Intrinsic Density and Intrinsically Small Sets

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Asymptotic Computability

Recent work in computability has introduced the idea of asymptotically computable sets, where a set A might not be computable, but there may be a computable function which computes A correctly "almost everywhere." That is, the asymptotic density of the error set is 0.

Definition

$\rho(A) = \lim_{n\to\infty} \frac{|A\restriction n|}{n}$ is the (asymptotic) density of A if it exists, where $\overline{\rho}(A)$ $(\underline{\rho}(A))$ denotes the upper (lower) density of A and is simply the limsup (liminf) of this sequence.

Types of Asymptotic Computability

By formally defining what is allowed to occur in the error set, we obtain the four notions of asymptotic computability.

- With effective dense computability, the function converges but refuses to answer.
- With generic computability, the function may diverge on the error set.
- With coarse computability, the function must be total but is incorrect on the error set.
- With dense computability, the function may diverge or be incorrect on the error set.

Potential Objection

It is natural to think of asymptotic computability as meaning "almost computable." However, there is a potentially troubling side effect with this line of thinking:

Theorem

(Jockusch and Schupp) Every Turing degree contains a set which is asymptotically computable.

Proof.

Let $A \subseteq \omega$. Then $F_A = \{n! : n \in A\}$ has density 0, and $F_A \equiv_T A$. However, F_A has asymptotic density 0, so the constant 0 function witnesses that F_A is asymptotically computable. $\square$

Avoiding Coding

This is suggesting that our error sets, computable ones in particular, may still be too large. Intrinsic density, permutation-invariant asymptotic density, was introduced to block these coding tricks.

Definition

Let $A \subseteq \omega$.

- $\overline{\boldsymbol{\rho}}(A) = \limsup_{\pi} \overline{\rho}(\pi(A))$, where π ranges over all computable permutations, is the absolute upper density of A. By taking the liminf of the lower densities, we get $\underline{\boldsymbol{\rho}}(A)$, the absolute lower density of A.
- If $\overline{\boldsymbol{\rho}}(A) = \underline{\boldsymbol{\rho}}(A)$, then we call this value $\boldsymbol{\rho}(A)$, the intrinsic density of A. If $\boldsymbol{\rho}(A) = 0$, then we say A is intrinsically small.

Examples and Non-Examples

Examples

- (Jockusch) r-cohesive (and therefore cohesive) sets are intrinsically small
- Given a collection of infinite sets $\{R_i\}_{i\in\omega}$, $\emptyset''$ together with the join of all R_i's can compute an intrinsically small set which is not disjoint from any R_i.
- Sufficiently random sets have intrinsic density $\frac{1}{2}$

Non-Examples

- 1-generics never have (intrinsic) density
- Infinite, co-infinite computable sets

Intrinsic Computability

Definition

(Astor) A set X is strongly intrinsically (asymptotically) computable if it is asymptotically computable with an intrinsically small error set.

Examples

- Computable sets (The error set is $\emptyset$)
- Maximal sets

Does this work?

Lemma

Any strongly intrinsically effectively densely computable set A is computable.

Proof.

Recall that in effective dense computability, the computable approximation converges to a special symbol that signifies refusal to answer. Therefore, the error set is just the inverse image of this symbol, which is computable.

Thus the error set is finite, so A is computable. □

A Less Trivial Example

Fortunately, we are not in general just providing a new definition of the computable sets.

Theorem

The high or DNC degrees are exactly the noncomputable degrees which contain a strong intrinsic coarsely computable set.

Theorem (Astor)

The Turing degrees which contain intrinsically small sets are exactly the high or DNC degrees.

Studying the Intrinsically Small Sets

The intrinsically small sets are closed under permutation by definition. They are also closed under unions and subsets. What about the join? What about other computable functions?

There are clearly more computable functions which preserve intrinsic smallness: For example, if X is intrinsically small, then $\{2n : n \in X\}$ is also intrinsically small. We'd like to understand how one might go about finding new intrinsically small sets from old ones.

Not All Computable Functions Work

Theorem

There is a total computable function f and an intrinsically small set X such that $f(X)$ does not have density 0.

The proof relies heavily on the fact that the inverse image of n under f has positive density for each n. This is not needed, though.

Theorem

There are total computable functions f and g such that $|f^{-1}(n)| < g(n)$ for all n, and an intrinsically small set X such that $f(X)$ does not have density 0.

Outline of the Proof

There is an intrinsically small set of hyperimmune-free degree.

Therefore there is a disjoint strong array which X fails to avoid. After a slight modification, the map which sends the elements of each cell to their index will be a suitable f for the theorem.

Some Functions Which Preserve Intrinsic Smallness

We can, however, describe some classes of functions which preserve intrinsic smallness.

Theorem

Let f be a computable, injective function with computable range. Then for any intrinsically small set X, $f(X)$ is also intrinsically small.

Corollary

If A and B are intrinsically small, then so is $A \oplus B$.

Proof.

Let $e(n) = 2n$ and $o(n) = 2n + 1$. Then if A and B are intrinsically small, $e(A)$ and $o(B)$ are intrinsically small, as is $e(A) \cup o(B)$. Therefore $A \oplus B = e(A) \cup o(B)$ is intrinsically small. □

An Improvement

We naturally would like to know if the computable range requirement can be removed. There is an improvement we can make.

Theorem

If f is a computable, injective function whose range has defined asymptotic density, then $\overline{\rho}(f(X)) = 0$ for any intrinsically small set X

Open Questions

Question

If f is a computable, injective function whose range has defined asymptotic density, is $f(X)$ intrinsically small for every intrinsically small set X?

Question

If f is a computable, injective function, is $\overline{\rho}(f(X)) = 0$ for every intrinsically small set X?

Intrinsic Smallness vs Hyperimmunity

Astor studied the relationship between intrinsic smallness and notions of immunity. Hyperimmune sets in particular always have intrinsic lower density 0, but in general hyperimmune sets are not intrinsically small and vice versa. This is true in every degree.

Theorem

Let X be a hyperimmune set. Then there is a hyperimmune set Y with $Y \equiv_T X$ and Y is as large as possible, i.e. $\overline{\rho}(Y) = 1$. If X is co-hypersimple, then so is Y.

Proof Sketch

Let X be hypersimple. Introduce infinitely many gaps computably such that the density at the start of the gaps approaches 1, but the density at the end of the gaps approaches 0. Now this set is not even immune, let alone hyperimmune.

However, if we enumerate the n-th gap into Y for $n \in X$, then we can show that a computable bound on $\overline{Y}$ allows a computable bound on $\overline{X}$. Therefore Y is hyperimmune.

Current and Future Work

There are plenty of remaining open questions about intrinsic density. Very little work has been done investigating Astor's other three notions of intrinsic computability. (These do not relate directly to intrinsically small sets like the strong variant does.)

Current work is being done investigating the relationship of intrinsic density to randomness and stochastity. A tangentially related interesting result is the following:

Theorem

(In Preparation) Every real in the unit interval is achieved as the intrinsic density of some set.