Q	**A***	2^*
3. $(P \wedge Q)$	$\wedge$**I** 1, 2	$1^*, 2^*$
4. $(Q \to (P \wedge Q))$	**CP** 2, 3	1^*
5. $(P \to (Q \to (P \wedge Q)))$	**CP** 1, 4	

Example 2.4.5. $\{(P \to R), (S \to \neg Q)\} \vdash ((P \wedge Q) \to (R \wedge \neg S))$

Conclusion	Rule	Assumptions
1. $(P \wedge Q)$	**A***	1^*
2. $(P \to R)$	**A**	2
3. P	$\wedge$**E** 1	1^*
4. R	**MP** 2, 3	$1^*, 2$
5. $(S \to \neg Q)$	**A**	5
6. Q	$\wedge$**E** 1	1^*
7. $\neg\neg Q$	**DN** 6	1^*
8.$\neg S$	**MT** 5, 7	$1^*, 5$
9. $(R \wedge \neg S)$	$\wedge$**I** 4, 8	$1^*, 2, 5$
10. $((P \wedge Q) \to (R \wedge \neg S))$	**CP**1, 9	2, 5

Example 2.4.6. $\{(P \to R)\} \vdash ((P \wedge Q) \to R)$

Conclusion	Rule	Assumptions
1. $(P \wedge Q)$	$\mathbf{A}^*$	1^*
2. P	$\wedge\mathbf{E}$ 1	1^*
3. $(P \to R)$	$\mathbf{A}$	3
4. R	**MP** $3, 2$	$1^*, 3$
5. $((P \wedge Q) \to R)$	**CP** $1, 4$	3

Example 2.4.7. Here is an argument:

If I wear green shoes then I also wear white socks. If I wear a purple shirt then I comb my hair. If I wear white socks and I comb my hair then I don't get to work early. I got to work early. Therefore I didn't wear green shoes and a purple shirt.

If you think through the argument, it should feel valid. We are going to translate this argument, and then prove it. First, we define our variables.

G: I wear green shoes.

W: I wear white socks.

P: I wear a purple shirt.

C: I comb my hair.

E: I get to work early.

With these variables, our argument is:

$$\{(G \to W), (P \to C), ((W \wedge C) \to \neg E), E\} \vdash \neg(G \wedge P)$$

It may take some experimenting to arrive at a workable plan for this proof. It is helpful to go back to the original paragraph. You might informally reason as follows: *If I did wear green shoes and a purple shirt then, using the first two implications, I would conclude that I wear white socks and I comb my hair. By the third implication, I don't get to work early. But I did get to work early, so that must not have been what happened.*

The key thing happening in this informal argument is that you first prove the implication $((G \wedge P) \to \neg E)$, and then combine this with E to conclude $\neg(G \wedge P)$. The subtlety is that the implication $((G \wedge P) \to \neg E)$ doesn't appear explicitly anywhere (even though it is a logical consequence).

To summarize: we prove $((G \wedge P) \to \neg E)$, using the assumptions. Having done this, we can finish in two more steps, using rules **DN** and **MT**.

Conclusion	Rule	Assumptions
1. $(G \wedge P)$	$\mathbf{A}^*$	1^*
2. G	$\wedge\mathbf{E}$ 1	1^*
3. P	$\wedge\mathbf{E}$ 1	1^*
4. $(G \to W)$	**A**	4
5. W	**MP** $4, 2$	$1^*, 4$
6. $(P \to C)$	**A**	6
7. C	**MP** $6, 3$	$1^*, 6$
8. $(W \wedge C)$	$\wedge\mathbf{I}$ $5, 7$	$1^*, 4, 6$
9. $((W \wedge C) \to \neg E)$	**A**	9
10. $\neg E$	**MP** $9, 8$	$1^*, 4, 6, 9$
11. $((G \wedge P) \to \neg E)$	**CP** $1, 10$	$4, 6, 9$
12. E	**A**	12
13. $\neg\neg E$	**DN** 12	12
14.$\neg(G \wedge P)$	**MT** $11, 13$	$4, 6, 9, 12$

2.5 Rules for disjunctions

The next two rules of inference involve $\vee$. The first rule lets us prove a disjunction, and the second lets us use a disjunction.

2.5.1 Rule of Inference: Disjunction Introduction

Rule 8: **Disjunction Introduction**, abbreviated $\vee\mathbf{I}$.

This rule says that from F we can conclude $F \vee G$; and from G we can conclude $F \vee G$. This gives two statements:

(a) $F \rhd (F \vee G)$

(b) $G \rhd (F \vee G)$

Directions for using $\vee\mathbf{I}$ *in proofs*:

Suppose that, at step n of our proof, we want to conclude $F \vee G$ by applying rule $\vee\mathbf{I}$ to F, which is a conclusion previously obtained at step i of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
n. $(F \vee G)$	$\vee\mathbf{I}$ i	(copy assumptions from row i)

The directions for concluding $F \vee G$ from G are analogous.

Example 2.5.1. $\{(P \wedge Q)\} \vdash (P \vee Q)$

Conclusion	Rule	Assumptions
1. $(P \wedge Q)$	**A**	1
2. P	$\wedge$**E** 1	1
3. $(P \vee Q)$	$\vee$**I** 2	1

2.5.2 Rule of Inference: Disjunction Elimination

The disjunction elimination rule has a slightly different flavor. Suppose we have a disjunction $(F \vee G)$, in other words we have shown that either F or G is true. In order to use this knowledge to prove something new, say a statement H, we must prove that $(F \to H)$ *and* $(G \to H)$ are both true.

Rule 9: **Disjunction Elimination**, abbreviated $\vee$**E**.

This rule says that from $(F \vee G)$, $(F \to H)$, and $(G \to H)$, we can conclude H, i.e.

$$(F \vee G), (F \to H), (G \to H) \rhd H$$

Strategy for using $\vee$**E**:

To prove a new statement H from $(F \vee G)$, using $\vee$**E**, we must do the following:

1. First prove $(F \vee G)$.
2. Prove $(F \to H)$. This may take several steps, and we will usually need to use **CP** to do this.
3. Prove $(G \to H)$. Again, this may take several steps and often uses **CP**.
4. Apply $\vee$**E** to conclude H.

You should think of this strategy as a "proof by cases". In order to conclude H from $(F \vee G)$ you need to prove:

- *Case* 1: F implies H
- *Case* 2: G implies H.

Directions for using $\vee$**E** *in proofs*:

Suppose that, at step n of our proof, we want to conclude H by applying rule $\vee$**E** to $(F \vee G)$, $(F \to H)$, and $(G \to H)$, which are conclusions previously obtained at steps i, j, and k, respectively, of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ H$	$\vee$**E** i, j, k	(copy assumptions from rows i, j, and k)

Example 2.5.2. $\{(P \vee Q)\} \vdash (Q \vee P)$

Conclusion	Rule	Assumptions
1. $(P \vee Q)$	**A**	1
2. P	**A***	2^*
3. $(Q \vee P)$	$\vee$**I** 2	2^*
4. $(P \rightarrow (Q \vee P))$	**CP** $2, 3$	
5. Q	**A***	5^*
6. $(Q \vee P)$	$\vee$**I** 5	5^*
7. $(Q \rightarrow (Q \vee P))$	**CP** $5, 6$	
8. $(Q \vee P)$	$\vee$**E** $1, 4, 7$	1

Example 2.5.3 (Informal Argument)**.** I want to argue that if you work on a homework problem with a study group, and you either know how to do the problem or you don't, then you will benefit from the experience. To do this, I might argue by cases as follows. In the first case, if you work in a study group and know how to do the problem, then you can explain the problem to the rest of the group and, in doing so, you obtain a deeper understanding of the problem and its explanation. So you benefit. In the second case, if you work in a study group and don't know how to do the problem, then someone else in the group can help you understand it. So you benefit.

Example 2.5.4 (Informal Argument)**.** To argue that a certain statement is true for all natural numbers n, we might give one argument proving if n is even, then the statement holds, and another argument proving if n is odd, the the statement holds as well. In view of the fact that every natural number is even or odd, this is enough to show the statement is true for all natural numbers.

We now return to some formal proofs using $\vee E$.

Example 2.5.5. $\{((P \wedge Q) \vee (P \wedge R))\} \vdash P$

Conclusion	Rule	Assumptions
1. $((P \wedge Q) \vee (P \wedge R))$	**A**	1
2. $(P \wedge Q)$	**A***	2^*
3. P	$\wedge$**E** 2	2^*
4. $((P \wedge Q) \rightarrow P)$	**CP** $2, 3$	
5. $(P \wedge R)$	**A***	5^*
6. P	$\wedge$**E** 5	5^*
7. $((P \wedge R) \rightarrow P)$	**CP** $5, 6$	
8. P	$\vee$**E** $1, 4, 7$	1

Example 2.5.6. $\{(P \wedge (Q \vee R))\} \vdash ((P \wedge Q) \vee (P \wedge R))$

Conclusion	Rule	Assumptions
1. $(P \wedge (Q \vee R))$	**A**	1
2. P	$\wedge$**E** 1	1
3. $(Q \vee R)$	$\wedge$**E** 1	1
4. Q	**A***	4^*
5. $(P \wedge Q)$	$\wedge$**I** 2, 4	$1, 4^*$
6. $((P \wedge Q) \vee (P \wedge R))$	$\vee$**I** 5	$1, 4^*$
7. $(Q \to ((P \wedge Q) \vee (P \wedge R)))$	**CP** 4, 6	1
8. R	**A***	8^*
9. $(P \wedge R)$	$\wedge$**I** 2, 8	$1, 8^*$
10. $(P \wedge Q) \vee (P \wedge R)$	$\vee$**I** 9	$1, 8^*$
11.$(R \to ((P \wedge Q) \vee (P \wedge R)))$	**CP** 8, 10	1
12.$(P \wedge Q) \vee (P \wedge R))$	$\vee$**E** 3, 7, 11	1

In the next example, one of the cases involved in an application of $\vee$**E** is already in our set of assumptions, and so we will not need to use **CP** to prove that case.

Example 2.5.7. $\{(P \vee Q), (P \to Q)\} \vdash Q$

Conclusion	Rule	Assumptions
1. $(P \vee Q)$	**A**	1
2. $(P \to Q)$	**A**	2
3. Q	**A***	3^*
4. $(Q \to Q)$	**CP** 3, 3	
5. Q	$\vee$**E** 1, 2, 4	1, 2

Example 2.5.8. $\{((P \to Q) \wedge (P \to R))\} \vdash (P \to (Q \wedge R))$

Conclusion	Rule	Assumptions
1. P	**A***	1^*
2. $((P \to Q) \wedge (P \to R))$	**A**	2
3. $(P \to Q)$	$\wedge$**E** 2	2
4. $(P \to R)$	$\wedge$**E** 2	2
5. Q	**MP** 3, 1	$1^*, 2$
6. R	**MP** 4, 1	$1^*, 2$
7. $(Q \wedge R)$	$\wedge$**I** 5, 6	$1^*, 2$
8. $(P \to (Q \wedge R))$	**CP** 1, 7	2

Example 2.5.9. $\{((P \to Q) \wedge (Q \to R))\} \vdash ((P \vee Q) \to R)$

Conclusion	Rule	Assumptions
1. $(P \vee Q)$	$\mathbf{A}^*$	1^*
2. $((P \to Q) \wedge (Q \to R))$	**A**	2
3. $(P \to Q)$	$\wedge\mathbf{E}$ 2	2
4. $(Q \to R)$	$\wedge\mathbf{E}$ 2	2
5. P	$\mathbf{A}^*$	5^*
6. Q	**MP** $3, 5$	$2, 5^*$
7. R	**MP** $4, 6$	$2, 5^*$
8. $(P \to R)$	**CP** $5, 7$	2
9. R	$\vee\mathbf{E}$ $1, 8, 4$	$1^*, 2$
10. $((P \vee Q) \to R)$	**CP** $1, 9$	2

2.6 Proof by contradiction

We are now ready to learn the final rule of inference.

2.6.1 Rule of Inference: Reductio ad absurdum

The idea behind this rule is that if we can prove an impossible conclusion from a statement F, then we can conclude $\neg F$. For us, an "impossible" conclusion is one of the form $(G \wedge \neg G)$, where G is any statement. Notice that this is equivalent to the law of the excluded middle. (See Subsection **??**)

Rule 10: **Reductio ad Absurdum**[4], abbreviated **RAA**.
This rule says that from $(F \to (G \wedge \neg G))$, we can conclude $\neg F$, i.e. $$(F \to (G \wedge \neg G)) \rhd \neg F.$$

A proof involving rule **RAA** is often called a "proof by contradiction".

Strategy for using **RAA**:

To prove a statement $\neg F$ using **RAA**, we must do the following:

1. Use $\mathbf{A}^*$ to assume F.
2. Prove a contradiction of the form $(G \wedge \neg G)$. This may take several steps, and will probably use the extra assumption F.
3. Use **CP** to obtain $(F \to (G \wedge \neg G))$ (and remove the extra assumption F).

[4] Latin for "reduction to absurdity"

4. Apply **RAA** to conclude $\neg F$.

Directions for using **RAA** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $\neg F$ by applying rule **RAA** to $(F \to (G \wedge \neg G))$, which is a conclusion previously obtained at step i of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
n. $\neg F$	**RAA** i	(copy assumptions from row i)

This strategy of proof by contradiction is used in informal arguments in English. Here is an argument of Lucretius, a Roman poet and philosopher in the first century B.C., showing that the universe is infinite.

Example 2.6.1 (Informal Example of **RAA**)**.**

> Suppose the universe is finite. Go to the boundary. From there, throw a dart straight out. Now, the dart is in the universe, but it is also not in the universe. This is a contradiction. Therefore, the universe must really be infinite.

Example 2.6.2 (Humorous Example of **RAA**)**.**

> Suppose not every positive whole number is interesting. Then there must be a least positive whole number which is not interesting. But being the least positive whole number which is not interesting is quite interesting in and of itself. So the number is both interesting and not interesting, a contradiction. Therefore every positive whole number is interesting.

Example 2.6.3. $\{P, \neg R\} \vdash \neg(P \to R)$

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $\neg R$	**A**	2
3. $(P \to R)$	**A***	3^*
4. R	**MP** $3, 1$	$1, 3^*$
5. $(R \wedge \neg R)$	$\wedge$**I** $2, 4$	$1, 2, 3^*$
6. $((P \to R) \to (R \wedge \neg R))$	**CP** $3, 5$	$1, 2$
7. $\neg(P \to R)$	**RAA** 6	$1, 2$

There is a shorter proof using **MT** instead of **RAA**.

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $\neg R$	**A**	2
3. $(P \to R)$	**A***	3^*
4. R	**MP** $3, 1$	$1, 3^*$
5. $((P \to R) \to R)$	**CP** $3, 4$	1
6. $\neg(P \to R)$	**MT** $5, 2$	$1, 2$

Example 2.6.4. $\{\neg(P \vee Q)\} \vdash \neg P$.

Conclusion	Rule	Assumptions
1. $\neg(P \vee Q)$	**A**	1
2. P	**A***	2^*
3. $(P \vee Q)$	$\vee$**I** 2	2^*
4. $((P \vee Q) \wedge \neg(P \vee Q))$	$\wedge$**I** 1, 3	$1, 2^*$
5. $(P \to ((P \vee Q) \wedge \neg(P \vee Q)))$	**CP** 2, 4	1
6. $\neg P$	**RAA** 5	1

Again, there is a proof using **MT** instead of **RAA**. Writing it down formally is a good exercise to check your understanding.

Example 2.6.5. $\neg P \vdash (P \to R)$

It is worth discussing the strategy and underlying logic of this problem. We want to prove $(P \to R)$, so the strategy is to assume P, prove R, and then use **CP**. After assuming P, we can prove R by first assuming $\neg R$, obtaining a contradiction, and then using **RAA** to conclude $\neg\neg R$. Finally, we use **DN** to obtain R. In this case, the assumption $\neg R$ will not actually be used in obtaining the contradiction. This because we have the given assumption of $\neg P$, and so once we add the extra assumption P, we are already in the realm of contradiction, in which case any conclusion can be obtained.

Conclusion	Rule	Assumptions
1. $\neg P$	**A**	1
2. P	**A***	2^*
3. $\neg R$	**A***	3^*
4. $(P \wedge \neg P)$	$\wedge$**I** 1, 2	$1, 2^*$
5. $(\neg R \to (P \wedge \neg P))$	**CP** 3, 4	$1, 2^*$
6. $\neg\neg R$	**RAA** 5	$1, 2^*$
7. R	**DN** 6	$1, 2^*$
8. $(P \to R)$	**CP** 2, 7	1

The underlying logic is *from a false premise, we can obtain any conclusion.* We can explain this using things we've already seen: Recall that we discovered that $(F \to G)$ is equivalent to $(\neg F \vee G)$ in Subsection 1.5. Then if F is never true, $\neg F$ is always true, and thus so is $(\neg F \vee G) = (F \to G)$. This is called the principle of explosion.

The next example is called the *law of excluded middle*, which we have discussed already.

Example 2.6.6. $\vdash (P \vee \neg P)$

Conclusion	Rule	Assumptions
1. $\neg(P \vee \neg P)$	$\mathbf{A}^*$	1^*
2. P	$\mathbf{A}^*$	2^*
3. $(P \vee \neg P)$	$\vee\mathbf{I}$ 2	2^*
4. $((P \vee \neg P) \wedge \neg(P \vee \neg P))$	$\wedge\mathbf{I}$ $3, 1$	$1^*, 2^*$
5. $(P \rightarrow ((P \vee \neg P) \wedge \neg(P \vee \neg P)))$	**CP** $2, 4$	1^*
6. $\neg P$	**RAA** 5	1^*
7. $(P \vee \neg P)$	$\vee\mathbf{I}$ 6	1^*
8. $((P \vee \neg P) \wedge \neg(P \vee \neg P))$	$\wedge\mathbf{I}$ $7, 1$	1^*
9. $(\neg(P \vee \neg P) \rightarrow ((P \vee \neg P) \wedge \neg(P \vee \neg P)))$	**CP** $1, 8$	
10. $\neg\neg(P \vee \neg P)$	**RAA** 9	
11. $(P \vee \neg P)$	**DN** 10	

Note that the conclusions in steps 3 and 7 are identical, but step 7 uses fewer assumptions than step 3. The same is true of steps 4 and 8.

2.7 Bi-conditional statements

There are no special rules of inference for the symbol $\leftrightarrow$. This is justified from the previously discussed observation that $\leftrightarrow$ is really an abbreviation. We feel free to use it in writing formulas. However, when we are doing proofs, we treat $(F \leftrightarrow G)$ as identical to $((F \rightarrow G) \wedge (G \rightarrow F))$. If we are given $(F \leftrightarrow G)$, then we may apply $\wedge\mathbf{E}$ just as we would given $((F \rightarrow G) \wedge (G \rightarrow F))$. If we are asked to prove $(F \leftrightarrow G)$, then we prove $((F \rightarrow G) \wedge (G \rightarrow F))$ and stop.

Example 2.7.1. $\{P, (P \rightarrow Q)\} \vdash (P \leftrightarrow Q)$

Conclusion	Rule	Assumptions
1. $(P \rightarrow Q)$	**A**	1
2. Q	$\mathbf{A}^*$	2^*
3.P	**A**	3
4. $(Q \rightarrow P)$	**CP** $2, 3$	3
4. $(P \rightarrow Q) \wedge (Q \rightarrow P)$	$\wedge\mathbf{I}$ $1, 4$	$1, 3$

Example 2.7.2. $\{(P \leftrightarrow Q), \neg P\} \vdash \neg Q$

Conclusion	Rule	Assumptions
1. $(P \leftrightarrow Q)$	**A**	1
2. $\neg P$	**A**	2
3. $(Q \rightarrow P)$	$\wedge\mathbf{E}$ 1	1
4. $\neg Q$	**MT** $3, 2$	$1, 2$

2.8 More examples of proofs

Example 2.8.1. $\{P\} \vdash (Q \to P)$

Conclusion	Rule	Assumptions
1. P	**A**	1
2. Q	**A***	2^*
3. $(Q \to P)$	**CP** $2, 1$	1

Example 2.8.2. $\{P\} \vdash (\neg P \to \neg Q)$

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $\neg P$	**A***	2^*
3. Q	**A***	3^*
4. $(P \wedge \neg P)$	$\wedge$**I** $1, 2$	$1, 2^*$
5. $(Q \to (P \wedge \neg P))$	**CP** $3, 4$	$1, 2^*$
6. $\neg Q$	**RAA** 5	$1, 2^*$
7. $(\neg P \to \neg Q)$	**CP** $2, 6$	1

Example 2.8.3. $\{(P \vee Q), \neg P\} \vdash Q$

Conclusion	Rule	Assumptions
1. $(P \vee Q)$	**A**	1
2. $\neg P$	**A**	2
3. P	**A***	3^*
4. $\neg Q$	**A***	4^*
5. $(P \wedge \neg P)$	$\wedge$**I** $3, 2$	$2, 3^*$
6. $(\neg Q \to (P \wedge \neg P))$	**CP** $4, 5$	$2, 3^*$
7. $\neg\neg Q$	**RAA** 6	$2, 3^*$
8. Q	**DN** 7	$2, 3^*$
9. $(P \to Q)$	**CP** $3, 8$	2
10. Q	**A***	10^*
11. $(Q \to Q)$	**CP** $10, 10$	
12. Q	$\vee$**E** $1, 9, 11$	$1, 2$

Example 2.8.4. $\{\neg(P \to Q)\} \vdash P$

Conclusion	Rule	Assumptions
1. $\neg(P \to Q)$	**A**	1
2. $\neg P$	**A***	2^*
3. P	**A***	3^*
4. $\neg Q$	**A***	4^*
5. $(P \wedge \neg P)$	$\wedge$**I** $3, 2$	$2^*, 3^*$
6. $(\neg Q \to (P \wedge \neg P))$	**CP** $4, 5$	$2^*, 3^*$
7. $\neg\neg Q$	**RAA** 6	$2^*, 3^*$
8. Q	**DN** 7	$2^*, 3^*$
9. $(P \to Q)$	**CP** $3, 8$	2^*
10. $((P \to Q) \wedge \neg(P \to Q))$	$\wedge$**I** $9, 1$	$1, 2^*$
11. $(\neg P \to ((P \to Q) \wedge \neg(P \to Q))$	**CP** $2, 10$	1
12. $\neg\neg P$	**RAA** 11	1
13. P	**DN** 12	1

Example 2.8.5. $\{\neg(P \vee Q)\} \vdash (\neg P \wedge \neg Q)$

Conclusion	Rule	Assumptions
1. $\neg(P \vee Q)$	**A**	1
2. P	**A***	2^*
3. $(P \vee Q)$	$\vee$**I** 2	2^*
4. $((P \vee Q) \wedge \neg(P \vee Q))$	$\wedge$**I** $3, 1$	$1, 2^*$
5. $(P \to ((P \vee Q) \wedge \neg(P \vee Q)))$	**CP** $2, 4$	1
6. $\neg P$	**RAA** 5	1
7. Q	**A***	7^*
8. $(P \vee Q)$	$\vee$**I** 7	7^*
9. $((P \vee Q) \wedge \neg(P \vee Q))$	$\wedge$**I** $8, 1$	$1, 7^*$
10. $(Q \to ((P \vee Q) \wedge \neg(P \vee Q)))$	**CP** $7, 9$	1
11. $\neg Q$	**RAA** 5	1
12. $(\neg P \wedge \neg Q)$	$\wedge$**I** $6, 11$	1

Example 2.8.6. We will translate an argument into propositional logic, and give a formal proof. Here is the argument:

> Quincy will study either Italian or Math. She will not study both. If she studies math, she will get an A. Therefore, if she does not get an A in math, then she studied Italian.

Define the propositional variables:
I: Quincy studies Italian.
M: Quincy studies math.
G: Quincy gets an A in math.
Assumptions: $(I \vee M)$, $\neg(I \wedge M)$, $(M \to G)$
Conclusion: $(\neg G \to I)$

Proof:

Conclusion	Rule	Assumptions
1. $\neg G$	$\mathbf{A}^*$	3^*
2. $(M \to G)$	$\mathbf{A}$	1
3. $\neg M$	$\mathbf{MT}$ $2,1$	$1^*, 2$
4. $(I \vee M)$	A	4
5. I	$\mathbf{A}^*$	5^*
6. $(I \to I)$	$\mathbf{CP}$ $5,5$	
7. M	$\mathbf{A}^*$	7^*
8. $\neg I$	$\mathbf{A}^*$	8^*
9. $(M \wedge \neg M)$	$\wedge\mathbf{I}$ $7,3$	$1^*, 2, 7^*$
10. $(\neg I \to (M \wedge \neg M)$	$\mathbf{CP}$ $8,9$	$1^*, 2, 7^*$
11. $\neg\neg I$	$\mathbf{RAA}$ 12	$1^*, 2, 7^*$
12. I	$\mathbf{DN}$ 11	$1^*, 2, 7^*$
13. $(M \to I)$	$\mathbf{CP}$ $7,12$	$1^*, 2$
14. I	$\vee\mathbf{E}$ $4,6,13$	$1^*, 2, 4$
15. $(\neg G \to I)$	$\mathbf{CP}$ $1,14$	$2, 4$

Note that the second assumption $\neg(I \wedge M)$ was not used in the proof. Does this make sense to you?

Example 2.8.7. *Argument*:

> The recording says either laurel or yanny. The recording cannot say both laurel and yanny. Therefore the recording says yanny if and only if it does not say laurel.

Define the variables:

Y: The recording says yanny.

L: The recording says laurel.

Assumptions: $(Y \vee L)$, $\neg(Y \wedge L)$
Conclusion: $(Y \leftrightarrow \neg L)$
Strategy:

1. We want to prove $(Y \leftrightarrow \neg L)$, which is an abbreviation for $(Y \to \neg L)$ and $(\neg L \to Y)$. So we will prove these two things separately (steps 1 through 21), and then apply $\wedge\mathbf{I}$ to obtain the desired conclusion (step 22).

2. We will use **CP** to prove both implications, $(Y \to \neg L)$ and $(\neg L \to Y)$.

3. To prove $(Y \to \neg L)$ (steps 1 through 8): Temporarily assume Y. We want $\neg L$, so we will temporarily assume L, and get a contradiction, and then apply **RAA**. In particular, if we assume L then we can get $(Y \wedge L)$ by $\wedge\mathbf{I}$ (remember we already temporarily assumed Y). But we have $\neg(Y \wedge L)$ in our original assumptions.

4. To prove $(\neg L \to Y)$ (steps 9 through 21). Temporarily assume $\neg L$. We want Y. We should be able to get this from the temporary assumption $\neg L$ and the original assumption $(Y \vee L)$ (much like in the last example, where we got I from $(I \vee M)$ and $\neg M$). This requires using $\vee$**E**. We have $(Y \vee L)$ and we want Y, so we need the implications $(Y \to L)$ (which we get trivially as usual, steps 11 and 12) and $(L \to Y)$ (which we get absurdly, in steps 13 through 19, because we have already temporarily assumed $\neg L$). Finally we have the necessarily formulas in steps 10, 12, and 19 to apply $\vee$**E** and get Y in step 20.

Proof:

Conclusion	Rule	Assumptions
1. Y	$\mathbf{A}^*$	1^*
2. L	$\mathbf{A}^*$	2^*
3. $(Y \wedge L)$	$\wedge\mathbf{I}$ $1, 2$	$1^*, 2^*$
4. $\neg(Y \wedge L)$	$\mathbf{A}$	4
5. $((Y \wedge L) \wedge \neg(Y \wedge L))$	$\wedge\mathbf{I}$ $3, 4$	$1^*, 2^*, 4$
6. $(L \to ((Y \wedge L) \wedge \neg(Y \wedge L)))$	$\mathbf{CP}$ $2, 5$	$1^*, 4$
7. $\neg L$	$\mathbf{RAA}$ 6	$1^*, 4$
8. $(Y \to \neg L)$	$\mathbf{CP}$ $1, 7$	4
9. $\neg L$	$\mathbf{A}^*$	9^*
10. $(Y \vee L)$	$\mathbf{A}$	10
11. Y	$\mathbf{A}^*$	11^*
12. $(Y \to Y)$	$\mathbf{CP}$ $11, 11$	
13. L	$\mathbf{A}^*$	13^*
14. $\neg Y$	$\mathbf{A}^*$	14^*
15. $(L \wedge \neg L)$	$\wedge\mathbf{I}$ $13, 9$	$9^*, 13^*$
16. $(\neg Y \to (L \wedge \neg L))$	$\mathbf{CP}$ $14, 15$	$9^*, 13^*$
17. $\neg\neg Y$	$\mathbf{RAA}$ 16	$9^*, 13^*$
18. Y	$\mathbf{DN}$ 17	$9^*, 13^*$
19. $(L \to Y)$	$\mathbf{CP}$ $13, 18$	9^*
20. Y	$\vee\mathbf{E}$ $10, 12, 19$	$9^*, 10$
21. $(\neg L \to Y)$	$\mathbf{CP}$ $9, 20$	10
22. $(Y \leftrightarrow \neg L)$	$\wedge\mathbf{I}$ $8, 21$	$4, 10$

2.9 Some useful proofs

The following proofs are used frequently as intermediate steps of larger proofs. For example, you may have some *wff* of the form $\neg(A \wedge B)$ at some step of a proof. Logically, you know that this statement is equivalent to $(\neg A \vee \neg B)$. But you have to copy the actual proof, so keeping a list of the useful proofs below will save time.

All of these useful proofs are *bi-directional* in the sense that they are examples where $\{F\} \vdash G$ and $\{G\} \vdash F$. Therefore we use the notation $F \equiv G$ to denote this.

1. $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$
2. $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$
3. $(A \wedge (B \vee C)) \equiv ((A \wedge B) \vee (A \wedge C))$
4. $(A \vee (B \wedge C)) \equiv ((A \vee B) \wedge (A \vee C))$
5. $(A \wedge (B \wedge C)) \equiv ((A \wedge B) \wedge C)$
6. $(A \wedge B) \equiv (B \wedge A)$
7. $(A \vee (B \vee C)) \equiv ((A \vee B) \vee C)$
8. $(A \vee B) \equiv (B \vee A)$.

The proofs of these are discussed at the end of the section.

The last few proofs, (5) through (8) are worth discussing. For example, we can iterate (5) and (6) to conclude that, given *wff*s $A_1, \ldots, A_n$, if F_1 is the result of taking conjunctions of $A_1, \ldots, A_n$ in some order, and F_2 is the result of taking the conjunction of $A_1, \ldots, A_n$ in some other order, then $F_1 \equiv F_2$. Therefore, since any combination can be proved from any other combination, we will be lazy and just write

$$(A_1 \wedge A_2 \wedge \ldots \wedge A_n)$$

to represent any of these possible combinations. If we obtain one combination in a proof, we will allow ourselves to replace it with any other combination, without any further justification. Moreover, if we have $(A_1 \wedge \ldots \wedge A_n)$ in a proof, then we allow ourselves to use $\wedge$**E** to conclude any particular A_i we want to focus on. If we obtain each of $A_1, \ldots, A_n$ at different steps of a proof, then we allow ourselves to use $\wedge$**I** to conclude $A_1, \ldots, A_n$ without specifying any particular combination (just be sure to list *all* assumptions behind the A_is).

The same remarks hold for disjunctions. Given *wff*s $A_1, \ldots, A_n$, we use

$$(A_1 \vee A_2 \vee \ldots \vee A_n)$$

to represent any possible way of combining $A_1, \ldots, A_n$ using only $\vee$. If we have some A_1 in a proof, then we can use $\vee$**I** to conclude $(A_1 \vee \ldots \vee A_n)$. If we have $(A_1 \vee \ldots \vee A_n)$ in a proof, and then we also prove $(A_1 \rightarrow B), \ldots, (A_n \rightarrow B)$, we can use $\vee$**I** to conclude B (again, be sure to list *all* assumptions underlying these *wff*s).

We now consider each of the above proofs.

1. (a) $\{\neg(A \wedge B)\} \vdash (\neg A \vee \neg B)$

Conclusion	Rule	Assumptions
1. $\neg(A \wedge B)$	**A**	1
2. $\neg(\neg A \vee \neg B)$	$\mathbf{A}^*$	2^*
3. $\neg A$	$\mathbf{A}^*$	3^*
4. $(\neg A \vee \neg B)$	$\vee\mathbf{I}$ 3	3^*
5. $((\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B))$	$\wedge\mathbf{I}$ $4, 2$	$2^*, 3^*$
6. $(\neg A \rightarrow ((\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)))$	**CP** $3, 5$	2^*
7. $\neg\neg A$	**RAA** 6	2^*
8. A	**DN** 7	2^*
9. $\neg B$	$\mathbf{A}^*$	9^*
10. $(\neg A \vee \neg B)$	$\vee\mathbf{I}$ 9	9^*
11. $((\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B))$	$\wedge\mathbf{I}$ $10, 2$	$2^*, 9^*$
12. $(\neg B \rightarrow ((\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)))$	**CP** $9, 11$	2^*
13. $\neg\neg B$	**RAA** 12	2^*
14. B	**DN** 13	2^*
15. $(A \wedge B)$	$\wedge\mathbf{I}$ $8, 14$	2^*
16. $((A \wedge B) \wedge \neg(A \wedge B))$	$\wedge\mathbf{I}$ $15, 1$	$1, 2^*$
17. $(\neg(\neg A \vee \neg B) \rightarrow ((A \wedge B) \wedge \neg(A \wedge B)))$	**CP** $2, 16$	1
18. $\neg\neg(\neg A \vee \neg B)$	**RAA** 17	1
19. $(\neg A \vee \neg B)$	**DN** 18	1

(b) $\{(\neg A \vee \neg B)\} \vdash \neg(A \wedge B)$

Conclusion	Rule	Assumptions
1. $(\neg A \vee \neg B)$	**A**	1
2. $(A \wedge B)$	$\mathbf{A}^*$	2^*
3. A	$\wedge\mathbf{E}$ 2	2^*
4. B	$\wedge\mathbf{E}$ 2	2^*
5. $\neg A$	$\mathbf{A}^*$	5^*
6. $(A \wedge \neg A)$	$\wedge\mathbf{I}$ $3, 5$	$2^*, 5^*$
7. $(\neg A \rightarrow (A \wedge \neg A))$	**CP** $5, 6$	2^*
8. $\neg B$	$\mathbf{A}^*$	8^*
9. A	$\mathbf{A}^*$	9^*
10. $(B \wedge \neg B)$	$\wedge\mathbf{I}$ $4, 8$	$2^*, 8^*$
11. $(A \rightarrow (B \wedge \neg B))$	**CP** $9, 10$	$2^*, 8^*$
12. $\neg A$	**RAA** 11	$2^*, 8^*$
13. $(A \wedge \neg A)$	$\wedge\mathbf{I}$ $3, 12$	$2^*, 8^*$
14. $(\neg B \rightarrow (A \wedge \neg A))$	$8, 13$	2^*
15. $(A \wedge \neg A)$	$\vee\mathbf{E}$ $1, 7, 14$	$1, 2^*$
16. $((A \wedge B) \rightarrow (A \wedge \neg A))$	**CP** $2, 15$	1
17. $\neg(A \wedge B)$	**RAA** 16	1

2. (*a*) $\{\neg(A \vee B)\} \vdash (\neg A \wedge \neg B)$ (Example 2.8.5)

Conclusion	Rule	Assumptions
1. $\neg(A \vee B)$	**A**	1
2. A	**A***	2^*
3. $(A \vee B)$	$\vee$**I** 2	2^*
4. $((A \vee B) \wedge \neg(A \vee B))$	$\wedge$**I** $3, 1$	$1, 2^*$
5. $(A \to ((A \vee B) \wedge \neg(A \vee B)))$	**CP** $2, 4$	1
6. $\neg A$	**RAA** 5	1
7. B	**A***	7^*
8. $(A \vee B)$	$\vee$**I** 7	7^*
9. $((A \vee B) \wedge \neg(A \vee B))$	$\wedge$**I** $8, 1$	$1, 7^*$
10. $(B \to ((A \vee B) \wedge \neg(A \vee B)))$	**CP** $7, 9$	1
11. $\neg B$	**RAA** 5	1
12. $(\neg A \wedge \neg B)$	$\wedge$**I** $6, 11$	1

(*b*) $\{(\neg A \wedge \neg B)\} \vdash \neg(A \vee B)$

Conclusion	Rule	Assumptions
1. $(\neg A \wedge \neg B)$	**A**	1
2. $\neg A$	$\wedge$**E** 1	1
3. $\neg B$	$\wedge$**E** 1	1
4. $(A \vee B)$	**A***	4^*
5. A	**A***	5^*
6. $(A \wedge \neg A)$	$\wedge$**I** $5, 2$	$1, 5^*$
7. $(A \to (A \wedge \neg A))$	**CP** $5, 6$	2
8. B	**A***	8^*
9. $\neg A$	**A***	9^*
10. $(B \wedge \neg B)$	$\wedge$**I** $8, 3$	$1, 8^*$
11. $(\neg A \to (B \wedge \neg B))$	**CP** $9, 10$	$1, 8^*$
12. $\neg\neg A$	**RAA** 11	$1, 8^*$
13. A	**DN** 12	$1, 8^*$
14. $(A \wedge \neg A)$	$\wedge$**I** $13, 2$	$1, 8^*$
15. $(B \to (A \wedge \neg A))$	**CP** $8, 14$	1
16. $(A \wedge \neg A)$	$\vee$**E** $4, 7, 15$	$1, 4^*$
17. $((A \vee B) \to (A \wedge \neg A))$	**CP** $4, 16$	1
18. $\neg(A \vee B)$	**RAA** 17	1

3. (a) $\{(A \wedge (B \vee C))\} \vdash ((A \wedge B) \vee (A \wedge C))$

Conclusion	Rule	Assumptions
1. $(A \wedge (B \vee C))$	**A**	1
2. A	$\wedge$**E** 1	1
3. $(B \vee C)$	$\wedge$**E** 1	1
4. B	**A***	4^*
5. $(A \wedge B)$	$\wedge$**I** $2, 4$	$1, 4^*$
6. $((A \wedge B) \vee (A \wedge C))$	$\vee$**I** 5	$1, 4^*$
7. $(B \to ((A \wedge B) \vee (A \wedge C)))$	**CP** $4, 6$	1
8. C	**A***	8^*
9. $(A \wedge C)$	$\wedge$**I** $2, 8$	$1, 8^*$
10. $((A \wedge B) \vee (A \wedge C))$	$\vee$**I** 9	$1, 8^*$
11. $(C \to ((A \wedge B) \vee (A \wedge C)))$	**CP** $8, 10$	1
12. $((A \wedge B) \vee (A \wedge C))$	$\vee$**E** $3, 7, 11$	1

(b) $\{((A \wedge B) \vee (A \wedge C))\} \vdash (A \wedge (B \vee C))$

Conclusion	Rule	Assumptions
1. $((A \wedge B) \vee (A \wedge C))$	**A**	1
2. $(A \wedge B)$	**A***	2^*
3. A	$\wedge$**E** 2	2^*
4. B	$\wedge$**E** 2	2^*
5. $(B \vee C)$	$\vee$**I** 4	2^*
6. $(A \wedge (B \vee C))$	$\wedge$**I** $3, 5$	2^*
7. $((A \wedge B) \to (A \wedge (B \vee C))$	**CP** $2, 6$	
8. $(A \wedge C)$	**A***	8^*
9. A	$\wedge$**E** 8	8^*
10. C	$\wedge$**E** 8	8^*
11. $(B \vee C)$	$\vee$**I** 10	8^*
12. $(A \wedge (B \vee C))$	$\wedge$**I** $9, 11$	8^*
13. $((A \wedge C) \to (A \wedge (B \vee C))$	**CP** $8, 12$	
14. $(A \wedge (B \vee C))$	$\vee$**E** $1, 7, 13$	1

4. (*a*) $\{(A \vee (B \wedge C))\} \vdash ((A \vee B) \wedge (A \vee C))$

Conclusion	Rule	Assumptions
1. $(A \vee (B \wedge C))$	**A**	1
2. A	$\mathbf{A}^*$	2^*
3. $(A \vee B)$	$\vee\mathbf{I}$ 2	2^*
4. $(A \vee C)$	$\vee\mathbf{I}$ 2	2^*
5. $((A \vee B) \wedge (A \vee C))$	$\wedge\mathbf{I}$ $3, 4$	2^*
6. $(A \rightarrow ((A \vee B) \wedge (A \vee C)))$	**CP** $2, 5$	
7. $(B \wedge C)$	$\mathbf{A}^*$	7^*
8. B	$\wedge\mathbf{E}$ 7	7^*
9. $(A \vee B)$	$\vee\mathbf{I}$ 8	7^*
10. C	$\wedge\mathbf{E}$ 7	7^*
11. $(A \vee C)$	$\vee\mathbf{I}$ 10	7^*
12. $((A \vee B) \wedge (A \vee C))$	$\wedge\mathbf{I}$ $9, 11$	7^*
13. $((B \wedge C) \rightarrow ((A \vee B) \wedge (A \vee C)))$	**CP** $7, 12$	
14. $((A \vee B) \wedge (A \vee C))$	$\vee\mathbf{E}$ $1, 6, 13$	1

(*b*) $\{((A \vee B) \wedge (A \vee C))\} \vdash (A \vee (B \wedge C))$ This proof is quite long, but it is not overly complicated once we break down the strategy.

i. We will argue by contradiction, that is assume $\neg(A \vee (B \wedge C))$ and reach an impossible conclusion to obtain $\neg\neg(A \vee (B \wedge C))$. We finish the proof by removing the double negation.

ii. To reach a contradiction, we shall first deduce $\neg A$ and $\neg(B \wedge C)$ from our initial assumption. For each, this is a simple application of **MT** and the fact shown above that $\neg((A \vee (B \wedge C))) \equiv (\neg A \wedge \neg(B \wedge C))$.

iii. We shall now use the given assumption $((A \vee B) \wedge (A \vee C))$ and $\neg A$ we obtained in the previous step to prove $(B \wedge C)$. To prove B, we shall use $\vee\mathbf{E}$ on $(A \vee B)$. Since we have $\neg A$, we shall use **RAA** when proving $(A \rightarrow B)$. We shall do the same thing to get C.

iv. All that remains is to use $\wedge\mathbf{I}$ to obtain $(B \wedge C)$, which together with the previously obtained $\neg(B \wedge C)$ is all we need to finish the proof.

Here is the proof in its entirety:

Conclusion	Rule	Assumptions
1. $\neg(A \vee (B \wedge C))$	$\mathbf{A}^*$	1^*
2. A	$\mathbf{A}^*$	2^*
3. $(A \vee (B \wedge C))$	$\vee\mathbf{I}$ 2	2^*
4. $(A \to (A \vee (B \wedge C)))$	$\mathbf{CP}$ $2, 4$	
5. $\neg A$	$\mathbf{MT}$ $4, 1$	1^*
6. $(B \wedge C)$	$\mathbf{A}^*$	6^*
7. $(A \vee (B \wedge C))$	$\vee\mathbf{I}$ 6	6^*
8. $((B \wedge C) \to (A \vee (B \wedge C)))$	$\mathbf{CP}$ $6, 7$	
9. $\neg(B \wedge C)$	$\mathbf{MT}$ $8, 1$	1^*
10. $((A \vee B) \wedge (A \vee C))$	$\mathbf{A}$	10
11. $(A \vee B)$	$\wedge\mathbf{E}$ 10	10
12. B	$\mathbf{A}^*$	12^*
13. $(B \to B)$	$\mathbf{CP}$ $12, 12$	
14. A	$\mathbf{A}^*$	14^*
15. $\neg B$	$\mathbf{A}^*$	15^*
16. $(A \wedge \neg A)$	$\wedge\mathbf{I}$ $14, 5$	$1^*, 14^*$
17. $(\neg B \to (A \wedge \neg A))$	$\mathbf{CP}$ $15, 16$	$1^*, 14^*$
18. $\neg\neg B$	$\mathbf{RAA}$17	$1^*, 14^*$
19. B	$\mathbf{DN}$18	$1^*, 14^*$
20. $(A \to B)$	$\mathbf{CP}$ $14, 19$	1^*
21. B	$\vee\mathbf{E}$ $11, 13, 20$	$1^*, 10$
22. $(A \vee C)$	$\wedge\mathbf{E}$ 10	10
23. C	$\mathbf{A}^*$	23^*
24. $(C \to C)$	$\mathbf{CP}$ $23, 23$	
25. A	$\mathbf{A}^*$	25^*
26. $\neg C$	$\mathbf{A}^*$	15^*
27. $(A \wedge \neg A)$	$\wedge\mathbf{I}$ $25, 5$	$1^*, 25^*$
28. $(\neg C \to (A \wedge \neg A))$	$\mathbf{CP}$ $26, 27$	$1^*, 25^*$
29. $\neg\neg C$	$\mathbf{RAA}$28	$1^*, 25^*$
30. C	$\mathbf{DN}$29	$1^*, 25^*$
31. $(A \to C)$	$\mathbf{CP}$ $25, 30$	1^*
32. C	$\vee\mathbf{E}$ $22, 24, 31$	$1^*, 10$
33. $(B \wedge C)$	$\wedge\mathbf{I}$ $21, 32$	$1^*, 10$
34. $((B \wedge C) \wedge \neg(B \wedge C))$	$\wedge\mathbf{I}$ $33, 9$	$1^*, 10$
35. $(\neg(A \vee (B \wedge C)) \to ((B \wedge C) \wedge \neg(B \wedge C)))$	$\mathbf{CP}$ $1, 35$	10
36. $\neg\neg(A \vee (B \wedge C))$	$\mathbf{RAA}$ 35	10
37. $(A \vee (B \wedge C))$	$\mathbf{DN}$ 36	

5. (*a*) $\{(A \wedge (B \wedge C))\} \vdash ((A \wedge B) \wedge C)$

Conclusion	Rule	Assumptions
1. $(A \wedge (B \wedge C))$	**A**	1
2. A	$\wedge$**E** 1	1
3. $(B \wedge C)$	$\wedge$**E** 1	1
4. B	$\wedge$**E** 3	1
5. C	$\wedge$**E** 3	1
6. $(A \wedge B)$	$\wedge$**I** 2, 4	1
7. $((A \wedge B) \wedge C)$	$\wedge$**I** 6, 5	1

(*b*) $\{((A \wedge B) \wedge C))\} \vdash (A \wedge (B \wedge C))$
The proof is basically identical to the proof for part (a).

6. (*a*) $\{(A \wedge B)\} \vdash (B \wedge A)$ (Example 2.4.1)

Conclusion	Rule	Assumptions
1. $(A \wedge B)$	A	1
2. A	$\wedge$**E** 1	1
3. B	$\wedge$**E** 1	1
4. $(B \wedge A)$	$\wedge$**I** 3, 2	1

(*b*) $\{(B \wedge A)\} \vdash (A \wedge B)$
The proof is basically identical to the proof for part (a).

7. (*a*) $\{(A \vee (B \vee C))\} \vdash ((A \vee B) \vee C)$

Conclusion	Rule	Assumptions
1. $(A \vee (B \vee C))$	**A**	1
2. A	**A***	2^*
3. $(A \vee B)$	$\vee$**I** 2	2^*
4. $((A \vee B) \vee C)$	$\vee$**I** 3	2^*
5. $(A \to ((A \vee B) \vee C))$	**CP** 2, 4	
6. $(B \vee C)$	**A***	6^*
7. B	**A***	7^*
8. $(A \vee B)$	$\vee$**I** 7	7^*
9. $((A \vee B) \vee C)$	$\vee$**I** 8	7^*
10. $(B \to ((A \vee B) \vee C)))$	**CP** 7, 9	
11. C	**A***	11^*
12. $((A \vee B) \vee C)$	$\vee$**I** 11	11^*
13. $(C \to ((A \vee B) \vee C)))$	**CP** 11, 12	
14. $((A \vee B) \vee C)$	$\vee$**E** 6, 10, 13	6^*
15. $((B \vee C) \to (A \vee (B \vee C)))$	**CP** 6, 14	
16. $(A \vee (B \vee C))$	$\vee$**E** 1, 5, 15	1

(*b*) $\{((A \vee B) \vee C)\} \vdash (A \vee (B \vee C))$
The proof is basically identical to the proof for part (a).

8. (*a*) $\{(A \vee B)\} \vdash (B \vee A)$ (Example 2.5.2)

Conclusion	Rule	Assumptions
1. $(A \vee B)$	**A**	1
2. A	$\mathbf{A}^*$	2^*
3. $(B \vee A)$	$\vee\mathbf{I}$ 2	2^*
4. $(A \to (B \vee A))$	**CP** $2, 3$	
5. B	$\mathbf{A}^*$	5^*
6. $(B \vee A)$	$\vee\mathbf{I}$ 5	5^*
7. $(B \to (B \vee A))$	**CP** $5, 6$	
8. $(B \vee A)$	$\vee\mathbf{E}$ $1, 4, 7$	1

(*b*) $\{(B \vee A)\} \vdash (A \vee B)$
The proof is basically identical to the proof for part (a).

Chapter 3

Truth

3.1 Definition of Truth

We are about to define truth for propositional logic. That is, we will give some rules that allow us to calculate the truth-value for a formula, *given truth values for the basic propositional variables.* These rules ignore all of the subtleties that are present in English. They have the virtue of being precise.

Rules for computing truth-values
1. $\neg G$ is true if and only if G is false. 2. $(G \wedge H)$ is true if and only if G and H are both true. 3. $(G \vee H)$ is true if and only if at least one of G, H is true. 4. $(G \rightarrow H)$ is true if and only if G is false or H is true.

We are treating $\leftrightarrow$ as an abbreviation, not as a basic connective. Nonetheless, we could add a rule for $\leftrightarrow$ by saying: $(G \leftrightarrow H)$ *is true if and only if* G *and* H *have the same truth value; i.e., both are true or both are false.*

Recall that a formation sequence for a formula G is a finite sequence of *wff* s, ending in G, such that each step is obtained by applying one of the clauses in the definition of *wff*. To calculate the truth value of a formula G, given truth values of the propositional variables, we work our way along a formation sequence, applying the rules above. Earlier, we wrote our formation sequences vertically, with numbered steps, and we indicated exactly how each step was obtained. Here, we omit the explanations, although we will make sure that our formation sequences could be explained properly. We will put the steps all on one horizontal line, and we will start by giving all of the propositional variables.

Sample formation sequence: For $(P \vee (\neg Q \wedge \neg P))$, we have the following formation sequence, written in the new way.

$$P \quad Q \quad \neg Q \quad \neg P \quad (\neg Q \wedge \neg P) \quad (P \vee (\neg Q \wedge \neg P))$$

Let us calculate the truth value for $(P \vee (\neg Q \wedge \neg P))$, in the case where P is false and Q is true. We see that $\neg Q$ is false and $\neg P$ is true. Then $(\neg Q \wedge \neg P)$ is false. Finally, $(P \vee (\neg Q \wedge \neg P))$ is false.

3.2 Truth tables

We get a great deal of information by considering all possible asignments of truth values for the propositional variables, and putting our calculations into a *truth table*. Here are the truth tables for the basic logical connectives.

Basic truth tables

1. The truth table for $\neg P$ is

P	$\neg P$
T	F
F	T

2. The truth table for $(P \wedge Q)$ is

P	Q	$(P \wedge Q)$
T	T	T
T	F	F
F	T	F
F	F	F

3. The truth table for $(P \vee Q)$ is

P	Q	$(P \vee Q)$
T	T	T
T	F	T
F	T	T
F	F	F

4. The truth table for $(P \to Q)$ is

P	Q	$(P \to Q)$
T	T	T
T	F	F
F	T	T
F	F	T

We can combine these basic truth tables to compute the truth tables for more complicated *wff* s.

Example 3.2.1. The truth table for $(P \vee (\neg Q \wedge \neg P))$ is

P	Q	$\neg Q$	$\neg P$	$(\neg Q \wedge \neg P)$	$(P \vee (\neg Q \wedge \neg P))$
T	T	F	F	F	T
T	F	T	F	F	T
F	T	F	T	F	F
F	F	T	T	T	T

Format for truth tables:

1. The upper row lists a formation sequence of the *wff* from left to right, starting with the propositional variables found in the *wff* (listed in alphabetical order). There is a horizontal line below this row.

2. There is a vertical line separating the propositional variables in the *wff* from the columns to the right, and another vertical line separating the *wff* (the final step of the formation sequence) from the columns to the left.

3. The truth values to the left of the first vertical line are all of the possible truth assignments for the variables. The number of rows generated by these possibilities will vary exponentially depending on the number of variables. (If n is the number of variables, there will be 2^n possible truth assignments.) Here are the possibilities for one, two, and three propositional variables.

P
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

You should follow the same pattern given in the previous examples each time you set up the propositional variables for a truth table: Start by making the first half of the rows true for the first propositional variable, and the second half false. For each successive column, split each group of rows where the previous variable had the same truth value in half. Make the first half true and the second half false. (Technically there is nothing special about this setup, other than everyone in the class using the same setup will make reading and digesting truth tables much easier.)

4. The entries of the truth table described in the last step are predetermined, that is to say will always look the same regardless of the *wff* you are analyzing. The rest of the entries depend on the particular *wff* and its formation sequence. You compute them in steps using the basic truth tables for logical connectives given above. The best way to get used to this pattern is by doing many examples.

Looking at the truth table in Example 3.2.1, we see that the formula $(P \lor (\neg Q \land \neg P))$ is true except when P is false and Q is true. We classify formulas according to whether they are always true, never true, or neither.

Definition 3.2.2. Let G be a *wff*.

1. G is *tautological* if it is always true—i.e., true on all lines of its truth table.
2. G is *inconsistent* if it is never true—i.e., false on all lines of its truth table.
3. G is *contingent* if it is sometimes true and sometimes false—i.e., true on at least one line and false on at least one line of its truth table.

Example 3.2.1 above is contingent. Let us consider further examples.

Example 3.2.3. The truth table for $(P \lor \neg P)$ is

P	$\neg P$	$(P \lor \neg P)$
T	F	T
F	T	T

This formula is tautological.

Example 3.2.4. The truth table for $(P \land \neg P)$ is

P	$\neg P$	$(P \land \neg P)$
T	F	F
F	T	F

This formula is inconsistent.

Example 3.2.5. The truth table for $(P \to (Q \to R))$ is

P	Q	R	$(Q \to R)$	$(P \to (Q \to R))$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

This formula is contingent.

Example 3.2.6. The truth table for $(P \vee (\neg Q \to \neg P))$ is

P	Q	$\neg Q$	$\neg P$	$(\neg Q \to \neg P)$	$(P \vee (\neg Q \to \neg P))$
T	T	F	F	T	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	T	T	T	T

This formula is tautological.

Example 3.2.7. The truth table for $((P \wedge Q) \wedge \neg(P \vee Q))$ is

P	Q	$P \wedge Q$	$P \vee Q$	$\neg(P \vee Q)$	$((P \wedge Q) \wedge \neg(P \vee Q))$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

This formula is inconsistent.

Example 3.2.8. The truth table for $((\neg P \to Q) \leftrightarrow (P \wedge Q))$ is

P	Q	$\neg P$	$(\neg P \to Q)$	$(P \wedge Q)$	$((\neg P \to Q) \leftrightarrow (P \wedge Q))$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	F	F	T

This formula is contingent.

3.3 Analysis of arguments using truth tables

In the last chapter, we used proofs to justify the validity of arguments. We will now use true tables to do this. Moreover, we will see that truth tables are an easy way to demonstrate that an argument is *not* valid. Eventually, we will discuss why the two methods (proofs vs. truth tables) are equivalent.

Suppose Γ is a set of assumptions and C is a conclusion. To analyze the validity of the argument "C follows from Γ" using a truth table, we make a combined table which includes all of the *wff*s in Γ, as well as the *wff* C. Reading left to right, the top row of this table should consist of:

1. The propositional variables found in all *wff*s involved in the argument.
2. Any intermediate *wff*s needing in the formation sequences for the *wff*s in the argument.
3. The *wff*s in Γ.
4. The *wff* C.

We separate each of the four groups above with a vertical line in the truth table (see the examples below). If the *wff*s involved in the argument are simple enough, you may not not need to include the intermediate parts in step 2.

Once this truth table has been completed, we will be able to determine if the argument is valid as follows:

1. If C is true whenever all of the *wff*s in Γ are true, then the argument is valid. In this case, we write $\Gamma \models C$ (read: "C is a logical consequence of Γ"). Note that there may be rows where the conclusion is false: if any of the assumptions is false in these rows, then this does not affect whether the argument is valid.
2. If there is a row in the truth table where all of the *wff*s in Γ are true, but C is false, then the argument is not valid. In this case, we write $\Gamma \not\models C$ (read: "C is not a logical consequence of Γ"). This row is called a *counterexample* to the argument.

Note that, when determining the validity of the argument, we *disregard* any rows in the truth table where at least one *wff* in Γ is false.

Let us consider an example. In Example 2.5.1, we gave a proof of the argument in the following example. We now verify the validity of the argument using a truth table.

Example 3.3.1. We use truth tables to determine whether $\{(P \wedge Q)\} \models (P \vee Q)$.

	variables		assumptions	conclusion
	P	Q	$(P \wedge Q)$	$(P \vee Q)$
1.	T	T	T	T
2.	T	F	F	T
3.	F	T	F	T
4.	F	F	F	F

In every row where the assumptions are true (just the first row), we see that the conclusion is also true. Therefore $\{(P \wedge Q)\} \models (P \vee Q)$.

Note that, in the last example, there are rows where the assumptions are false and the conclusion is true (in particular, the second and third rows). Once again, this *does not* imply that the argument is not valid.

Example 3.3.2. We use truth tables to determine whether

$$\{(P \to R), \neg R, (Q \to \neg R)\} \models \neg(P \vee Q).$$

	variables			intermediate	assumptions			conclusion
	P	Q	R	$(P \vee Q)$	$(P \to R)$	$\neg R$	$(Q \to \neg R)$	$\neg(P \vee Q)$
1.	T	T	T	T	T	F	F	F
2.	T	T	F	T	F	T	T	F
3.	T	F	T	T	T	F	T	F
4.	T	F	F	T	F	T	T	F
5.	F	T	T	T	T	F	F	F
6.	F	T	F	T	T	T	T	F
7.	F	F	T	F	T	F	T	T
8.	F	F	F	F	T	T	T	T

In the sixth row (when P and R are false, and Q is true), we see that all of the assumptions are true, but the conclusion is false. In other words, this row is a counterexample to the argument. Therefore we conclude

$$\{(P \to R), \neg R, (Q \to \neg R)\} \not\models \neg(P \vee Q).$$

Note that, in the previous example, we did not include the *wff* $\neg R$ among the "intermediate" columns. This is because $\neg R$ was an assumption, and so such a column would have been redundant.

From the last example, we see that truth tables can be used to justify that an argument is *not* valid. We did not do examples of this in the chapter on proofs because, at the moment, demonstrating that a proof *does not* exist is much harder than demonstrating one does exist. Note also that, in order to verify that an argument is not valid using a truth table, it is enough to find *a single line* which serves as a counterexample. In other words, it suffices to write down one line where the assumptions of the argument are all true and the conclusion is false. Therefore if you can reason through the argument, and determine which line is likely to be a counterexample, then you may be able to avoid checking *all* lines of the truth table. This is especially helpful when there are a lot of propositional variables. For instance, the assumptions in the next example involve four propositional variables, and so the whole truth table has 16 lines ($16 = 2^4$). We will avoid checking all 16 lines by simply reasoning out a counterexample.

Example 3.3.3. We determine whether $\{(P \vee Q), (P \to S), (S \to P)\} \models (Q \vee R)$.

A counterexample to this argument is a situation where the assumptions are true, but the conclusion $(Q \vee R)$ is false. This is equivalent to saying that Q and R are *both* false. So if we can find a situation where Q are R are both false, but all assumptions are true, then this will serve as a counterexample to the argument. Such a situation can be obtained by letting P and S both be true, while Q and R are both false. We can verify this with one line of the truth table.

variables				assumptions			conclusion
P	Q	R	S	$(P \vee Q)$	$(P \to S)$	$(S \to P)$	$(Q \vee R)$
T	F	F	T	T	T	T	F

Therefore we conclude $\{(P \vee Q), (P \to S), (S \to P)\} \not\models (Q \vee R)$.

Example 3.3.4. We give a truth table to show $\{(Q \to (P \to R)), \neg R, Q\} \models \neg P$.

	variables			intermediate	assumptions			conclusion
	P	Q	R	$(P \to R)$	$(Q \to (P \to R))$	$\neg R$	Q	$\neg P$
1.	T	T	T	T	T	F	T	F
2.	T	T	F	F	F	T	T	F
3.	T	F	T	T	T	F	F	F
4.	T	F	F	F	T	T	F	F
5.	F	T	T	T	T	F	T	T
6.	F	T	F	T	T	T	T	T
7.	F	F	T	T	T	F	F	T
8.	F	F	F	T	T	T	F	T

Note that, in this example, the *wff* Q appears in two columns, since it is both a variable and an assumption.

Chapter 4

Soundness and Completeness

4.1 Two notions of validity

We have now discuss two ways of determining whether an argument is valid. To review, suppose we have a set of assumptions Γ and a conclusion C. We say:

1. Γ *proves* C, written $\Gamma \vdash C$, if there is a proof of C from the assumptions in Γ, which follows the proof system we learned in Chapter 2.

2. C *is a logical consequence of* Γ, written $\Gamma \models C$, if C is true whenever the assumptions in Γ are true. Here the meaning of "true" is according to the definition in Chapter 3, and the verification is done with a truth table.

The goal of this chapter is to show that these two methods of verifying arguments are equivalent. This is really two statements:

Soundness Theorem: If $\Gamma \vdash C$ then $\Gamma \models C$. That is, if we can prove something, then it is true.

Completeness Theorem: If $\Gamma \models C$ then $\Gamma \vdash C$. That is, if something is true, then a proof exists.

As we discussed in the introduction to the course, the words "soundness" and "completeness" are adjectives used to describe the proof system in Chapter 2. The first says that the proof system is "sound" in the sense that if we can prove an argument then it must be valid in the sense of truth. The second says that if an argument is valid according to the definition of truth, then there is a proof of it.

The Completeness Theorem has significant consequences in mathematics, especially in cases where it appears that there are infinitely many assumptions in the argument (i.e. Γ is an infinite set). This most salient in predicate logic, and so we will delay further discussion of this point until Part II of the course.

At this point, a natural question is why we would bother learning two different analyses of arguments. Therefore, it is worth observing some strengths and weakness of the two methods.

Proofs	
Strengths	Weaknesses
inform underlying logic	can be hard to find
often faster (shorter)	complicated to present
can be adapted to other logics (e.g. predicate logic)	hard to show an argument is *not* valid (i.e. there is *no proof*)

Truth Tables	
Strengths	Weaknesses
easy to complete	not very enlightening
easy to program	often slower (longer)
gives explicit counterexample when argument is not valid	cannot be adapted to many logics (e.g. predicate logic)

4.2 Examples: proofs vs. truth tables

Before discussing *Soundness* and *Completeness*, it is worth doing a number of examples in which we decide which of the two methods (proofs or truth tables) will be the best. In these examples we will use the *contrapositives* of the Soundness and Completeness Theorems. The *contrapositive* of an implication "If P then Q." is the logically equivalent statement "If not Q then not P." (Think **MT**) In symbols:

$$\{(P \to Q)\} \models (\neg Q \to \neg P) \quad \text{and} \quad \{(\neg Q \to \neg P)\} \models (P \to Q)$$

Contrapositive of Soundness: If $\Gamma \not\models C$ then $\Gamma \nvdash C$.

Contrapositive of Completeness: If $\Gamma \nvdash C$ then $\Gamma \not\models C$.

Example 4.2.1. We determine whether $\{(P \to Q), Q\} \vdash P$.

The notation suggests that we want to look for a proof. However, if we take a moment to think about the argument, it should become clear that argument is not valid. In this case, the only tool we have to show that there is no proof, is to show $\{(P \to Q), Q\} \not\models P$ (using a truth table), and then apply *Soundness*. So we complete the truth table.

	variables		assumptions		conclusion
	P	Q	$(P \to Q)$	Q	P
1.	T	T	T	T	T
2.	T	F	F	F	T
3.	F	T	T	T	F
4.	F	F	T	F	F

The desired counterexample can be found in row 3.

Example 4.2.2. We determine whether

$$\{(\neg((P \wedge R) \to (Q \vee S)) \to (A \vee B)), \neg(A \vee B), (P \wedge R), ((Q \vee S) \to M)\} \models M$$

The notation suggests completing a truth table, which would have 128 lines (there are 7 variables). On the other hand, if we think about the argument, it seems valid. Altogether, it will be faster to write a proof of

$$\{(\neg((P \wedge R) \to (Q \vee S)) \to (A \vee B)), \neg(A \vee B), (P \wedge R), ((Q \vee S) \to M)\} \vdash M$$

and then apply *Soundness*.

Conclusion	Rule	Assumptions
1. $(\neg((P \wedge R) \to (Q \vee S)) \to (A \vee B))$	**A**	1
2. $\neg(A \vee B)$	**A**	2
3. $\neg\neg((P \wedge R) \to (Q \vee S))$	**MT** $1, 2$	$1, 2$
4. $((P \wedge R) \to (Q \vee S))$	**DN** 3	$1, 2$
5. $(P \wedge R)$	**A**	5
6. $(Q \vee S)$	**MP** $4, 5$	$1, 2, 5$
7. $((Q \vee S) \to M)$	**A**	7
8. M	**MP** $7, 6$	$1, 2, 5, 7$

This proof is much faster (only 8 lines versus a truth table with 128 rows, 17 columns, and 2,176 entries).

The next example uses an argument from the first chapter.

Example 4.2.3. Consider the following argument.

> If the river floods, then our entire wheat crop will be destroyed. If the wheat crop is destroyed, then our community will be bankrupt. The river will flood if there is an early thaw. In any case, there will be heavy rains later in the summer. Therefore, if there is an early thaw, then our entire community will be bankrupt or there will be heavy rains later in the summer.

We will analyze the validity of this argument. First, we set variables and write down the assumptions and conclusion.

R: The river floods.

D: The wheat crop will be destroyed.

E: There is an early thaw.

B: The community will be bankrupt.

H: There are heavy rains later in the summer.

Assumptions

1. $(R \to D)$
2. $(D \to B)$
3. $(E \to R)$
4. H

Conclusion: $(E \to (B \vee H))$

Now think through the argument logically. Although there are what seem to be extraneous or disconnected pieces, it should feel sound. Once again, a truth table would have 32 lines, so we give a proof instead.

Conclusion	Rule	Assumptions
1. E	$\mathbf{A}^*$	1^*
2. H	$\mathbf{A}$	2
3. $(B \vee H)$	$\vee\mathbf{I}$ 2	2
4. $(E \to (B \vee H))$	$\mathbf{CP}$ 1, 3	2

Example 4.2.4. Consider the following argument.

> Tom embezzled money from his employer *Cybercore Dynamics*. If he is caught, then he will go to jail and his assistant will be fired. Tom's assistant is being fired. Therefore, Tom is going to jail.

We want to justify or refute the validity of this argument. Reading through the argument, it sounds problematic. While the assumptions say that Tom going to jail will result in his assistant being fired, this doesn't necessarily mean that the assistant being fired will mean Tom goes to jail. The assistant could be getting fired for other reasons.

Since we think this argument is *not* valid, we should use a truth table to look for a counterexample. We need to identify the assumptions of the argument, and the conclusion, and then translate these statements to *wff*s.

Variables

E: Tom embezzled money from his employer,

C: Tom is caught.

J: Tom goes to jail.

A: Tom's assistant is fired.

Assumptions

1. E
2. $(C \to (J \wedge A))$
3. A

Conclusion: J

Altogether, we have decided to use a truth table to show

$$\{E, (C \to (J \wedge A)), A\} \not\models J$$

Since there are 4 variables, the full truth table would have sixteen lines. So it will be faster to use our logical reasoning above to isolate the counterexample. We want a situation in which J is false, but the assumptions are all true. Since we reasoned that assistant could be getting fired for reasons other than Tom being caught for embezzling, we should expect the counterexample to occur in a case where Tom is *not* caught for embezzling (i.e. when C is false). Since all assumptions need to be true in order to obtain a counterexample, we need E and A to be true. So our predicated counterexample is when E and A are both true, but C is false. The last thing to do is verify this row in the truth table.

variables				intermediate	assumptions			conclusion
A	C	E	J	$(J \wedge A)$	E	$(C \to (J \wedge A))$	A	J
T	F	T	F	F	T	T	T	F

We consider two more *valid* arguments. In each case, we give the proof *and* the truth table. You can decide which method you prefer.

Example 4.2.5.
Argument: If the first appeal or second appeal is successful, then the original decision will be reversed. Hence, if the original decision stands, there was not a successful first appeal or a successful second appeal.

D: The original decision is reversed.

A: The first appeal succeeds.

B: The second appeal succeeds.

Assumption: $((A \vee B) \to D)$
Conclusion: $(\neg D \to (\neg A \wedge \neg B))$
Proof:

Conclusion	Rule	Assumptions
1. $((A \vee B) \to D)$	A	1
2. $\neg D$	A^*	2^*
3. A	A^*	3^*
4. $(A \vee B)$	$\vee$**I** 3	3^*
5. D	**MP** $1, 4$	$1, 3^*$
6. $(D \wedge \neg D)$	$\wedge$**I** $5, 2$	$1, 2^*, 3^*$
7. $(A \to (D \wedge \neg D))$	**CP** $3, 6$	$1, 2^*$
8. $\neg A$	**RAA** 7	$1, 2^*$
9. B	$\mathbf{A}^*$	9^*
10. $(A \vee B)$	$\vee$**I** 9	9^*
11. D	**MP** $1, 10$	$1, 9^*$
12. $(D \wedge \neg D)$	$\wedge$**I** $11, 2$	$1, 2^*, 9^*$
13. $(B \to (D \wedge \neg D))$	**CP** $9, 12$	$1, 2^*$
14. $\neg B$	**RAA** 7	$1, 2^*$
15. $(\neg A \wedge \neg B)$	$\wedge$**I** $8, 14$	$1, 2^*$
16. $(\neg D \to (\neg A \wedge \neg B))$	**CP** $2, 16$	1

Truth table:

	variables			intermediate					assumption	conclusion
	A	B	D	$\neg A$	$\neg B$	$\neg D$	$(A \vee B)$	$(\neg A \wedge \neg B)$	$((A \vee B) \to D)$	$(\neg D \to (\neg A \wedge \neg B))$
1.	T	T	T	F	F	F	T	F	T	T
2.	T	T	F	F	F	T	T	F	F	F
3.	T	F	T	F	T	F	T	F	T	T
4.	T	F	F	F	T	T	T	F	F	F
5.	F	T	T	T	F	F	T	F	T	T
6.	F	T	F	T	F	T	T	F	F	F
7.	F	F	T	T	T	F	F	T	T	T
8.	F	F	F	T	T	T	F	T	T	T

Example 4.2.6. The following argument was discussed in a first-year philosophy course at Notre Dame.

Argument: If God can create a mountain that He cannot climb, then He is not omnipotent. If God cannot create such a mountain, then He is not omnipotent. Therefore, God is not omnipotent.

C: God can create a mountain that He cannot climb.

O: God is omnipotent.

Assumptions: $(C \to \neg O), (\neg C \to \neg O)$
Conclusion: $\neg O$
Proof:

Conclusion	Rule	Assumptions
1. $(C \to \neg O)$	**A**	1
2. $(\neg C \to \neg O)$	**A**	2
3. O	$\mathbf{A}^*$	3^*
4. $\neg\neg O$	**DN** 3	3^*
5. $\neg C$	**MT** $1, 4$	$1, 3^*$
6. $\neg\neg C$	**MT** $2, 4$	$2, 3^*$
7. $(\neg C \wedge \neg\neg C)$	$\wedge$**I** $5, 6$	$1, 2, 3^*$
8. $(O \to (\neg C \wedge \neg\neg C))$	**CP** $3, 7$	$1, 2$
9. $\neg O$	**RAA** 8	$1, 2$

	variables		intermediate	assumptions		conclusion
	C	O	$\neg C$	$(C \to \neg O)$	$(\neg C \to \neg O)$	$\neg O$
1.	T	T	F	F	T	F
2.	T	F	F	T	T	T
3.	F	T	T	T	F	F
4.	F	F	T	T	T	T

By arguing about the behavior of truth tables, we can also give a "meta-proof" that the truth table corresponding the previous argument does not contain a counterexample, *without actually exhibiting the truth table.* Such an argument might go as follows.

Suppose there is a counterexample in the truth table. Then the conclusion must be false, and so O is true in this line. But the two assumptions must also be true. Both assumptions are implications, in which the second term is $\neg O$, which is false on this line. Therefore the first term of the implications must also be false, in order for the implications themselves to be true. In other words both C and $\neg C$ must be false on this line. This is not possible, so therefore there can be no counterexample in the truth table.

Notice that the previous argument involved *proof by contradiction.*

4.3 More examples

4.3.1 Refining assumptions

In some arguments, especially those in which one is trying to determine the *cause* of something, it is necessary to begin with a simple form of the argument, and then add complexity by refining the assumptions. The following example came from a seminar course at Notre Dame.

Example 4.3.1.
Initial argument: The driveway is wet. There are only three possible causes: sprinklers, a flood, or rain. Therefore, one of the three must have happened.

We start by assigning variables.

D: The driveway is wet.

S: The sprinklers were on.

F: There was a flood.

R: There was rain.

Initial assumptions: D, $(D \to (S \vee F \vee R))$
Initial conclusion: $(S \vee F \vee R)$

$\{D, (D \to (S \vee F \vee R))\} \vdash (S \vee F \vee R)$ is valid. (The proof should be easy to see.)

In order to determine the *actual* cause of the wet driveway, we can rule out some possibilities by refining the argument.

Refined argument: The driveway is wet. There are only three possible causes: sprinklers, a flood, or rain. If there has been a flood, the basement would also be wet, but it is not. If it has just rained, then the driveway across the street would also be wet, but it is not. So, the cause of our wet driveway must be the sprinklers.

Added variables:

A: The driveway across the street is wet.

B: The basement is wet.

Added assumptions: $F \to B$, $\neg B$, $(R \to A)$, $\neg A$
Refined conclusion: S
Altogether, the new argument is

$$\{D, (D \to (S \vee F \vee R)), (F \to B), \neg B, (R \to A), \neg A\} \vdash S$$

It shouldn't be hard to convince yourself that this argument is valid. The full truth table would contain 64 rows, and so we will omit it. A formal proof is also quite long (29 steps), and is given below. As an exercise, you should try to give a "meta-argument" there there can be no counterexample in the truth table (similar to what was done in Example 4.2.6. You could also look for a more efficient proof (if one exists).

Proof of: $\{D, (D \to (S \vee F \vee R)), (F \to B), \neg B, (R \to A), \neg A\} \vdash S$

Conclusion	Rule	Assumptions
1. D	**A**	1
2. $(D \to (S \vee F \vee R))$	**A**	2
3. $(S \vee F \vee R)$	**MP** $2, 1$	$1, 2$
4. $(F \to B)$	**A**	4
5. $\neg B$	**A**	5
6. $\neg F$	**MT** $4, 5$	$4, 5$
7. $(R \to A)$	**A**	7
8. $\neg A$	**A**	8
9. $\neg R$	**MT** $7, 8$	$7, 8$
10. S	$\mathbf{A}^*$	10^*
11. $(S \to S)$	**CP** $10, 10$	
12. F	$\mathbf{A}^*$	12^*
13. $\neg S$	$\mathbf{A}^*$	13^*
14. $(F \wedge \neg F)$	$\wedge\mathbf{I}$ $12, 6$	$4, 5, 12^*$
15. $(\neg S \to (F \wedge \neg F))$	**CP** $13, 14$	$4, 5, 12^*$
16. $\neg\neg S$	**RAA** 15	$4, 5, 12^*$
17. S	**DN** 16	$4, 5, 12^*$
18. $(F \to S)$	**CP** $12, 17$	$4, 5$
19. R	$\mathbf{A}^*$	19^*
20. $\neg S$	$\mathbf{A}^*$	20^*
21. $(R \wedge \neg R)$	$\wedge\mathbf{I}$ $19, 9$	$7, 8, 19^*$
22. $(\neg S \to (R \wedge \neg R))$	**CP** $20, 21$	$7, 8, 19^*$
23. $\neg\neg S$	**RAA** 22	$7, 8, 19^*$
24. S	**DN** 23	$7, 8, 19^*$
25. $(R \to S)$	**CP** $19, 24$	$7, 8$
26. $(F \vee R)$	$\mathbf{A}^*$	26^*
27. S	$\vee\mathbf{E}$ $26, 18, 25$	$4, 5, 7, 8, 26^*$
28. $((F \vee R) \to S)$	**CP** $26, 27$	$4, 5, 7, 8$
29. S	$\vee\mathbf{E}$ $3, 11, 28$	$1, 2, 4, 5, 7, 8$

4.3.2 Tacit assumptions

Many arguments have *tacit* assumptions, that is to say, assumptions which are not explicitly stated. In mathematics, this is often because these assumptions are obvious or universally accepted as true. In real life, tacit assumptions can appear for the same reasons, but also for the more dangerous reason that the person making the argument wants to use an assumption they know you may not accept, and therefore wishes to disguise this assumption.

The following is a mathematical example.

Example 4.3.2. In the following x denotes a real number.
Argument: If $x \geq 0$, then $x^2 \geq 0$. Also, if $x < 0$, then $x^2 \geq 0$. So, $x^2 \geq 0$.

Variables:

P: $x \geq 0$

N: $x < 0$

Q: $x^2 \geq 0$

Explicit assumptions:

1. $(P \to Q)$
2. $(N \to Q)$

Conclusion: Q

We want to analyze the argument $\{(P \to Q), (N \to Q)\} \vdash Q$. But this doesn't seem very likely to be valid. Indeed, we would have the following counterexample in the truth table.

variables			assumptions		conclusion
P	N	Q	$(P \to Q)$	$(N \to Q)$	Q
F	F	F	T	T	F

On the other hand, the original argument sounds valid. Looking more closely, we see that the counterexample is one in which P and N are both false, i.e. $x \geq 0$ and $x < 0$ are both false. We are missing the tacit assumption that x is either positive or negative.

Tacit assumption: $(P \vee N)$

Altogether, our actual argument is $\{(P \to Q), (N \to Q), (P \vee N)\} \vdash Q$. The proof is easy:

Conclusion	Rule	Assumptions
1. $(P \vee N)$	**A**	1
2. $(P \to Q)$	**A**	2
3. $(N \to Q)$	**A**	3
4. Q	$\vee$**E** $1, 2, 3$	$1, 2, 3$

4.3.3 An example from the LSAT

The following question comes from the LSAT sample test of June, 2007.

> Suppose I have promised to keep a confidence and someone asks me a question that I cannot answer truthfully without thereby breaking the promise. Obviously, I cannot both keep and break the same promise. Therefore, one cannot be obliged both to answer all questions truthfully and to keep all promises.

Which one of the following arguments is most similar in its reasoning to the argument above?

(A) If creditors have legitimate claims against a business and the business has the resources to pay those debts, then the business is obliged to pay them. Also, if a business has obligations to pay debts, then a court will force it to pay them. But the courts did not force this business to pay its debts, so either the creditors did not have legitimate claims or the business did not have suffcient resources.

(B) If we put a lot of effort into making this report look good, the client might think we did so because we believed our proposal would not stand on its own merits. On the other hand, if we do not try to make the report look good, the client might think we are not serious about her business. So, whatever we do, we risk her criticism.

(C) It is claimed that we have the unencumbered right to say whatever we want. It is also claimed that we have the obligation to be civil to others. But civility requires that we not always say what we want. So, it cannot be true both that we have the unencumbered right to say whatever we want and that we have the duty to be civil.

(D) If we extend our business hours, we will either have to hire new employees or have existing employees work overtime. But both new employees and additional overtime would dramatically increase our labor costs. We cannot afford to increase labor costs, so we will have to keep our business hours as they stand.

We may translate the original argument into propositional logic, using the variables:

A: I answer truthfully.

K: I keep the promise.

Translated, the original argument is $\{(A \to \neg K)\} \vdash \neg(A \wedge K)$.

Now we translate the arguments in each of (A), (B), (C), and (D).

(A) L: Creditors have legitimate claims against a business.

R: The business has the resources to pay the debts.

P: The business is obliged to pay the the debts.

F: The courts did not force the business to pay its debts.

Argument: $\{((L \wedge R) \to P), (P \to F), \neg F\} \vdash (\neg L \vee \neg R)$

(B) E: We put effort into making the report look good.

M: The client thinks we believe the proposal stands on its own merits.

S: The client thinks we are serious about her business.

Argument: $\{(E \to \neg M), (\neg E \to \neg S)\} \vdash ((E \vee \neg E) \to (\neg M \vee \neg S))$

(C) W: We say whatever we want.

C: We are civil.

Argument: $\{(C \to \neg W)\} \vdash \neg(W \wedge C)$

(D) E: We extend our business ours.

H: We have to hire new workers.

O: We have existing employees work overtime.

I: Our labor costs increase.

Argument: $\{(E \to (H \vee O)), ((H \vee O) \to I), \neg I\} \vdash \neg E$

The answer is clearly (C). Is this argument valid?

4.3.4 Puzzles

One fun application of logic is puzzles. Let's see how we can use the tools we have developed to solve the following recreational puzzles.

Example 4.3.3. The people on the planet GB are either green or blue. Green people **always** tell the truth and blue people **never** tell the truth.

1. We meet two people: Anna and Bill.
   Anna says: *If I am green then so is Bill.*
   What color is Anna? What color is Bill?

   Let's translate these statements into symbols.

   A: Anna is green.

   B: Bill is green.

We will use a truth table to figure out the truth values of A and B. First, we need to determine what information we know.

Anna's claim is $(A \to B)$. If Anna is green then she always tells the truth, and so this statement is true. So we know $(A \to (A \to B))$. On the other hand, if $(A \to B)$ is true then Anna is telling the truth about this claim, and so she must be green. In other words $((A \to B) \to A)$ is true. Altogether, we know $(A \leftrightarrow (A \to B))$ is true. Let's consider the truth table for this *wff*.

A	B	$(A \to B)$	$(A \leftrightarrow (A \to B))$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

The only truth assignment of the variables, which results in the truth of this *wff*, is when A and B are both true. So we conclude that Anna and Bill are both green.

2. We meet two people: Anna and Bill.
Anna says: *We are both blue.*
What color is Anna? What color is Bill?

Now, Anna's claim is $(\neg A \wedge \neg B)$. By the same argument as before, Anna's claim is true if and only if she is green, i.e. we know $(A \leftrightarrow (\neg A \wedge \neg B))$ is true. We can give an informal argument: If Anna is green then her claim is true, which is impossible. So Anna must be blue. But then her claim is false, and so Bill must be green.

Altogether, if we complete the truth table for the *wff* $(A \leftrightarrow (\neg A \wedge \neg B))$, then we expect the only line in which this *wff* is true is the one where A is false and B is true. Indeed:

A	B	$(\neg A \wedge \neg B)$	$(A \leftrightarrow (\neg A \wedge \neg B))$
T	T	F	F
T	F	F	F
F	T	F	T
F	F	T	F

3. (Where's Waldo?) We meet two people: Anna and Bill.
Anna says: *If we are both green, then Waldo is on the planet.*
Bill says: *That is true.*
Is Waldo on the planet?

In addition to A and B as above, let W stand for "Waldo is on the planet". Anna's claim is $((A \wedge B) \to W)$. Bill's claim is that Anna's claim is true.

Therefore Bill's claim is true if and only if Anna's claim is true. Altogether, our assumptions are:

$$(A \leftrightarrow ((A \wedge B) \rightarrow W)) \quad \text{and} \quad (B \leftrightarrow ((A \wedge B) \rightarrow W))$$

We argue as follows. If A and B are both true then $(A \wedge B)$ is true and, from our assumptions, we know $((A \wedge B) \rightarrow W)$ is true, which together mean W is true. Therefore we know $((A \wedge B) \rightarrow W)$ is true. From the assumptions above, we conclude that A and B are both true. Therefore W is true.

This was an informal proof of

$$\{(A \leftrightarrow ((A \wedge B) \rightarrow W)), (B \leftrightarrow ((A \wedge B) \rightarrow W))\} \vdash W$$

Here is the formal proof:

Conclusion	Rule	Assumptions
1. $(A \leftrightarrow ((A \wedge B) \rightarrow W))$	**A**	1
2. $(B \leftrightarrow ((A \wedge B) \rightarrow W))$	**A**	2
3. $(A \wedge B)$	**A***	3^*
4. A	$\wedge$**E** 3	3^*
5. $(A \rightarrow ((A \wedge B) \rightarrow W))$	$\wedge$**E** 1	1
6. $((A \wedge B) \rightarrow W))$	**MP** $5, 4$	$1, 3^*$
7. W	**MP** $6, 3$	$1, 3^*$
8. $((A \wedge B) \rightarrow W)$	**CP** $3, 7$	1
9. $(((A \wedge B) \rightarrow W) \rightarrow A)$	$\wedge$**E** 1	1
10. $(((A \wedge B) \rightarrow W) \rightarrow B)$	$\wedge$**E** 2	2
11. A	**MP** $9, 8$	1
12. B	**MP** $10, 8$	2
13. $(A \wedge B)$	$\wedge$**I** $11, 12$	$1, 2$
14. W	**MP** $8, 13$	$1, 2$

The proof follows the informal argument given above.

4.4 Verifying Soundness

Soundness shouldn't be that hard to believe. (Furthermore, how useful is a proof system if it is not sound? Do we want to have a proof system that can prove false things?) If we look back at our definition of truth and our rules of inference, we can see that if the inputs for a rule of inference are true, then the conclusion will also be true. To apply this, suppose we have some set Γ of assumptions and we have some *wff* C with $\Gamma \vdash C$. Then we work through each step of the proof and verify (using a truth table, or an English argument) that whenever all of the assumptions are true, the conclusion at that step is true. If we can do this at every step, then we can certainly do so at the last step, where the only assumptions used are in Γ and the conclusion is C.

Example 4.4.1. We shall verify Soundness for the following proof of $\{P, \neg Q\} \vdash \neg(P \to Q)$.

Conclusion	Rule	Assumptions
1. P	**A**	1
2. $\neg Q$	**A**	2
3. $(P \to Q)$	$\mathbf{A}^*$	3^*
4. Q	**MP** $3, 1$	$1, 3^*$
5. $(Q \wedge \neg Q)$	$\wedge$**I** $4, 2$	$1, 2, 3^*$
6. $((P \to Q) \to (Q \wedge \neg Q))$	**CP** $3, 5$	$1, 2$
7. $\neg(P \to Q)$	**RAA** 6	$1, 2$

Now we need to eplain why the conclusion at a given step is true if all of the assumptions behind it are true for every step. For each of the first three steps, this is clear because we are using Rule **A**.

- If P is true, then P is true.
- If $\neg Q$ is true, then $\neg Q$ is true.
- If $(P \to Q)$ is true, then $(P \to Q)$ is true.

Next is step four. Consider this truth table:

P	Q	$(P \to Q)$	Q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	F

In all of the rows where both P and $(P \to Q)$ are true (just the first row), Q is true. Thus Q is true whenever all of the assumptions are true. This argument will work whenever we use **MP** in a proof by the definition of truth: Suppose we apply **MP** to rows i and j, that is row i has $(A \to B)$ in the Conclusion column, and row j has A in the Conclusion column. Then whenever the assumptions on row i are true, $(A \to B)$ is true, and whenever the assumptions on row j are true, A is true. Since the assumptions on the row where we apply **MP** are the assumptions from rows i and j, both $(A \to B)$ and A are true. Since by definition $(A \to B)$ is true if and only if B is true whenever A is, B must be true. (A similar appeal to the definition of truth for $\to$ can be used to verify any step where we use **MT** so long as we have verified the previous steps.)

To verify step five, we can simply recognize that we defined $(A \land B)$ to be true if and only if A and B are both true. If assumption 2 is true, $\neg Q$ is true since we have already verified row two. If assumptions 1 and 3* are both true, then Q is true since we have already verified row four. Therefore $(Q \land \neg Q)$ is true by the definition of truth. (As in the case for **MP**, this argument will work whenever we use $\land$**I** so long as we have already verified the steps involved. Simply appealing to the definition of truth in a similar fashion can also be used to verify any step where we use $\land$**E** or $\lor$**I** if the previous steps have been verified.)

Next is step six. Again, we shall rely on how we defined truth for $(A \to B)$. Consider $((P \to Q) \to (Q \land \neg Q))$ and assume assumptions 1 and 2 are true.

- If $(P \to Q)$ is false, then the whole statement is true by our definition of truth.
- Therefore, suppose $(P \to Q)$ is true. Then this is assumption 3*, and we verified row five above. This means that $(Q \land \neg Q)$ is true since all three assumptions on row five are true, so $((P \to Q) \to (Q \land \neg Q))$ is true.

Therefore, in both cases we have that $((P \to Q) \to (Q \land \neg Q))$ is true if assumptions 1 and 2 are true, so it is always true when the assumptions are because one of the cases must apply. (Again, the same argument can be used whenever we use **CP** in a proof.)

Finally, we shall verify step seven. If assumptions 1 and 2 are true, then we showed previously that $((P \to Q) \to (Q \land \neg Q))$ is true. For this to be true, $(Q \land \neg Q)$ must be true whenever $(P \to Q)$ is. However, consider the following truth table:

Q	$\neg Q$	$(Q \land \neg Q)$
T	F	F
F	T	F

Since $(Q \land \neg Q)$ is never true, let alone when assumptions 1 and 2 are, then $(P \to Q)$ cannot be true either. By our definition of truth for $\neg$, $(P \to Q)$ is not true exactly when $\neg(P \to Q)$ is true. (We can apply this same reason to any correct use of **RAA** in a proof.)

Therefore we have verified that whenever P is true and $\neg Q$ is true, that $\neg(P \to Q)$ is true. Furthermore, we have done so using general arguments that can be applied to arbitrary applications of these rules of inference in other proofs as well. Once we know how to argue for all of the rules of inference, then we will be able to see that Soundness is indeed true.

For an implication $(F \to G)$, the contrapositive statement is $(\neg G \to \neg F)$. In fact, using some notation from before, $(F \to G) \equiv (\neg G \to \neg F)$. The contrapositive of Soundness is very useful: if $\Gamma \not\models C$, then $\Gamma \nvdash C$. That is, if C is not true whenever all of the assumptions in Γ are true, then we cannot prove C from Γ. This is the best way to show that something is NOT proved by a set of assumptions.

4.5 Explanation of Completeness

Recall Completeness:

Completeness Theorem: If $\Gamma \models C$ then $\Gamma \vdash C$. That is, if something is true, then a proof exists.

Completeness may be much less obvious than Soundness at first glance. As mentioned earlier, in fact, there are proof systems which are Sound but not Complete. Propositional logic is simple enough, however, that Completeness is not all that unclear once we see what is happening. To explain why our proof system is Complete, we shall show that any tautology can be proved from no assumptions, i.e. $\models C$ implies $\vdash C$. Then we shall show that we can actually reduce the general case to this specific case.

Theorem 4.5.1. Completeness Theorem (Special Case): If $\models C$, then $\vdash C$. That is, if C is always true, then we can prove it using no assumptions.

First, let's see how this special case is enough to prove the Completeness Theorem. Suppose we have $\Gamma \models C$, where $\Gamma = \{F_1, F_2, \ldots, F_n\}$. Then $\models ((F_1 \wedge F_2 \wedge \cdots \wedge F_n) \to G)$ by the definition of truth. Thus we can apply the special case of the Completeness Theorem, i.e. $\vdash ((F_1 \wedge F_2 \wedge \cdots \wedge F_n) \to G)$. Then we can take any proof of $((F_1 \wedge F_2 \wedge \cdots \wedge F_n) \to G)$ from no assumptions and add $F_1,\ F_2,\ \ldots,\ F_n$ using n applications of Rule **A**. Now we can use multiple applications of $\wedge$**I** to obtain $(F_1 \wedge F_2 \wedge \cdots \wedge F_n)$. Finally, apply **MP** to obtain G. This shows that $\Gamma \vdash G$.

Therefore, we just need to justify the special case of Completeness. Suppose $\vdash C$ and $P_1, P_2, \ldots, P_n$ are all of the propositional variables that appear in C. We say a *wff* F is *complete* for these propositional variables if it represents a truth assignment for these variables, i.e. it is a conjunction of n things, where for each i, exactly one of P_i or $\neg P_i$ is in the conjunction. For example, all the complete *wff*s for P_1 and P_2 are

$$(P_1 \wedge P_2),\ (P_1 \wedge \neg P_2),\ (\neg P_1 \wedge P_2),\ (\neg P_1 \wedge \neg P_2)$$

A *wff* of this form represents the row of the truth table where P_i is false for all of the P_i that appear as $\neg P_i$ in the *wff* and the rest are true. Below we are going to show that complete *wff*s either prove or disprove (i.e. prove the negation of) any *wff* made from only $P_1, P_2, \ldots, P_n$.

Let F be a complete *wff* for $P_1, P_2, \ldots, P_n$. Then if $\neg P_i$ is one of the terms in F, we see that $\{F\} \vdash \neg P_i$ by an easy two step proof. Similarly, if $\neg P_i$ isn't one of the things in the conjunction, then P_i is, so $\{F\} \vdash P_i$. Now suppose that D and E are *wff*s made out of $P_1, P_2, \ldots, P_n$ and for each one, $\{F\}$ either proves it and its negation. Then we can show that $\{F\}$ either proves or disproves $\neg D$, $(E \vee D)$, $(E \wedge D)$, and $(E \to D)$. Therefore we can see that $\{F\}$ proves or disproves every *wff* involving only $P_1, P_2, \ldots, P_n$ by following a formation sequence for it.

Recall that C is tautological. By the previous paragraph, either $\{F\} \vdash C$ or $\{F\} \vdash \neg C$. If $\{F\} \vdash \neg C$, then by the Soundness Theorem, $\{F\} \models \neg C$. This means that in the row of the truth table represented by F (the one where it is true), $\neg C$ is true. In particular, C is false in this row. This is a contradiction, however, as C being tautological means it is true in every row of the truth table. Therefore, we must have $\{F\} \vdash C$, i.e. a proof of C using only the assumption F. Then we can replace the use of Rule **A** by Rule **A*** to assume F, then prove C. Once we reach C, we can use **CP** to prove $(F \to C)$ from no assumptions. Thus $\vdash (F \to C)$.

Since each complete *wff* corresponds to a row in a truth table with n propositional variables, there are 2^n complete *wff*s. Let G be the disjunction of all of them. Notice that G is tautological, as for each row exactly one complete *wff* F is true in that row, and G is asserting that at least one such F is true. Furthermore, by repeating portions of the proofs from from Example 2.6.6 (law of the excluded middle) and Subsection 2.9 statements 3. and 4. (distribution of conjunctions/disjunctions), we can construct a proof of G. That is, we have $\vdash G$.

Now we are ready to finish the special case: We have $\vdash G$, where G is the disjunction of all the *wff*s F that are complete for $P_1, P_2, \ldots, P_n$. For each F, we have $\vdash (F \to C)$. So we can write a proof of C from no assumptions as follows:

- Prove G from no assumptions.
- Add on the proof of $(F \to C)$ from no assumptions for each complete *wff* F.
- Conclude C using $\vee$**E** applied to G and each of the $(F \to C)$s.

Part II

Predicate Logic

Chapter 5

Beginning predicate logic

5.1 Shortcomings of propositional logic

We now know how to analyze any argument, provided that we can successfully translate it into propositional logic. However, there are some arguments that do not translate into propositional logic. Consider the following.

Example 5.1.1. (famous)
Assumptions:

1. All men are mortal.
2. Socrates is a man.

Conclusion: Socrates is mortal.

Example 5.1.2. (not famous, but of the same form)
Assumptions:

1. All Notre Dame women are smart.
2. Natalia is a Notre Dame woman.

Conclusion: Natalia is smart.

Arguments of this form are called "syllogisms". The arguments seem correct. However, if we try to translate into propositional logic, we have trouble. In propositional logic, complicated statements are built up from basic ones using the logical connectives "and", "or" "implies", and "not". The arguments above have none of these logical connectives. Thus, we cannot take any of the statements apart. So, when we attempt to translate Example 1 into propositional logic, we have to use a different propositional variable for each statement.

A: All men are mortal.

S: Socrates is a man.

M: Socrates is mortal.

The argument then becomes: $A, S \models M$. But, as an argument in propositional logic, this is clearly not valid. In trying to translate into propositional logic, we lose everything that is important in the syllogism. Propositional logic does not have a way to talk about *properties* (e.g. being mortal or being a smart person). It doesn't have *names* for special objects (e.g. Socrates or Natalia). Finally, there is no way to say in propositional logic that a property holds *for all* objects of a certain kind.

Predicate logic has the expressive power that we need to address these shortcomings of propositional logic. For propositional logic, we described the symbols, gave an inductive definition of *well-formed formula*, or *wff*, did some translation, developed a proof system, defined truth, and stated Completeness and Soundness Theorems. We will do the same for predicate logic.

5.2 Symbols of predicate logic

Predicate languages have the following symbols. Some are the same as for propositional languages, and others are new.

logical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$ (with the same interpretations)

parentheses and commas (for organization)

quantifiers:

$\forall$ (for all)

$\exists$ (there exists)

equality: $=$ (interpreted as saying two objects are the same)

predicate (or relation) symbols: $A, B, C, P, Q, R \ldots$ (interpreted as properties of objects)

individual variables: $u, v, w, x, y, z, \ldots$ (placeholders for objects)

individual constants: $a, b, c, p, q, r, \ldots$ (interpreted as specific objects)

Remark 5.2.1. We use capital letters for relation symbols, and lower case letters for variables and constants. We will try to consistently use different letters for constants than for variables (e.g. early letters in the alphabet for constants, later letters for variables). But often the best thing to do is just specify in any given problem what kind of symbol the letter is being used for.

Each relation symbol applies to a fixed number of inputs at a time. (This is called the *arity* of the relation.) Here are some examples:

- 1-place relation symbols (or *unary* relations) are interpreted as properties of single objects such as "x is mortal" or "x is a smart person" or "x is an even number".
- 2-place relation symbols (or *binary* relations) are interpreted as relations on two objects such as "x is less than y" or "x and y are friends"
- 3-place relation symbols (or *ternary* relations) are interpreted as relations on three objects such as "x is a city between the cities y and z".

Continuing this way, we have 4-place relation symbols, 5-place relation symbols, etc... The arity of any particular relation symbol in our language will be made evident by the way we construct well-formed formulas using the symbol.

Note that, depending on how we interpret a relation, the *order* of the objects may matter. For example if "x is less than y" is not the same as "y is less than x", while "x and y are both even numbers" is the same as "y and x are both even numbers".

5.3 Definition of well-formed formula

We need some preliminary definitions before we can define well-formed formulas.

Definition 5.3.1. A *term* is a variable or a constant.

In each case, a term can be interpreted as an object we are interested in. We use variables to allow the possibility of talking about *arbitrary* objects, and we use constants to talk about *specific* objects.

Definition 5.3.2. An *atomic formula* is a string of symbols of one of the following two forms:

1. $t = t'$, where t and t' are terms
2. $P(t_1, \ldots, t_n)$, where P is an n-place predicate symbol and $t_1, \ldots, t_n$ are terms.

In each case, an atomic formula can be interpreted as an assertion about the terms involved. For example $t = t'$ asserts "t and t' are equal", while $P(t_1, \ldots, t_n)$ asserts "the n-place relation P holds on the objects $t_1, \ldots, t_n$". However, the meaning of this assertion will only make sense once we "plug in" objects for any variables occurring in the formula. We will discuss this in detail in the next section.

Remark 5.3.3. Atomic formulas are the analog in predicate logic of propositional variables in propositional logic. In particular, atomic formulas are the basic assertions that we will combine, using logical connectives and quantifiers, in order to make more complicated assertions.

Example 5.3.4. Here are some examples of atomic formulas. Let v, w, x be variables and let a be a constant.

- $x = y$
- $v = a$
- $M(x)$
- $M(a)$
- $L(v, w)$
- $B(a, x, y)$

Note, in particular, that an atomic formula can involve a mixture of variables and constants (as in the last example).

We now define the class of *well-formed formulas* (*wff*s) for predicate logic.

Definition 5.3.5.

(1) An atomic formula is a *wff*.

(2) If F is a *wff*, then so is $\neg F$.

(3) If F and G are *wff*s, then so are $(F \wedge G)$, $(F \vee G)$, and $(F \to G)$.

(4) If F is a *wff* and v is a variable, then $\exists v F$ and $\forall v F$ are *wff*s.

(5) A string of symbols is a *wff* if and only if it can be obtained by finitely many applications of conditions 1, 2, 3, and 4.

Here is an example of a more complicated *wff* in predicate logic:

$$\exists y \forall x (M(x) \to L(y, x))$$

We will discuss more in the next section how to *interpret* what this *wff* is saying. But it can be abstractly read as follows.

> "There exists a y such that for all x, if the relation M holds on x then the relation L holds on x and y."

As in propositional logic, to show that a string of symbols F is a *wff*, we give a *formation sequence*, building it up from the simplest parts (the atomic formulas), and adding connectives and quantifiers one at a time.

Example 5.3.6. Give a formation sequence for $\exists y \forall x (M(x) \to L(y, x))$.

1. $M(x)$ is a *wff* by condition 1 (atomic formula).
2. $L(y, x)$ is a *wff* by condition 1 (atomic formula).
3. $(M(x) \to L(y, x))$ is a *wff* by condition 3 applied to steps 1 and 2.
4. $\forall x (M(x) \to L(y, x))$ is a *wff* by condition 4 applied to step 3.
5. $\exists y \forall x (M(x) \to L(y, x))$ is a *wff* by condition 4 applied to step 4.

It is important to note that atomic formulas cannot be broken into simpler formulas. In particular, variables and constants are not themselves formulas. Also, $x =$ is not a formula.

We are going to use an abbreviation for applying the negation symbol $\neg$ to an atomic formula of the form $t = t'$, where t and t' are terms. If we follow the rules exactly then the negation of $t = t'$ looks like $\neg t = t'$, which is confusing to read. Therefore, we will use $t \neq t'$ to stand for the negation of $t = t'$. Also, the formal rules don't require us to put parentheses around an atomic formula of the form $t = t'$. For example $(x = y \wedge M(x))$ is a *wff*. However, we will allow ourselves to put parentheses in extra places anytime we think it will improve readability of the formula in question, e.g. $((x = y) \wedge M(x))$.

Example 5.3.7. Give a formation sequence showing that $(x \neq s \wedge \forall x \neg R(x, a))$ is a *wff*. (a, s are constants and R is a 2-place predicate symbol.)

1. $x = s$ is a *wff* by condition 1.
2. $R(x, a)$ is a *wff* by condition 1.
3. $x \neq s$ is a *wff* by condition 2 applied to step 1.
4. $\neg R(x, a)$ is a *wff* by condition 2 applied to step 2.
5. $\forall x \neg R(x, a)$ is a *wff* condition 4 applied to step 4.
6. $(x \neq s \wedge \forall x \neg R(x, a))$ is a *wff* by condition 3 applied to steps 3 and 5.

As in propositional logic, formation sequences are not unique. Here is another one for the same *wff*.

1. $x = a$ is a *wff* by condition 1.
2. $x \neq a$ is a *wff* by condition 2 applied to step 1.
3. $R(x, a)$ is a *wff* by condition 1.
4. $\neg R(x, a)$ is a *wff* by condition 2 applied to step 3.
5. $\forall x \neg R(x, a)$ is a *wff* condition 4 applied to step 4.
6. $(x \neq s \wedge \forall x \neg R(x, a))$ is a *wff* by condition 3 applied to steps 2 and 5.

Once again, you may want to add extra parentheses to improve readability. For example, you may think that the *wff* in the last example would be easier to read if it were:

$$((x \neq s) \wedge \forall x(\neg R(x, a)))$$

As long as you don't break any of rules in the definition of a *wff* (e.g. don't *remove* parentheses that are required by the rules) you are free to be flexible with adding parentheses. In general, the rule of thumb is to only add parentheses around things that are themselves a *wff*, e.g. around $x = y$ to make $(x = y)$, but not around $x =$.

5.4 Translation

When translating sentences and arguments to propositional logic, we had to make a choice of propositional variables. Translation in predicate logic can be more subtle. We need to choose relations symbols, with appropriately chosen places, and also choose constant symbols for any distinguished objects in the argument.

Example 5.4.1. Recall the famous syllogism.

> All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

Relations

$M(x)$: x is a man.

$D(x)$: x is mortal.

Constants

s: Socrates

Assumptions

"All men are mortal." $\forall x(M(x) \to D(x))$

"Socrates is a man." $M(s)$

Conclusion

"Socrates is mortal." $D(s)$

Note that there is some subtlety in translating a sentence like "all men are mortal" into an expression involving logical connections (e.g. $\to$). The formula $\forall x(M(x) \to D(x))$ would be more literally translated as: "For all objects x, if x is a man then x is mortal." This is just a more awkward way to say: "All men are mortal."

Example 5.4.2. Here is another famous argument, due to Descartes.

> I think. Therefore, I am.

Relations

$T(x)$: x thinks.

$B(x)$: x is.

Constants

i: I

Assumptions

"I think." $T(i)$

Conclusion

"I am." $B(i)$

As translated, the premise and the conclusion are not related. Note also that there are grammatical issues (e.g. we don't say "I is" when translating $B(i)$.

One common interpretation of the argument is that Descartes is not just making an assertion about himself, but about all people.

"If a person thinks, then a person exists." $\forall x(T(x) \rightarrow B(x))$

We will look at some simple examples.

Example 5.4.3. Consider the following symbols.

Relations

$D(x)$: x is a dog.

$F(x)$: x has fleas.

Constants

s: Spot (a dog)

1. "All dogs have fleas."
$$\forall x(D(x) \rightarrow F(x))$$
Literally: "For all x, if x is a dog then x has fleas."
2. "Some dogs have fleas."
$$\exists x(D(x) \wedge F(x))$$
Literally: "There exists an x such that x is a dog and x has fleas."
3. "If Spot has fleas, then all dogs do."
$$(F(s) \rightarrow \forall x(D(x) \rightarrow F(x)))$$
4. "Spot is not the only dog."
$$\exists x(D(x) \wedge x \neq s)$$
Literally: "There exists an x such that x is a dog and x is not the same as Spot."
5. "There are at least two dogs."
$$\exists x \exists y(D(x) \wedge D(y) \wedge x \neq y)$$
6. "There are exactly two dogs."
$$\exists x \exists y(D(x) \wedge D(y) \wedge x \neq y \wedge \forall z((D(z) \rightarrow (z = x \vee z = y))))$$

Remark 5.4.4.

1. Whenever we decide to use a quantifier in a translation, we have to introduce a new variable which is different from the previously used variables. For example we used three variables x, y, z in the last translation of the previous example.
2. The quantifier "$\forall x$" is translated literally as "for all x". The quantifier "$\exists x$" can be translated literally as "for some x", or we often say "there exists an x such that".

5.5 Learning the idioms for predicate logic

While it may not be as natural to translate statements into predicate logic as it is to translate statements into propositional logic, with practice you can learn how to get around the limitations of predicate logic.

Example 5.5.1.

Relations	*Constants*
$D(x)$: x is a dog.	s: Spot
$L(x, y)$: x likes y.	

1. "Spot likes everyone."
$$\forall x L(s, x)$$
2. "Someone likes Spot."
$$\exists x L(x, s)$$
3. "Spot likes everyone who likes him."
$$\forall x(L(x, s) \to L(s, x))$$
4. "Spot likes himself."
$$L(s, s)$$
5. "Spot likes every dog that likes itself."
$$\forall x((D(x) \wedge L(x, x)) \to L(s, x))$$

Example 5.5.2.

Relations	*Constants*
$N(x)$: x is a Notre Dame student.	m: mathematics
$C(x)$: x is a Clemson student.	
$S(x)$: x is a school subject.	
$G(x, y)$: x is good at y.	

1. "Math is not the only school subject.
$$\exists x(S(x) \wedge x \neq m)$$
2. "Every Notre Dame student is good at math."
$$\forall x(N(x) \to G(x, m))$$

3. "Some Clemson students are good at math."

$$\exists x(C(x) \wedge G(x, m))$$

4. "Not all Clemson students are good at math."

$$\neg \forall x(C(x) \rightarrow G(x, m))$$

   Alternatively: $\exists x(C(x) \wedge \neg G(x, m))$

5. "Every Notre Dame student is good at some subject."

$$\forall x(N(x) \rightarrow \exists y(S(y) \wedge G(x, y)))$$

6. "Every Notre Dame student is good at every subject."

$$\forall x(N(x) \rightarrow \forall y(S(y) \rightarrow G(x, y)))$$

7. "Some Clemson students are good at every subject."

$$\exists x(C(x) \wedge \forall y(S(y) \rightarrow G(x, y)))$$

8. "All Notre Dame students are good at any subject that all Clemson students are good at."

$$\forall x(N(x) \rightarrow \forall y((S(y) \wedge \forall z(C(z) \rightarrow G(z, y))) \rightarrow G(x, y)))$$

Example 5.5.3. The order of quantifiers matters. Think about the difference between the following statements.

1. "Everyone likes someone."
2. "Someone likes everyone."

Let $L(x, y)$ by a 2-place relation symbol for "x likes y". We translate each sentence above.

1. $\forall x \exists y L(x, y)$
2. $\exists x \forall y L(x, y)$

We could add more variants.

(a) "Everyone is liked by someone." $\forall x \exists y L(y, x)$

(b) "Someone is liked by everyone." $\exists x \forall y L(y, x)$

Example 5.5.4. We can use equality to say that two things are the same, or different, or to indicate the number of things with some property.

Relations

$N(x)$: x is a Notre Dame student.

$G(x)$: x is graduating this year.

1. "There exactly two Notre Dame Students."

$$\exists x \exists y (x \neq y \wedge N(x) \wedge N(y) \wedge \forall z (N(z) \rightarrow (z = x \vee z = y)))$$

   The following is a trick for writing this with fewer symbols.

$$\exists x \exists y (x \neq y \wedge \forall z (N(z) \leftrightarrow (z = x \vee z = y)))$$

2. The trick in the last step is useful if we have to say that there are a certain number of objects with several properties.

   "Exactly two Notre Dame students are graduating this year."

$$\exists x \exists y (x \neq y \wedge \forall z ((N(z) \wedge G(z)) \leftrightarrow (z = x \vee z = y)))$$

5.6 Free variables and sentences

Some *wff*s say something about one or more variables. Other *wff*s just make an assertion about the properties, relations, and special individuals named by our predicates and constants.

Example 5.6.1. Think of variables as ranging over the natural numbers $0, 1, 2, \ldots$. Consider the 2-place predicate symbol $L(x, y)$ for "x is less than y".

1. $L(x, y)$

   This formula says something about x and y. It is true if we plug in 3 for x and 7 for y. It is not true if we plug in 4 for x and 2 for y.

2. $\exists x L(x, y)$

   This formula asserts that there exists something that is less than y. It is true if we plug in 3 for x, and not true if we plug in 0 for x. Note that the variable y is no longer "free" to be replaced by a number, since we have added a quantifier referring to this variable.

3. $\forall x \exists y L(x, y)$

   This formula says that there is no greatest number; i.e., for any number, there is a larger one. Since we have introduced quantifiers referring to all variables in the formula, there are no variables to plug numbers into. The formula is now describing a property of the whole set of numbers we are considering. Moreover, this property is true.

4. $\forall y \exists x L(x, y)$

   This formula says that for each number, there is a smaller one. Once again the formula is a statement about the set of numbers, and there are no "free" variables to plug numbers into. In this case however, the statement is false, since we are not including any numbers smaller than 0. If we changed context to consider all integers–positive and negative–then this statement would be true.

Motivated by the previous examples, we want to distinguish variables in formulas which are "free" to be replaced by objects, which are not "free" because they are referred to by a quantifier. We also want to distinguish formulas with no free variables.

Definition 5.6.2. The *scope* of a quantifier $\forall v$ or $\exists v$ in a *wff* F (really, of a particular occurrence of the quantifier) is the *wff* that the quantifier is applied to in a formation sequence for F. A more natural way to describe this is that the scope is the smallest *wff* appearing to the right of the quantifier. (Here parentheses are critical.)

Example 5.6.3. Let G be a 1-place relation, and let M be a 2-place relation.

1. In the formula $\forall x(G(x) \to \exists y M(x, y))$, the scope of $\forall x$ is $(G(x) \to \exists y M(x, y))$, and the scope of $\exists y$ is $M(x, y)$.

2. In the formula $(M(x, z) \wedge \forall z(M(z, y) \vee M(y, z)))$, the scope of the quantifier $\forall z$ is $(M(z, y) \vee M(y, z))$.

3. In the formula $(\exists y(G(y) \wedge M(x, y)) \vee G(x))$, the scope of the quantifier $\exists y$ is $(G(y) \wedge M(x, y))$.

4. In the formula $\forall x(G(x) \to (\exists y G(y) \wedge M(x, z)))$, the scope of the quantifier $\forall x$ is $(G(x) \to (\exists y G(y) \wedge M(x, z)))$, and the scope of $\exists y$ is $G(y)$.

5. In the formula $(\exists x G(x) \wedge \exists x \neg G(x))$, the scope of the first quantifier $\exists x$ is $G(x)$, and the scope of the second $\exists x$ is $\neg G(x)$.

Definition 5.6.4.

1. An occurrence of a variable v in a *wff* F is *bound* if it is in the scope of a quantifier $\forall v$ or $\exists v$.

2. An occurrence of a variable v in a *wff* F is *free* if is not bound.

3. A *sentence* is a *wff* in which no variable occurs freely.

Example 5.6.5. For each of the following *wff*s, we determine whether the *wff* is a sentence, and if not, we state the free occurrences of variables.

1. $L(x, y)$ is not a sentence–the variables x and y occur freely.
2. $\exists x L(x, y)$ is not a sentence–the variable y occurs freely. Note that $\exists z L(z, y)$ says the same thing about y; the particular choice of variable being quantified doesn't matter, so long as it is different from y.
3. $\forall x \exists y L(x, y)$ is a sentence. Note that $\forall y \exists z L(y, z)$ says the same thing.
4. $\forall y \exists x L(x, y)$ is a sentence.

Example 5.6.6. Let G be a 1-place relation, P a 2-place relation, and let a, b be constant symbols.

1. $(G(a) \wedge \forall x P(x, b))$ is a sentence.
2. $\forall y(\exists z G(z) \rightarrow P(x, y))$ is not a sentence– the variable x occurs freely.
3. $\exists x P(x, b) \vee \forall y(G(y) \wedge P(x, y)))$ is a sentence.

Example 5.6.7. As previously noticed, we may change variables with bound occurrences without changing the meaning of the *wff*. Here are more examples.

$S(x)$: x is a sport.

$L(x, y)$: x likes y.

We write a formula expressing "x likes some sport"

$$\exists y(S(y) \wedge L(x, y))$$

We could change the variable y to z and say the same thing.

$$\exists z(S(z) \wedge L(x, z))$$

Note that if we change y to x in the original *wff* we obtain

$$\exists x(S(x) \wedge L(x, x))$$

This is now a sentence, which says: "there is a sport that likes itself." The meaning of the formula has changed completely.

Remark 5.6.8. Consider the *wff*

$$(Q(x) \wedge \exists x P(x, y))$$

This *wff* is not a sentence, as the first occurrence of x is a free occurrence. On the other hand, the second occurrence of x is bound by the quantifier $\exists x$. While this is not technically a problem, it does introduce room for error. As we noticed above, we can change a bound variable in the scope of some quantifier without changing the meaning of the *wff*. For example, if we do this with the previous *wff* then we obtain

$$(Q(x) \wedge \exists z P(z, y))$$

By changing variables bound by quantifiers, we can always ensure that no variable has both free and bound occurrences in the same formula. This way, it makes sense to talk about the *free variables* of a formula, without any confusion.

In general, a *wff* with free variables says something about the variables. A sentence makes an assertion about a whole system. We will give meaning to a formula by thinking of the variables as ranging over a certain class of objects, letting the predicates stand for certain relations, and letting the constants name specific objects. A *wff* with free variables may be "satisfied" by assignments of certain objects for the free variables, and not others. A sentence will just be "true" or "false". We will make this precise later.

Example 5.6.9. Let $P(x, y)$ be a relation interpreted as "x is a parent of y."

1. $P(x, y)$ says "x is a parent of y." Both x and y are free variables.

2. $\exists x P(x, y)$ says "y has a parent." Here y is a free variable. The other variable, x, is not mentioned when we translate into English. We could write $\exists z P(z, y)$ and say the same thing.

3. $\exists y P(x, y)$ says "x is a parent." Here x is a free variable. The other variable, y, is not mentioned in the translation. We we would translate $\exists z P(x, z)$ in the same way.

4. $\forall y \exists x P(x, y)$ says "everyone has a parent." This formula is a sentence. The English translation does not mention any variables. We could write $\forall x \exists y P(y, x)$ and say the same thing.

5. $\forall x \exists y P(x, y)$ says "everyone is a parent." This formula is also a sentence.

5.6.1 Substituting constants for free variables

We are about to start proofs in predicate logic. In our proofs, we will use only *sentences*, not formulas with free variables. We will obtain some of these sentences by "substituting" constants for free variables. So, let us focus on that process.

First, suppose $P(x, y)$ is a 2-place relation symbol. Then $P(x, y)$ is itself a *wff*, and it has two free variables x and y. We can think of more complicated *wff*s as relations holding on their free variables. Therefore, we write "$F(x)$ is a *wff*" to mean that this *wff* contains exactly one free variable x. Similarly, we write "$G(x, y)$ is a *wff*" to mean that this *wff* contains exactly two free variables x and y. We do the same thing for *wff*s with any arbitrary number of variables.

Suppose $A(x)$ is a *wff* (in the free variable x), and c is a constant symbol. Then $A(c)$ is the sentence that results from substituting the constant c for the free occurrences of x in the formula $A(x)$. Similarly, we may write $B(x, y)$ to indicate that B is a formula with at most x and y occurring freely. Then $B(c, y)$ is the formula that results from substituting the constant c for the free occurrences of x. (We do the same thing with other variables and constants.)

Example 5.6.10. Let $F(x)$ be the formula $(P(x) \to Q(x))$. Then $F(c)$ is the sentence $(P(c) \to Q(c))$.

Example 5.6.11. If $G(x, y)$ be the formula $(P(y) \wedge \exists z R(x, z))$, then $G(x, c)$ is $(P(c) \wedge \exists z L(x, z))$, and $G(b, c)$ is the sentence $(P(c) \wedge \exists z L(b, z))$.

Note that x and y both occur freely in $G(x, y)$, x occurs free in $G(x, c)$, and $G(b, c)$ is a sentence.

Chapter 6

Proofs

As in propositional logic, we will use the notation $\{F_1, \ldots, F_n\} \vdash G$ to mean that there is a proof of the sentence G from the sentences $F_1, \ldots, F_n$.

Our proof system will include all the rules from propositional logic:

$$\mathbf{A}, \mathbf{MP}, \mathbf{MT}, \mathbf{DN}, \mathbf{CP}, \wedge\mathbf{I}, \wedge\mathbf{E}, \vee\mathbf{I}, \vee\mathbf{E}, \mathbf{RAA}$$

The only difference is that now we apply the rules to sentences in predicate logic, rather than to *wff*s of propositional logic.

6.1 Basic proofs

The easiest proofs in predicate logic are ones where there are no quantifiers and no equalities involved in the sentences. Such proofs work very much the same as in propositional logic. The atomic formulas behave like propositional variables in proofs you are familiar with.

Example 6.1.1. $\{P(c), (P(c) \to Q(c))\} \vdash Q(c)$.

Conclusion	Rule	Assumptions
1. $P(c)$	**A**	1
2. $(P(c) \to Q(c))$	**A**	2
3. $Q(c)$	**MP** $1, 2$	$1, 2$

Remark 6.1.2. IMPORTANT! In addition to the assumptions and conclusion of the initial argument being sentences, *every single* wff *in the "Conclusion" column of the proof must be a sentence.*

Example 6.1.3. $\{((A(c) \vee B(c,d)) \to (\neg A(c) \wedge \neg B(c,d)))\} \vdash \neg B(c,d)$.

Conclusion	Rule	Assumptions
1. $B(c,d)$	$\mathbf{A}^*$	1^*
2. $((A(c) \vee B(c,d))$	$\vee\mathbf{I}$ 1	1^*
3. $((A(c) \vee B(c,d)) \to (\neg A(c) \wedge \neg B(c,d)))$	$\mathbf{A}$	3
4. $(\neg A(c) \wedge \neg B(c,d))$	**MP** 3, 2	$1^*, 3$
5. $\neg B(c,d)$	$\wedge\mathbf{E}$ 4	$1^*, 3$
6. $(B(c,d) \wedge \neg B(c,d))$	$\wedge\mathbf{I}$ 1, 5	$1^*, 3$
7. $(B(c,d) \to (B(c,d) \wedge \neg B(c,d)))$	**CP** 1, 6	3
8. $\neg B(c,d)$	**RAA** 7	3

Example 6.1.4. The following is Descartes' divisibility argument for substance dualism.

> If the mind and body are one and the same substance, they have all of the same features. The body is divisible. The mind is indivisible. Therefore, the mind and body must be different substances.

Relations

$S(x,y)$: "x and y are the same substance"

$D(x)$: "x is divisible"

Constants

m: mind

b: body

We translate the above argument. The first sentence makes a stronger claim than what is needed for the rest of the argument. All we need is "If the mind and body are the same substance then one is divisible if and only if the other is." This way the argument translates as:

$$\{(S(m,b) \to (D(b) \leftrightarrow D(m))), D(b), \neg D(m)\} \vdash \neg S(m,b)$$

Conclusion	Rule	Assumptions
1. $S(m,b)$	$\mathbf{A}^*$	1^*
2. $(S(m,b) \to (D(b) \leftrightarrow D(m)))$	$\mathbf{A}$	2
3. $(D(b) \leftrightarrow D(m))$	**MP** 2, 1	$1^*, 2$
4. $(D(b) \to D(m))$	$\wedge\mathbf{E}$ 3	$1^*, 2$
5. $D(b)$	$\mathbf{A}$	5
6. $D(m)$	**MP** 4, 5	$1^*, 2, 5$
7. $\neg D(m)$	$\mathbf{A}$	7
8. $(D(m) \wedge \neg D(m))$	$\wedge\mathbf{I}$ 6, 7	$1^*, 2, 5, 7$
9. $(S(m,b) \to (D(m) \wedge \neg D(m)))$	**CP** 1, 8	2, 5, 7
10. $\neg S(m,b)$	**RAA** 9	2, 5, 7

6.2 Rules for universal quantifiers

In addition to the ten old rules for the propositional connectives, we need new rules for quantifiers and equality.

6.2.1 Rule of Inference: Universal Elimination

Rule 11: **Universal Elimination**, abbreviated **UE**.
If $F(x)$ is a *wff* and c is a constant then from $\forall x F(x)$ we can conclude $F(c)$. In symbols: $$\forall x F(x) \rhd F(c)$$

Directions for using **UE** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $F(c)$ by applying rule **UE** to $\forall x F(x)$, which is a conclusion previously obtained at step i of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ F(c)$	**UE** i	(copy assumptions from row i)

Example 6.2.1. Recall the famous syllogism.

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

Relations

$M(x)$: x is a man.

$D(x)$: x is mortal.

Constants

s: Socrates

Then the three sentences translated are:

$\forall x(M(x) \to D(x))$

$M(s)$

$D(s)$

We write the argument using theses sentences:

$$\{\forall x(M(x) \to D(x)), M(s)\} \vdash D(s)$$

Proof:

Conclusion	Rule	Assumptions
1. $\forall x(M(x) \to D(x))$	**A**	1
2. $(M(s) \to D(s))$	**UE** 1	1
3. $M(s)$	**A**	3
4. $D(s)$	**MP** $2, 3,$	$1, 3$

Here are several more examples.

Example 6.2.2. $\{\forall x \forall y(Q(x,y) \to Q(y,x)), Q(c,d)\} \vdash Q(d,c)$.

Conclusion	Rule	Assumptions
1. $\forall x \forall y(Q(x,y) \to Q(y,x))$	**A**	1
2. $\forall y(Q(c,y) \to Q(y,c))$	**UE** 1	1
3. $(Q(c,d) \to Q(d,c))$	**UE** 2	1
4. $Q(c,d)$	**A**	4
5. $Q(d,c)$	**MP** $3,4$	$1,4$

Example 6.2.3. $\{\neg P(m)\} \vdash \neg \forall x P(x)$.

Conclusion	Rule	Assumptions
1. $\forall x P(x)$	**A***	1^*
2. $P(m)$	**UE** 1	1^*
3. $\neg P(m)$	**A**	3
4. $(P(m) \wedge \neg P(m))$	$\wedge$**I** $2,3$	$1^*,3$
5. $(\forall x P(x) \to (P(m) \wedge \neg P(m)))$	**CP** $1,4$	3
6. $\neg \forall x P(x)$	**RAA** 5	3

Example 6.2.4. $\{\forall y P(y), \forall z Q(z)\} \vdash (P(s) \wedge Q(s))$

Conclusion	Rule	Assumptions
1. $\forall y P(y)$	**A**	1
2. $P(s)$	**UE** 1	1
3. $\forall z Q(z)$	**A**	3
4. $Q(s)$	**UE** 3	3
5. $(P(s) \wedge Q(s))$	$\wedge$**I** $2,4$	$1,3$

Example 6.2.5. $\{\forall x(P(x) \vee Q(x)), \forall x(P(x) \to Q(x))\} \vdash Q(c)$.

Conclusion	Rule	Assumptions
1. $\forall x((P(x) \vee Q(x))$	**A**	1
2. $(P(c) \vee Q(c))$	**UE** 1	1
3. $\forall x(P(x) \to Q(x))$	**A**3	
4. $(P(c) \to Q(c))$	**UE** 3	3
5. $Q(c)$	**A***	5^*
6. $((Q(c) \to Q(c))$	**CP** $5,5$	
7. $Q(c)$	$\vee$**E** $2,4,6$	$1,3$

6.2.2 Rule of Inference: Universal Introduction

The next rule allows us to prove universal statements of the form $\forall x F(x)$, where $F(x)$ is a *wff*. For this rule, we need to define the notion of an *arbitrary constant*: if in some step of a proof we decide we want to prove a universal formula $\forall x F(x)$, then we say a constant is *arbitrary* does not appear in any assumptions and does not appear in $\forall x F(x)$.

Rule 12: **Universal Introduction**, abbreviated **UI**.
This rule says that if $F(x)$ is a *wff* and c is an arbitrary constant, then from $F(c)$ we can conclude $\forall x F(x)$. In symbols, if c is arbitrary then: $$F(c) \rhd \forall x F(x)$$

Strategy for using **UI**:

Suppose, at some step of a proof, we decide we want to prove a universal statement of the form $\forall x F(x)$.

1. Choose an arbitrary constant c. (I.e., a constant not in any previous step of the proof, not in any assumption, and not in $\forall x F(x)$)
2. Prove $F(c)$.
3. Apply **UI** to conclude $\forall x F(x)$.

Directions for using **UI** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $\forall x F(x)$ by applying rule **UI** to $F(c)$, which is previously obtained at step i of the proof, where c is an arbitrary constant. Then we add the following row:

Conclusion	Rule	Assumptions
n. $\forall x F(x)$	**UI** i	(copy assumptions from row i)

Example 6.2.6. $\{\forall x P(x), \forall x(P(x) \to Q(x))\} \vdash \forall x Q(x)$.

Conclusion	Rule	Assumptions
1. $\forall x P(x)$	**A**	1
2. $P(c)$	**UE** 1	1
3. $\forall x(P(x) \to Q(x))$	**A**	3
4. $(P(c) \to Q(c))$	**UE** 3	3
5. $Q(c)$	**MP** 4, 2	1, 3
6. $\forall x Q(x)$	**UI** 5	1, 3

When using **UI** it is *extremely important* to be sure that you are applying the rule to a line of the proof involving an arbitrary constant. For example suppose you are concluding $\forall x F(x)$ by applying **UI** to a line of your proof containing $F(c)$. To double-check that c is actually arbitrary, you should do the following:

- Check that c does not appear in $\forall x F(x)$.
- Check that c does not appear in any of the underlying assumptions of the step in which you obtain $F(c)$.

Example 6.2.7. $\{\forall x(P(x) \to Q(x))\} \vdash (\forall x P(x) \to \forall x Q(x))$.

Conclusion	Rule	Assumptions
1. $\forall x P(x)$	$\mathbf{A}^*$	1^*
2. $P(c)$	**UE** 1	1^*
3. $\forall x(P(x) \to Q(x))$	**A**	3
4. $(P(c) \to Q(c))$	**UE** 3	3
5. $Q(c)$	**MP** $4, 2$	$1^*, 3$
6. $\forall x Q(x)$	**UI** 5	$1^*, 3$
7. $(\forall x P(x) \to \forall x Q(x))$	**CP** $1, 6$	3

Example 6.2.8. $\{\forall x(P(x) \to Q(x))\} \vdash (\forall x \neg Q(x) \to \forall x \neg P(x))$.

Conclusion	Rule	Assumptions
1. $\forall x \neg Q(x)$	$\mathbf{A}^*$	1^*
2. $\neg Q(c)$	**UE** 1	1^*
3. $\forall x(P(x) \to Q(x))$	**A**	3
4. $(P(c) \to Q(c))$	**UE** 3	3
5. $\neg P(c)$	**MT** $4, 2$	$1^*, 3$
6. $\forall x \neg P(x)$	**UI** 5	$1^*, 3$
7. $(\forall x \neg Q(x) \to \forall x \neg P(x))$	**CP** $1, 6$	3

Example 6.2.9. $\{\forall x \forall y P(x, y)\} \vdash \forall x P(x, x)$.

Conclusion	Rule	Assumptions
1. $\forall x \forall y P(x, y)$	**A**	1
2. $\forall y P(c, y)$	**UE** 1	1
3. $P(c, c)$	**UE** 2	1
4. $\forall x P(x, x)$	**UI** 3	1

Example 6.2.10. $\{\forall x \forall y P(x, y)\} \vdash \forall y \forall x P(x, y)$.

Conclusion	Rule	Assumptions
1. $\forall x \forall y P(x, y)$	**A**	1
2. $\forall y P(c, y)$	**UE**	1
3. $P(c, d)$	**UE** 2	1
4. $\forall x P(x, d)$	**UI** 2	1
5. $\forall y \forall x P(x, y)$	**UI** 4	1

Example 6.2.11. Show that $\{\forall x(P(x) \vee Q(x)), \forall x \neg P(x)\} \vdash \forall x Q(x)$

Conclusion	Rule	Assumptions
1. $\forall x(P(x) \vee Q(x))$	**A**	1
2. $(P(c) \vee Q(c))$	**UE** 1	1
3. $\forall x \neg P(x)$	**A**	3
4. $\neg P(c)$	**UE** 3	3
5. $Q(c)$	**A***	5^*
6. $((Q(c) \to Q(c))$	**CP** $5, 5$	
7. $P(c)$	**A***	7^*
8. $\neg Q(c)$	**A***	8^*
9. $(P(c) \wedge \neg P(c))$	$\wedge$**I** $7, 4$	$3, 7^*$
10. $(\neg Q(c) \to (P(c) \wedge \neg P(c)))$	**CP** $8, 9$	$3, 7^*$
11. $\neg\neg Q(c)$	**RAA** 10	$3, 7^*$
12. $Q(c)$	**DN** 11	$3, 7^*$
13. $(P(c) \to Q(c))$	**CP** $7, 12$	3
14. $Q(c)$	$\vee$**E** $2, 13, 6$	$1, 3$
15. $\forall x Q(x)$	**UI** 14	$1, 3$

Example 6.2.12. $\{(\forall x P(x) \vee \forall x Q(x))\} \vdash \forall x(P(x) \vee Q(x))$.

Conclusion	Rule	Assumptions
1. $(\forall x P(x) \vee \forall x Q(x))$	**A**	1
2. $\forall x P(x)$	**A**$^*2^*$	
3. $P(c)$	**UE** 2	2^*
4. $(P(c) \vee Q(c))$	$\vee$**I** 3	2^*
5. $(\forall x P(x) \to (P(c) \vee Q(c)))$	**CP** $2, 4$	
6. $\forall x Q(x)$	**A***	6^*
7. $Q(c)$	**UE** 6	6^*
8. $(P(c) \vee Q(c))$	$\vee$**I** 7	6^*
9. $(\forall x Q(x) \to (P(c) \vee Q(c)))$	**CP** $6, 8$	
10. $(P(c) \vee Q(c))$	$\vee$**E** $1, 5, 9$	1
11. $\forall x(P(x) \vee Q(x))$	**UI** 10	1

Example 6.2.13. $\vdash (\forall x \forall y R(x, y) \to \forall x R(x, c))$

Conclusion	Rule	Assumptions
1. $\forall x \forall y R(x, y)$	**A***	1^*
2. $\forall y R(a, y)$	**UE** 1	1^*
3. $R(a, c)$	**UE** 2	1^*
4. $\forall x R(x, c)$	**UI** 3	1^*
5. $(\forall x \forall y R(x, y) \to \forall x R(x, c))$	**CP** $1, 4$	

Example 6.2.14. The following is a *false proof* of $\{P(c)\} \vdash \forall x P(x)$.

Conclusion	Rule	Assumptions
1. $P(c)$	**A**	1
2. $\forall x P(x)$	**UI** 1	1

The line we are applying **UI** to (line 1) contains an underlying assumption (also line 1), which involves the constant c being replaced by a quantifier. So c is not an arbitrary constant, and so we are *not* justified in applying **UI** on line 2.

6.3 Rules for existential quantifiers

6.3.1 Rule of Inference: Existential Introduction

Rule 13: **Existential Introduction**, abbreviated **EI**.

This rule says that from $F(c)$ we can conclude $\exists x F(x)$. In symbols:

$$F(c) \rhd \exists x F(x)$$

In this case c does not need to be an arbitrary constant.

Directions for using **EI** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $\exists x F(x)$ by applying rule **EI** to $F(c)$, which is previously obtained at step i of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
n. $\exists x F(x)$	**EI** i	(copy assumptions from row i)

Example 6.3.1. $\{\forall x P(x)\} \vdash \exists x P(x)$

Conclusion	Rule	Assumptions
1. $\forall x P(x)$	**A**	1
2. $P(c)$	**UE** 1	1
3. $\exists x P(x)$	**EI** 2	1

Example 6.3.2. $\{\forall x P(x,c)\} \vdash \forall x \exists y P(x,y)$.

Conclusion	Rule	Assumptions
1. $\forall x P(x,c)$	**A**	1
2. $P(d,c)$	**UE** 1	1
3. $\exists y P(d,y)$	**EI** 2	1
4. $\forall x \exists y P(x,y)$	**UI** 3	1

Example 6.3.3. $\{P(c,c)\} \vdash \exists x P(c,x)$

Conclusion	Rule	Assumptions
1. $P(c,c)$	**A**	1
2. $\exists x P(c,x)$	**EI** 1	1

Example 6.3.4. $\{P(c,c)\} \vdash \exists x P(x,x)$

Conclusion	Rule	Assumptions
1. $P(c,c)$	**A**	1
2. $\exists x P(x,x)$	**EI** 1	1

6.3.2 Rule of Inference: Existential Elimination

Rule 14: **Existential Elimination**, abbreviated **EE**.

If $F(x)$ is a *wff* H is a sentence, and c is arbitrary then:

$$\exists x F(x), (F(c) \to H) \rhd H$$

Here c is arbitrary if it does not appear in $\exists x F(x)$, H, or any assumptions used to obtain $\exists x F(x)$ or $(F(c) \to H)$.

Strategy for using **EE**:

Suppose, at some step of a proof, we have $\exists x F(x)$ and we want to prove H. We proceed as follows.

1. Pick an arbitrary constant c (not in $F(x)$, H, or any assumptions).
2. Temporarily assume $F(c)$ with $\mathbf{A}^*$.
3. Prove H.
4. Apply **CP** to conclude $(F(c) \to H)$.
5. Apply **EE** to conclude H.

Directions for using **EE** *in proofs*:

Suppose that, at step n of our proof, we want to conclude H by applying rule **EE** to $\exists x F(x)$ and $(F(c) \to H)$, which are previously obtained at steps i and j of the proof, where c is an arbitrary constant. Then we add the following row:

Conclusion	Rule	Assumptions
n. H	**EE** i, j	(copy assumptions from rows i and j)

Example 6.3.5. $\{\exists xP(x), \forall x(P(x) \to Q(x))\} \vdash \exists xQ(x)$.

Conclusion	Rule	Assumptions
1. $\exists xP(x)$	**A**	1
2. $P(c)$	**A***	2^*
3. $\forall x(P(x) \to Q(x))$	**A**	3
4. $(P(c) \to Q(c))$	**UE** 3	3
5. $Q(c)$	**MP** $4, 2$	$2^*, 3$
6. $\exists xQ(x)$	**EI** 5	$2^*, 3$
7. $(P(c) \to \exists xQ(x))$	**CP** $2, 6$	3
8. $\exists xQ(x)$	**EE** $1, 7$	$1, 3$

Once again, when using **EE** it is *extremely important* to be sure that you are using an arbitrary constant. For example suppose you are concluding G by applying **EE** to lines of the proof containing $\exists xF(x)$ and $(F(c) \to H)$. To double-check that c is actually arbitrary, you should do the following:

- Check that c does not appear in $\exists xF(x)$.
- Check that c does not appear in H.
- Check that c does not appear in any of the underlying assumptions of the steps in which you obtain $\exists xF(x)$ or $(F(c) \to H)$.

Example 6.3.6. $\{\exists xP(x), \forall x(\neg Q(x) \to \neg P(x))\} \vdash \exists xQ(x)$.

Conclusion	Rule	Assumptions
1. $\exists xP(x)$	**A**	1
2. $P(c)$	**A***	2
3. $\forall x(\neg Q(x) \to \neg P(x))$	**A**	3
4. $(\neg Q(c) \to \neg P(c))$	**UE** 3	3
5. $\neg\neg P(c)$	**DN** 2	2^*
6. $\neg\neg Q(c)$	**MT** $4, 5$	$2^*, 3$
7. $Q(c)$	**DN** 6	$2^*, 3$
8. $\exists xQ(x)$	**EI** 7	$2^*, 3$
9. $(P(c) \to \exists xQ(x))$	**CP** $2, 8$	3
10. $\exists xQ(x)$	**EE** $1, 9$	$1, 3$

Example 6.3.7. Show that $\{\forall x\exists yL(x, y)\} \vdash \exists xL(c, x)$

Conclusion	Rule	Assumptions
1. $\forall x\exists yL(x, y)$	**A**	1
2. $\exists yL(c, y)$	**UE** 1	1
3. $L(c, d)$	**A***	3^*
4. $\exists xL(c, x)$	**EI** 3	3^*
5. $(L(c, d) \to \exists xL(c, x))$	**CP** $3, 4$	
6. $\exists xL(c, x)$	**EE** $2, 5$	1

Example 6.3.8. $\{\exists x \forall y L(x, y)\} \vdash \exists x L(x, c)$

Conclusion	Rule	Assumptions
1. $\exists x \forall y L(x, y)$	**A**	1
2. $\forall y L(d, y)$	**A***	2^*
3. $L(d, c)$	**UE** 2	2^*
4. $\exists x L(x, c)$	**EI** 3	2^*
5. $(\forall y L(d, y) \to \exists x L(x, c))$	**CP** $2, 4$	
6. $\exists x L(x, c)$	**EE** $1, 5$	1

Recall that reversing the order of quantifiers changes the meaning of a sentence. Consider the example of a relation $L(x, y)$ for "x likes y". The sentence $\exists x \forall y L(x, y)$ says "There is someone who likes everyone". The sentence $\forall y \exists x L(x, y)$ says "Everyone is liked by someone". These two sentences say different things, but the first sentence implies the second sentence. Here is a formal proof.

Example 6.3.9. $\{\exists x \forall y L(x, y)\} \vdash \forall y \exists x L(x, y)$

Conclusion	Rule	Assumptions
1. $\exists x \forall y L(x, y)$	**A**	1
2. $\forall y L(c, y)$	**A***	2^*
3. $L(c, d)$	**UE** 2	2^*
4. $\exists x L(x, d)$	**EI** 3	2^*
5. $\forall y \exists x L(x, y)$	**UI** 4	2^*
6. $(\forall y L(c, y) \to \forall y \exists x L(x, y))$	**CP** $2, 5$	
7. $\forall y \exists x L(x, y)$	**EE** $1, 6$	1

On the other hand, $\forall y \exists x L(x, y)$ *does not imply* $\exists x \forall y L(x, y)$. But this example illustrates where we can go wrong by not being careful with arbitrary constants.

Example 6.3.10. The following is a FALSE PROOF of $\{\forall y \exists x L(x, y)\} \vdash \exists x \forall y L(x, y)$.

Conclusion	Rule	Assumptions
1. $\forall y \exists x L(x, y)$	**A**	1
2. $\exists x L(x, c)$	**UE** 1	1
3. $L(d, c)$	**A***	3^*
4. $\forall y L(d, y)$	**UI** 3	3^*
5. $\exists x \forall y L(x, y)$	**EI** 4	3^*
6. $(L(d, c) \to \exists x \forall y L(x, y))$	**CP** $3, 5$	
7. $\exists x \forall y L(x, y)$	**EE** $2, 6$	1

The line we are applying **UI** to (line 3) contains an underlying assumption (also line 3), which involves the constant c being replaced by a quantifier. So c is not an arbitrary constant, and so we are *not* justified in applying **UI** here. In general, we cannot apply **UI** to an assumption: The constant symbol will be appearing in an assumption, and therefore not be arbitrary.

In line 2 we chose c to be arbitrary, but then, when chose d in line 3, we are now in a situation where d depends on c. Therefore applying **UI** to line 3 is not valid, since c is no longer arbitrary in line 3.

Example 6.3.11. The following is a FALSE PROOF of

$$\{M(s), \exists x W(x)\} \vdash \exists x(M(x) \wedge W(x))$$

Conclusion	Rule	Assumptions
1. $M(s)$	**A**	1
2. $\exists x W(x)$	**A**	2
3. $W(s)$	$\mathbf{A}^*$	3^*
4. $(M(s) \wedge W(s))$	$\wedge$**I** $1,3$	$1,3^*$
5. $\exists x(M(x) \wedge W(x))$	**EI** 4	$1,3^*$
6. $(W(s) \to \exists x(M(x) \wedge W(x)))$	**CP** $3,5$	1
7. $\exists x(M(x) \wedge W(x))$	**EE** $2,6$	$1,2$

One of the lines we are applying **EE** to (namely, line 6) uses an underlying assumption (line 1) containing the "arbitrary" constant s (so s is not really arbitrary). We are *not* justified in applying **EE** in line 7.

We know $M(s)$ and $\exists x W(x)$, but s is not arbitrary (it appears in our assumptions) and so we do not know that $W(s)$ necessarily holds. This makes the temporary assumption in line 3 problematic.

6.4 Rules for equality

6.4.1 Rule of Inference: Identity Introduction

Rule 15: **Identity Introduction**, abbreviated =**I**.
For any constant c, we conclude $c = c$ from no assumptions. In symbols, $$\rhd\ c = c$$

Directions for using =**I** *in proofs*:

Suppose that, at step n of our proof, we want to conclude $c = c$ by applying rule =**I** where c is a constant. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ c = c$	=**I**	(no assumptions)

6.4.2 Rule of Inference: Identity Elimination

Rule 15: **Identity Elimination**, abbreviated =**E**.
For any constants c, d and any sentence F, suppose F' is the result of replacing some occurrences of c in F by d. Then: $$F, c = d \rhd F'$$

Directions for using =**I** *in proofs*:

Suppose that, at step n of our proof, we want to conclude F' by applying rule =**E** to F and $c = d$, which are previously obtained at steps i and j of the proof. Then we add the following row:

Conclusion	Rule	Assumptions
$n.\ F'$	=**E** i, j,	(copy assumptions from rows i and j)

Caution: The order of the equality $c = d$ matters. It is not good enough to have F and $d = c$, when we want to replace occurrences of c in F by d.

Example 6.4.1. $\{c = d\} \vdash d = c$

Conclusion	Rule	Assumptions
1. $c = d$	**A**	1
2. $c = c$	=**I**	
3. $d = c$	=**E** $2, 1$	1

In this application of =**E**, F is $c = c$ and we combine with $c = d$ to replace the first occurrence of c in F by d (so F' is $d = c$).

Example 6.4.2. $\{b = c, c = d\} \vdash b = d$

Conclusion	Rule	Assumptions
1. $b = c$	**A**	1
2. $c = d$	**A**	2
3. $b = d$	=**E** $1, 2$	$1, 2$

In this application of =**E**, F is $b = c$ and we combine with $c = d$ to replace the occurrence of c in F by d (so F' is $b = d$).

Example 6.4.3. $\vdash \forall x \forall y((P(x) \wedge x = y) \rightarrow P(y))$

Conclusion	Rule	Assumptions
1. $(P(c) \wedge c = d)$	**A***	1^*
2. $P(c)$	$\wedge$**E** 1	1^*
3. $c = d$	$\wedge$**E** 1	1^*
4. $P(d)$	=**E** $2, 3$	1^*
5. $((P(c) \wedge c = d) \rightarrow P(d))$	**CP** $1, 4$	
6. $\forall y((P(c) \wedge c = y) \rightarrow P(y))$	**UI** 5	
7. $\forall x \forall y((P(x) \wedge x = y) \rightarrow P(y))$	**UI** 6	

Example 6.4.4. Prove $\vdash \forall x\ x = x$.

Conclusion	Rule	Assumptions
1. $c = c$	$=$**I**	
2. $\forall x\ x = x$	**UI** 1	

Example 6.4.5. $\{F(c), d = c\} \vdash F(d)$

Conclusion	Rule	Assumptions
1. $F(c)$	**A**	1
2. $d = c$	**A**	2
3. $d = d$	$=$**I**	
4. $c = d$	$=$**E** $3, 2$	2
5. $F(d)$	$=$**E** $1, 4$	$1, 2$

Now that we know what to do with the equality symbol, we can give a more sensible translation and proof of Descartes' divisibility argument for substance dualism (considered in Example 6.1.4).

Example 6.4.6. Consider the following argument.

> If the mind and body are one and the same substance, they have all of the same features. The body is divisible. The mind is indivisible. Therefore, the mind and body must be different substances.

We choose the following symbols.

Relations	*Constants*
$D(x)$: "x is divisible"	m: mind
	b: body

In the argument, the sentence "If the mind and body are one and the same substance, they have all of the same features" is just a special case of saying that if two objects are the same then they satisfy the same unary relations. This is something that we can prove in general, from no assumptions. Therefore it does not need to be explicitly stated as an assumption in this particular argument. We only need to translate: "The body is divisible. The mind is indivisible. Therefore, the mind and body must be different substances." Here is the translation:

$$\{D(b), \neg D(m)\} \vdash b \neq m$$

Here is the proof (remember that $b \neq m$ is an abbreviation for $\neg(b = m)$):

Conclusion	Rule	Assumptions
1. $b = m$	$\mathbf{A}^*$	1^*
2. $D(b)$	**A**	2
3. $D(m)$	$=\mathbf{E}\ 2, 1$	$1^*, 2$
4. $\neg D(m)$	**A**	4
5. $(D(m) \wedge \neg D(m))$	$\wedge\mathbf{I}\ 3, 4$	$1^*, 2, 4$
6. $(b = m \rightarrow (D(m) \wedge \neg D(m)))$	**CP** $1, 5$	$2, 4$
7. $b \neq m$	**RAA** 6	$2, 4$

Chapter 7

Structures and Truth

In propositional logic, the only fundamental ingredients of our languages were propositional variables. Therefore, a formal definition of truth only relied on specific truth assignments of the variables. In predicate logic, our fundamental ingredients are atomic formulas, which make assertions about objects. More complicated sentences describe qualities about the set of objects as a whole. Therefore, in order to define whether these assertions are true or false, we first need to specific collection of objects we are studying.

7.1 Languages and structures

Recall Example 5.6.1, in which we considered variables ranging over the natural numbers $0, 1, 2, \ldots$. We let $L(x, y)$ be a relation symbol interpreted as "x is less than y". In this case $\exists x L(x, 0)$ is false. If we changed our structure to be all integers, positive and negative, then $\exists x L(x, 0)$ is true. Altogether, in order to define truth for formulas and sentences in predicate logic, it is vital to first specify what structure of objects is being studied. We will use the word *set* to mean a collection of objects. We use curly braces to denote sets, e.g. $\{\alpha, \beta, \gamma\}$ is notation for the set containing the objects α, β, and γ. We often give names to sets, e.g. $A = \{\alpha, \beta, \gamma\}$. We write $\alpha \in A$ to mean "α is an element of A".

Definition 7.1.1. Suppose we have a predicate language L, which contains relation symbols and constant symbols. A *structure* $\mathcal{A}$ for the language L is a *universe* and a set of *interpretations*:

- The *universe of* $\mathcal{A}$ is some non-empty set A that contains all of the objects we will consider, such as the positive integers or the students in a class.
- If P is an n-place relation symbol in L, then the *interpretation of* P *in* $\mathcal{A}$, denoted by P^*, is a set of lists of elements of the universe that each has n elements. These lists will be all of the inputs that will make P "true." (Here order matters in general, although it might not turn out to matter in specific examples.)

- If c is a constant symbol in L, then the *interpretation of c in $\mathcal{A}$*, denoted by c^*, is a designated element of the universe. For example, in our famous syllogism, s was a constant symbol and s^* was Socrates.

This definition is rather abstract, so we should consider some examples. We will use lowercase Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ for elements of the universe that are not named by constants.

Example 7.1.2. Let $L = \{P(x), Q(x, y), c\}$.

1. $\mathcal{A} = (A, P^*, Q^*, c^*)$ where
   - A is the set of people in this class.
   - P^* is the set of people who wear glasses.
   - Q^* is the set of pairs of people (α, β) such that α is older than β.
   - c^* is the instructor.
2. $\mathcal{A} = (A, P^*, Q^*, c^*)$ where
   - A is the set of all integers.
   - P^* is the set of all even integers.
   - Q^* is the set of pairs of integers (α, β) such that α is divisible by β. (e.g. $(8, 4)$, $(25, 5)$ are in Q^*; but $(4, 8)$, $(25, 7)$ are not in Q^*)
   - c^* is 0.
3. $\mathcal{A} = (A, P^*, Q^*, c^*)$ where
   - A is the set $\{\alpha, \beta, \gamma, \delta\}$.
   - P^* is the set $\{\alpha, \delta\}$.
   - Q^* is the set of pairs $\{(\beta, \gamma), (\gamma, \beta), (\delta, \alpha)\}$.
   - c^* is γ.

Example 7.1.3. Suppose $L = \{P(x), N(x, y), B(x, y, z), w, s\}$. Then a structure is a sequence $\mathcal{A} = (A, P^*, N^*, B^*, w^*, s^*)$ where A is a nonempty set, P^* is a set of elements of A, N^* is a set of pairs (α, β) of elements of A, N^* is a set of triples (α, β, γ) of elements of A, and w^*, s^* are fixed elements of A.

For a specific example, let A be the set of cities in the US. Let P^* be the set of cities with a population of at least 500,000. Let N^* be the set of pairs (α, β) of cities such that the distance from α to β is less than 500 miles. Let B^* be the set of triples (α, β, γ) of cities such that γ is between α and β. Let w^* be Washington D.C., and let s^* be Seattle. New York is in P^*, but South Bend is not in P^*. The element s^* is in P^*. The pair (South Bend, Chicago) is in N^*, while (South Bend, New York) is not. The pair (w^*, s^*) is not in P^*. The triple (South Bend, Portage, Chicago) is in B^*, while (Portage, Chicago, South Bend) is not.

7.2 Satisfaction and truth

For a sentence F, such as $\forall x P(x)$ or $\exists x P(x)$, we will ask whether F is *true* or *false* in a given structure. For the structure in Example 7.1.2(1), the sentence $\exists x P(x)$ is true—there are students who wear glasses. The sentence $\forall x P(x)$ is false in this structure—it is not the case that all students wear glasses.

For a formula $F(x_1, \ldots, x_n)$, we will ask whether F is *satisfied* in a given structure when we substitute particular elements of the universe for the free variables $x_1, \ldots, x_n$. For the structure in Example 7.1.2(2), the formula $(P(x) \wedge \neg P(y))$ is satisfied when we substitute 2 for x, and 3 for y. It is not satisfied when we substitute 3 for x, and 2 for y.

For the structure in Example 7.1.2(3), the formula $P(x)$ is satisfied when we substitute α for x, but not when we substitute β. The sentence $\exists x P(x)$ is true in the structure, while the sentence $\forall x P(x)$ is false in the structure.

For a general structure $\mathcal{A}$ in a language L, and a *wff* $F(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$, we want to similarly define what it means for $F(x_1, \ldots, x_n)$ to be *true* when we plug in elements $\alpha_1, \ldots, \alpha_n$ in A. We will denote this by $\mathcal{A} \models F(\alpha_1, \ldots, \alpha_n)$. In the following definition, when we write $F(x_1, \ldots, x_n)$ to mean that the free variables from F are from *among* $x_1, \ldots, x_n$ (but perhaps F doesn't actually use all of them).

Definition 7.2.1. Fix a structure $\mathcal{A}$ in a language L. We define satisfaction of formulas in $\mathcal{A}$ as follows.

1. If P is an n-place relation symbol and $\alpha_1, \ldots, \alpha_n \in A$ then

   $\mathcal{A} \models P(\alpha_1, \ldots, \alpha_n)$ if and only if the sequence $(\alpha_1, \ldots, \alpha_n)$ is in the set P^*.

2. If $F(x, y)$ is the formula $x = y$ and $\alpha, \beta \in A$, then

   $$\mathcal{A} \models F(\alpha, \beta) \text{ if and only if } \alpha = \beta.$$

3. Given a *wff* $F(x_1, \ldots, x_n)$, a constant symbol c, and $\alpha_2, \ldots, \alpha_n \in A$,

   $$\mathcal{A} \models F(c, \alpha_2, \ldots, \alpha_n) \text{ if and only if } \mathcal{A} \models F(c^*, \alpha_2, \ldots, \alpha_n).$$

4. Given a *wff* $F(x_1, \ldots, x_n)$ and $\alpha_1, \ldots, \alpha_n \in A$,

   $$\mathcal{A} \models \neg F(\alpha_1, \ldots, \alpha_n) \text{ if and only if } \mathcal{A} \not\models F(\alpha_1, \ldots, \alpha_n).$$

5. Given *wff*s $F(x_1, \ldots, x_n)$, $G(x_1, \ldots, x_n)$ and $\alpha_1, \ldots, \alpha_n \in A$,

   $\mathcal{A} \models (F(\alpha_1, \ldots, \alpha_n) \wedge G(\alpha_1, \ldots, \alpha_n))$ if and only if
   $\mathcal{A} \models F(\alpha_1, \ldots, \alpha_n)$ and $\mathcal{A} \models G(\alpha_1, \ldots, \alpha_n)$.
   $\mathcal{A} \models (F(\alpha_1, \ldots, \alpha_n) \vee G(\alpha_1, \ldots, \alpha_n))$ if and only if
   $\mathcal{A} \models F(\alpha_1, \ldots, \alpha_n)$ or $\mathcal{A} \models G(\alpha_1, \ldots, \alpha_n)$.
   $\mathcal{A} \models (F(\alpha_1, \ldots, \alpha_n) \to G(\alpha_1, \ldots, \alpha_n))$ if and only if
   "if $\mathcal{A} \models F(\alpha_1, \ldots, \alpha_n)$ then $\mathcal{A} \models G(\alpha_1, \ldots, \alpha_n)$".

6. Given a *wff* $F(x_1, \ldots, x_n, z)$ and $\alpha_1, \ldots, \alpha_n \in A$,

$$\mathcal{A} \models \forall z F(\alpha_1, \ldots, \alpha_n, z) \text{ if and only if } \mathcal{A} \models F(\alpha_1, \ldots, \alpha_n, \beta) \text{ for all } \beta \in A.$$

$$\mathcal{A} \models \exists z F(\alpha_1, \ldots, \alpha_n, z) \text{ if and only if } \mathcal{A} \models F(\alpha_1, \ldots, \alpha_n, \beta) \text{ for some } \beta \in A.$$

7. If F is a sentence and $\mathcal{A} \models F$ then we say F **is true in** $\mathcal{A}$.

Example 7.2.2. Consider the structure $\mathcal{A} = (A, P^*, Q^*)$, where $A = \{\alpha, \beta\}$, $P^* = \{\alpha\}$, $Q^* = \{\beta\}$.

1. $\forall x(P(x) \vee Q(x))$ is true in $\mathcal{A}$.

   We need to check whether each element of A satisfies $(P(x) \vee Q(x))$. For α, we have $\mathcal{A} \models P(\alpha)$ and so $\mathcal{A} \models (P(\alpha) \vee Q(\alpha))$. For β, we have $\mathcal{A} \models Q(\beta)$ and so $\mathcal{A} \models (P(\beta) \vee Q(\beta))$. We have checked all of the elements, so $\forall x(P(x) \vee Q(x))$ is true in $\mathcal{A}$.

2. $\forall x P(x) \vee \forall x Q(x)$ is not true in $\mathcal{A}$.

   We need to see that neither disjunct is true. We start with $\forall x P(x)$. Note $\mathcal{A} \models P(\alpha)$, but $\mathcal{A} \not\models P(\beta)$. Therefore, $\mathcal{A} \not\models \forall x P(x)$. Similarly, $\mathcal{A} \not\models \forall x Q(x)$. Neither disjunct is true, and so the sentence $\forall x P(x) \vee \forall x Q(x)$ is false in $\mathcal{A}$.

3. $\exists x(P(x) \wedge Q(x))$ is not true in $\mathcal{A}$.

   We check that no element of A satisfies $(P(x) \wedge Q(x))$. For α, we have $\mathcal{A} \not\models Q(\alpha)$ so $\mathcal{A} \not\models (P(\alpha) \wedge Q(\alpha))$. For β, we have $\mathcal{A} \not\models P(\beta)$ so $\mathcal{A} \not\models (P(\beta) \wedge Q(\beta))$. Therefore, $\exists x(P(x) \wedge Q(x))$ is false in $\mathcal{A}$.

4. $\forall x \forall y((P(x) \wedge P(y)) \rightarrow x = y)$ is true in $\mathcal{A}$.

   Since there are two variables x and y with universal quantifiers, we have to check all possibilities for substituting x by α or β, and substituting y by α or β.

   (i) If $x = \alpha$ and $y = \alpha$, we have $((P(\alpha) \wedge P(\alpha)) \rightarrow \alpha = \alpha)$. Since $\mathcal{A} \models (P(\alpha) \wedge P(\alpha))$ and $\mathcal{A} \models \alpha = \alpha$, the implication is true in $\mathcal{A}$.

   (ii) If $x = \beta$ and $y = \beta$, we have $((P(\beta) \wedge P(\beta)) \rightarrow \beta = \beta)$. Since $\mathcal{A} \not\models (P(\beta) \wedge P(\beta))$, the implication is true in $\mathcal{A}$.

   (iii) If $x = \alpha$ and $y = \beta$, we have $((P(\alpha) \wedge P(\beta)) \rightarrow \alpha = \beta)$. Since $\mathcal{A} \not\models (P(\alpha) \wedge P(\beta))$, the implication is true in $\mathcal{A}$.

   (iv) If $x = \beta$ and $y = \alpha$, we have $((P(\beta) \wedge P(\alpha)) \rightarrow \beta = \alpha)$. Since $\mathcal{A} \not\models (P(\beta) \wedge P(\alpha))$, the implication is true in $\mathcal{A}$.

5. $\forall x \forall y((P(x) \wedge Q(y)) \rightarrow x = y)$ is not true in $\mathcal{A}$.

   Substituting α for x and β for y, we have $\mathcal{A} \models (P(\alpha) \wedge Q(\beta))$, but $\mathcal{A} \not\models \alpha = \beta$. So $\mathcal{A} \not\models ((P(\alpha) \wedge Q(\beta)) \rightarrow \alpha = \beta)$.

Example 7.2.3. Consider a language with a 3-place relation symbol $P(x, y, z)$. In each of the following examples we will interpret this relation as $x + y = z$. Consider the following sentences.

(i) $\forall x \forall y \exists z P(x, y, z)$

(ii) $\forall x \forall z \exists y P(x, y, z)$

(iii) $\forall z \exists x P(x, x, z)$

(iv) $\exists x \forall z P(x, z, z)$

Let $\mathbb{N}$ be the set of *natural numbers* $\{0, 1, 2, \ldots\}$. Let $\mathbb{Z}$ be the set of *integers* $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Let $\mathbb{Q}$ be the set of *rational numbers* (i.e. fractions $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$).

Statement (i) is true in $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. For any a, b in one of these sets, $a + b$ is in the same set and we have $P(a, b, a + b)$.

Statement (ii) is false in $\mathbb{N}$. For example $\mathbb{N} \not\models \exists y P(2, y, 1)$ since -1 is not in $\mathbb{N}$. On the other hand, statement (ii) is true in the other two structures. For any a, c in $\mathbb{Z}$ or $\mathbb{Q}$, we have $P(a, c - a, c)$.

Statement (iii) is false in $\mathbb{N}$ and $\mathbb{Z}$. In both cases these structures to not satisfy $\exists x P(x, x, 3)$. However $\mathbb{Q}$ satisfies statement (iii) since, for any c, we have $P(\frac{c}{2}, \frac{c}{2}, c)$.

Statement (iv) is true in all three structures. For any c in one of these sets, we have $P(0, c, c)$. Note if we let $\mathbb{Z}^+$ be the set of *positive integers* $\{1, 2, 3, \ldots\}$, then this structure does not satisfy statement (iv).

7.2.1 Tautologies

In propositional logic, a tautology was a formula true in every row of its truth table. In predicate logic, we have structures instead of rows of truth tables. But we can define tautologies in an analogous way.

Definition 7.2.4. For a sentence F, we say that F is a *tautology*, written $\models F$, if F is true in all structures for the given language.

Example 7.2.5. The following sentences are tautologies. This follows easily from the definition of satisfaction.

1. $\forall x(x = x)$,
2. $\forall x(P(x) \vee \neg P(x))$,
3. $\forall x \forall y(x = y \rightarrow (P(x) \leftrightarrow P(y)))$.

Example 7.2.6. The sentence $\exists x(P(x) \wedge Q(x))$ is not a tautology. To see this, we give an example of a structure in which the sentence is false. Let $A = \{\alpha, \beta\}$, $P^* = \{\alpha\}$ and let $Q^* = \{\beta\}$.

Definition 7.2.7. Formulas $F(x_1, \ldots, x_n)$ and $G(x_1, \ldots, x_n)$ are *logically equivalent* if the sentence

$$\mathcal{A} \models \forall x_1 \ldots \forall x_n (F(x_1, \ldots, x_n) \leftrightarrow G(x_1, \ldots, x_n))$$

is a tautology. We write $F(x_1, \ldots, x_n) \equiv G(x_1, \ldots, x_n)$ to denote that these formulas are logically equivalent. Note that if F and G are sentences then $F \equiv G$ if and only if, the sentence $(F \leftrightarrow G)$ is a tautology.

We now give some useful facts about distributing negation signs in formulas. In each fact below, the formulas $F, G, \ldots$ may have free variables (which we are not explicitly listing).

Useful facts about negation
1. $\neg(F \wedge G) \equiv (\neg F \vee \neg G)$ 2. $\neg(F \vee G) \equiv (\neg F \wedge \neg G)$ 3. $\neg(F \rightarrow G) \equiv (F \wedge \neg G)$ 4. $\neg \forall x F(x) \equiv \exists x \neg F(x)$ 5. $\neg \exists x F(x) \equiv \forall x \neg F(x)$

Let us see how to use these facts. For each of the following formulas, we find a logically equivalent one with the negations brought inside next to the atomic formulas.

Example 7.2.8.

$$\begin{aligned} \neg \exists x (P(x) \rightarrow Q(x)) &\equiv \forall x \neg (P(x) \rightarrow Q(x)) \\ &\equiv \forall x (P(x) \wedge \neg Q(x)) \end{aligned}$$

$$\begin{aligned} \neg \forall x (P(x) \vee Q(x)) &\equiv \exists x \neg (P(x) \vee Q(x)) \\ &\equiv \exists x (\neg P(x) \wedge \neg Q(x)) \end{aligned}$$

$$\begin{aligned} \neg \forall x \exists y \forall z B(x, y, z) &\equiv \exists x \neg \exists y \forall z B(x, y, z) \\ &\equiv \exists x \forall y \neg \forall z B(x, y, z) \\ &\equiv \exists x \forall y \exists z \neg B(x, y, z) \end{aligned}$$

Chapter 8

Soundness and Completeness

8.1 Validity of arguments in predicate logic

In propositional logic, we wrote $\Gamma \models C$ to mean that C is true in any row of a truth table where all *wff*s in Γ are true. If we again use structures to take the place of rows in truth tables, we obtain the following definition.

Definition 8.1.1. Let Γ be a set of sentences and C a sentence in a predicate language L. We say C **is a logical consequence of** Γ (or Γ **implies** C) , written $\Gamma \models C$, if C is true in any L-structure in which all sentences in Γ are true.

As in the case of propositional logic, there are soundness and completeness theorems. Let Γ be a set of assumptions and C a conclusion.

Soundness Theorem: If $\Gamma \vdash C$ then $\Gamma \models C$.

Completeness Theorem: If $\Gamma \models C$ then $\Gamma \vdash C$.

As in propositional logic, we can use these theorems to show that certain arguments are invalid, and thus no proof can be found. In propositional logic, one could conceivably forego the proof system entirely when analyzing arguments, since calculations with truth tables are completely effective and always provide the answer. In predicate logic, we have no truth tables and thus is becomes more difficult to determine *directly* that $\Gamma \models C$. So, to determine the validity of an argument, we use two approaches simultaneously.

Approach I. If we think that $\Gamma \models C$ then we look for a proof.

Approach II. If we think that $\Gamma \not\models C$, then we look for a *counterexample*. This is a structure $\mathcal{A}$ in which the assumptions in Γ are all true and the conclusion C is false.

When we try to analyze a particular argument, we know that one of the two approaches will work, but we don't know which one. We have to be prepared to switch from one to the other. We start by looking for a proof (or a counterexample), and if we don't find it, then we switch and look for a counterexample (or a proof). If we still have trouble, we switch again.

8.2 Analysis of arguments

We will analyze some arguments in predicate logic. To show that an argument is valid, we give a proof. To show that an argument is invalid, we give a "counterexample", where this is a structure in which the assumptions are all true and the conclusion is false. In any case, we must first decide whether we think the argument is valid or invalid, and proceed appropriately.

Example 8.2.1. Determine whether $\{(\exists x P(x) \wedge \exists x Q(x))\} \models \exists x(P(x) \wedge Q(x))$.

After translating the underlying meaning of these sentences, the argument sounds invalid. So we look for a counterexample. We need a structure $\mathcal{A} = (A, P^*, Q^*)$ such that

(*i*) $\mathcal{A} \models (\exists x P(x) \wedge \exists x Q(x))$

(*ii*) $\mathcal{A} \not\models \exists x(P(x) \wedge Q(x))$

In general, there could be many different structures that work as a counterexample. We only need one. Let $A = \{\alpha, \beta\}$, $P^* = \{\alpha\}$, and $Q^* = \{\beta\}$.

To verify (*i*), we have $\mathcal{A} \models P(\alpha)$ and so $\mathcal{A} \models \exists x P(x)$. We have $\mathcal{A} \models Q(\beta)$ and so $\mathcal{A} \models \exists x Q(x)$. Altogether, $\mathcal{A} \models (\exists x P(x) \wedge \exists x Q(x))$.

To verify (*ii*), we must check that no substitution for x of an element of A satisfies the resulting formula. We have $\mathcal{A} \not\models P(\beta)$ and so $\mathcal{A} \not\models (P(\beta) \wedge Q(\beta))$. We have $\mathcal{A} \not\models Q(\alpha)$ and so $\mathcal{A} \not\models (P(\alpha) \wedge Q(\alpha))$. Since α and β are all of the elements in A, this verifies $\mathcal{A} \not\models \exists x(P(x) \wedge Q(x))$.

Example 8.2.2. Determine whether $\{\forall x(P(x) \rightarrow Q(x))\} \models (\exists x P(x) \rightarrow \exists x Q(x))$.

This argument sounds valid, so we look for a proof.

Conclusion	Rule	Assumptions
1. $\exists x P(x)$	**A***	1^*
2. $P(c)$	**A***	2^*
3. $\forall x(P(x) \rightarrow Q(x))$	**A**	3
4. $(P(c) \rightarrow Q(c))$	**UE** 3	3
5. $Q(c)$	**MP** $4, 2$	$2^*, 3$
6. $\exists x Q(x)$	**EI** 5	$2^*, 3$
7. $(P(c) \rightarrow \exists x Q(x))$	**CP** $2, 6$	3
8. $\exists x Q(x)$	**EE** $1, 7$	$1^*, 3$
9. $(\exists x P(x) \rightarrow \exists x Q(x))$	**CP** $1, 8$	3

Example 8.2.3. Determine whether $\{\forall x(\neg P(x) \to Q(x)), \exists x P(x)\} \models \exists x \neg Q(x)$.

We construct a counterexample $\mathcal{A} = (A, P^*, Q^*)$ such that

(i) $\mathcal{A} \models \forall x(\neg P(x) \to Q(x))$

(ii) $\mathcal{A} \models \exists x(P(x))$

(iii) $\mathcal{A} \not\models \exists x \neg Q(x)$

To satisfy (ii) we need at least one element in P^*; to satisfy (iii) we need all elements to be in Q^*; to satisfy (i) we need every element not in P^* to be in Q^*. Here is an example. Let $A = \{\alpha\}$, $P^* = \{\alpha\}$, and $Q^* = \{\alpha\}$. You may think of other counterexamples.

Example 8.2.4. Determine whether

$$\{\forall x(P(x) \vee Q(x)), \exists x \neg P(x)\} \models \exists x Q(x).$$

Conclusion	Rule	Assumptions
1. $\forall x(P(x) \vee Q(x))$	**A**	1
2. $\exists x \neg P(x)$	**A**	2
3. $\neg P(c)$	**A***	3^*
4. $(P(c) \vee Q(c))$	**UE** 1	1
5. $P(c)$	**A***	5^*
6. $\neg Q(c)$	**A***	6^*
7. $(P(c) \wedge \neg P(c))$	$\wedge$**I** $5, 3$	$3^*, 5^*$
8. $(\neg Q(c) \to (P(c) \wedge \neg P(c)))$	**CP** $6, 7$	$3^*, 5^*$
9. $\neg\neg Q(c)$	**RAA** 8	$3^*, 5^*$
10. $Q(c)$	**DN** 9	$3^*, 5^*$
11. $(P(c) \to Q(c))$	**CP** $5, 10$	3^*
12. $Q(c)$	**A***	12^*
13. $(Q(c) \to Q(c))$	**CP** $12, 12$	
14. $Q(c)$	$\vee$**E** $4, 11, 13$	$1, 3^*$
15. $\exists x Q(x)$	**EI** 14	$1, 3^*$
16. $(\neg P(c) \to \exists x Q(x))$	**CP** $3, 15$	1
17. $\exists x Q(x)$	**EE** $2, 16,$	$1, 2$

Example 8.2.5. Determine whether

$$\{\forall x(P(x) \vee Q(x)), \exists x \neg P(x), \exists x \neg Q(x))\} \models \forall x(P(x) \to \neg Q(x)).$$

We give a counterexample $\mathcal{A} = (A, P^*, Q^*)$. We need every element of A to be in P^* or Q^*. We need at least one element not in P^*, and at least one element not in Q^*. Finally, want $\forall x(P(x) \to \neg Q(x))$ to be false, i.e. we want $\mathcal{A} \models \neg\forall x(P(x) \to \neg Q(x))$. To parse this it could be useful to think of the way negations distribute.

$$\neg\forall x(P(x) \to \neg Q(x)) \equiv \exists x \neg(P(x) \to \neg Q(x)) \equiv \exists x(P(x) \wedge Q(x))$$

So we need an element of A to be in both P^* and Q^*. All of this is accomplished if we set $A = \{\alpha, \beta, \gamma\}$, $P^* = \{\alpha, \beta\}$, $Q^* = \{\beta, \gamma\}$.

8.3 Analysis of arguments in English

In this section, we will combine everything we have learned to give a full analysis of arguments written in English. In particular, given an English argument we will do the following.

1. Choose an appropriate predicate language in which the argument can most accurately be translated to predicate logic. Give the interpretations of the symbols in the language you have chosen.
2. Translate the argument to predicate logic using the language chosen above. Clearly distinguish the assumptions from the conclusion.
3. Determine whether the argument is valid. If it is valid, give a formal proof in the proof system of predicate logic. If it is not valid, give a counterexample, i.e. a structure in which the assumptions of the argument (as *wff*s in predicate logic) are true and the conclusion (as a *wff* in predicate logic) is false.

Example 8.3.1. Analyze the following argument.

> No politicians are fair. Some judges are fair. Therefore, some judges are not politicians.

Language

$P(x)$: x is a politician.

$J(x)$: x is a judge.

$F(x)$: x is fair.

Translation

$$\{\forall x(P(x) \to \neg F(x)), \exists x(J(x) \land F(x))\} \models \exists x(J(x) \land \neg P(x))$$

Analysis: The argument sounds valid, so we attempt a proof.

Conclusion	Rule	Assumptions
1. $\exists x(J(x) \land F(x))$	**A**	1
2. $(J(c) \land F(c))$	$\mathbf{A}^*$	2^*
3. $J(c)$	$\land$**E** 2	2^*
4. $F(c)$	$\land$**E** 2	2^*
5. $\forall x(P(x) \to \neg F(x))$	**A**	5
6. $(P(c) \to \neg F(c))$	**UE** 5	5
7. $\neg\neg F(c)$	**DN** 4	2^*
8. $\neg P(c)$	**MT** $6, 7$	$2^*, 5$
9. $(J(c) \land \neg P(c))$	$\land$**I** $3, 8$	$2^*, 5$
10. $\exists x(J(x) \land \neg P(x))$	**EI** 9	$2^*, 5$
11. $((J(c) \land F(c)) \to \exists x(J(x) \land \neg P(x)))$	**CP** $2, 10$	5
12. $\exists x(J(x) \land \neg P(x))$	**EE** $1, 11$	$1, 5$

Remark 8.3.2. In the last example, you may have decided to translate "no politicians are fair" as: $\neg\exists x(P(x) \wedge F(x))$. This is of course equivalent to the translation above, but would result in a slightly longer proof.

Conclusion	Rule	Assumptions
1. $\exists x(J(x) \wedge F(x))$	**A**	1
2. $(J(c) \wedge F(c))$	$\mathbf{A}^*$	2^*
3. $J(c)$	$\wedge$**E** 2	2^*
4. $F(c)$	$\wedge$**E** 2	2^*
5. $P(c)$	$\mathbf{A}^*$	5^*
6. $(P(c) \wedge F(c))$	$\wedge$**I** $5, 4$	$2^*, 5^*$
7. $\exists x(P(x) \wedge F(x))$	**EI** 6	$2^*, 5^*$
8. $\neg\exists x(P(x) \wedge F(x))$	**A**	8
9. $(\exists x(P(x) \wedge F(x)) \wedge \neg\exists x(P(x) \wedge F(x))$	$\wedge$**I** $7, 8$	$2^*, 5^*, 8$
10. $(P(c) \rightarrow (\exists x(P(x) \wedge F(x)) \wedge \neg\exists x(P(x) \wedge F(x)))$	**CP** $5, 9$	$2^*, 8$
11. $\neg P(c)$	**RAA** 11	$2^*, 8$
12. $(J(c) \wedge \neg P(c))$	$\wedge$**I** $3, 11$	$2^*, 8$
13. $\exists x(J(x) \wedge \neg P(x))$	**EI** 12	$2^*, 8$
14. $((J(c) \wedge F(c)) \rightarrow \exists x(J(x) \wedge \neg P(x)))$	**CP** $2, 13$	8
15. $\exists x(J(x) \wedge \neg P(x))$	**EE** $1, 14$	$1, 8$

Example 8.3.3. Analyze the following argument.

> Some judges are fair. All judges are politicians. Therefore some politicians are fair.

Language

$P(x)$: x is a politician.

$J(x)$: x is a judge.

$F(x)$: x is fair.

Translation

$$\{\exists x(J(x) \wedge F(x)), \forall x(J(x) \rightarrow P(x))\} \models \exists x(P(x) \wedge F(x))$$

Analysis: The argument sounds valid, so we attempt a proof.

Conclusion	Rule	Assumptions
1. $\exists x(J(x) \wedge F(x))$	**A**	1
2. $(J(c) \wedge F(c))$	$\mathbf{A}^*$	2^*
3. $J(c)$	$\wedge$**E** 2	2^*
4. $F(c)$	$\wedge$**E** 2	2^*
5. $\forall x(J(x) \rightarrow P(x))$	**A**	5
6. $(J(c) \rightarrow P(c))$	**UE** 5	5
7. $P(c)$	**MP** $6, 3$	$2^*, 5$
8. $(P(c) \wedge F(c))$	$\wedge$**I** $7, 4$	$2^*, 5$
9. $\exists x(P(x) \wedge F(x))$	**EI** 8	$2^*, 5$
10. $((J(c) \wedge F(c)) \rightarrow \exists x(P(x) \wedge F(x)))$	**CP** $2, 9$	5
11. $\exists x(P(x) \wedge F(x))$	**EE** $1, 10$	$1, 5$

Example 8.3.4. Analyze the following argument.

> No Angry Birds players play chess. Some mathematicians play chess and some do not. Therefore, some mathematicians play neither chess nor Angry Birds.

Language

$B(x)$: x plays Angry Birds.

$C(x)$: x plays chess.

$M(x)$: x is a mathematician.

Translation

$$\{\neg\exists x(B(x) \wedge C(x)), (\exists x(M(x) \wedge C(x)) \wedge \exists x(M(x) \wedge \neg C(x)))\} \\ \models \exists x(M(x) \wedge \neg B(x) \wedge \neg C(x))$$

Analysis: The argument sounds invalid, so we construct a counterexample.
We want a structure $\mathcal{A} = (A, B^*, C^*, M^*)$ such that:

(*i*) $\mathcal{A} \models \neg\exists x(B(x) \wedge C(x))$ (i.e. nothing is in both B^* and C^*).

(*ii*) $\mathcal{A} \models (\exists x(M(x) \wedge C(x)) \wedge \exists x(M(x) \wedge \neg C(x)))$ (i.e. there is something in M^* and C^*, and something in M^* and not in C^*).

(*iii*) $\mathcal{A} \not\models \exists x(M(x) \wedge \neg B(x) \wedge \neg C(x))$, which is equivalent to:
$\mathcal{A} \models \forall x(\neg M(x) \vee B(x) \vee C(x))$ (i.e. everything is either not in M^* or in B^* or in C^*).

As usual, the are many structures satisfying these conditions. Here is one possibility:

$A = \{\alpha, \beta\}$

$B^* = \{\beta\}$

$C^* = \{\alpha\}$

$M^* = \{\alpha, \beta\}$

Remark 8.3.5. Note that, in the last example, the structure chosen for a counterexample has nothing to do with Angry Birds, chess, or mathematicians. This is because, as usual, we are not interested in the meaning of the specific ingredients of the argument, nor with whether the assumptions are actually true. We are only interested in analyzing the underlying structure of the argument to see if the argument is logically sound. Therefore, an abstract structure like the one we constructed is sufficient to demonstrate the fact that in general this argument is not valid.

On the other hand, you should be able to use your counterexample to describe a situation specific to the actual argument, which illustrates why it is invalid. For example, here we would imagine a situation where there are only two people in the universe, both are mathematicians, one plays Angry Birds but not chess, and the other plays chess but not Angry Birds.

Example 8.3.6. Analyze the following argument.

> No riverboat pilots are writers. Samuel Clemens was a riverboat pilot. Mark Twain was a writer. Therefore, Samuel Clemens was not Mark Twain.

Language

$R(x)$: x is a riverboat pilot.	c: Samuel Clemens
$W(x)$: x is a writer.	t: Mark Twain

Translation

$$\{\neg\exists x(R(x) \wedge W(x)), R(c), W(t)\} \models c \neq t$$

Analysis: The argument sounds valid, so we attempt a proof.

Conclusion	Rule	Assumptions
1. $c = t$	**A***	1^*
2. $R(c)$	**A**	2
3. $R(t)$	=**E** $2, 1$	$1^*, 2$
4. $W(t)$	**A**	4
5. $(R(t) \wedge W(t))$	$\wedge$**I** $3, 4$	$1^*, 2, 4$
6. $\exists x(R(x) \wedge W(x))$	**EI** 5	$1^*, 2, 4$
7. $\neg\exists x(R(x) \wedge W(x))$	**A**	7
8. $(\exists x(R(x) \wedge W(x)) \wedge \neg\exists x(R(x) \wedge W(x)))$	$\wedge$**I** $6, 7$	$1^*, 2, 4, 7$
9. $(c = t \to (\exists x(R(x) \wedge W(x)) \wedge \neg\exists x(R(x) \wedge W(x))))$	**CP** $1, 8$	$2, 4, 7$
10. $c \neq t$	**RAA** 9	$2, 4, 7$

Remark 8.3.7. In the last example, you may have instead translated the sentence "no riverboat pilots are writers" as: $\forall x(R(x) \to \neg W(x))$. Once again, this

is equivalent to the translation above, but results in a slightly different proof.

Conclusion	Rule	Assumptions
1. $c = t$	**A***	1^*
2. $R(c)$	**A**	2
3. $R(t)$	$=$**E** $2, 1$	$1^*, 2$
4. $\forall x(R(x) \to \neg W(x))$	**A**	4
5. $(R(t) \to \neg W(t))$	**UE** 4	4
6. $\neg W(t)$	**MP** $5, 3$	$1^*, 2, 4$
7. $W(t)$	**A**	7
8.$(W(t) \wedge \neg W(t))$	$\wedge$**I** $7, 6$	$1^*, 2, 4, 7$
9. $(c = t \to (W(t) \wedge \neg W(t)))$	**CP** $1, 8$	$2, 4, 7$
10. $c \neq t$	**RAA** 9	$2, 4, 7$

Example 8.3.8. Analyze the following argument.

> Matt and Nancy are the only ones studying at Reckers. Matt and Nancy are both working on Italian. Therefore, everyone studying at Reckers is working on Italian.

Language

$R(x)$: x is studying at Reckers.	m: Matt
$I(x)$: x is working on Italian.	n: Nancy

Translation

$$\{\forall x(R(x) \leftrightarrow (x = m \vee x = n)), (I(m) \wedge I(n))\} \models \forall x(R(x) \to I(x))$$

Analysis: The argument sounds valid, so we attempt a proof.

Conclusion	Rule	Assumptions
1. $R(c)$	$\mathbf{A}^*$	1^*
2. $\forall x(R(x) \leftrightarrow (x = m \vee x = n))$	$\mathbf{A}$	2
3. $(R(c) \leftrightarrow (c = m \vee c = n))$	$\mathbf{UE}$ 2	2
4. $(R(c) \rightarrow (c = m \vee c = n))$	$\wedge\mathbf{E}$ 3	2
5. $(c = m \vee c = n)$	$\mathbf{MP}$ $4, 1$	$1^*, 2$
6. $(I(m) \wedge I(n))$	$\mathbf{A}$	6
7. $I(m)$	$\wedge\mathbf{E}$ 6	6
8. $I(n)$	$\wedge\mathbf{E}$ 6	6
9. $c = c$	$=\mathbf{I}$	
10. $c = m$	$\mathbf{A}^*$	10^*
11. $m = c$	$=\mathbf{E}$ $9, 10$	10^*
12. $I(c)$	$=\mathbf{E}$ $7, 11$	$6, 10^*$
13. $(c = m \rightarrow I(c))$	$\mathbf{CP}$ $10, 12$	6
14. $c = n$	$\mathbf{A}^*$	14^*
15. $n = c$	$=\mathbf{E}$ $9, 14$	14^*
16. $I(c)$	$=\mathbf{E}$ $8, 15$	$6, 14^*$
17. $(c = n \rightarrow I(c))$	$\mathbf{CP}$ $14, 16$	6
18. $I(c)$	$\vee\mathbf{E}$ $5, 13, 17$	$1^*, 2, 6$
19. $(R(c) \rightarrow I(c))$	$\mathbf{CP}$ $1, 18$	$2, 6$
20. $\forall x(R(x) \rightarrow I(x))$	$\mathbf{UI}$ 19	$2, 6$

Remark 8.3.9. In the last proof, we had to spend a few extra steps reversing $c = m$ to $m = c$ and $c = n$ to $n = c$. If we had anticipated this, we could have translated the first assumption instead as: $\forall x(R(x) \leftrightarrow (m = x \vee n = x))$. This makes for a slightly shorter proof.

Conclusion	Rule	Assumptions
1. $R(c)$	$\mathbf{A}^*$	1^*
2. $\forall x(R(x) \leftrightarrow (m = x \vee n = x))$	$\mathbf{A}$	2
3. $(R(c) \leftrightarrow (m = c \vee n = c))$	$\mathbf{UE}$ 2	2
4. $(R(c) \rightarrow (m = c \vee n = c))$	$\wedge\mathbf{E}$ 3	2
5. $(m = c \vee n = c)$	$\mathbf{MP}$ $4, 1$	$1^*, 2$
6. $(I(m) \wedge I(n))$	$\mathbf{A}$	6
7. $I(m)$	$\wedge\mathbf{E}$ 6	6
8. $I(n)$	$\wedge\mathbf{E}$ 6	6
9. $m = c$	$\mathbf{A}^*$	9^*
10. $I(c)$	$=\mathbf{E}$ $7, 9$	$6, 9^*$
11. $(m = c \rightarrow I(c))$	$\mathbf{CP}$ $9, 10$	6
12. $n = c$	$\mathbf{A}^*$	12^*
13. $I(c)$	$=\mathbf{E}$ $8, 12$	$6, 12^*$
14. $(n = c \rightarrow I(c))$	$\mathbf{CP}$ $12, 13$	6
15. $I(c)$	$\vee\mathbf{E}$ $5, 11, 14$	$1^*, 2, 6$
16. $(R(c) \rightarrow I(c))$	$\mathbf{CP}$ $1, 15$	$2, 6$
17. $\forall x(R(x) \rightarrow I(x))$	$\mathbf{UI}$ 16	$2, 6$

Example 8.3.10. Analyze the following argument (which is very loosely based on an opinion piece by Milton Friedman in the *New York Times*).

> No government can bring order to Afghanistan without a huge investment of money. The Russians invested a great deal, and, even so, they were unsuccessful. If we invest much more money in Afghanistan, then we will be unable to do anything about health care. Therefore, if we want to improve health care, we will not be able to bring order to Afghanistan.

Language

$I(x)$: x invests a great deal of money in Afghanistan.

$O(x)$: x brings order to Afghanistan.

$H(x)$: x improves health care.

r: Russian government

u: United States government

Translation

$$\{\forall x(O(x) \to I(x)), (I(r) \wedge \neg O(r)), (I(u) \to \neg H(u))\} \models (H(u) \to \neg O(u))$$

Analysis: The argument sounds valid, so we attempt a proof.

Conclusion	Rule	Assumptions
1. $H(u)$	$\mathbf{A}^*$	1^*
2. $\neg\neg H(u)$	**DN** 1	1^*
3. $(I(u) \to \neg H(u))$	**A**	3
4. $\neg I(u)$	**MT** 3, 2	$1^*, 3$
5. $\forall x(O(x) \to I(x))$	**A**	5
6. $(O(u) \to I(u))$	**UE** 5	5
7. $\neg O(u)$	**MT** 6, 4	$1^*, 3, 5$
8. $(H(u) \to \neg O(u))$	**CP** 1, 7	3, 5

Remark 8.3.11. In the last example, the assumption $(I(r) \wedge \neg O(r))$ was not used. Indeed, the crux of the argument has nothing to do with the Russian government.

Example 8.3.12. Analyze the following argument.

> Everyone had exams either Tuesday or Thursday. Therefore, either everyone had exams Tuesday, or everyone had exams Thursday.

Language

$T(x)$: x had exams Tuesday.

$R(x)$: x has exams Thursday.

Translation

$$\{\forall x(T(x) \vee R(x))\} \models (\forall x T(x) \vee \forall x R(x))$$

Analysis: The argument sounds invalid, so we construct a counterexample.
We want a structure $\mathcal{A} = (A, T^*, T^*)$ satisfying the following conditions.

(*i*) $\mathcal{A} \models \forall x(T(x) \vee R(x))$ (i.e. everything is either in T^* or in R^*)

(*ii*) $\mathcal{A} \not\models (\forall x T(x) \vee \forall x R(x))$, which is equivalent to
$\mathcal{A} \models (\exists x \neg T(x) \wedge \exists x \neg R(x))$ (i.e. there is something not in T^* and something not in R^*)

Here is an example.

$A = \{\alpha, \beta\}$

$T^* = \{\alpha\}$

$R^* = \{\beta\}$

Example 8.3.13. Analyze the following argument.

> Everyone who is highly successful must be both bright and hard-working. All Notre Dame students are bright. Some Notre Dame students are highly successful. Therefore, some Notre Dame students are hard-working.

Language

$S(x)$: x is highly successful.

$B(x)$: x is bright.

$H(x)$: x is hard-working.

$N(x)$: x is a Notre Dame student.

Translation

$$\{\forall x(S(x) \rightarrow (B(x) \wedge H(x))), \forall x(N(x) \rightarrow B(x)), \exists x(N(x) \wedge S(x))\} \models \exists x(N(x) \wedge H(x))$$

Analysis: The argument sounds valid, so we give a proof. Note that the proof does not use the assumption that all Notre Dame students are bright.

Conclusion	Rule	Assumptions
1. $\exists x(N(x) \wedge S(x))$	**A**	1
2. $(N(c) \wedge S(c))$	**A***	2^*
3. $N(c)$	$\wedge$**E** 2	2^*
4. $S(c)$	$\wedge$**E** 2	2^*
5. $\forall x(S(x) \to (B(x) \wedge H(x)))$	**A**	5
6. $(S(c) \to (B(c) \wedge H(c)))$	**UE** 5	5
7. $(B(c) \wedge H(c))$	**MP** $6, 4$	$2^*, 5$
8. $H(c)$	$\wedge$**E** 7	$2^*, 5$
9. $(N(c) \wedge H(c))$	$\wedge$**I** $3, 8$	$2^*, 5$
10. $\exists x(N(x) \wedge H(x))$	**EI** 9	$2^*, 5$
11. $((N(c) \wedge S(c)) \to \exists x(N(x) \wedge H(x)))$	**CP** $2, 10$	5
12. $\exists x(N(x) \wedge H(x))$	**EE** $1, 11$	$1, 5$

Example 8.3.14. Analyze the following argument.

> All riverboat pilots are writers. Mark Twain is a writer. Samuel Clemens is not a riverboat pilot. Therefore Mark Twain and Samuel Clemens are not the same person.

Language

$R(x)$: x is a riverboat pilot.	c: Samuel Clemens
$W(x)$: x is a writer.	t: Mark Twain

Translation

$$\{\forall x(R(x) \to W(x)), W(t), \neg R(c)\} \models t \neq c$$

Analysis: The argument sounds invalid, so we construct a counterexample.

We want a structure $\mathcal{A} = (A, R^*, W^*, c^*, t^*)$ satisfying the following conditions.

(*i*) $\mathcal{A} \models \forall x(R(x) \to W(x))$ (i.e. everything in R^* is in W^*).

(*ii*) $\mathcal{A} \models W(t)$ (i.e. t^* is in W^*)

(*iii*) $\mathcal{A} \models \neg R(c)$ (i.e. c^* is not in R^*)

(*iv*) $\mathcal{A} \not\models t \neq c$, which is equivalent to
$\mathcal{A} \models t = c$ (i.e. t^* and c^* are the same object).

Here is an example.

$A = \{\alpha\}$

$R^* = \emptyset$

$W^* = \{\alpha\}$

$c^* = \alpha$

$t^* = \alpha$

Example 8.3.15. Analyze the following argument.

> Anyone who isn't a writer is a riverboat pilot. Mark Twain is the only writer who is a riverboat pilot. Therefore everyone is a riverboat pilot.

Language

$R(x)$: x is a riverboat pilot. t: Mark Twain

$W(x)$: x is a writer.

Translation

$$\{\forall x(\neg W(x) \to R(x)), \forall x((R(x) \land W(x)) \leftrightarrow x = t)\} \models \forall x R(x)$$

Analysis: The argument sounds invalid, so we construct a counterexample.

We want a structure $\mathcal{A} = (A, R^*, W^*, t^*)$ satisfying the following conditions.

(*i*) $\mathcal{A} \models \forall x(\neg W(x) \to R(x))$ (i.e. anything not in W^* is in R^*).

(*ii*) $\mathcal{A} \models \forall x((R(x) \land W(x)) \leftrightarrow x = t)$ (i.e. t^* is the only object in both R^* and W^*).

(*iii*) $\mathcal{A} \not\models \forall x R(x)$ (i.e. there is something that is not in R^*).

Here is an example.

$A = \{\alpha, \beta\}$

$R^* = \{\alpha\}$

$W^* = \{\alpha, \beta\}$

$t^* = \alpha$

Example 8.3.16. Analyze the following argument.

> No one is both a riverboat pilot and a writer. Mark Twain is a writer. Samuel Clemens is a riverboat pilot. There is someone who is not a riverboat pilot. There is someone who is neither a riverboat pilot nor a writer.

Language

$R(x)$: x is a riverboat pilot.

$W(x)$: x is a writer.

c: Samuel Clemens

t: Mark Twain

Translation

$$\{\neg\exists x(R(x) \wedge W(x)), W(t), R(c), \exists x \neg R(x)\} \models \exists x(\neg R(x) \wedge \neg W(x))$$

Analysis: The argument sounds invalid, so we construct a counterexample.

We want a structure $\mathcal{A} = (A, R^*, W^*, c^*, t^*)$ satisfying the following conditions.

(i) $\mathcal{A} \models \neg\exists x(R(x) \wedge W(x))$ (i.e. there is nothing in both R^* and W^*).

(ii) $\mathcal{A} \models W(t)$ (i.e. t^* is in W^*).

(iii) $\mathcal{A} \models R(c)$ (i.e. c^* is in R^*).

(iv) $\mathcal{A} \models \exists x \neg R(x)$ (i.e. there is something not in R^*).

(v) $\mathcal{A} \not\models \exists x(\neg R(x) \wedge \neg W(x))$, which is equivalent to
$\mathcal{A} \models \forall x(R(x) \vee W(x))$ (i.e. everything is either in R^* or in W^*).

Here is an example.

$A = \{\alpha, \beta\}$

$R^* = \{\alpha\}$

$W^* = \{\beta\}$

$c^* = \alpha$

$t^* = \beta$