Noncomputable Coding, Density, and Stochasticity

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Asymptotic Density

Recall that the (asymptotic) density of X is

$$\rho(X) = \lim_{n \to \infty} \frac{|X \upharpoonright n|}{n}$$

if this limit exists. $\overline{\rho}(X)$ and $\underline{\rho}(X)$, the upper and lower density, are the limit superior and limit inferior respectively.

Intrinsic Density

Intrinsic Density is the computable permutation invariant form of asymptotic density, that is A has intrinsic density α if $\rho(\pi(A)) = \alpha$ for all computable permutations π. We denote it by $P(A)$ if it exists. A is intrinsically small if $P(A) = 0$.

The absolute upper density, $\overline{P}(A)$, is the limit superior of $\rho(\pi(A))$ over all computable permutations π. The absolute lower density, $\underline{P}(A)$, is the limit inferior.

Randomness

What values of $r \in [0, 1]$ are achieved as the intrinsic density of a set?

The short answer is everything: 0 and 1 are well known. If μ_r is the Bernoulli measure with parameter r (i.e. the r-biased coin flip measure), then standard arguments show that every μ_r-1-random set has intrinsic density r. (It also holds for Schnorr randomness, but the proof takes more work.)

Motivation

Can we use sets of a given intrinsic density to understand those of a different one?

As we shall see, sets with intrinsic density r are a strict super set of the μ_r-randoms and have fundamentally different closure properties as a class. (The join preserves intrinsic density.) We shall introduce noncomputable coding methods, the `into` and `within` operations, to change intrinsic densities and construct an example for every real $r \in (0, 1)$ without appealing to μ_r-randomness.

Changing Density via Noncomputable Coding

To understand intrinsic density, we set out to find some method of combining sets A and B such that the intrinsic density of the resulting set is some function of the intrinsic densities of A and B. Classical operations, notably the join, do not work. Therefore we turn to the `into` operation.

The Into Operation

Definition

Given two sets

$$A = \{a_0 < a_1 < a_2 < \dots\}$$

and

$$B = \{b_0 < b_1 < b_2 < \dots\}$$

we define the set $B \triangleright A$, or "B into A," to be

$$\{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$$

This will be a fundamental tool for changing densities.

Examples

- If E is the set of evens, then $E \triangleright E$ is the set of multiples of 4: $e_n = 2n$, so
$$e_{e_n} = e_{2n} = 2(2n) = 4n$$
- If O is the set of odds, then $E \triangleright O$ is the set of naturals congruent to 1 mod 4: $o_n = 2n + 1$ and $e_n = 2n$, so
$$o_{e_n} = o_{2n} = 2(2n) + 1 = 4n + 1$$
- For any A and B,
$$A \oplus B = (A \triangleright E) \sqcup (B \triangleright O)$$

Basic Properties

- $A = A \triangleright \omega$
- $A = \omega \triangleright A$
- $B \triangleright A \subseteq A$
- $(B \triangleright A) \sqcup (\overline{B} \triangleright A) = A$
- $\triangleright$ is associative

Into and Asymptotic Density

Lemma

- $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(A)\overline{\rho}(B)$
- $\underline{\rho}(B \triangleright A) \geq \underline{\rho}(A)\underline{\rho}(B)$

Corollary

$\rho(B \triangleright A) = \rho(B)\rho(A)$

Into and Intrinsic Density

For a set X, P_X represents intrinsic density relative to X, i.e. invariance under all X-computable permutations.

Theorem

If $P(A) = \alpha$ *and* $P_A(B) = \beta$, *then* $P(B \triangleright A) = \alpha\beta$.

Powers of Two

Lemma

There is a countable, disjoint sequence of sets $\{A_i\}_{i\in\omega}$ *such that* $P(A_i) = \frac{1}{2^{i+1}}$. *Furthermore,* $\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$.

Proof Sketch.

Let X be 1-Random. Then by the general form of Van Lambalgen's theorem and the fact that 1-Randoms have intrinsic density $\frac{1}{2}$, the columns $X^{[n]}$ of X give us countably many sets all with intrinsic density relative to the rest. Then define $B_0 = \omega$, $A_n = \overline{X^{[n]}} \triangleright B_n$, and $B_{n+1} = X^{[n]} \triangleright B_n$. $\square$

Arbitrary Intrinsic Density

Let $r \in (0,1)$ and $\{A_i\}_{i\in\omega}$ be as in the previous lemma. We identify B_r with the set of bits which are 1 in the binary expansion of r. Then $\bigcup_{i\in B_r} A_i$ will have intrinsic density $\Sigma_{i\in B_r} \frac{1}{2^{i+1}} = r$.

Corollary

If $r \in (0,1)$ computes a 1-Random, then r computes a set of intrinsic density r.

MWC Stochasticity

Informally, we say a set X is von Mises-Wald-Church stochastic, or MWC stochastic, for r if every infinite subsequence selected from it by a process which can only use information about whether or not $i \in X$ for $i < n$ to decide whether or not to include n in the subsequence has asymptotic density r.

Into and Stochasticity

Lemma

If A is MWC-stochastic for α relative to B and B is MWC-stochastic for β relative to A, then $B \triangleright A$ is MWC-stochastic for $\alpha\beta$.

Into and Randomness

Question

If A is μ_α-random relative to B and B is μ_β-random relative to A, is $B \triangleright A$ $\mu_{\alpha\beta}$-random?

The `Within` Operation

The `into` operation has a natural dual.

Definition

Given two sets

$$A = \{a_0 < a_1 < a_2 < \dots\}$$

and

$$B = \{b_0 < b_1 < b_2 < \dots\}$$

we define the set $B \triangleleft A$, or "B within A", to be $\{n : a_n \in B\}$. In other words, $B \cap A$ is some subset of A, so there is some X such that $X \triangleright A = B \cap A$. In this case, $B \triangleleft A = X$.

Example

- Let T be the multiple of three. Then $T \triangleleft E = T$:

$$T \triangleleft E = \{n : 2n \in T\} = \{n : 6|2n\} = \{n : 3|n\}$$

- $T \triangleleft O$ is the set of naturals congruent to 1 mod 3:

$$T \triangleleft O = \{n : 2n + 1 \in T\} = \{n : 2n \equiv 2 \bmod 3\} = \{n : n \equiv 1 \bmod 3\}$$

`Within`: Basic Properties

- $\omega = A \triangleleft A$
- $(B \triangleleft A) \sqcup (\overline{B} \triangleleft A) = \omega$
- If $B \subseteq A$, then $(B \triangleleft A) \triangleright A = B \cap A = B$.
- $\triangleleft$ is not associative: Let E be the evens, O the odds, and N the set of naturals congruent to 2 mod 4. Then

$$(O \triangleleft N) \triangleleft E = \emptyset \triangleleft E = \emptyset$$

but

$$O \triangleleft (N \triangleleft E) = O \triangleleft O = \omega$$

Applications of `Within`

Let C be a computable set.

Lemma

If A has intrinsic density α, then so does $A \triangleleft C$.

Lemma

If A has MWC-stochasticity for α, then so does $A \triangleleft C$.

Lemma

If A is μ_α-random, then so is $A \triangleleft C$.