NONCOMPUTABLE CODING, DENSITY, AND STOCHASTICITY

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Abstract. Intrinsic density was introduced by Astor to study asymptotic computability. Intrinsically small sets, those of intrinsic density zero, serve as the basis for generalizing classical asymptotic computability to its permutation-invariant form. It was already known that intrinsic density corresponds with a weakening of KL-stochasticity, and we set out to study which intrinsic densities are possible through the lenses of computability theory and stochasticity. Computable coding methods cannot be used to change intrinsic density, so we shall introduce the `into` and `within` set operations to create examples of sets with defined intrinsic density and construct examples of sets with arbitrary density without simply appealing to randomness. These operations turn out to be surprisingly useful as a form of noncomputable coding, and shall be our primary focus. After exploring their applications to intrinsic density, we shall also find applications to the study of Turing degrees and the classical notions of Church stochasticity and MWC stochasticity.

Keywords: intrinsic density, stochasticity, randomness, computability, martingale

1. Introduction

Intrinsic density originated in an attempt to resolve some objections about classical asymptotic computability, which uses sets of density zero as error sets in computations. We briefly recall the notion of (asymptotic) density in the natural numbers:

Definition 1.1. *Let $A \subseteq \omega$.*

- *The density of A at n is $\rho_n(A) = \frac{|A \upharpoonright n|}{n}$, where $A \upharpoonright n = A \cap \{0, 1, 2, \ldots, n-1\}$.*
- *The upper density of A is $\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A)$.*
- *The lower density of A is $\underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A)$.*
- *If $\overline{\rho}(A) = \underline{\rho}(A) = \alpha$, we call α the density of A and denote it by $\rho(A)$.*

Remark. We shall follow the convention, unless otherwise stated, that capital English letters represent sets of natural numbers and the lowercase variant, indexed by a subscript of natural numbers, represents the elements of the set. As an example, if E is the set of even numbers, then $e_n = 2n$. If F is the set of factorials, then $f_n = (n+1)!$. (f_n is not $n!$ because $0! = 1! = 1$.) Recall that the principal function for a set A, p_A, is defined via $p_A(n) = a_n$.

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Using this representation, it is not hard to see the following characterization of upper and lower density:

Lemma 1.2. *Let* $A \subseteq \omega$ *be* $\{a_0 < a_1 < a_2 < \dots\}$. *Then*

- $\overline{\rho}(A) = \limsup_{n\to\infty} \frac{n+1}{a_n+1}$
- $\underline{\rho}(A) = \liminf_{n\to\infty} \frac{n}{a_n}$

Proof. Note that if $A \restriction n+1$ has a 0 in the final bit, then

$$\rho_n(A) = \frac{|A \restriction n|}{n} > \frac{|A \restriction n|}{n+1} = \rho_{n+1}(A)$$

Therefore, to compute the upper density it suffices to check only those numbers n for which $A \restriction n$ has a 1 as its last bit. Those numbers are exactly $a_n + 1$ by the definition of a_n, and $|A \restriction a_n + 1| = n+1$. Therefore $\{\frac{n+1}{a_n+1}\}_{n\in\omega}$ is a subsequence of $\{\rho_n(A)\}_{n\in\omega}$ which dominates the original sequence, so $\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A) = \limsup_{n\to\infty} \frac{n+1}{a_n+1}$.

Similarly, to compute the lower density it suffices to check only the numbers n such that the final digit of $A \restriction n$ is a 0, but the final digit of $A \restriction n+1$ is a 1. (That is, if there is a consecutive block of zeroes in the characteristic function of A, we only need to check the density at the end of the block when computing lower density, as each intermediate point of the zero block has a higher density than the end.) These numbers are exactly a_n by definition, and $|A \restriction a_n| = n$. Therefore $\{\frac{n}{a_n}\}_{n\in\omega}$ is a subsequence of $\{\rho_n(A)\}_{n\in\omega}$ which is dominated by the original sequence, so $\underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A) = \liminf_{n\to\infty} \frac{n}{a_n}$. □

One potential objection with using sets of density 0 as error sets is that there are many computable sets of density zero, so one is able to make any information that one desires be "almost" computable by hiding it within a small computable set. To combat this, Astor [2] introduced intrinsic density, which requires that sets have the same asymptotic density under any computable permutation:

Definition 1.3.

- *The absolute upper density of* A *is*

$$\overline{P}(A) = \sup\{\overline{\rho}(\pi(A)) : \pi \text{ a computable permutation}\}$$

- *The absolute lower density of* A *is*

$$\underline{P}(A) = \inf\{\underline{\rho}(\pi(A)) : \pi \text{ a computable permutation}\}$$

- *If* $\overline{P}(A) = \underline{P}(A) = \alpha$*, we call* α *the intrinsic density of* A *and denote it by* $P(A)$.

Interestingly, this turns out to be a robust measure of lack of information. If a set X has intrinsic density, then we cannot computably shrink or enlarge parts of it with a permutation to change the density. If we knew where elements of X could be found, then we could build a permutation that sent them to a set of density 1 or 0. This intuition has a formal counterpart: Astor [3] proved that any set which has intrinsic density must be of high or DNC degree, i.e. must be sufficiently noncomputable. Sets of intrinsic density 0, also known as intrinsically small sets, were explored by Astor in [2] and [3] and the author in [15]. Exploring intrinsic smallness inherently explores sets of intrinsic density 1 because intrinsic density is preserved by complementation.

As seen in Lemma 1.12 below, appealing to randomness will yield intrinsic density r for any $r \in (0,1)$. However, as intrinsic density is itself a poor notion of randomness (for example if $P(A) = r$, then $P(A \oplus A) = r$), we wish to study the achievable intrinsic densities through the lens of computability theory without solely appealing to randomness. Our goal is to combine sets with intrinsic density in such a fashion that the resulting set has a different intrinsic density. Unfortunately, computable coding methods cannot do this, which we will formalize in Section 2. This motivates our development of new tools for noncomputable coding, the `into` and `within` operations on sets, which we shall introduce in Section 3. These operations shall turn out to be highly effective at coding sets in noncomputable fashion and are central to our results. We shall additionally study their applications to Turing degrees in Section 3 and other notions of density in Section 5. First, however, we shall review the necessary concepts.

As hinted above, intrinsic densities between 0 and 1 are linked to stochasticity and randomness. Here we shall provide a brief review of these from the perspective of computability theory. Stochasticity and randomness are closely related notions which also measure lack of information (i.e. how much information an observer lacks), and turn out to have strong ties to intrinsic density. Stochasticity represents the idea that we cannot select bits from an infinite sequence of 0's and 1's in such a way that the ratio of 1's to the number of bits is not the same as the ratio for the original sequence.

One can think of this as having an infinite sequence X of 0-1-valued coins, where we also think of X as a set under the identification $X = \{n : \text{The } n\text{-th coin is 1-valued}\}$. X has some asymptotic density $r \in [0,1]$ We try to use some selection process to pick coins from X to build a new sequence of coins Y with $\rho(Y) \neq r$. If we are successful, then X is not stochastic. Changing the ways we are allowed to select coins gives us different notions of stochasticity. We review the noteworthy notions of stochasticity from the literature.

A monotone *selection function* is a partial function $f : 2^{<\omega} \to \{0,1\}$. That is, f looks at a finite binary string and decides if it wants to select (i.e. return 1) the following bit or not based on the previous bits. Given a selection function f, it induces a map $\hat{f} : 2^\omega \to 2^\omega$ that is defined via $\hat{f}(A) = \{n : f(A \upharpoonright n) \downarrow = 1\}$ for all A. (We shall abuse notation and allow f to represent both a monotone selection function and the induced map $\hat{f}$ on Cantor space.) We say A is von Mises-Wald-Church stochastic for r if $\rho(\{n : p_{f(A)}(n) \in A\}) = r$ for all computable monotone selection functions. If we restrict this to only the total f, then the corresponding notion is called Church stochasticity. In both cases, we may use the results of the first n bits to computably determine whether or not we want to select the $n+1$-st bit, but all of the bits we select must be counted in order.

Using our coin analogy, for Church stochasticity, all of the coins have been covered by cups. We must choose whether or not to add the first coin to our new sequence before looking under any cups. Then we look under the first cup and check the value, and we use this information moving forward. Having revealed the first n coins, we must choose whether or not to select the $n+1$st coin (i.e. determine if we think it is 1-valued) prior to revealing it.

For MWC stochasticity, at each step we provide a program that will decide whether or not to select the $n+1$st coin based on the results of the first n coins. We then run the program and look under the cup. We do not need to wait for the program to halt (and it may never halt) before continuing on to the next coin, but we can never go back and feed the program more information or change it any way. Even though our selection process for the n-th coin may halt after the value of the n-th coin is known, it could not have had access to that information in its calculation.

Another historically important notion of stochasticity is KL-stochasticity, or Kolmogorov-Loveland stochasticity. This notion is similar to MWC-stochasticity, however we are allowed to select bits out of order rather than being forced to choose whether or not to select the n-th bit only after seeing the first n bits. One particularly important weakening of KL-stochasticity for our study of intrinsic density is injection stochasticity. A set A is injection stochastic for r if $\rho(f^{-1}(A)) = r$ for all total computable injective f. Permutation stochasticity, as expected, is the subclass where f is required to be a permutation. Using this definition, Astor first observed the following:

Lemma 1.4 (Astor [2] Lemma 4.2). *A set A is r-injection stochastic if and only if it is r-permutation stochastic.*

Proof. If A is r-injection stochastic, it is trivially r-permutation stochastic.

Suppose that A is r-permutation stochastic. Then $\rho(\pi(A)) = r$ for every computable permutation π. Let f be a total computable injective function and let $F = \{n! : n \in \omega\}$. Define π_f via $\pi_f(n) = f(n)$ if $n \notin F$ and $f(n)$ is not in $\pi_f([0, n))$, and the least element of the complement of $\pi_f([0, n))$ otherwise. As π_f is a computable permutation, so is π_f^{-1} and thus $\rho(\pi_f^{-1}(A)) = r$.

Now notice that $\pi_f^{-1}(A) \upharpoonright n$ differs from $f^{-1}(A) \upharpoonright n$ by at most $2|F \upharpoonright n|$, as there can only be disagreement on F and $f^{-1}(\pi_f(F))$. In fact, there are two types of disagreement. In the first, we specifically mapped $\pi_f(n!)$ to something other than $f(n!)$, which can only happen within F. In the second, k is not a factorial but $f(k) \in \pi_f([0, k))$ because of some $n! < k$. Thus the set of disagreements has density zero because F does, so

$$\rho(f^{-1}(A)) = \rho(\pi_f^{-1}(A)) = r$$

□

Remark. *In his version of the proof, Astor used the squares instead of the factorials. In general for all of our arguments which use a computable density* 0 *set, any set with such properties will suffice. We shall endeavor to use the factorials in all such proofs for clarity.*

It is immediate from the definition that r-permutation stochasticity is exactly intrinsic density r. Therefore, this lemma shows that r-injection stochasticity also corresponds to intrinsic density r. Unlike stochasticity, intrinsic density is defined without fixing r ahead of time. Motivated by this, we shall use $\mathcal{C}$-density r to mean $\mathcal{C}$ stochasticity for r.

Remark. *While computability theory most commonly studies stochasticity with regards to $\frac{1}{2}$, stochasticity with regards to parameters other than $\frac{1}{2}$ has been studied before. For example, see Kjos-Janssen, Taveneaux, and Thapen [12]. However, our use of the term density as opposed to stochasticity is to differentiate our intentions: stochasticity is generally studied by fixing some $r \in [0,1]$ and a notion of stochasticity and then studying the class of sets which are stochastic for r. (In the context of randomness, this corresponds to fixing a measure from the outset.) Density, on the other hand, does not fix r and studies the class of sets which are stochastic for some r. This is a larger class that often has its own interesting properties. While the same sets appear in both settings, we are really studying the class containing these sets.*

We shall develop our tools of study in Section 3 to tackle intrinsic (injection-) densities, and we shall apply those tools to MWC- and Church-densities in Section 5. One important trait of Church-density is that if A has Church-density α, then $\rho(A) = \alpha$ because the selection function $\hat{1}$ which selects every bit is a total computable monotone selection function. (It follows immediately from the fact that MWC-density is defined for a larger class of selection functions that the same is true of MWC-density.)

Closely related, Randomness is well-studied and more well-known than stochasticity, so we shall only provide a cursory overview. (For a more in-depth review of randomness as well as stochasticity, see Downey-Hirschfeldt [6].) While there are many notions of randomness, we shall only need 1-Randomness, also known as Martin-Löf Randomness, for our purposes. There are many equivalent ways of defining randomness, and we shall recall two. In computability theory most randomness is studied with respect to the Lebesgue measure, so we shall start with the more familiar form before generalizing the definitions to arbitrary measures.

Definition 1.5. *A martingale is a function $m : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that*

$$m(\sigma) = \frac{1}{2}m(\sigma 0) + \frac{1}{2}m(\sigma 1)$$

for all σ. A supermartingale is a function $s : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ with

$$s(\sigma) \geq \frac{1}{2}s(\sigma 0) + \frac{1}{2}s(\sigma 1)$$

for all σ. A (super)martingale m succeeds on a set X if $\limsup_{n\to\infty} m(X \upharpoonright n) = \infty$. X is 1-Random if no computably enumerable supermartingale succeeds on it.

Martingales capture the unpredictability of random sets: we could not win arbitrarily large amounts of money betting on the bits of X in any c.e. or computable way. An alternative yet equivalent formulation of randomness is the measure-theoretic approach, which is based upon the intuition that if a set is random then it should avoid all small sets which can be described with computable approximations.

Definition 1.6. *A Martin-Löf (ML) test is a sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of uniformly Σ^0_1 classes with $\mu(\mathcal{U}_i) \leq 2^{-i}$ for all i. (Here μ is the usual Lebesgue measure on Cantor space.) A set X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_\rangle$. X is 1-Random if it passes every Martin-Löf test.*

While historically the study of algorithmic randomness began with respect to the Lebesgue or "fair coin" measure, much work has focused on studying randomness with respect to other measures. It is not difficult to see how Definition 1.6 generalizes to an arbitrary computable measure.

Definition 1.7. *Let ν be a computable measure on Cantor space. A ν-Martin-Löf test is a sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of uniformly Σ^0_1 classes with $\nu(\mathcal{U}_i) \leq 2^{-i}$ for all i. A set X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_\rangle$. X is 1-Random with respect to ν if it passes every ν-Martin-Löf test.*

Note that effectivity concerns are all that keeps one from generalizing this to arbitrary measures. Investigating ways to address this problem has proven to be a rich area of study. Given an arbitrary measure μ, Reimann and Slaman [17] defined randomness with respect to μ as being random with respect to some representation of μ. Conversely, Levin [13], Gács [7], and Hoyrup and Rojas [8] utilized the notion of uniform tests to give an alternate definition. Day and Miller [5] proved that these approaches are in fact the same.

One can generalize the equivalence of Definition 1.5 and Definition 1.6 to obtain a definition for randomness with respect to a measure for martingales to match 1.7.

Definition 1.8. *Let μ be a computable measure. Given a finite binary string σ, $[\sigma] \subseteq 2^\omega$ represents the basic open set of extensions of σ, and $\mu(\sigma) = \mu([\sigma])$. A μ-martingale is a function $m : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ such that*

$$\mu(\sigma)m(\sigma) = \mu(\sigma 0)m(\sigma 0) + \mu(\sigma 1)m(\sigma 1)$$

for all σ. A μ-supermartingale is a function $s : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ with

$$\mu(\sigma)s(\sigma) \geq \mu(\sigma 0)s(\sigma 0) + \mu(\sigma 1)s(\sigma 1)$$

for all σ. A μ-(super)martingale m succeeds on a set X if $\limsup_{n\to\infty} m(X \upharpoonright n) = \infty$. A set X is μ-1-Random if no computably enumerable μ-supermartingale succeeds on it.

As in the case of definition 1.7, there are some effectivity concerns in regards to non-computable measures, but they will not affect our work.

We are primarily concerned with the following special class of measures.

Definition 1.9. *Let $0 < r < 1$ be a real number. The Bernoulli measure with parameter r, μ_r, is the measure on Cantor space such that for any $\sigma \in 2^{<\omega}$,*

$$\mu_r(\sigma) = r^{|\{n<|\sigma|:\sigma(n)=1\}|}(1-r)^{|\{n<|\sigma|:\sigma(n)=0\}|}$$

We say X is r-1-Random if it is μ_r-1-Random.

Note that $\mu_{\frac{1}{2}}$ is the usual Lebesgue measure. More generally, μ_r is the r-biased "coin flip" measure, or the measure induced by the Bernoulli probability with parameter r. As we are only working with Bernoulli measures, we shall use the Reimann-Slaman definition of randomness for noncomputable measures and rely only on the fact that every representation of μ_r can compute r, i.e. [17] Propositions 2.2 and 2.3.

Remark. *Stochasticity and randomness are closely related. Cholak asked the interesting question if they can be viewed as the same, i.e. is there a natural notion*

$\mathcal{C}$ of stochasticity for which $\mathcal{C}$-density r corresponds to μ_r-Randomness? Bienvenu pointed out that this is essentially answered by a result of Vovk [21] *(With related work done by Bienvenu* [4] *proving the same for weaker notions of randomness) in the negative for any reasonably natural notion: If $\{p_i\}_{i\in\omega}$ is a sequence of reals, then the generalized Bernoulli measure ν for this sequence is given by*

$$\nu(\sigma) = \prod_{\sigma(i)=1} p_i \cdot \prod_{\sigma(i)=0} (1-p_i)$$

If this sequence converges to $\frac{1}{2}$ with $\Sigma_{i=0}^{\infty}(p_i - \frac{1}{2})^2 = \infty$, then the ν-random sets and the Martin-Löf-Random sets are disjoint. Thus under any natural notion of stochasticity $\mathcal{C}$ the selected bits will be given by independent random variables of probability arbitrarily close to probability $\frac{1}{2}$ for all but finitely many. Therefore both Martin-Löf -randoms and ν-randoms would have $\mathcal{C}$-density $\frac{1}{2}$.

As a quick side note, it is possible that one could avoid representations altogether: By a result of Kjos-Hanssen [11], the so-called *Hippocratic* r-Random sets, defined via Hippocratic ML-tests (essentially ML-tests which can be accessed without seeing information about r and μ_r), are exactly the r-1-Random sets. However, it is not known if Hippocratic r-Random sets defined via Hippocratic martingales are the same as the Hippocratic r-Random sets defined via Hippocratic ML-tests. The standard argument for transforming an ML-test into a martingale and vice versa does not go through in the Hippocratic case: we need access to r. Further evidence suggests Hippocratic martingales may give rise to a separate notion: Kjos-Hanssen, Taveneaux, and Thapen [12] showed that Hippocratic r-computably random sets, formulated using Hippocratic martingales, are not the same as the r-computably random sets. Because of this uncertainty, we stick to the implicit use of representations in our work as we shall use both ML-tests and martingales for convenience and rely on the fact that these yield the same notion of randomness.

A slight modification of the standard proof that randomness for (super)martingales is the same as randomness for Martin-Löf tests (as found in Downey-Hirschfeldt [6] Section 6.3.1, referencing work of Ville [20] and Schnorr [18]) shows that 1-Randomness with respect to μ is equivalent to μ-1-Randomness. (We shall give the argument for the computable case, and the non-computable case will follow from proper relativization.)

Theorem 1.10 (Essentially Ville [20])**.** *Let μ be a computable measure. Let m be a μ-(super)martingale.*

- *If $\sigma \in 2^{<\omega}$ and S is a prefix-free set of extensions of σ, then*

$$\sum_{\tau\in S} \mu(\tau)m(\tau) \le \mu(\sigma)m(\sigma)$$

- *Let $R_n = \{X : \exists k\ m(X \restriction k) \ge n\}$. Then $\mu(R_n) \le \frac{m(\emptyset)}{n}$.*

Proof. • Note that it suffices to only consider finite sets S, as if S is infinite and $\Sigma_{\tau\in S}\, \mu(\tau)m(\tau) > \mu(\sigma)m(\sigma)$, there is some finite subset of S also exhibiting this property.

We argue by induction on $|S|$. For $|S| = 1$, let $\tau \succeq \sigma$, i.e. $\tau = \sigma\gamma$ for some $\gamma \in 2^{<\omega}$. Note by induction and the definition of a μ-(super)martingale that $\mu(\gamma)m(\tau) \leq m(\sigma)$. Therefore

$$\mu(\tau)m(\tau) = \mu(\sigma)\mu(\gamma)m(\tau) \leq \mu(\sigma)m(\sigma)$$

Now suppose $|S| = k+1$ and the induction hypothesis holds for all $i \leq k$. Let $\gamma \succeq \sigma$ be maximal such that $\tau \succeq \gamma$ for all $\tau \in S$. Then let $S_0 \subseteq S$ be the set of all $\tau \in S$ with $\tau \succeq \gamma 0$ and let $S_1 = S \setminus S_0$. (Note that for all $\tau \in S_1$, $\tau \succeq \gamma 1$.) Therefore, as γ is maximal such that all $\tau \in S$ are extensions of γ, both $|S_0| \leq k$ and $|S_1| \leq k$. Therefore, the induction hypothesis implies that

$$\sum_{\tau \in S_0} \mu(\tau)m(\tau) \leq \mu(\gamma 0)m(\gamma 0)$$

and

$$\sum_{\tau \in S_1} \mu(\tau)m(\tau) \leq \mu(\gamma 1)m(\gamma 1)$$

Therefore

$$\sum_{\tau \in S} \mu(\tau)m(\tau) = \sum_{\tau \in S_0} \mu(\tau)m(\tau) + \sum_{\tau \in S_1} \mu(\tau)m(\tau) \leq \mu(\gamma 0)m(\gamma 0) + \mu(\gamma 1)m(\gamma 1)$$

By the properties of a μ-(super)martingale we have

$$\mu(\gamma 0)m(\gamma 0) + \mu(\gamma 1)m(\gamma 1) \leq \mu(\gamma)m(\gamma)$$

and therefore

$$\sum_{\tau \in S} \mu(\tau)m(\tau) \leq \mu(\gamma)m(\gamma)$$

The base case proved that $\mu(\gamma)m(\gamma) \leq \mu(\sigma)m(\sigma)$, so this concludes the induction.

- Let S be a prefix-free set which induces the Σ^0_1-class R_n with all $\tau \in S$ satisfying $m(\tau) \geq n$. By definition,

$$\mu(R_n) = \sum_{\tau \in S} \mu(\tau)$$

As each $\tau \in S$ satisfies $m(\tau) \geq n$,

$$\sum_{\tau \in S} \mu(\tau) \leq \sum_{\tau \in S} \frac{m(\tau)}{n}\mu(\tau)$$

Finally, we may apply the first part with $\sigma = \emptyset$ to obtain

$$\sum_{\tau \in S} \frac{m(\tau)}{n}\mu(\tau) \leq \frac{\mu(\emptyset)m(\emptyset)}{n} = \frac{m(\emptyset)}{n}$$

□

Theorem 1.11 (Essentially Schnorr [18])**.** *X is μ-1-Random if and only if it is 1-Random with respect to μ.*

Proof. Let m be a c.e. μ-(super)martingale. Without loss of generality, assume $m(\emptyset) = 1$. Let $U_n = \{X : \exists k\ m(X \upharpoonright k) \geq 2^n\}$. This is a μ-Martin-Löf test by Theorem 1.10, and it is immediate that $X \in \bigcap_{n \in \omega} U_n$ if and only if m succeeds on X.

Let $\{U_n\}_{n \in \omega}$ be a μ-Martin-Löf test with $\{S_n\}_{n \in \omega}$ the uniform sequence of c.e. prefix-free finite binary strings which induces $\{U_n\}_{n \in \omega}$. We shall define c.e. μ-martingales m_n via the following procedure: If we see σ enter S_n at some stage, then add 1 to $m_n(\tau)$ for all $\tau \succeq \sigma$. For $\gamma \prec \sigma$, add $\frac{\mu(\sigma)}{\mu(\gamma)}$ to $m_n(\gamma)$ if $\mu(\gamma)$ is nonzero, and 0 otherwise. Then it is immediate from this definition that $m_e : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is a c.e. function. Furthermore, note that it is a μ-martingale: let $\sigma \in 2^{<\omega}$. We must show that $\mu(\sigma)m_n(\sigma) = \mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 1)$.

As S_n is prefix-free, if $\sigma \succeq \tau \in S_n$, then

$$\mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 0) = \mu(\sigma 1) + \mu(\sigma 0) = \mu(\sigma) = \mu(\sigma)m_n(\sigma)$$

by construction. Otherwise, if $\mu(\tau) = 0$ for some $\tau \preceq \sigma$, then $\mu(\sigma) = \mu(\sigma 0) = \mu(\sigma 1) = 0$ and we are done. Therefore, we may assume $\mu(\tau) > 0$ for all $\tau \preceq \sigma$. Then

$$m_n(\sigma) = \sum_{\tau \in S_n, \tau \succ \sigma} \frac{\mu(\tau)}{\mu(\sigma)} = \frac{1}{\mu(\sigma)} \sum_{\tau \in S_n, \tau \succ \sigma} \mu(\tau)$$

by definition. Note that for $i = 0, 1$,

$$m_n(\sigma i) = \sum_{\tau \in S_n, \tau \succeq \sigma i} \frac{\mu(\tau)}{\mu(\sigma i)}$$

as if $\sigma i \in S_n$ then $m_n(\sigma i) = 1 = \frac{\mu(\sigma i)}{\mu(\sigma i)}$. Therefore

$$\mu(\sigma 1)m_n(\sigma 1) + \mu(\sigma 0)m_n(\sigma 0) = \mu(\sigma 1)\Big(\sum_{\tau \in S_n, \tau \succeq \sigma 1} \frac{\mu(\tau)}{\mu(\sigma 1)}\Big) + \mu(\sigma 0)\Big(\sum_{\tau \in S_n, \tau \succeq \sigma 0} \frac{\mu(\tau)}{\mu(\sigma 0)}\Big)$$

Factoring out the denominators, we get

$$\frac{\mu(\sigma 1)}{\mu(\sigma 1)}\Big(\sum_{\tau \in S_n, \tau \succeq \sigma 1} \mu(\tau)\Big) + \frac{\mu(\sigma 0)}{\mu(\sigma 0)}\Big(\sum_{\tau \in S_n, \tau \succeq \sigma 0} \mu(\tau)\Big)$$

$$\Big(\sum_{\tau \in S_n, \tau \succeq \sigma 1} \mu(\tau)\Big) + \Big(\sum_{\tau \in S_n, \tau \succeq \sigma 0} \mu(\tau)\Big) = \sum_{\tau \in S_n, \tau \succeq \sigma} \mu(\tau)$$

Thus,

$$\frac{\mu(\sigma)}{\mu(\sigma)} \sum_{\tau \in S_n, \tau \succeq \sigma 1} \mu(\tau) = \mu(\sigma) \sum_{\tau \in S_n, \tau \succ \sigma} \frac{\mu(\tau)}{\mu(\sigma)} = \mu(\sigma)m_n(\sigma)$$

Thus m_n is a μ-martingale, and $\{m_n\}_{n\in\omega}$ is a uniformly c.e. collection of μ-martingales. Furthermore, $m_n(\emptyset) = \Sigma_{\tau \in S_n} \mu(\tau) \leq 2^{-n}$, so $m = \Sigma_{n\in\omega} m_n$ is a c.e. μ-martingale by a slight modification of Proposition 6.3.2 of Downey-Hirschfeldt [6]. Finally, it follows that m succeeds on X if and only if $X \in \bigcap_{n\in\omega} U_n$. □

Astor [2] proved that 1-Random sets have density $\frac{1}{2}$ by referring to Propositions 3.2.13 and 3.2.16 of Nies [16], which state that 1-Randoms must have density $\frac{1}{2}$ and that they are closed under permutations. In fact the more general result that r-1-Randoms have intrinsic density r is true, and we provide a simple proof here for convenience. The techniques are simple modifications to those found in Nies [16] and Downey-Hirschfeldt [6].

Lemma 1.12. *Let $r \in (0, 1)$. If X is r-1-Random, then X has intrinsic density r.*

Proof. We shall first show that r-random sets must have density r. This is natural when one considers the martingale approach to randomness: If we expect the ratio of ones to be larger than r, then we shall bet more of our capital on ones. If we do so carefully, then our betting strategy will succeed on sets with sufficiently large upper density. Prior work has been done studying the relationship between (martin)gales and the density of a set, especially relating to dimension. For example, see Lutz [14]. We shall give a straightforward calculus proof that is sufficient for our purposes. If r is not computable, then we will implicitly work relative to a given representation of μ_r, which can compute r.

Formally, we define a family of martingales such that at least one will succeed on any set with upper density greater than r. Let $0 < \alpha < 1 - r$ be rational and consider the martingale $M_\alpha : 2^{<\omega} \to \mathbb{Q}$ defined via:

- $M_\alpha(\emptyset) = 1$
- $M_\alpha(\sigma 0) = (1 - \frac{\alpha}{1-r})M_\alpha(\sigma)$
- $M_\alpha(\sigma 1) = (1 + \frac{\alpha}{r})M_\alpha(\sigma)$

It is immediate that M_α is a computable r-martingale from definition. If $n_\sigma = |\{k < |\sigma| : \sigma(k) = 1\}|$,

$$M_\alpha(\sigma) = (1 + \frac{\alpha}{r})^{n_\sigma}(1 - \frac{\alpha}{1-r})^{|\sigma| - n_\sigma}$$

Let $r < \epsilon \leq 1$. If $\rho_{|\sigma|}(\sigma) \geq \epsilon$, then $n_\sigma \geq \epsilon|\sigma|$ and

$$M_\alpha(\sigma) \geq (1 + \frac{\alpha}{r})^{\epsilon|\sigma|}(1 - \frac{\alpha}{1-r})^{(1-\epsilon)|\sigma|} = ((1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon})^{|\sigma|}$$

Notice that for a fixed ϵ, an exercise in calculus shows that α can be chosen such that $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$: As $\alpha < 1 - r$, $1 - \frac{\alpha}{1-r} > 0$, so we can take the logarithm. $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$ if and only if

$$\epsilon \log(1 + \frac{\alpha}{r}) + (1 - \epsilon)\log(1 - \frac{\alpha}{1-r}) > 0$$

Rearranging, this occurs if and only if

$$\log(1 - \frac{\alpha}{1-r}) > \epsilon(\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r}))$$

As $1 - \frac{\alpha}{1-r} < 1$ and $1 + \frac{\alpha}{r} > 1$,

$$\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r}) < 0$$

and the previous expression can be rearranged to obtain

$$\frac{\log(1 - \frac{\alpha}{1-r})}{\log(1 - \frac{\alpha}{1-r}) - \log(1 + \frac{\alpha}{r})} < \epsilon$$

By L'Hôpital's Rule, the limit of the left hand side as α approaches 0 is r. As $\epsilon > r$, there is α close enough to 0 such that this is true, and thus such that $(1 + \frac{\alpha}{r})^{\epsilon}(1 - \frac{\alpha}{1-r})^{1-\epsilon} > 1$ is true.

For such an α, M_α succeeds on any set X whose upper density is greater than ϵ, as this implies that there are infinitely many n such that $M_\alpha(X \upharpoonright n) \geq ((1 + \alpha)^{\epsilon}(1 - \alpha)^{1-\epsilon})^n$. Therefore, for any X with $\overline{\rho}(X) > R$, there is an $\epsilon > R$ with $\overline{\rho}(X) \geq \epsilon$. The corresponding M_α thus succeeds on X. Additionally, for any set X with lower density less than R, the same analysis can be applied to the complement. By switching the roles of $(1 + \frac{\alpha}{R})$ and $(1 - \frac{\alpha}{1-R})$ in the construction of M_α, we obtain an r-martingale which succeeds on X. Therefore any r-1-Random set must have density r.

Now we shall show that r-1-Random sets are also closed under permutation, completing the proof. Here the classical notion of martingales does not work as well, as permutations do not select bits monotonically in general like martingales do. However, it is not difficult to see that permutations preserve μ_r, so we shall prove this result using the measure notion of randomness, which is enough due to Theorem 1.11.

Given $\sigma \in 2^{<\omega}$, consider $[\sigma] = \{X \in 2^\omega : \sigma \preceq X\}$. For π a computable permutation, let

$$[\pi(\sigma)] = \{X \in 2^\omega : X(\pi(n)) = \sigma(n) \text{ for all } n < |\sigma|\}$$

Notice that $[\pi(\sigma)]$ is open. Furthermore, let $k = max_{n<|\sigma|}\{\pi(n)\}$. Then

$$P_\sigma = \{\tau \in 2^{k+1} : \tau(\pi(n)) = \sigma(n) \text{ for all } n < |\sigma|\}$$

is a prefix-free set which defines $[\pi(\sigma)]$. Then for all σ it follows from the definition of $[\pi(\sigma)]$ that

$$\mu_r([\pi(\sigma)]) = \sum_{\tau \in P_\sigma} \mu_r(\tau) = \mu_r(\sigma) \sum_{\gamma \in 2^{k+1-|\sigma|}} \mu_r(\gamma) = \mu_r([\sigma])$$

If $\{\mathcal{U}_i\}_{i\in\omega}$ is a μ_r-Martin-Löf test, then let $\mathcal{V}_i$ be defined via

$$\mathcal{V}_i = \bigcup_{\sigma \in U_i} [\pi(\sigma)]$$

By the above, $\mu_r(\mathcal{V}_i) = \mu_r(\mathcal{U}_i)$, so $\{\mathcal{V}_i\}_{i\in\omega}$ is also a μ_r-Martin-Löf test because π is computable. X passes $\{\mathcal{U}_i\}_{i\in\omega}$ if and only if $\pi(X)$ passes $\{\mathcal{V}_i\}_{i\in\omega}$ by definition. Therefore if Y is not r-1-Random, then $\pi^{-1}(Y)$ is not r-1-Random either. Thus the r-1-Randoms are closed under computable permutation as desired. □

Remark. *The previous lemma also holds for computable and Schnorr randomness, however the proofs are more complex and outside our purview for this paper. The above proof that r-1-Random sets have density r proves the same for computably r-Random sets as M_α is computable, however proving that computable randoms are closed under computable permutations is more difficult. For such a proof, see Nies* [16] *7.6.24. Our proof of closure under computable permutations applies to Schnorr randoms without modification, however proving that r-Schnorr randoms have density r is more difficult: see 3.5.21 of Nies* [16].

We therefore obtain every real in the unit interval as the intrinsic density of some set. However, as we shall see, there is a large gap between intrinsic density and randomness. We would like to construct or find sets with arbitrary intrinsic density without needing to appeal to full randomness to better understand them. (We shall still use some randomness, but only regular Martin-Löf randomness, and only for convenience.) To achieve this, we shall introduce the `into` and `within` operations in Section 3 to develop new sets with defined intrinsic density from old ones.

We would like to take a set A of intrinsic density α and a set B of intrinsic density β and somehow code B and A in such a way that we are left with a set which has new intrinsic density obtained as some function of α and β. However, we shall show in Section 2 that we cannot hope to code things in a nice computable way that allows us to recover the original sets, as intrinsic density was defined with the intention of blocking computable coding in the setting of asymptotic computability.

Our main technique will involve proving that two sets A and B cannot have different intrinsic densities by creating a computable permutation which sends A to B modulo a set of density zero. The following lemma shows that if we can do this, then the density of the image of A is the same as the density of B, and therefore that they cannot have different intrinsic densities.

Lemma 1.13. *If $\overline{\rho}(H) = 0$, then $\overline{\rho}(X \setminus H) = \overline{\rho}(X \cup H) = \overline{\rho}(X)$ and $\underline{\rho}(X \setminus H) = \underline{\rho}(X \cup H) = \underline{\rho}(X)$.*

Proof. Notice that

$$\rho_n(X) = \rho_n(X \setminus H) + \rho_n(X \cap H)$$

By definition. Therefore

$$\overline{\rho}(X) = \limsup_{n\to\infty} \rho_n(X) = \limsup_{n\to\infty} \rho_n(X \setminus H) + \rho_n(X \cap H)$$

By subadditivity of the limit superior,

$$\overline{\rho}(X) \leq \limsup_{n\to\infty} \rho_n(X \setminus H) + \limsup_{n\to\infty} \rho_n(X \cap H)$$

As $\overline{\rho}(H) = 0$ and $X \cap H \subseteq H$,

$$\overline{\rho}(X) \leq \limsup_{n\to\infty} \rho_n(X \setminus H) = \overline{\rho}(X \setminus H)$$

However, $\overline{\rho}(X \setminus H) \leq \overline{\rho}(X)$ because $X \setminus H \subseteq X$, so $\overline{\rho}(X) = \overline{\rho}(X \setminus H)$ as desired.

The argument for the union and the argument for lower density are functionally identical. (For the union we use $X \cup H$, X, and $H \setminus X$ in place of X, $X \setminus H$, and $X \cap H$ respectively.) □

We begin by illustrating why the classical operations for combining two sets fail to yield new intrinsic densities in Section 2, motivating the creation of new tools. Section 3 will introduce the two key operations, `into` and `within`, for obtaining sets with defined intrinsic densities and use them to construct a set of arbitrary intrinsic density $r \in (0, 1)$ from any Martin-Löf random and r. Section 3 shall also provide some applications for these tools in the study of Turing degrees, including answering an open question of the author from [15]. We shall close some natural gaps that arise in Sections 2 and 3 in Section 4, then we will conclude in Section 5 by applying our techniques from Section 3 to study MWC and Church density.

2. The Failure of Classical Coding

2.1. **The Join.** As mentioned previously, we would like to find some operation that takes sets A and B of intrinsic density α and β respectively and outputs a new set with intrinsic density which is given as a function of α and β. The most common operation for combining two sets in computability theory is the join. It is easy to show that if A has density α and B has density β, then $A \oplus B$ has density $\frac{\alpha+\beta}{2}$. However, this is not so simple in the case of intrinsic density.

Lemma 2.1. *If $P(A) \neq P(B)$, then $A \oplus B$ does not have intrinsic density.*

Proof. We shall proceed by showing that there is a computable permutation which sends $A \oplus B$ to A modulo a set of density 0, and similarly for B. Then the upper (and lower) density of $A \oplus B$ under these permutations will match that of A and B respectively. Therefore if these densities are different, the density of $A \oplus B$ is not invariant under computable permutation.

Let $F = \{n! : n \in \omega\}$ and $G = \overline{F}$. For any fixed computable permutation π, there is another computable permutation $\hat{\pi}$ defined via enumerating the odds onto the factorials in order and enumerating the evens onto the nonfactorials according to the ordering induced by π. That is, $\hat{\pi}(2n+1) = f_n$ and $\hat{\pi}(2n) = g_{\pi(n)}$.

Then as F has density 0, Lemma 1.13 shows

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus B) \setminus F)$$

As the image of the odds under $\hat{\pi}$ is a subset of F,

$$\hat{\pi}(A \oplus B) \setminus F = \hat{\pi}(A \oplus \emptyset)$$

and

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus \emptyset))$$

Notice that $\hat{\pi}(A \oplus \emptyset)$ is just $\pi(A)$ with each element n increased by $|F \upharpoonright n|$. Thus

$$\rho_n(\pi(A)) \geq \rho_n(\hat{\pi}(A \oplus \emptyset)) \geq \frac{|\pi(A) \upharpoonright n| - |F \upharpoonright n|}{n}$$

As F is the factorials, the final expression tends to $\rho_n(\pi(A))$ in the limit, so we see that

$$\overline{\rho}(\hat{\pi}(A \oplus \emptyset)) = \overline{\rho}(\pi(A))$$

and

$$\overline{\rho}(\hat{\pi}(A \oplus B)) = \overline{\rho}(\hat{\pi}(A \oplus \emptyset)) = \overline{\rho}(\pi(A))$$

$\underline{\rho}(\hat{\pi}(A \oplus B)) = \underline{\rho}(\pi(A))$ by a nearly identical argument.

In particular, $\overline{P}(A \oplus B) \geq \overline{P}(A)$ and $\underline{P}(A \oplus B) \leq \underline{P}(A)$ because we are taking the limit superior and inferior over all computable permutations, of which $\hat{\pi}$ is but one. (Basically, $\hat{\pi}$ sends $A \oplus B$ to $\pi(A)$ modulo a set of density zero, so the intrinsic upper (lower) density of $A \oplus B$ cannot be smaller (larger) than the intrinsic upper (lower) density of A.) Reversing the use of the evens and the odds in the definition of $\hat{\pi}$, we get that the same is true for B in place of A, so $\underline{P}(A \oplus B) \leq min(\underline{P}(A), \underline{P}(B))$ and $\overline{P}(A \oplus B) \geq max(\overline{P}(A), \overline{P}(B))$. Therefore if $P(A) \neq P(B)$, $\underline{P}(A \oplus B) \neq \overline{P}(A \oplus B)$. □

2.2. **The Cartesian Product.** Another classical candidate would be the Cartesian product $A \times B$. However, this is even less reliable than the join. Whether or not $A \times B$ even has *asymptotic* density related to the density of A and the density of B can depend on the selected pairing function. For example, if $\langle , \rangle : \omega^2 \to \omega$ is a pairing function, consider the function $f : \omega^2 \to \omega$ defined via

$$f(i, n) = \langle i - 1, n \rangle!$$

for $i > 0$ and

$$f(0, n) = s_n$$

where S is the set of nonfactorials. Then f has all of the properties we desire in a pairing function, i.e. it is a computable bijection with computable inverse between ω^2 and ω. Using f as a pairing function, $A \times B$ (as a set of codes for pairs $\langle a, b \rangle$, $a \in A$ and $b \in B$) would have density equal to that of B if $0 \in A$ and density 0 otherwise. Removing or adding a single element from A never changes the density, let alone the intrinsic density, but we could toggle the upper density of $A \times B$ between 0 and $\overline{\rho}(B)$ by toggling whether or not 0 is in A.

Even if we fix a pairing function $\langle , \rangle$ which does respect the density of A and B, the above f shows that this will not extend to intrinsic density: As f and $\langle , \rangle$ are both computable and have computable inverse, there is a permutation π such that $\pi(\langle n, m \rangle) = f(n, m)$. Then $\pi(A \times B)$ will be as in the previous paragraph, so $A \times B$ cannot have intrinsic density determined by the intrinsic densities of A and B.

These methods seem like they should generalize to any attempt at "nicely" coding A and B into computable sets in such a way that we can easily recover them. This intuition will be formalized in Theorem 3.3.

3. `Into`, `Within`, and Intrinsic Density

We already know that every real in the unit interval is achieved as an intrinsic density by finding a set with the correct type of randomness. (For intrinsic density 0 and 1, randomness will only give the trivial examples $\emptyset$ and ω. However, nontrivial examples are known to exist from the work in [2], [3], and [15].) However, the reliance on randomness here is not ideal: intrinsic density is itself not a good notion of randomness as there are sets with defined intrinsic density which can be computed by arbitrarily small subsets: Let A be 1-Random and let $X_0 = A$ and $X_{n+1} = X_n \oplus X_n$. By Lemma 1.12 and Theorem 4.1 below, X_k will be a set of intrinsic density $\frac{1}{2}$, but $\{n : 2^k n \in X_k\} = A$, so there is a subset with density $\frac{1}{2^{k+1}}$ which computes all of X_k. Therefore our goal is to create tools for working with intrinsic density that work solely at the level of intrinsic density rather than the stronger level of randomness.

The methods of Section 2 illustrate why coding methods that enumerate a set onto a computable one are insufficient for our purposes. As long as we computably know where one of our sets A is being coded, there is a permutation which can make the resulting set look like A modulo a set of density 0, so the best case scenario is that the resulting set can have the same intrinsic density as the original sets. We therefore must use a coding method which is not inherently computable to achieve our goals of changing the density. For example, Astor [2] proved that if A has intrinsic density α and B is 1-Random relative to A, then $A \cap B$ has intrinsic density $\frac{\alpha}{2}$. We shall generalize this approach and prove that it works with a much weaker requirement of relative intrinsic density rather than relative randomness.

The following coding methods are natural and computable in A and B, but do not allow us to recover A or B easily, and so do not fall prey to the methods of the previous section. The remainder of the paper shall be dedicated to their study. We shall show in this section that these tools generalize the previous known results and do not require appeals to randomness. We shall then apply them to achieve every real in the unit interval. (This part will use randomness, but only as a convenience to obtain sets with the right properties: the core theorems are stated without need for randomness.)

Definition 3.1. *Let A and B be sets of natural numbers.*

- *$B \triangleright A$, or B into A, is*
$$\{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$$
That is, $B \triangleright A$ is the subset of A obtained by taking the "B-th elements of A."
- *$B \triangleleft A$, or B within A, is*
$$\{n : a_n \in B\}$$
That is, $B \triangleleft A$ is the set X such that $X \triangleright A = A \cap B$.

With $A \triangleright B$, we are simply thinking of A as a copy of ω as a well-order and $B \triangleright A$ is the subset corresponding to B under the order preserving isomorphism between A and ω. The intuition for why this might work for our purposes is that if a computable permutation on ω could change the size of a copy of B living inside A, then it must have been able to change the size of B or A to begin with. We shall see below that this intuition is correct and $B \triangleright A$ will work elegantly with intrinsic density, multiplying the intrinsic densities of A and B so long as some conditions are met.

We first make a few elementary observations:

- For all A, $A = A \triangleright \omega = \omega \triangleright A = A \triangleleft \omega$.
- For all A and B and any i, a_i is either in B or $\overline{B}$. Therefore i is either in $B \triangleleft A$ or $\overline{B} \triangleleft A$ respectively, so $(B \triangleleft A) \sqcup (\overline{B} \triangleleft A) = \omega$.
- If A is intrinsically small, then so is $X \triangleright A$ for any X, as intrinsic smallness is closed under subsets. The same is not true for $X \triangleleft A$, as in general it is not necessarily a subset of A or X.
- If $B \cap C = \emptyset$, then $(B \triangleright A) \cap (C \triangleright A) = \emptyset$. Furthermore, $A = (X \triangleright A) \sqcup (\overline{X} \triangleright A)$.
- A set A has MWC-density r if $\rho(A \triangleleft f(A)) = r$ for all partial computable monotone selection functions f.
- $\triangleright$ is associative, i.e. $B \triangleright (A \triangleright C) = (B \triangleright A) \triangleright C$: By definition, $(A \triangleright C) = \{c_{a_0} < c_{a_1} < c_{a_2} < \dots\}$ and thus
$$B \triangleright (A \triangleright C) = \{c_{a_{b_0}} < c_{a_{b_1}} < c_{a_{b_2}} < \dots\}$$
Similarly, $(B \triangleright A) = \{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$, and therefore by definition
$$(B \triangleright A) \triangleright C = \{c_{a_{b_0}} < c_{a_{b_1}} < c_{a_{b_2}} < \dots\}$$
- $\triangleleft$ is not associative: Consider the set of evens E, the set of odds O, and the set N of evens which are not multiples of 4. Then
$$(O \triangleleft N) \triangleleft E = \emptyset \triangleleft N = \emptyset$$
However,
$$O \triangleleft (N \triangleleft E) = O \triangleleft O = \omega$$
- $\triangleright$ and $\triangleleft$ do not associate with each other in general:
$$B \triangleright (A \triangleleft (B \triangleright A)) = B \triangleright \omega = B$$
but
$$(B \triangleright A) \triangleleft (B \triangleright A) = \omega$$
Similarly, $B \triangleleft (A \triangleright B) = \omega$, but $(B \triangleleft A) \triangleright B$ is a subset of B.

The following theorem is quite intuitive and allows us to use a single set with defined intrinsic density to find new ones, however these new sets will have the same intrinsic density as the original set. We shall first prove a technical lemma to aid in our proofs.

Lemma 3.2. *Let $f_0, f_1, \dots, f_k$ be a finite collection of injective computable functions and let C be a computable set. Then there is a computable set $H \subseteq C$ such that $\overline{\rho}(f_i(H)) = 0$ for all i.*

Proof. Let $h_0 = c_0$. Then given h_n, define h_{n+1} to be the least element c of C with $f_i(c) \geq h_n!$ for all i. Set $H = \{h_0 < h_1 < h_2 < \dots\}$. Then $\overline{\rho}(f_i(H)) = 0$ for all i because $|f_i(H) \upharpoonright n| \leq |\{n! : n \in \omega\} \upharpoonright n|$. □

Theorem 3.3. *Let C be computable and $P(A) = \alpha$. Then $P(A \triangleleft C) = \alpha$.*

Proof. Under the map which takes c_n to n, $A \cap C$ is mapped to $A \triangleleft C$. However unless C is ω, this is not a permutation. Using Lemma 3.2, we are able to massage this map into a permutation which takes c_n to n modulo a set of density 0. Then under this permutation, $A \cap C$ (and A) goes to $A \triangleleft C$ modulo a set of density 0. Therefore if $A \triangleleft C$ did not have intrinsic density α, A could not either by Lemma 1.13.

Formally, assume $P(A \triangleleft C) \neq \alpha$. Suppose π is a computable permutation with $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$. Let $f : C \to \omega$ be defined via $f(c_n) = n$. Then $f(A \cap C) = A \triangleleft C$:

$$A \cap C \xrightarrow{f} A \triangleleft C \xrightarrow{\pi} \pi(A \triangleleft C)$$

By Lemma 3.2, there is $H \subseteq C$ computable with $\overline{\rho}(\pi(f(H))) = 0$. Define $\pi_f : \omega \to \omega$ via $\pi_f(n) = f(n)$ for $n \in C \setminus H$, and for $n \in \overline{C} \sqcup H$ define $\pi_f(n)$ to be the least element of $f(H)$ not equal to $\pi_f(j)$ for some $j < n$. As f agrees with π_f on $C \setminus H$,

$$\pi_f((A \cap C) \setminus H) = f(A \cap C) \setminus f(H) = (A \triangleleft C) \setminus f(H)$$

Therefore by applying π,

$$\pi(\pi_f((A \cap C) \setminus H)) = \pi((A \triangleleft C) \setminus f(H)) = \pi(A \triangleleft C) \setminus \pi(f(H))$$

Using the above equality,

$$\overline{\rho}(\pi(\pi_f((A \cap C) \setminus H))) = \overline{\rho}(\pi(A \triangleleft C) \setminus \pi(f(H)))$$

As $\overline{\rho}(\pi(f(H))) = 0$, we can apply Lemma 1.13 and see

$$\overline{\rho}(\pi(A \triangleleft C) \setminus \pi(f(H))) = \overline{\rho}(\pi(A \triangleleft C))$$

As $(A \cap C) \setminus H \subseteq A$,

$$\overline{\rho}(\pi(\pi_f(A))) \geq \overline{\rho}(\pi(\pi_f((A \cap C) \setminus H))) = \overline{\rho}(\pi(A \triangleleft C))$$

However, we assumed that $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$, so $\overline{\rho}(\pi(\pi_f(A))) > \alpha$. As $\pi \circ \pi_f$ is a computable permutation, this implies $P(A) \neq \alpha$.

This proves that if π is a computable permutation with $\overline{\rho}(\pi(A \triangleleft C)) > \alpha$, then $P(A) \neq \alpha$. If there is no such permutation, there must be a computable permutation π with $\underline{\rho}(\pi(A \triangleleft C)) < \alpha$ because we assumed that $P(A \triangleleft C) \neq \alpha$. Then because

$$(\pi(A \triangleleft C)) \sqcup (\pi(\overline{A} \triangleleft C)) = \pi((A \triangleleft C) \sqcup (\overline{A} \triangleleft C)) = \pi(\omega) = \omega$$

we have $\rho_n(\pi(\overline{A} \triangleleft C)) = 1 - \rho_n(\pi(A \triangleleft C))$ for all n. Therefore by the subtraction properties of the limit superior,

$$\overline{\rho}(\pi(\overline{A} \triangleleft C)) \geq 1 - \underline{\rho}(\pi(A \triangleleft C))$$

As we assumed $\underline{\rho}(\pi(A \triangleleft C)) < \alpha$,

$$1 - \underline{\rho}(\pi(A \triangleleft C)) > 1 - \alpha$$

Thus $\overline{\rho}(\pi(\overline{A} \triangleleft C)) > 1-\alpha$. We now apply the previous case to get that $P(\overline{A}) \neq 1-\alpha$, which automatically implies $P(A) \neq \alpha$. □

We obtain an alternate proof of Lemma 2.1 as a corollary of this result.

Corollary 3.4. *(Lemma 2.1) If $P(A) \neq P(B)$, then $A \oplus B$ does not have intrinsic density.*

Proof. Suppose $A \oplus B$ has intrinsic density γ. Let E be the set of even numbers and O the set of odd numbers. By Theorem 3.3,

$$P((A \oplus B) \triangleleft E) = P((A \oplus B) \triangleleft O) = \gamma$$

However $(A \oplus B) \triangleleft E = A$ and $(A \oplus B) \triangleleft O = B$, so $P(A) = P(B) = \gamma$. □

In addition to giving a much simpler proof of Lemma 2.1, this result is confirming what we might suspect given the results of Section 2: we cannot achieve sets of new intrinsic density by enumerating sets of intrinsic density along computable sets, as the resulting set must have the same intrinsic density if it has intrinsic density at all. Therefore we need to turn our attention to coding within noncomputable sets.

We now make an observation about the asymptotic density of $B \triangleright A$, which will be critical for investigating its intrinsic density.

Lemma 3.5.

- $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(B)\overline{\rho}(A)$.
- $\underline{\rho}(B \triangleright A) \geq \underline{\rho}(B)\underline{\rho}(A)$.

Proof. By Lemma 1.2,

$$\overline{\rho}(B \triangleright A) = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot 1 = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot \frac{b_n+1}{b_n+1}$$

By the submultiplicativity of the limit superior,

$$\overline{\rho}(B \triangleright A) \leq (\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})(\limsup_{n\to\infty} \frac{n+1}{b_n+1}) = (\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})\overline{\rho}(B)$$

Now $\{\frac{b_n+1}{a_{b_n}+1}\}_{n\in\omega}$ is a subsequence of $\{\frac{n+1}{a_n+1}\}_{n\in\omega}$, so

$$\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1} \leq \limsup_{n\to\infty} \frac{n+1}{a_n+1} = \overline{\rho}(A)$$

Therefore $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(B)\overline{\rho}(A)$ as desired.

The case for the limit inferior is nearly identical, reversing $\leq$ to $\geq$ and using supermultiplicativity along with the corresponding identity from Lemma 1.2. □

Corollary 3.6. *If $\rho(A) = \alpha$ and $\rho(B) = \beta$, then $\rho(B \triangleright A) = \alpha\beta$.*

Therefore, if $B \triangleright A$ has intrinsic density, its intrinsic density must be the product of the densities of A and B. Our next goal is to prove that $B \triangleright A$ does indeed have defined intrinsic density with sufficient assumptions on A and B. Recall that a set X has Y-intrinsic density, or intrinsic density relative to Y, if its density is invariant under all Y-computable permutations as opposed to just the computable ones. We use $P_Y(X)$ to denote the Y-intrinsic density of X if it exists.

Theorem 3.7. *If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(B \triangleright A) = \alpha\beta$.*

Proof. The proof is very similar to the proof of Theorem 3.3, however we shall present it fully here without referring to techniques from that proof, as it is quite technical. Here the idea is that for any fixed computable permutation π, there is an A-computable permutation which sends B to $\pi(B \triangleright A) \triangleleft \pi(A)$ modulo a set of density 0. Therefore if π witnesses that $B \triangleright A$ does not have intrinsic density $\alpha\beta$, i.e. $\pi(B \triangleright A)$ does not have density $\alpha\beta$, and A has intrinsic density α, Lemma 3.5 will show that $\pi(B \triangleright A) \triangleleft \pi(A)$ does not have density β, and thus B does not have A-intrinsic density β.

Formally, assume $P(A) = \alpha$. Assume that $P(B \triangleright A) \neq \alpha\beta$. We shall show that $P_A(B) \neq \beta$. First suppose that there is some computable permutation π such that $\overline{\rho}(\pi(B \triangleright A)) > \alpha\beta$. We shall let $\pi(A) = \{p_0 < p_1 < p_2 < \dots\}$. Let $f : A \to \omega$ be defined via $f(a_n) = n$ and $g : \pi(A) \to \omega$ via $g(p_n) = n$, i.e. f maps A to its indices and g maps $\pi(A)$ to its indices. Then $f(B \triangleright A) = B$ and $g(\pi(B \triangleright A)) = \pi(B \triangleright A) \triangleleft \pi(A)$:

$$\begin{array}{ccc} B \triangleright A & \xrightarrow{\ \pi\ } & \pi(B \triangleright A) \\ \big\downarrow f & & \big\downarrow g \\ B & & \pi(B \triangleright A) \triangleleft \pi(A) \end{array}$$

Note by Lemma 3.5 that $\overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) > \beta$: From the definition,

$$(\pi(B \triangleright A) \triangleleft \pi(A)) \triangleright \pi(A)) = \pi(B \triangleright A)$$

and $\overline{\rho}(B \triangleright A) > \alpha\beta$ by assumption. $\overline{\rho}(\pi(A)) = \alpha$ because $P(A) = \alpha$, so $\overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) \leq \beta$ would contradict Lemma 3.5.

From this point forward we shall let

$$X = \pi(B \triangleright A) \triangleleft \pi(A)$$

for the sake of readability.

By Lemma 3.2 relativized to A and applied to $g \circ \pi$, there is an A-computable set $H \subseteq A$ such that:

$$\overline{\rho}(g(\pi(H))) = 0$$

We shall now define permutations which preserve the properties of f and g outside of H. Define $\pi_f : \omega \to \omega$ via $\pi_f(k) = f(k)$ for $k \in A \setminus H$, and for $k \in \overline{A} \sqcup H$, let $\pi_f(k)$ be the least element of $f(H)$ not equal to $\pi_f(m)$ for some $m < k$. Define $\pi_g : \omega \to \omega$ similarly using $\pi(A)$, $\pi(H)$, and $g(\pi(H))$ in place of A, H, and $f(H)$ respectively. Then π_f and π_g are A-computable because H, f, and g are, and it is a permutation because f and g are bijections (from A and $\pi(A)$ to ω respectively) which have been modified to be total without violating injectivity or surjectivity.

Now we shall compute $\pi_g(\pi(\pi_f^{-1}(B \setminus f(H))))$. As $f(B \triangleright A) = B$ and f agrees with π_f on $\overline{H}$,

$$\pi_f^{-1}(B \setminus f(H)) = (B \triangleright A) \setminus H$$

Furthermore

$$\pi((B \triangleright A) \setminus H) = \pi(B \triangleright A) \setminus \pi(H)$$

As $g(\pi(B \triangleright A)) = X$ and π_g agrees with g on $\overline{\pi(H)}$,

$$\pi_g(\pi(B \triangleright A) \setminus \pi(H)) = g(\pi(B \triangleright A)) \setminus g(\pi(H)) = X \setminus g(\pi(H))$$

Thus $\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))) = X \setminus g(\pi(H))$. As $\overline{\rho}(g(\pi(H)) = 0$, Lemma 1.13 shows

$$\overline{\rho}(X \setminus g(\pi(H))) = \overline{\rho}(X)$$

By the definition of X,

$$\overline{\rho}(X) = \overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A))$$

which is greater than β by the above. As $B \setminus f(H) \subseteq B$,

$$\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))) \subseteq \pi_g(\pi(\pi_f^{-1}(B)))$$

and thus

$$\overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B)))) \geq \overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B \setminus f(H)))))$$

Therefore

$$\overline{\rho}(\pi_g(\pi(\pi_f^{-1}(B)))) \geq \overline{\rho}(\pi(B \triangleright A) \triangleleft \pi(A)) > \beta$$

As $\pi_g \circ \pi \circ \pi_f^{-1}$ is an A-computable permutation, $P_A(B) \neq \beta$.

Therefore we have proved that if there is some computable permutation π such that $\overline{\rho}(\pi(B \triangleright A)) > \alpha\beta$, then $P_A(B) \neq \beta$. If there is no such permutation, then there must be a computable permutation π such that $\underline{\rho}(\pi(B \triangleright A)) < \alpha\beta$ because we assumed $P(B \triangleright A) \neq \alpha\beta$. As $A = (B \triangleright A) \sqcup (\overline{B} \triangleright A)$, $\pi(A) = \pi(B \triangleright A) \sqcup \pi(\overline{B} \triangleright A)$. Therefore

$$\overline{\rho}(\pi(\overline{B} \triangleright A)) = \overline{\rho}(\pi(A) \setminus \pi(B \triangleright A))$$

The fact that $\rho_n(\pi(A)) = \rho_n(\pi(B \triangleright A)) + \rho_n(\pi(A) \setminus \pi(B \triangleright A))$ combined with the properties of the limit superior with regards to subtraction implies

$$\overline{\rho}(\pi(A) \setminus \pi(B \triangleright A)) \geq \overline{\rho}(\pi(A)) - \underline{\rho}(\pi(B \triangleright A))$$

We know that $\overline{\rho}(\pi(A)) = \alpha$ because $P(A) = \alpha$. As we assumed that $\underline{\rho}(\pi(B \triangleright A)) < \alpha\beta$,

$$\overline{\rho}(\pi(\alpha)) - \underline{\rho}(\pi(B \triangleright A)) > \alpha - \alpha\beta = \alpha(1 - \beta)$$

Bringing this together,

$$\overline{\rho}(\pi(\overline{B} \triangleright A)) > \alpha(1 - \beta)$$

Thus we can apply the first case of the proof to show that $P_A(\overline{B}) \neq 1 - \beta$, which automatically implies $P_A(B) \neq \beta$, so we are done. □

Notice that the previous two theorems prove a more general form of Astor's result:

Corollary 3.8. *If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(A \cap B) = \alpha\beta$.*

Remark. *Astor* [2] *proved this for the special case when A has intrinsic density α and B is 1-Random relative to A, which by Lemma 1.12 implies B has A-intrinsic density $\frac{1}{2}$.*

Proof. By definition,

$$A \cap B = (B \triangleleft A) \triangleright A$$

As $P_A(B) = \beta$, Theorem 3.3 relativized to A shows that $P_A(B \triangleleft A) = \beta$. Therefore we can apply Theorem 3.7 to A and $B \triangleleft A$ to get that

$$P((B \triangleleft A) \triangleright A) = P(A \cap B) = \alpha\beta$$

□

With these tools in hand, we may now look towards constructing a set of arbitrary intrinsic density. To do this, we would like to have a countable collection of sets which all have intrinsic density relative to each other so that we may apply Theorem 3.7 repeatedly.

Lemma 3.9. *There is a countable, disjoint sequence of sets $\{A_i\}_{i \in \omega}$ such that $P(A_i) = \frac{1}{2^{i+1}}$. Furthermore,* $\lim_{n \to \infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$.

Proof. Recall that given a set X, $X^{[i]}$ denotes the i-th column of X, i.e. $\{n : \langle i, n \rangle \in X\}$. Let $X \subseteq \omega$ be 1-Random. Then for all i, $X^{[i]}$ is 1-Random relative to $\bigoplus_{j \neq i} X^{[j]}$. (Essentially Van Lambalgen [19], Downey-Hirschfeldt [6] Corollary 6.9.6) Note that the proof of Lemma 1.12 relativizes to the fact that Z-1-Randoms have Z-intrinsic density $\frac{1}{2}$ easily. In particular, taking a single 1-Random automatically gives us infinitely many mutually 1-Random sets. Using these together with Theorem 3.7, we can construct the desired sequence, where the mutual randomness ensures us that the conditions of the theorem are met.

Let $B_0 = \omega$. Given B_n, let

$$A_n = \overline{X^{[n]}} \triangleright B_n$$

and

$$B_{n+1} = X^{[n]} \triangleright B_n$$

Note that for all i, $B_{i+1} \subseteq B_i$ and $A_i \cap B_{i+1} = \emptyset$, as $B_{i+1} = X^{[i]} \triangleright B_i$ and $A_i = \overline{X^{[i]}} \triangleright B_i$. Then for $i < j$, $A_i \cap A_j = \emptyset$ because $A_j \subseteq B_j \subseteq B_{i+1}$. Thus $\{A_i\}_{i \in \omega}$ is disjoint. We now verify that $P(A_i) = \frac{1}{2^{i+1}}$ and $P(B_i) = \frac{1}{2^i}$ by induction.

$P(B_0) = P(\omega) = 1$, and B_0 is computable. Suppose that B_i is $\bigoplus_{j<i} X^{[j]}$-computable and that $P(B_i) = \frac{1}{2^i}$. Then $B_{i+1} = X^{[i]} \triangleright B_i$ is $B_i \oplus X^{[i]}$-computable, and therefore $\bigoplus_{j<i+1} X^{[j]}$-computable. Then by the above, both $X^{[i]}$ and $\overline{X^{[i]}}$ are 1-Random relative to B_i. Therefore $P_{B_i}(X^{[i]}) = P_{B_i}(\overline{X^{[i]}}) = \frac{1}{2}$ by the relativization of Lemma 1.12. Thus by Theorem 3.7,

$$P(A_i) = P(\overline{X^{[i]}} \triangleright B_i) = P(\overline{X^{[i]}})P(B_i) = \frac{1}{2} \cdot \frac{1}{2^i} = \frac{1}{2^{i+1}}$$

A nearly identical argument for $P(B_{i+1})$ verifies $P(B_{i+1}) = \frac{1}{2^{i+1}}$, which completes the induction.

Finally, note that $\lim_{n \to \infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$ must be true for any such collection of sets, as $\lim_{n \to \infty} P(\bigsqcup_{i \leq n} A_i) = 1$. □

Jockusch and Schupp [10] proved that asymptotic density enjoys a restricted form of countable additivity: if there is a countable sequence $\{S_i\}_{i\in\omega}$ of disjoint sets such that $\rho(S_i)$ exists for all i and

$$\lim_{n\to\infty} \overline{\rho}(\bigsqcup_{i>n} S_i) = 0$$

then

$$\rho(\bigsqcup_{i\in\omega} S_i) = \sum_{i=0}^{\infty} \rho(S_i)$$

The intrinsic density analog of this results follows immediately from the fact that permutations preserve disjoint unions. That is, if there is a countable sequence $\{S_i\}_{i\in\omega}$ of disjoint sets such that $P(S_i)$ exists for all i and

$$\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} S_i) = 0$$

then

$$P(\bigsqcup_{i\in\omega} S_i) = \sum_{i=0}^{\infty} P(S_i)$$

This together with the previous lemma allows us to construct a set with intrinsic density r for any $r \in (0,1)$.

Corollary 3.10. *Every real in $(0,1)$ is realized as the intrinsic density of some set of natural numbers.*

Proof. Let $r \in (0,1)$. Let $B_r \subseteq \omega$ be the set whose characteristic function is identified with the binary expansion that gives r, i.e. the set of all n such that the n-th bit in the binary expansion for r is a one. Now let $\{A_i\}_{i\in\omega}$ be as in Lemma 3.9. Let $X_r = \bigsqcup_{n\in B_r} A_n$. Note that

$$\lim_{n\to\infty} \overline{P}(\bigsqcup_{i\in B_r, i>n} A_i) = 0$$

Because $\bigsqcup_{i\in B_r, i>n} A_i \subseteq \bigsqcup_{i>n} A_i$ and $\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$. By the fact that countable unions sum intrinsic densities and the definition of X_r,

$$P(X_r) = \sum_{n\in B_r} P(A_n) = \sum_{n\in B_r} \frac{1}{2^{n+1}}$$

By the definition of the binary expansion,

$$P(X_r) = \sum_{n\in B_r} \frac{1}{2^{n+1}} = r$$

□

Notice that Corollary 3.10 can be relativized to a fixed oracle Y. Lemma 1.12 and Van Lambalgen's theorem both relativize, so Lemma 3.9 does as well. This suffices to relativize Corollary 3.10: in general, B_r cannot be taken to have any special relationship with Y, as it is unique for a fixed r. However, whether or not B_r is computable from the oracle has no basis on X_r because each A_n has Y-intrinsic density.

3.1. **Noncomputable Coding and the Turing Degrees.** One downside of the into operation is that the degree of $A \triangleright B$ is not necessarily equal to the degree of $A \oplus B$. This is because $A \triangleright B$ cannot necessarily compute A or B. However, given B as an oracle, $A \triangleright B$ can easily compute $A = \{n : b_n \in A \triangleright B\}$. Therefore, combining the into operation with the join allows us to prove results about Turing degrees.

Lemma 3.11. *Suppose $P \subseteq 2^\omega$ is closed under subsets and closed under self join, i.e. if $X \in P$ then $X \oplus X \in P$ also. Then the P-degrees are closed upwards.*

Proof. Suppose A computes B, with $B \in P$. Then $B \oplus B \in P$ as P is closed under self join. Furthermore, $B \oplus (A \triangleright B) \in P$ because it is a subset of $B \oplus B$. Furthermore, $B \oplus (A \triangleright B) \equiv_T A$ as A computes B and thus computes $B \oplus (A \triangleright B)$, which in turn computes A as above. Therefore $(B \oplus (A \triangleright B)$ witnesses that the degree of A is a P-degree. □

An example of the application of this result is an easy proof of the classic fact that the hyperimmune degrees are closed upwards: Hyperimmune sets are closed under subsets (As the principal function of a subset is greater than the principal function of the original set, so if the former is computably dominated then so is the latter) and are closed under self join (if $f(n)$ computably dominates $p_(B \oplus B)(n)$, then $f(2n)$ dominates $p_B(n)$.)

More interestingly, this answers Question 4.1 from [15].

Corollary 3.12. *For any X, the Turing degrees of X-intrinsically small (i.e. intrinsic density* 0*) sets are exactly the X-high or X-DNC degrees.*

Proof. As noted in [3], if noncomputable A is not X-high or X-DNC, then A does not have X-intrinsic density at all. Furthermore, Astor also proved that every high or DNC set computes an intrinsically small set, and this proof relativizes as noted in [15]. Thus every X-high or X-DNC degree computes an X-intrinsically small set. Therefore it suffices to show that the degrees of X-intrinsically small sets are closed upwards.

By the relativized form of [15] Corollary 2.7 or Theorem 4.1 below, the X-intrinsically small sets are closed under self join. They are clearly closed under subsets by the definition of intrinsic smallness. Therefore these degrees are closed upwards by Lemma 3.11, completing the proof. □

It is interesting to revisit what was known about this question prior. Astor proved the non-relativized version in [3], however he relied upon the result of Jockusch [9] that any collection of sets which is closed under subsets and contains an arithmetical set is closed upwards in the Turing degrees. As noted in [15] there are X for which there is no arithmetical X-intrinsically small set (for example, $\emptyset^{(\omega)}$), so for these X the proof does not relativize. Using the above result, though, this difficulty can be avoided.

An important note about Lemma 3.11 is that it is not a strengthening of Jockusch's result: In fact, Jockusch proved this theorem to show that the cohesive degrees are closed upwards, and cohesive sets are quite easily seen to not be closed under self join. In practice, it is likely that most natural phenomena being studied will have an arithmetical example, and thus Jockusch's result will apply. Therefore it is likely that Lemma 3.11 will mostly be useful in the same manner as above: to prove the relativized version of a theorem where the relativization ensures there is no arithmetical example.

4. Filling the Gaps for Intrinsic Density

Our work in the previous two sections left some gaps, which we address here using the `into` and `within` operations.

We showed in Section 2 that if the join of two sets has intrinsic density, then each set has the same intrinsic density. We now have the machinery to prove the converse.

Theorem 4.1. *Suppose $P(A) = P(B) = \alpha$. Then $P(A \oplus B) = \alpha$.*

Proof. We shall use a technical lemma to complete the proof. Let E represent the even numbers, and let O represent the odd numbers. Lemma 4.1.1 will prove that, for any computable permutation π,

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha$$

To show this we shall give a computable permutation which sends A to $\pi(A \oplus B) \triangleleft \pi(E)$ modulo a set of density zero. We will do this by first showing there is an obvious computable injective function which takes A to $\pi(A \oplus B) \triangleleft \pi(E)$, then use the techniques from Section 3 to massage it into a suitable permutation. We can use the same method to send B to $\pi(A \oplus B) \triangleleft \pi(O)$ modulo a set of density 0.

From there, we will use Lemma 4.1.1 to show that $\rho(\pi(A \oplus B)) = \alpha$, proving the theorem. Note that we cannot use Theorem 3.3 to obtain the above facts because we are trying to prove that $A \oplus B$ has intrinsic density α.

Lemma 4.1.1. *Let π be a computable permutation and let A and B be as in the statement of Theorem 4.1. Then*

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha$$

Proof. Let $h : \pi(E) \to \omega$ send the n-th element of $\pi(E)$ to n (i.e. the inverse of the principal function), and let $d : \omega \to E$ be defined via $d(n) = 2n$. Then notice that $d(A) = A \oplus \emptyset$. Furthermore, observe that for any $X \subseteq \pi(E)$, $h(X) = X \triangleleft \pi(E)$ by the definition of h and the within operation. Therefore

$$h(\pi(d(A))) = h(\pi(A \oplus \emptyset)) = \pi(A \oplus \emptyset) \triangleleft \pi(E)$$

As $\pi(A \oplus B) \cap \pi(E) \subseteq \pi(A \oplus \emptyset)$,

$$\pi(A \oplus \emptyset) \triangleleft \pi(E) = \pi(A \oplus B) \triangleleft \pi(E)$$

Thus $h(\pi(d(A))) = \pi(A \oplus B) \triangleleft \pi(E)$. We shall now use the techniques of Section 3 to change h and d into permutations which preserve the relevant densities.

By Lemma 3.2, there is a computable set $H \subseteq \pi(E)$ with $\overline{\rho}(h(H)) = 0$. Now define the computable permutation π_h via $\pi_h(n) = h(n)$ for $n \in \pi(E) \setminus H$, and have π_h enumerate $\pi(O) \sqcup H$ onto $h(H)$ in order. Similarly, define the computable permutation π_d via $\pi_d(n) = d(n)$ for $n \in \omega \setminus d^{-1}(\pi^{-1}(H))$, and have π_d enumerate $d^{-1}(\pi^{-1}(H))$ onto $O \sqcup \pi^{-1}(H)$.

As π_d agrees with d on $\overline{d^{-1}(\pi^{-1}(H))}$, we now see that

$$\pi_d(A \setminus \pi_d^{-1}(\pi^{-1}(H))) = (A \oplus \emptyset) \setminus \pi^{-1}(H)$$

Furthermore, applying π shows that

$$\pi(\pi_d(A \setminus \pi_d^{-1}(\pi^{-1}(H)))) = \pi((A \oplus \emptyset) \setminus \pi^{-1}(H)) = \pi(A \oplus \emptyset) \setminus H$$

As π_h agrees with h on $\pi(E) \setminus H$ and $h(\pi(A \oplus \emptyset)) = \pi(A \oplus B) \triangleleft \pi(E)$, we have

$$\pi_h(\pi(A \oplus \emptyset) \setminus H) = (\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H)$$

Therefore $(\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H) \subseteq \pi_h(\pi(\pi_d(A)))$ and $\pi_h(\pi(\pi_d(A))) \subseteq (\pi(A \oplus B) \triangleleft \pi(E)) \cup h(H)$.

By choice of H, $\overline{\rho}(h(H)) = 0$, so Lemma 1.13 shows that

$$\overline{\rho}(\pi_h(\pi(\pi_d(A)))) = \overline{\rho}((\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H)) = \overline{\rho}(\pi(A \oplus B) \triangleleft \pi(E))$$

and

$$\underline{\rho}(\pi_h(\pi(\pi_d(A)))) = \underline{\rho}((\pi(A \oplus B) \triangleleft \pi(E)) \setminus h(H)) = \underline{\rho}(\pi(A \oplus B) \triangleleft \pi(E))$$

Therefore, as $P(A) = \alpha$ and $\pi_h \circ \pi \circ \pi_d$ is a computable permutation,

$$\rho(\pi(A \oplus B) \triangleleft \pi(E)) = \alpha$$

A nearly identical argument with O in place of E and B in place of A shows similarly that

$$\rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha$$

□

We shall now show that this implies that $\rho(\pi(A \oplus B)) = \alpha$. Consider $\rho_n(\pi(A \oplus B))$. By definition,

$$\rho_n(\pi(A \oplus B)) = \frac{|\pi(A \oplus B) \upharpoonright n|}{n}$$

As $\omega = \pi(E) \sqcup \pi(O)$,

$$\frac{|\pi(A \oplus B) \upharpoonright n|}{n} = \frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n| + |\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{n}$$

The latter expression can be rewritten as

$$\frac{|\pi(E) \upharpoonright n|}{|\pi(E) \upharpoonright n|} \cdot \frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n|}{n} + \frac{|\pi(O) \upharpoonright n|}{|\pi(O) \upharpoonright n|} \cdot \frac{|\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{n}$$

Let m be the largest number such that the m-th element of $\pi(E)$ is less than n, and let k be the analogous number for $\pi(O)$. Now notice that

$$\frac{|\pi(A \oplus B) \cap \pi(E) \upharpoonright n|}{|\pi(E) \upharpoonright n|} = \rho_m(\pi(A \oplus B) \triangleleft \pi(E))$$

and

$$\frac{|\pi(A \oplus B) \cap \pi(O) \upharpoonright n|}{|\pi(O) \upharpoonright n|} = \rho_k(\pi(A \oplus B) \triangleleft \pi(O))$$

by the definition of the within operation. Therefore, we can rewrite $\rho_n(\pi(A \oplus B))$ as

$$\rho_m(\pi(A \oplus B) \triangleleft \pi(E)) \cdot \rho_n(\pi(E)) + \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) \cdot \rho_n(\pi(O))$$

Using the fact that $\rho_n(\pi(E)) + \rho_n(\pi(O)) = 1$,

$$\rho_n(\pi(A\oplus B)) = \rho_m(\pi(A\oplus B)\triangleleft\pi(E))\cdot\rho_n(\pi(E))+\rho_k(\pi(A\oplus B)\triangleleft\pi(O))\cdot(1-\rho_n(\pi(E)))$$

Rearranging, this is equal to

$$\rho_k(\pi(A \oplus B) \triangleleft \pi(O)) + \rho_n(\pi(E)) \cdot (\rho_m(\pi(A \oplus B) \triangleleft \pi(E)) - \rho_k(\pi(A \oplus B) \triangleleft \pi(O)))$$

Taking the limit as n goes to infinity, m and k both go to infinity. Thus

$$\rho_m(\pi(A \oplus B) \triangleleft \pi(E)) - \rho_k(\pi(A \oplus B) \triangleleft \pi(O))$$

goes to 0 by Lemma 4.1.1. As $\rho_n(\pi(E))$ is bounded between 0 and 1 by definition, the second term vanishes. Therefore

$$\lim_{n\to\infty} \rho_n(\pi(A \oplus B)) = \lim_{n\to\infty} \rho_k(\pi(A \oplus B) \triangleleft \pi(O)) = \rho(\pi(A \oplus B) \triangleleft \pi(O)) = \alpha$$

as desired. □

Lemma 2.1 and Theorem 4.1 can easily be generalized.

Definition 4.2. *Let H be a computable, infinite, co-infinite set. Then the H-join of A and B, denoted by $A \oplus_H B$, is*

$$(A \triangleright H) \sqcup (B \triangleright \overline{H})$$

Notice that $A \oplus B = A \oplus_E B$. Furthermore, there is a computable permutation π that sends E to H and O to $\overline{H}$ in order. Therefore $\pi(A \oplus B) = A \oplus_H B$, so the generalizations of Lemma 2.1 and Theorem 4.1 follow without needing to rework the proofs.

Recall that Theorem 3.7 says if $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(B \triangleright A) = \alpha\beta$. Whether or not either of these conditions can be weakened or dropped is a natural question. It is immediate that we cannot drop the requirement that A has intrinsic density: $P_A(\omega) = 1$ for any A, so ω always satisfies the requirements on B, but $\omega \triangleright A = A$, so A must have intrinsic density. Similarly, $B \triangleright \omega = B$ for any B, so B must have intrinsic density. Therefore the only possible weakening of Theorem 3.7 would be to require $P(B) = \beta$ as opposed to $P_A(B) = \beta$. However, this fails in a strong way.

Lemma 4.3. *Let $P(A) = \frac{1}{2}$. Then $P(A \oplus \overline{A}) = \frac{1}{2}$ but $A \triangleright (A \oplus \overline{A})$ does not have intrinsic density.*

Proof. Note that $A\oplus\overline{A}$ has intrinsic density $\frac{1}{2}$ by Theorem 4.1 as $P(A) = \frac{1}{2}$ implies $P(\overline{A}) = \frac{1}{2}$.

Let E represent the set of even numbers. Notice that $A \oplus \overline{A}$ contains exactly one of $2k$ or $2k+1$ for all $k \in \omega$. Therefore the n-th element of $A\oplus\overline{A}$ is $2n$ if $n \in A$ and $2n + 1$ if $n \notin A$. Thus

$$E \triangleleft (A \oplus \overline{A}) = A$$

by definition. By the properties of the within operation,

$$A \triangleright (A \oplus \overline{A}) = (E \triangleleft (A \oplus \overline{A})) \triangleright (A \oplus \overline{A}) = E \cap (A \oplus \overline{A}) = A \oplus \emptyset$$

By Lemma 2.1, however, $A \oplus \emptyset$ does not have intrinsic density. □

Note that we cannot generalize this result to $A \oplus_H \overline{A}$ in general, specifically it is not always true that $H \triangleleft (A \oplus_H \overline{A}) = A$: consider H the set of naturals congruent to 2 modulo 3, and let A be a set containing 0 but not containing 1. Then $0 \notin H \triangleleft (A \oplus_H \overline{A})$ because $p_{A \oplus_H \overline{A}}(0) = 1$ and $1 \notin H$. Thus $H \triangleleft (A \oplus_H \overline{A}) \neq A$ as witnessed by 0.

Recall that Theorem 3.3 says that if $P(A) = \alpha$ and C is computable, then $A \triangleleft C$ also has intrinsic density α. It is natural to wonder if this is symmetric: does $C \triangleleft A$ have intrinsic density? The proof of Lemma 4.3 shows that it is possible for $C \triangleleft A$ to have intrinsic density. However, this is not true in general, as $C \triangleleft \omega = C$. It is not obvious what can be said, if anything, about when $C \triangleleft A$ has intrinsic density. Future work exploring this may reveal something interesting about the structure of sets with intrinsic density: let $P(A) > 0$, C be coinfinite, computable with $P(C \triangleleft A) > 0$. Such sets would witness the failure of the weak version of Theorem 3.7, as $(C \triangleleft A) \triangleright A = C \cap A$ and no subset of a computable set can have intrinsic density greater than zero.

5. Applications to Classical stochasticity

We shall now apply the tools of Section 3 to MWC-density and Church-density. It turns out that the `into` and `within` operations behave similarly for MWC and Church densities as they do for intrinsic density, however other operations are less well behaved. As per the remark following Definition 3.1, we need to measure $A \triangleleft f(A)$ for all computable monotone selection functions f to check MWC-density, and the total such f to check Church-density. (Recall that $f(A) = \{n : f(A \upharpoonright n) \downarrow = 1\}$.) Throughout this section, we shall focus on MWC-density, however, all of our results will go through for Church-density as well: we will often be given a monotone selection function and need to modify it to suit our needs. Our modification will never make a total monotone selection function not total, so the result will hold in the Church-density case as well.

From a general perspective, as argued by Bienvenu [Personal Communication], we can see from simply computing measures that sets with Church-density r must exist for $r \in (0,1)$, as the set of all such sets has measure 1. He also noted that this argument relativizes to $\emptyset'$, which implies the same is true for MWC-density r sets.

From a computability theory perspective, we can make this very explicit. As in the intrinsic density case, sets sufficiently random with respect to μ_r will have MWC-density r. (As in Lemma 1.12, this was previously known and follows from standard arguments. We provide a proof for convenience.)

Lemma 5.1. *Let $r \in (0,1)$. If X is r-1-Random, then X has MWC-density r.*

Proof. We shall argue by contrapositive. Let f be a monotone selection function and suppose that $\overline{\rho}(X \triangleleft f(X)) > \epsilon > r$ for some rational ϵ. It is sufficient to consider this case, as if $\underline{\rho}(X \triangleleft f(X)) < r$ then $\overline{\rho}(\overline{X} \triangleleft \overline{f}(\overline{X})) > 1 - r$, where $\overline{f}$ is the monotone selection function obtained by flipping the bits then applying f. Additionally, without loss of generality we may assume in the partial case that $f(X \upharpoonright n) \downarrow$ for all n: given f, we know that infinitely often $\rho_n(X \triangleleft f(X)) > \epsilon$. Therefore whenever we have some σ witnessing this fact by stage s, we may force $f(\tau)$ to converge to 0 for all $\tau \preceq \sigma$ which have not converged by stage s. As in Lemma 1.11, if r is noncomputable then we are implicitly working relative to an arbitrary representation for μ_r, which can compute r.

We shall construct an r-(super)martingale which succeeds on X using f. Let α be as in the proof of Lemma 1.12 for r and ϵ, i.e. such that $(1+\frac{\alpha}{r})^\epsilon(1-\frac{\alpha}{1-r})^{1-\epsilon} > 1$. Define $m : 2^{<\omega} \to \{0,1\}$ as follows:

- $m(\emptyset) = 1$.
- If $f(\sigma) = 1$, let $m(\sigma 1) = (1+\frac{\alpha}{r})m(\sigma)$ and $m(\sigma 0) = (1-\frac{\alpha}{1-r})m(\sigma)$.
- If $f(\sigma) = 0$, let $m(\sigma 1) = m(\sigma 0) = m(\sigma)$.
- If $f(\sigma)\uparrow$, let $m(\sigma 1) = m(\sigma 0) = 0$.

Note that m is a c.e. r-(super)martingale. Furthermore, as $f(X \upharpoonright k)\downarrow$ for all k, $m(X \upharpoonright n) \neq 0$ for all n. Thus

$$m(X \upharpoonright n) = (1+\frac{\alpha}{r})^{|X\upharpoonright n|}(1-\frac{\alpha}{1-r})^{n-|X\upharpoonright n|}$$

If $\rho_s(X \triangleleft f(X)) > \epsilon$, then let $n = p_{f(X)}(s)$, the s-th element of $f(X)$. Then

$$|(X \triangleleft f(X)) \upharpoonright s| = |\{k < n : f(X \upharpoonright k) = 1 \text{ and } k \in X\}| \geq \epsilon s$$

so it follows that

$$m(X \upharpoonright n) \geq (1+\frac{\alpha}{r})^{\epsilon s}(1-\frac{\alpha}{1-r})^{(1-\epsilon)s} = ((1+\frac{\alpha}{r})^\epsilon(1-\frac{\alpha}{1-r})^{1-\epsilon})^s$$

By choice of α, $(1+\frac{\alpha}{r})^\epsilon(1-\frac{\alpha}{1-r})^{1-\epsilon} > 1$, so $\sup_{n\to\infty} m(X \upharpoonright n) = \infty$ because there are infinitely many such s. Thus m succeeds on X and therefore X is not r-1-Random. □

Remark. *If f is a total computable selection function, then the above m is a computable r-martingale and therefore this shows that computable randoms have Church density r. Ambos-Spies* [1] *exhibited a computable random which is not MWC stochastic, so in general m may be properly c.e. Wang* [22] *showed that Schnorr randoms need not be Church stochastic, so obviously the above result does not extend to r-Schnorr randoms.*

This shows that every real in the interior of the unit interval is achieved. ω and $\emptyset$ technically complete the whole unit interval, albeit trivially. However, nontrivial sets of MWC-density 0 should exist simply by being "small enough" to appear small under countably many selection functions. We shall address this before turning back to the `into` and `within` operations.

Lemma 5.2. *There exists an infinite set $A \leq_T \emptyset'$ such that A has MWC-density (and therefore Church-density) 0.*

Proof. The construction is similar in principle to the jump strategy for constructing intrinsically small sets from [15]. However, the details are more technical due to the fact that monotone selection rules can change their behavior on different inputs whereas permutations cannot. We cannot simply choose large enough elements to enter our set, as a given monotone selection rule may refuse to act until it sees an element enter the set. We utilize the power of the jump to determine if, for a given monotone selection function f, it is possible to force a large gap into $A \triangleleft f(A)$ and ensure the density is small. If it is not possible, then we do not allow anything into $A \triangleleft f(A)$ at all until a large gap appears naturally. If no such gap appears, then

$A \triangleleft f(A)$ will be finite and we succeed.

Formally, let f_i be an enumeration of the partial computable monotone selection functions. The basic module for ensuring that $\overline{\rho}(A \triangleleft f_i(A)) = 0$ for this specific f_i is as follows: After seeing the n-th 1 enter $A \triangleleft f_i(A)$ at σ_s, we do not allow another 1 to enter until we see n^2 0's enter. (Notice that convergence is not an issue, as the jump can determine if $f_i(\sigma)\downarrow$ uniformly in σ and i.) We will attempt to achieve this by picking some m such that $f_i(\sigma_s 0^k) = 1$ for n^2 k's less than m and setting $\sigma_{s+1} = \sigma_s 0^m 10$. The jump can determine if such an m exists.

Suppose we have defined $\sigma \preceq A$ and there is no m such that $f_i(\sigma 0^m) = 1$. Then we cannot force anything into $A \triangleleft f_i(A)$ without adding extra 1's to A, potentially adding some 1's to $f_j(A)$ for some $j \neq i$. To fix this issue, we say f_i is paused for σ if there does not exist an m such that $f_i(\sigma 0^m) = 1$. As mentioned above, $\emptyset'$ can determine if f_i is paused for σ. When determining how to extend σ_s to $\sigma_{s+1} = \sigma_s 0^m 10$, if $f_i(\sigma_s 0^m 1) = 1$, then σ_{s+1} puts a 0 into $A \triangleleft f_i(A)$. If not, then nothing changes. In both cases, no 1's are added to $A \triangleleft f_i(A)$ by σ_s. We continue and ask if f_i is paused for σ_{s+1}. Either we will eventually see enough 0's enter $A \triangleleft f_i(A)$ after some number of stages and be allowed to add a 1, or this will not happen and $f_i(A)$ will be finite. We succeed in both cases.

We say f_i is almost paused for σ if there is some k such that f_i is paused for $\sigma 0^k$ and $\sigma 0^k$ does not put enough zeroes into $A \triangleleft f_i(A)$. Here, to say f_i is paused means we cannot force another 0 into $A \triangleleft f_i(A)$ by only adding 0's to A. To say f_i is almost paused means we may be able to force some zeroes into $A \triangleleft f_i(A)$, but we cannot force enough zeroes into $A \triangleleft f_i(A)$. (Being almost paused resembles a Σ^0_2 question, but the bound on the number of zeroes necessary reduces it to a question the jump can answer: we can ask if f_i is paused for σ_s: If so, then it is almost paused. If not, then extend to $\sigma_s 0^k$, where k witnesses that f_i is not paused for σ_s. This adds a zero to $A \triangleleft f_i(A)$. Now ask if f_i is paused for $\sigma_s 0^k$ and repeat. Eventually we will either reach a point where it is paused or we will put enough zeroes into $A \triangleleft f_i(A)$.)

Finally, we describe the construction using this module on all i simultaneously: at stage s, we consider only the $i \leq s$. Using the jump, determine those i which are almost paused at stage s and ignore them. For the remaining i, we may choose m large enough such that $\sigma_{s+1} = \sigma_s 0^m 10$ puts enough zeroes into $A \triangleleft f_i(A)$ to ensure n^2 zeroes are enumerated before the $n+1$-st 1, where n is the current number of ones in $A \triangleleft f_i(A)$. As we are ignoring all of the almost paused selection functions, we can always extend to σ_{s+1} and thus A is infinite. Furthermore, $\overline{\rho}(A \triangleleft f_i(A)) = 0$ for all i as either $f_i(A)$ is finite or $\rho_n(A \triangleleft f_i(A)) \leq \frac{k+1}{k^2+k+1}$ for increasing k. (If there are $k+1$ ones in $|A \triangleleft f_i(A) \upharpoonright n|$, then there are at least k^2 zeroes between the final two.) □

This still leaves open MWC-density 1, as it is not a priori obvious that MWC-density behaves with complements as intrinsic density does. We can easily prove that it does, however, thus obtaining a non-trivial example of an MWC-density 1 set.

Lemma 5.3. *Let A have MWC-density α. Then $\overline{A}$ has MWC-density $1-\alpha$.*

Proof. Let f be a computable monotone selection function. Define $\overline{f}: 2^{<\omega} \to \{0,1\}$ via $\overline{f}(\sigma) = f(1-\sigma)$, where $1-\sigma = \tau \in 2^{|\sigma|}$ with $\tau(n) = 1-\sigma(n)$ for all $n < |\sigma|$. Then $f(\overline{A} \upharpoonright n) = \overline{f}(A \upharpoonright n)$. As A has MWC-density α and $\overline{f}$ is a computable monotone selection function, either $\overline{f}(A)$ is finite or $\rho(A \triangleleft \overline{f}(A)) = \alpha$. If the former, then $f(\overline{A})$ is also finite. If the latter, then

$$\overline{A} \triangleleft f(\overline{A}) = \overline{A} \triangleleft \overline{f}(A) = \overline{A \triangleleft \overline{f}(A)}$$

Therefore

$$\rho(\overline{A} \triangleleft f(\overline{A})) = \rho(\overline{A \triangleleft \overline{f}(A)}) = 1 - \rho(A \triangleleft \overline{f}(A)) = 1-\alpha$$

as desired. □

Having obtained the whole unit interval in nontrivial fashion, we now turn to investigating MWC-density analogs of results from Section 3. The `into` and `within` operations perform in nearly the same fashion.

Lemma 5.4. *Suppose C is computable and A has MWC-density α. Then $A \triangleleft C$ has MWC-density α.*

Proof. Let f be a computable monotone selection function. Define $\hat{C}: 2^{<\omega} \to 2^{<\omega}$ via $\hat{C}(\sigma) = \tau$ with $\tau \in 2^{max(n:c_n<|\sigma|)+1}$ and $\tau(i) = \sigma(c_i)$ for all $i < |\tau|$. Notice that $\hat{C}(X \upharpoonright c_n) = (X \triangleleft C) \upharpoonright n$ by definition.

Now define $f_C: 2^{<\omega} \to \{0,1\}$ via $f_C(\sigma) = 1$ if and only if $|\sigma| = c_i$ for some i and $f(\hat{C}(\sigma)) = 1$. As C is computable, $\hat{C}$ is computable and thus f_C is a computable monotone selection function. We now show that $A \triangleleft f_C(A) = (A \triangleleft C) \triangleleft f(A \triangleleft C)$.

We shall show that $(A \triangleleft C) \triangleleft f(A \triangleleft C) \subseteq A \triangleleft f_C(A)$ with a sequence of if and only ifs, therefore proving the reverse as well. n is in $(A \triangleleft C) \triangleleft f(A \triangleleft C)$ if and only if the n-th element of $f(A \triangleleft C)$ is in $A \triangleleft C$, i.e. the n-th k with $f((A \triangleleft C) \upharpoonright k) = 1$ is in $A \triangleleft C$. This occurs if and only if $c_k \in A$. Now note that $f_C(A)$ is the set of all c_i such that $f(\hat{C}(A \upharpoonright c_i)) = f((A \triangleleft C) \upharpoonright i) = 1$, so k is as above if and only if $c_k \in f_C(A)$ and $c_k \in A$. Note that c_k must be the n-th element of $f_C(A)$ because k was the n-th number with $f((A \triangleleft C) \upharpoonright k) = 1$, so $n \in A \triangleleft f_C(A)$.

As A has MWC-density α,

$$\rho((A \triangleleft C) \triangleleft f(A \triangleleft C)) = \rho(A \triangleleft f_C(A)) = \alpha$$

As f was arbitrary, $A \triangleleft C$ also has MWC-density α. □

To prove the analog of Theorem 3.7 for MWC-density, we require more relativization. We shall see that this is a theme with MWC-density compared to intrinsic density. Unlike intrinsic density, where the selection and interpretation functions act independently of the input set, MWC-density can change the selected bits based on finitely much of the input set. This means that if B is related to A in some predictable fashion, then a monotone selection rule may be able to use information from B to predict bits of A. Assuming the sets have MWC-density relative to each other will avoid this issue as using B as an oracle will allow us to simulate an input set involving B, and vice versa for A. We shall see some consequences of this distinction after Theorem 5.6. To prove this theorem, however, we shall require the following technical observation. The proof is merely obtained by unraveling definitions, but we provide it for clarity as the definitions can be cumbersome.

Lemma 5.5. *Let A, B and C be sets. Then*

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = A \triangleleft (B \cap C)$$

Proof. By definition,

$$A \triangleleft (B \cap C) = \{n : p_{B\cap C}(n) \in A\}$$

That is, it is the set of all n such that the n-th element of $B \cap C$ is in A.

Similarly, by definition

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = \{n : p_{B\triangleleft C}(n) \in A \triangleleft C\}$$

That is, it is the set of all n such that the n-th element of $B \triangleleft C$ is in $A \triangleleft C$. However, if $k \in A \triangleleft C$ for some k, this by definition means $c_k \in A$. Therefore if $n \in (A \triangleleft C) \triangleleft (B \triangleleft C)$, this translates to $c_{p_{B\triangleleft C}(n)} \in A$. As $p_{B\triangleleft C}(n)$ is the n-th element of $B \triangleleft C$, $c_{p_{B\triangleleft C}(n)}$ is the n-th element of C which is in B. Another way to phrase this is that $c_{p_{B\triangleleft C}(n)}$ is the n-th element of $B \cap C$. This confirms that the sets are identical. □

It is worth noting a corollary of this lemma which we will not need yet is not obvious at first glance: As intersection is symmetric,

$$A \triangleleft (B \cap C) = A \triangleleft (C \cap B)$$

Therefore applying Lemma 5.5 once on each side tells us that

$$(A \triangleleft C) \triangleleft (B \triangleleft C) = (A \triangleleft B) \triangleleft (C \triangleleft B)$$

Now we are ready to prove the analog of Theorem 3.7.

Theorem 5.6. *Suppose that A has MWC-density α relative to B and B has MWC-density β relative to A. Then $B \triangleright A$ has MWC-density $\alpha\beta$.*

Proof. The proof is similar to the proof of Theorem 3.7, however there is an extra consideration for MWC-density because the selected bits can depend on the input. In Theorem 3.7, $\pi(B \triangleright A)$ is a subset of $\pi(A)$, so we send B to $\pi(B \triangleright A) \triangleleft \pi(A)$ (modulo a set of density zero) and apply Lemma 3.5. However, we don't know in general if $A \triangleleft f(A)$ contains $(B \triangleright A) \triangleleft f(B \triangleright A)$ because $f(B \triangleright A)$ need not be a subset of $f(A)$, so we first construct a B-computable monotone selection function f_B such that $f_B(A) = f(B \triangleright A)$ and therefore $A \triangleleft f_B(A)$ is a superset of $(B \triangleright A) \triangleleft f(B \triangleright A)$. Then because A has MWC-density α relative to B, $A \triangleleft f_B(A)$ will have density α. From there we shall borrow the proof idea of Theorem 3.7, namely we shall construct an A-computable monotone selection function f_A such that

$$B \triangleleft f_A(B) = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f_B(A))$$

Again, as B has MWC-density β with respect to A, $B \triangleleft f_A(B)$ will have density β. We may then apply Lemma 3.5 to show that $(B \triangleright A) \triangleleft f(B \triangleright A)$ has density $\alpha\beta$ as desired.

Formally, let f be a computable monotone selection function. If $f(B \triangleright A)$ is finite or undefined, we are done. If not, define $f_B : 2^{<\omega} \to \{0, 1\}$ via $f_B(\sigma) = f(B \triangleright \sigma)$, where $B \triangleright \sigma \in 2^{|\sigma|}$ is defined as one might expect: $B \triangleright \sigma(n) = 1$ if and only if $\sigma(n) = 1$ and

n is the b_i'th m such that $\sigma(m) = 1$ for some $i \in \omega$. As $(X \triangleright Y) \upharpoonright n = X \triangleright (Y \upharpoonright n)$, it is immediate that

$$f_B(A) = \{n : f_B(A \upharpoonright n) = 1\} = \{n : f(B \triangleright (A \upharpoonright n)) = 1\} =$$

$$\{n : f((B \triangleright A) \upharpoonright n) = 1\} = f(B \triangleright A)$$

Therefore, as $B \triangleright A \subseteq A$,

$$(B \triangleright A) \triangleleft f(B \triangleright A) = (B \triangleright A) \triangleleft f_B(A) \subseteq A \triangleleft f_B(A)$$

Let

$$X = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f(B \triangleright A))$$

We shall construct an A-computable monotone selection function f_A such that $B \triangleleft f_A(B) = X$ via Lemma 5.5.

Let $f_A : 2^{<\omega} \to \{0, 1\}$ be defined via $f_A(\sigma) = f(\sigma \triangleright A)$, where $\sigma \triangleright A = \tau \in 2^{a_{|\sigma|}}$ is defined via $\tau(n) = 1$ if and only if $n = a_m$ and $\sigma(m) = 1$ for some $m < |\sigma|$. We now claim that $B \triangleleft f_A(B) = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A))$.

If $n \in (B \triangleleft A) \triangleleft (A \cap f(B \triangleright A))$, then the n-th element of $A \cap f(B \triangleright A)$ is in $B \triangleright A$ by the definition of the within operation. This implies it is of the form a_m for $m \in B$, where m is the n-th number k such that $a_k \in A \cap f(B \triangleright A)$. As a_m is in $f(B \triangleright A)$, by the definition of f_A this implies that m is the n-th number with

$$f((B \triangleright A) \upharpoonright a_m) = f((B \upharpoonright m) \triangleright A) = f_A(B \upharpoonright m) = 1$$

Thus m is the n-th element of $f_A(B)$, and it lies in B, so $m = p_{f_A(B)}(n) \in B$. Therefore $n \in B \triangleleft f_A(B)$. As n was arbitrary,

$$(B \triangleleft A) \triangleleft (A \cap f(B \triangleright A)) \subseteq B \triangleleft f_A(B)$$

This argument reverses, so $B = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A))$.

Therefore,

$$X = ((B \triangleright A) \triangleleft f(B \triangleright A)) \triangleleft (A \triangleleft f(B \triangleright A)) = (B \triangleright A) \triangleleft (A \cap f(B \triangleright A)) = B \triangleleft f_A(B)$$

The first equality is by definition, the second is by Lemma 5.5, and the final is from the previous paragraph. This implies

$$X \triangleright (A \triangleleft f_B(A)) = (B \triangleleft f_A(B)) \triangleright (A \triangleleft f_B(A))$$

As A has MWC-density α with respect to B and $f_B(A) = f(B \triangleright A)$, $\rho(A \triangleleft f(B \triangleright A)) = \alpha$. As B has MWC-density β with respect to A, $\rho(B \triangleleft f_A(B)) = \beta$. Therefore by Lemma 3.5,

$$\rho((B \triangleleft f_A(B)) \triangleright (A \triangleleft f(B \triangleright A))) = \rho(B \triangleleft f_A(B))\rho(A \triangleleft f(B \triangleright A)) = \alpha\beta$$

Finally, recall from the definition of X that $X \triangleright (A \triangleleft f(B \triangleright A)) = (B \triangleright A) \triangleleft f(B \triangleright A)$. Therefore

$$\rho((B \triangleright A) \triangleleft f(B \triangleright A)) = \rho(X \triangleright (A \triangleleft f(B \triangleright A))) = \alpha\beta$$

as desired. □

Following Theorem 3.7, we were able to obtain as an easy corollary that if A has intrinsic density α and B has intrinsic density β relative to A, then $A \cap B$ has intrinsic density $\alpha\beta$. The proof simply observed that $B \triangleleft A$ had intrinsic density β relative to A via the relativized form of Theorem 3.3 and then applied Theorem 3.7 because $(B \triangleleft A) \triangleright A = A \cap B$.

Unfortunately, the same proof is not guaranteed to work in the MWC-density case. Theorem 5.6 requires relativization in both directions, and while the relativized form of Theorem 5.4 ensures that $B \triangleleft A$ has MWC-density β relative to A, it does not ensure that A has MWC-density α relative to $B \triangleleft A$, so we cannot apply Theorem 5.6 as we wish. This remains an open question which we shall state fully in Question 6.4.

Fortunately, we can recover the intersection property for relatively MWC-dense sets using an alternate proof.

Lemma 5.7. *If A has MWC-density α relative to B and B has MWC-density β relative to A, then $A \cap B$ has MWC-density $\alpha\beta$.*

Proof. Let f be a computable monotone selection function. If $f(A \cap B)$ is finite, then we are done. Otherwise, consider $(A \cap B) \triangleleft f(A \cap B)$. Define the B-computable monotone selection function $f_B : 2^{<\omega} \to \{0, 1\}$ via $f_B(\sigma) = 1$ if and only if $f(\sigma \cap B) = 1$, where $\sigma \cap B = \tau \in 2^{|\sigma|}$ is given by $\tau(n) = 1$ if and only if $\sigma(n) = 1$ and $B(n) = 1$. Then clearly $f_B(A) = f(A \cap B)$, so $A \triangleleft f_B(A) = A \triangleleft f(A \cap B)$. As A has MWC-density α relative to B,

$$\rho(A \triangleleft f_B(A)) = \rho(A \triangleleft f(A \cap B)) = \alpha$$

We shall now construct an A-computable monotone selection function f_A such that

$$B \triangleleft f_A(B) = (B \triangleleft f(A \cap B)) \triangleleft (A \triangleleft f(A \cap B))$$

via Lemma 5.5.

Define $f_A : 2^{<\omega} \to \{0, 1\}$ via $f_A(\sigma) = 1$ if and only if $f(A \cap \sigma) = 1$ and $|\sigma| \in A$, where $A \cap \sigma$ is defined similarly to $\sigma \cap B$ in the obvious way. Then it follows immediately that $f_A(B) = A \cap f(A \cap B)$, so

$$B \triangleleft f_A(B) = B \triangleleft (A \cap f(A \cap B)$$

By Lemma 5.5,

$$B \triangleleft (A \cap f(A \cap B) = (B \triangleleft f(A \cap B)) \triangleleft (A \triangleleft f(A \cap B))$$

Therefore by the properties of the within operation we have that

$$(B \triangleleft f_A(B)) \triangleright (A \triangleleft f_B(A)) = (B \triangleleft f(A \cap B)) \triangleleft (A \triangleleft f(A \cap B)) \triangleright (A \triangleleft f(A \cap B)) =$$

$$(B \triangleleft f(A \cap B)) \cap (A \triangleleft f(A \cap B)) = (A \cap B) \triangleleft f(A \cap B)$$

By Lemma 3.5,

$$\rho((B \triangleleft f_A(B)) \triangleright (A \triangleleft f_B(A)) = \rho((A \cap B) \triangleleft f(A \cap B)) = \alpha\beta$$

As f was arbitrary, $A \cap B$ has MWC-density $\alpha\beta$. $\square$

As we have seen, the `into` and `within` operations behave similarly for both MWC-density and intrinsic density. However, this seems to be a statement on the usefulness of these tools rather than a statement about the similarities between the two types of density. Below we shall see that more common set operations behave quite differently between the two notions.

Where Lemma 4.1 says that in a specific sense intrinsic density is ignorant of (computable) internal structure, the opposite is true of MWC-density. In fact, the analog of Lemma 4.1 for MWC-density fails in very strong fashion.

Lemma 5.8. *Suppose that A has MWC-density α for $0 \leq \alpha < 1$. Then $A \oplus A$ does not have MWC-density.*

Proof. Let E be the set of even numbers and let O be the set of odd numbers. Define $f : 2^{<\omega} \to \{0,1\}$ via $f(\sigma) = 1$ if $|\sigma| \in O$ and $\sigma(|\sigma| - 1) = 1$ and $f(\sigma) = 0$ otherwise. Then for any A, $f(A \oplus A) = A \triangleright O$. Therefore

$$(A \oplus A) \triangleleft f(A \oplus A) = (A \oplus A) \triangleleft (A \triangleright O) = \omega$$

so

$$\rho((A \oplus A) \triangleleft f(A \oplus A)) = 1$$

However as A has MWC-density $\alpha < 1$, it has density α and $A \oplus A$ has density α. Therefore $A \oplus A$ cannot have MWC-density as its asymptotic density does not match the density of $(A \oplus A) \triangleleft f(A \oplus A)$. ☐

Not only does the join fail to behave for MWC-density, but we shall in fact see that the union does not behave either. The difficulty lies in the fact that the bits selected by f on $A \sqcup B$ need not be the union of the bits selected by f on A and the bits selected by f on B in general. On one hand, it is not difficult to prove that if A has MWC-density α relative to B and B has MWC-density β relative to A with A and B disjoint, then $A \sqcup B$ has MWC-density $\alpha + \beta$: given a monotone selection function f, there is a B-computable monotone selection function f_B such that $f_B(A) = f(A \sqcup B)$ and similarly there is an A-computable monotone selection function f_A such that $f_A(B) = f(A \sqcup B)$. Then

$$(A \sqcup B) \triangleleft f(A \sqcup B) = (A \triangleleft f(A \sqcup B)) \sqcup (B \triangleleft f(A \sqcup B)) = (A \triangleleft f_B(A)) \sqcup (B \triangleleft f_A(B))$$

by the properties of the within operation. Therefore

$$\rho((A \sqcup B) \triangleleft f(A \sqcup B)) = \rho(A \triangleleft f_B(A)) + \rho(B \triangleleft f_A(B)) = \alpha + \beta$$

However, Lemma 5.7 ensures that $A \cap B = \emptyset$ implies that one of A or B has MWC-density 0 under these assumptions, so this result cannot be used to obtain new MWC-densities as the disjoint unions of sets with others.

One may think to drop the requirements that A and B have MWC-density relative to one another, therefore disallowing the use of Lemma 5.7 and avoiding this problem. However, the union still need not have MWC-density. The following lemmas will allow us to construct such an example.

Lemma 5.9. *If A has MWC-density 0 and g is an increasing, total, computable function, then $B = \{a_n + g(n) : n \in \omega\}$ also has MWC-density 0.*

Proof. We argue by contrapositive. Let f be a monotone selection function such that $\overline{\rho}(B \triangleleft f(B)) > 0$. We shall construct a monotone selection function $\hat{f}$ such that $\overline{\rho}(A \triangleleft \hat{f}(A)) \geq \overline{\rho}(B \triangleleft f(B)) > 0$.

Given $\sigma \in 2^{<\omega}$, let $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ represent all indices on which σ is 1. Define $g(\sigma)$ to be $\tau \in 2^{|\sigma|+g(k+1)}$ with $\tau(i) = 1$ if and only if $i = \sigma_j + g(j)$ for some $j \leq k$. Finally, define $\hat{f} : 2^{<\omega} \to \{0, 1\}$ via $\hat{f}(\sigma) = 1$ if and only if $f(g(\sigma)) = 1$. Suppose that $n \in B \triangleleft f(B)$. Then $p_{f(B)}(n) = a_k + g(k)$ for some $k \in \omega$. In particular, $f(B \upharpoonright a_k + g(k)) = 1$. Therefore, as $g(A \upharpoonright a_k) = B \upharpoonright a_k + g(k)$ by the definition of $g(\sigma)$, we have

$$\hat{f}(A \upharpoonright a_k) = f(g(A \upharpoonright a_k)) = f(B \upharpoonright a_k + g(k)) = 1$$

Finally, notice that the m such that $p_{\hat{f}(A)}(m) = a_k$ must be less than or equal to n because each element of $\hat{f}(A)$ corresponds to an element of $f(B)$ but not necessarily vice versa. It follows that $\overline{\rho}(A \triangleleft \hat{f}(A)) \geq \overline{\rho}(B \triangleleft f(B))$, as each element of $B \triangleleft f(B)$ corresponds to an element of $A \triangleleft \hat{f}(A)$ which is no larger. As $\overline{\rho}(B \triangleleft f(B)) > 0$, we are done. □

Lemma 5.10. *There exists a set A such that A and $A \triangleright \overline{A}$ both have MWC-density, but $A \sqcup (A \triangleright \overline{A})$ does not.*

Proof. By Lemma 5.2, there is an infinite set X with MWC-density 0. By Lemma 5.9, $A = \{x_n + n^2 : n \in \omega\}$ also has MWC-density 0. Notice that

$$a_{n+1} - a_n = x_{n+1} + (n+1)^2 - x_n - n^2 = x_{n+1} - x_n + 2n + 1 > 2n + 1$$

It follows that the a_n-th element of $\overline{A}$ is $a_n + n + 1$. (The only way this could not be the case is if $a_{n+1} \leq a_n + n + 1$.) Therefore $A \triangleright \overline{A} = \{a_n + n + 1 : n \in \omega\}$, so it has MWC-density 0 by Lemma 5.9.

Let $f : 2^{<\omega} \to \{0, 1\}$ be defined via $f(\sigma) = 1$ if and only if $m < |\sigma|$ is the largest number with $\sigma(m) = 1$, σ has $2k+1$ 1's, and $|\sigma| = m + k + 1$. It is immediate that f is a total monotone selection function, and furthermore $f(A \sqcup (A \triangleright \overline{A})) = A \triangleright \overline{A}$: by the above, $A \sqcup (A \triangleright \overline{A})$ alternates between elements of A and elements of $A \triangleright \overline{A}$. The elements of A signal where elements of $A \triangleright \overline{A}$ will sit, allowing f to select exactly those elements. Therefore

$$(A \sqcup (A \triangleright \overline{A})) \triangleleft f(A \sqcup (A \triangleright \overline{A})) = (A \sqcup (A \triangleright \overline{A})) \triangleleft (A \triangleright \overline{A}) = \omega$$

Thus $A \sqcup (A \triangleright \overline{A})$ does not have MWC-density 0. However, it has density 0 as the union of two sets of density 0, so it does not have MWC-density. □

This shows that disjoint unions in general need not sum MWC-densities. However, our example is of two sets with MWC-density 0, and rely on the fact that they are spread out nontrivially. Is it possible to find an example with sets of positive MWC-density? It turns out that the answer is yes. Bienvenu [Personal Communication] shared the following argument: We shall construct disjoint A and B with both having MWC-density $\frac{1}{2}$ but $A \sqcup B$ does not have MWC-density 1. With probability $\frac{1}{n}$, keep both $2n$ and $2n+1$ out of both A and B. For all naturals m not explicitly excluded, with independent probability $\frac{1}{2}$ put m into A and put it into B if it does not enter A. Then with probability 1, A and B both have MWC-density $\frac{1}{2}$ but $A \sqcup B$ does not have MWC-density 1 because the monotone selection function which selects any bit following to bits that appear as why 10 will always select a 0. (This will be infinite by the effective version of the second Borel-Cantelli lemma.)

Another potential solution to the problem of misbehaving unions is to remove the requirement that the sets be disjoint: if A has MWC-density α relative to B and B has MWC-density β relative to A, then must $A \cup B$ have MWC-density $\alpha+\beta-\alpha\beta$? (The inclusion-exclusion principle implies that $\rho_n(A \cup B) = \rho_n(A) + \rho_n(B) - \rho_n(A \cap B)$. Together with Lemma 5.7, this suggests that the MWC-density of $A \cup B$ must be $\alpha + \beta - \alpha\beta$ if it has MWC-density at all.) It turns out that this is true.

Lemma 5.11. *Suppose A has MWC-density α relative to B and B has MWC-density β relative to A. Then $A \cup B$ has MWC-density $\alpha + \beta - \alpha\beta$.*

Proof. Let f be a computable monotone selection function. If $f(A \cup B)$ is finite, we are done. Otherwise, consider $(A \cup B) \lhd f(A \cup B)$. By definition,

$$\rho((A \cup B) \lhd f(A \cup B)) = \lim_{n\to\infty} \rho_n((A \cup B) \lhd f(A \cup B))$$

By the inclusion-exclusion principle and the properties of the within operation,

$$\rho_n((A \cup B) \lhd f(A \cup B)) = \rho_n(A \lhd f(A \cup B)) + \rho_n(B \lhd f(A \cup B)) - \rho_n((A \cap B) \lhd f(A \cup B))$$

Let $f_A : 2^{<\omega} \to \{0,1\}$ be defined via $f(\sigma) = 1$ if and only if $f(\sigma \cup A) = 1$, where $\sigma \cup A = \tau \in 2^{|\sigma|}$ with $\tau(n) = 1$ if and only if $\sigma(n) = 1$ or $A(n) = 1$. Let f_B be defined similarly for B in place of A.

As A has MWC-density α relative to B and B has MWC-density β relative to A,

$$\rho(A \lhd f_B(A)) = \rho(A \lhd f(A \cup B)) = \lim_{n\to\infty} \rho_n(A \lhd f(A \cup B)) = \alpha$$

and

$$\rho(B \lhd f_A(B)) = \rho(B \lhd f(A \cup B)) = \lim_{n\to\infty} \rho_n(B \lhd f(A \cup B)) = \beta$$

Therefore, what remains is to use an argument similar to that for Lemma 5.7 to handle the intersection.

Define $\hat{f_A} : 2^{<\omega} \to \{0,1\}$ via $\hat{f_A}(\sigma) = 1$ if and only if $f_A(\sigma) = 1$ and $|\sigma| \in A$. Then it follows immediately that

$$\hat{f_A}(B) = A \cap f_A(B) = A \cap f(A \cup B)$$

so

$$B \lhd \hat{f_A}(B) = B \lhd (A \cap f(A \cup B)$$

By Lemma 5.5,

$$B \lhd (A \cap f(A \cup B) = (B \lhd f(A \cup B)) \lhd (A \lhd f(A \cup B))$$

Therefore by the same argument as in Lemma 5.7 we have

$$\rho((B \lhd f_A(B)) \rhd (A \lhd f_B(A)) = \rho((A \cap B) \lhd f(A \cup B)) = \alpha\beta$$

Thus we have $\lim_{n\to\infty} \rho_n((A \cap B) \lhd f(A \cup B)) = \alpha\beta$, and it follows that

$$\rho((A \cup B) \lhd f(A \cup B)) =$$

$$\lim_{n\to\infty} \rho_n(A \lhd f(A \cup B)) + \lim_{n\to\infty} \rho_n(B \lhd f(A \cup B)) - \lim_{n\to\infty} \rho_n((A \cap B) \lhd f(A \cup B)) = \alpha+\beta-\alpha\beta$$

as desired. As f was arbitrary, $A \cup B$ has MWC-density $\alpha + \beta - \alpha\beta$. □

In addition to the general union, we can show that a specific type of disjoint union combines MWC-densities in the same way. The format and disjointness of this special form is more useful for our attempts to translate the proof of Corollary 3.10 to MWC-density. While Lemma 5.1 showed that MWC-density achieves every real in the unit interval, we would still like to be able to generalize Corollary 3.10. We shall discuss the attempts to translate this proof into the MWC-density case and why they fall short. We start by introducing our special case of disjoint union.

Lemma 5.12. *Suppose that A has MWC-density α relative to B and B has MWC-density β relative to A. Then $A \sqcup (B \triangleright \overline{A})$ has MWC-density $\alpha + \beta(1-\alpha) = \alpha + \beta - \alpha\beta$.*

Proof. Let f be a monotone selection function. We wish to show that

$$\rho((A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \alpha + \beta - \alpha\beta$$

By the properties of the within operation,

$$(A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A})) = (A \triangleleft f(A \sqcup (B \triangleright \overline{A}))) \sqcup ((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A})))$$

so

$$\rho((A \sqcup (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho((A \triangleleft f(A \sqcup (B \triangleright \overline{A})))) + \rho(((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))))$$

Therefore, we shall first construct a B-computable monotone selection function f_B such that $f_B(A) = f(A \sqcup (B \triangleright \overline{A}))$. Then

$$A \triangleleft f(A \sqcup (B \triangleright \overline{A})) = A \triangleleft f_B(A)$$

and therefore because A has MWC-density α with respect to B, we have

$$\rho(A \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho(A \triangleleft f_B(A)) = \alpha$$

Define $f_B : 2^{<\omega} \to \{0,1\}$ via $f_B(\sigma) = 1$ if and only if $f(\sigma \sqcup (B \triangleright \overline{\sigma})) = 1$, where $\sigma \sqcup (B \triangleright \overline{\sigma})$ is defined to be $\tau \in 2^{|\sigma|}$ with $\tau(k) = 1$ if and only if $\sigma(k) = 1$ or k is the b_i-th 0 in σ for some i. From this definition, it is immediate that $f_B(A) = f(A \sqcup (B \triangleright \overline{A}))$ as desired.

It remains to show that

$$\rho((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \beta(1-\alpha) = \beta - \beta\alpha$$

We would like to use Theorem 5.6 here, however we cannot because $B \triangleright \overline{A}$ w not have MWC-density relative to A. To fix this, we will mimic the proof of Theorem 5.6, that is we shall construct a B-computable monotone selection function g_B such that $g_B(\overline{A}) = f(A \sqcup (B \triangleright \overline{A}))$. Then $\overline{A} \triangleleft g_B(\overline{A})$ will be a superset of $(B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))$ with density $1-\alpha$ because A has MWC-density α relative to B. Then there is some X such that

$$X \triangleright (\overline{A} \triangleleft g_B(\overline{A})) = (B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))$$

Finally, it suffices to construct an A-computable monotone selection function g_A such that $B \triangleleft g_A(B) = X$: X will then have density β due to the fact that B has MWC-density β relative to A and Lemma 3.5 will ensure that

$$\rho((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) = \rho(X \triangleright (\overline{A} \triangleleft g_B(\overline{A}))) = \beta(1-\alpha)$$

as desired.

Define $g_B : 2^{<\omega} \to \{0,1\}$ via $g_B(\sigma) = 1$ if and only if $f_B(\overline{\sigma}) = 1$, where $\overline{\sigma}$ is defined to be $\tau \in 2^{|\sigma|}$ with $\tau(k) = 1$ if and only if $\sigma(k) = 0$. Then

$$g_B(\overline{A}) = f_B(A) = f(A \sqcup (B \triangleright \overline{A}))$$

Let $g_A : 2^{<\omega} \to \{0,1\}$ be defined via $g_A(\sigma) = f(A \sqcup (\sigma \triangleright \overline{A})$, where $A \sqcup (\sigma \triangleright \overline{A}) = \tau \in 2^{p_{\overline{A}}(|\sigma|)}$ is defined via $\tau(n) = 1$ if and only if $n \in A$ or $n = p_{\overline{A}}(k)$ for some $k < |\sigma|$ and $\sigma(k) = 1$. We now claim that $B \triangleleft g_A(B) = X$.

Recall that X is

$$((B \triangleright \overline{A}) \triangleleft f(A \sqcup (B \triangleright \overline{A}))) \triangleleft (\overline{A} \triangleleft g_B(\overline{A}))$$

As mentioned above, $g_B(\overline{A}) = f(A \sqcup (B \triangleright \overline{A}))$, so we may apply Lemma 5.5 to obtain

$$X = (B \triangleright \overline{A}) \triangleleft (\overline{A} \cap f(A \sqcup (B \triangleright \overline{A})))$$

Suppose $n \in X$. By the definition of X,

$$p_{\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))}(n) \in B \triangleright \overline{A}$$

That is, the n-th element of $\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))$ is in $B \triangleright \overline{A}$. Therefore, it is of the form $p_{\overline{A}}(b_k)$ for some k. Furthermore, $p_{\overline{A}}(b_k) \in f(A \sqcup (B \triangleright \overline{A}))$, so by definition $f((A \sqcup (B \triangleright \overline{A})) \upharpoonright p_{\overline{A}}(b_k)) = 1$. This then implies, by the definition of g_A, that $g_A(B \upharpoonright b_k) = 1$. Therefore $b_k \in g_A(B)$, and $p^{-1}_{g_A(B)}(b_k) \in B \triangleleft f_A(B)$. Finally, note that $p^{-1}_{g_A(B)}(b_k) = n$ because every element of $g_A(B)$ is an element of $\overline{A} \cap f(A \sqcup (B \triangleright \overline{A}))$ by definition, and b_k corresponds to the n-th such one. Therefore $n \in B \triangleleft g_A(B)$. This argument reverses, so $B \triangleleft g_A(B) = X$. □

Note that if A has MWC-density α relative to B and $B \triangleleft \overline{A}$ (whether this latter relativization is implied by the other conditions is essentially Question 6.4) and B has MWC-density β relative to A, then Lemma 5.11 can be obtained as an easy corollary of Lemma 5.12:

$$A \cup B = A \sqcup (B \cap \overline{A}) = A \sqcup ((B \triangleleft \overline{A}) \triangleright \overline{A})$$

Lemma 5.1 relativizes in straightforward fashion. As a result, the proof of Lemma 3.9 immediately lifts to prove an analog for MWC-density: There is a disjoint sequence of sets $\{A_i\}_{i \in \omega}$ such that each A_i has MWC-density $\frac{1}{2^{i+1}}$ relative to the others which can be obtained using the into operation and Van Lambalgen's Theorem. (Theorem 5.6 requires more relativization than Theorem 3.7, but the fact that Theorem 5.6 itself relativizes ensures that the same proof technique applies.)

Unfortunately, the fact that unions do not preserve MWC-density in general means that given a real r, we do not know that the infinite union of the A_i's corresponding to the binary expansion of r will have MWC-Density. In the finite case, however, Lemma 5.12 will ensure the union has the desired MWC-density.

Lemma 5.13. *Let X be 1-Random and let $\{A_i\}_{i \in \omega}$ be constructed from X as in Lemma 3.9. If D is a finite set of natural numbers, then $\bigsqcup_{i \in D} A_i$ is X-computable and has MWC-density $\Sigma_{i \in D} \frac{1}{2^{i+1}}$.*

Proof. Essentially, each $\bigsqcup_{i \in D} A_i$ is composed of finitely many unions of the form found in Lemma 5.12 and finitely many applications of the into operation. Van

Lambalgen's theorem will ensure we have all of the necessary relativizations necessary to use Lemma 5.12 and Theorem 5.6 to reduce the number of unions by one. Combined with induction on the size of the union, this will prove the result.

Recall that we defined $A_0 = \overline{X^{[0]}}$ and

$$A_i = \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$$

for $i > 0$. Therefore

$$\bigsqcup_{i \in D} A_i = \bigsqcup_{i \in D} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$$

(If $i = 0$ or $i = 1$ then we take $X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$ to mean ω and $X^{[0]}$ respectively to ensure that this does indeed match the definition of A_i from Lemma 3.9.)

We argue by induction on the size of D. If D is a singleton, then its member is of the form $\overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$ for some i. By Van Lambalgen's Theorem, each $X^{[j]}$ is 1-Random relative to the join of the others, and therefore by Lemma 5.1 each has MWC-density $\frac{1}{2}$ relative to the join of the others. Thus $\overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}$ has MWC-density $\frac{1}{2^{i+1}}$ by Theorem 5.6. This concludes the base case.

Now suppose it holds that for any 1-Random X and any finite set D of size less than or equal to n, $\bigsqcup_{i \in D} A_i$ has MWC-density $\Sigma_{i \in D} \frac{1}{2^{i+1}}$. Now suppose D has size $n + 1$. First consider the case when $0 \in D$. Then using the fact that $(A \sqcup B) \triangleright C = (A \triangleright C) \sqcup (B \triangleright C)$ and the associativity of the into operation,

$$\bigsqcup_{i \in D} A_i = \overline{X^{[0]}} \sqcup (\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}) =$$

$$\overline{X^{[0]}} \sqcup ((\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]}) \triangleright X^{[0]})$$

Let Y be defined via $Y^{[i]} = X^{[i+1]}$. Y is 1-Random relative to $X^{[0]}$ by Van Lambalgen's Theorem. Thus by the relativized induction hypothesis,

$$\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]} = \bigsqcup_{i \in D, i > 0} \overline{Y^{[i-1]}} \triangleright Y^{[i-2]} \triangleright \ldots Y^{[0]}$$

has MWC-density $\Sigma_{i \in D, i > 0} \frac{1}{2^i}$ relative to $X^{[0]}$. Finally, Lemma 5.12 then implies that

$$\overline{X^{[0]}} \sqcup ((\bigsqcup_{i \in D, i > 0} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots X^{[1]}) \triangleright X^{[0]})$$

has MWC-density

$$\frac{1}{2} + (\sum_{i \in D, i > 0} \frac{1}{2^i})(1 - \frac{1}{2}) = \frac{1}{2} + (\sum_{i \in D, i > 0} \frac{1}{2^{i+1}}) = \sum_{i \in D} \frac{1}{2^{i+1}}$$

as desired.

Now suppose that $j > 0$ is the least element of D. Then we have

$$\bigsqcup_{i \in D} A_i = (\overline{X^{[j]}} \triangleright X^{[j-1]} \triangleright \ldots \triangleright X^{[0]}) \sqcup (\bigsqcup_{i \in D, i > j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[0]}) =$$

$$(\overline{X^{[j]}} \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})) \sqcup (\bigsqcup_{i\in D, i>j} (\overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]}) \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})) =$$
$$(\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]})) \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})$$

Let Y be defined via $Y^{[i]} = Y^{[i+j]}$ and $\hat{D} = \{n - j : n \in D\}$. Then Y is 1-Random by Van Lambalgen's Theorem and $\hat{D}$ is a set of size n which contains 0. Therefore we can apply the relativized version of the previous case to see that

$$\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]}) = \overline{Y^{[0]}} \sqcup (\bigsqcup_{i\in \hat{D}, i>0} \overline{Y^{[i]}} \triangleright Y^{[i-1]} \triangleright \ldots \triangleright Y^{[0]})$$

has MWC-density

$$\sum_{i\in\hat{D}} \frac{1}{2^{i+1}} = \sum_{i\in D} \frac{1}{2^{i+1-j}} = \sum_{i\in D} \frac{2^j}{2^{i+1}}$$

relative to $X^{[j-1]} \triangleright \ldots \triangleright X^{[0]}$. As $X^{[j-1]} \triangleright \ldots \triangleright X^{[0]}$ has MWC-density $\frac{1}{2^j}$ relative to $\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]})$ by Van Lambalgen's Theorem and multiple iterations of the relativized form of Theorem 5.6, it follows that

$$(\overline{X^{[j]}} \sqcup (\bigsqcup_{i\in D, i>j} \overline{X^{[i]}} \triangleright X^{[i-1]} \triangleright \ldots \triangleright X^{[j]})) \triangleright (X^{[j-1]} \triangleright \ldots \triangleright X^{[0]})$$

has MWC-density

$$(\sum_{i\in D} \frac{2^j}{2^{i+1}}) \frac{1}{2^j} = \sum_{i\in D} \frac{1}{2^{i+1}}$$

as desired. This completes the induction. □

Unfortunately, it remains open whether or not this can be extended to infinite unions of this form, which is Question 6.5 below. The difficulty lies once again in the fact that the input set can change which bits are and are not selected. In theory, given any $0 < r < 1$ and the set coding its binary expansion B_r as in Corollary 3.10, for any $\epsilon > 0$ there exists an N such that $\bigsqcup_{n\in B_r, n<N} A_n \triangleleft f(\bigsqcup_{n\in B_r, n<N} A_n)$ has MWC-density within ϵ of r. If we could impose a nice enough uniformity condition on the A_n's, then we may be able to assert that the change from adding the remaining A_n's is no more than ϵ. In practice, however, elements of A_k may change which bits are selected by f in non-uniform fashion so that the density of $\bigsqcup_{n\in B_r, n<N} A_n \triangleleft f(\bigsqcup_{n\in B_r, n<N} A_n)$ is meaningless compared to the density of $\bigsqcup_{n\in B_r} A_n \triangleleft f(\bigsqcup_{n\in B_r} A_n)$.

6. Closing Remarks and Questions

We set out to study which reals in the unit interval could be achieved as the intrinsic density of some set without appealing to the stronger property of randomness. To study this, we introduced the `into` and `within` operations. These turned out to be useful tools for coding sets in noncomputable fashion. They formed a calculus of sorts for intrinsic density, which allowed us to construct sets of arbitrary intrinsic density from any 1-Random. We were also able to prove that the same tools apply to MWC-density and Church-density with slightly stronger requirements, where more common notions like the join and union failed to behave as we would like. The into operation also showed potential as a tool for studying classes of Turing degrees.

We believe there is significant room for future work. We did not investigate full KL-density in this paper. The methods used for studying MWC-density may work there as well, although a priori the non-monotonic nature of KL-density requires more care. Additionally, it is unknown if the `into` and `within` operations work for randomness: i.e. if A is μ_r-random relative to B and B is μ_s-random relative to A, is $B \triangleright A$ μ_{rs}-random? So far we have exploited the fact that stochasticity is determined by analyzing $i(A) \triangleleft s(A)$ and being able to utilize the connection between the `into` and `within` operations. Different methods seem to be necessary to study how the into operation interacts with martingales and/or ML-tests.

There is a notable open question based on our results. In subsection 3.1, we proved that the Turing degrees of X-intrinsically small sets (and thus sets of intrinsic density 1) are closed upwards for all X, and therefore that they coincide with the X-DNC or X-high degrees. However, it is not clear what the case is for the degrees of sets with intrinsic density r for $r \in (0,1)$ even in the non-relativized case. To begin with, the proof that every high or DNC set computes an intrinsically small set does not generalize to intrinsic density r.

Additionally, we cannot use Lemma 3.11 to show that the degrees are closed upwards as sets of intrinsic density r are not closed under subsets for $r > 0$. However, there is a similar argument that shows potential. Suppose $B \geq_T A$ and $P(A) = r \in (0,1)$. Then A is of high or DNC degree as it has defined intrinsic density, so B is also of high or DNC degree. Therefore there is $\hat{B} \equiv_T B$ with $P(\hat{B}) = 1$. If $P_A(\hat{B}) = 1$, then $P(\hat{B} \triangleright A) = r$ by Theorem 3.7. Therefore $A \oplus (\hat{B} \triangleright A) \equiv_T B$ as in Lemma 3.11, and $P(A \oplus (\hat{B} \triangleright A)) = r$ by Theorem 4.1. In other words, if B is sufficiently powerful (high or DNC) relative to a set of intrinsic density r it computes, then it is Turing equivalent to a set of intrinsic density r. However, for any $B \geq_T A$ not A-high and not A-DNC, we know that $\hat{B}$ has $P(\hat{B}) = 1$ but that $P_A(B)$ does not exist. Therefore we cannot apply Theorem 3.7. This motivates the following open question.

Question 6.1. *Suppose $P(A) = r$ and $P(B) = 1$. Is it the case that $P(B \triangleright A) = r$?*

This a weak version of Theorem 3.7. The counterexample to the general weak form shown in Lemma 4.3 had $P(A) = \frac{1}{2}$, so it does not apply to this special case. If this is true, then we could apply it in place of Theorem 3.7 to complete the above argument.

Alternatively, we can formulate this question differently: If $P(A) = r$ and B is intrinsically small, is $B \triangleright A$ intrinsically small? It is worthwhile to note that we cannot hope for this to be true for arbitrary A: there is an intrinsically small set B of hyperimmune free degree. Therefore there is total computable f which dominates p_B. Define $c_n = \Sigma_{i \leq n} f(i)$. Now define

$$A = [0, b_0) \cup \{c_0\} \cup \bigcup_{i \geq 0} (c_i, c_i + (b_{i+1} - b_i)) \cup \{c_{i+1}\}$$

Then we will have $B \triangleright A = C$ is not intrinsically small. Thus there must be some requirement on A for this to be true.

We close by stating some open questions of a more technical nature relating to our various results.

For intrinsic density, we proved that $P(A \triangleleft C) = \alpha$ if C is computable and $P(A) = \alpha$. It is known that $C \triangleleft A$ does not necessarily have intrinsic density in general as witnessed by $A = \omega$, which leads to the following question.

Question 6.2. *Are there conditions on A such that, for computable C, $C \triangleleft A$ has intrinsic density?*

For a discussion on the applications of this question, see the end of Section 4.

In proving Theorem 5.6, we used more relativization than was necessary in the intrinsic density analog Theorem 3.7. However, it is not known whether this is necessary or merely useful.

Question 6.3. *Is the relativization optimal in Theorem 5.6? That is, are there sets A and B such that B has MWC-density β relative to A and A has MWC-density α but $B \triangleright A$ does not have MWC-density $\alpha\beta$?*

The same proof that showed the relativization used in Theorem 3.7 is optimal will not work for Theorem 5.6 because $A \oplus \overline{A}$ will not have MWC-density.

We could not directly lift the proof that the intersection of two intrinsically dense sets multiplied the intrinsic densities of the sets to the case of MWC-density due to different relativization requirements between Theorem 3.7 and its analog Theorem 5.6. A positive resolution to the following question would allow us to do this.

Question 6.4. *If A has MWC-density α and C is computable, does A have MWC-density α relative to $C \triangleleft A$? If so, does this relativize? If this is not true, is it at least the case that whenever A has MWC-density α relative to B and B has MWC-density relative to A, does A have MWC-density α relative to $B \triangleleft A$?*

Our usual techniques do not suffice to answer this question, as they are focused on using oracles, or relativized information, to answer questions about non-relativized MWC-density. This question requires us to answer a question about MWC-density relative to a specific set using non-relativized information.

If this is true and relativizes, or the weaker formulation is true, then whenever A has MWC-density α relative to B and B has MWC-density β relative to A, $A \cap B$ would have MWC-density $\alpha\beta$ as a corollary of Theorem 5.6 and $A \cup B$ would have MWC-density $\alpha + \beta - \alpha\beta$ as a corollary of Lemma 5.12. (Recall that both of these facts are true, but they required separate proofs.)

At the end of Section 5, we discussed the difficulty in translating the proof of Corollary 3.10 into the intrinsic density case.

Question 6.5. *Given a sequence $\{A_n\}_{n\in\omega}$ as constructed in Lemma 3.9, let $0 < r < 1$ and let B_r be the set representing its binary sequence. Does $\bigsqcup_{n\in B_r} A_n$ have MWC-density r? If not in general, are there additional requirements we can put on the sequence to force this to be true?*

We say X is range sotchastic for r if $f\rho(f(A) \triangleleft range(f)) = r$ for all total computable injective functions f.

Question 6.6. *Is it the case that every set of intrinsic density α has range-density α? That is, for any set A with intrinsic density α, is it the case that $f(A) \triangleleft range(f)$ has density α for all total computable injective functions f?*

Note that this is similar to a question asked by the author in [15]: is it the case that for every intrinsically small set A and total computable injective function f, $f(A)$ is intrinsically small?

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