Strategic Uncertainty and the Ex-post Nash Property in Large Games

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Outline

- Non-cooperative games: the role of independence
- Problem of defining a mixed strategy (profile) in non-atomic games: mutual independence and measurability of a continuum of random variables
- We will focus on large non-atomic (strategic) games:
    - Two kinds of randomized strategy profile:
      behavior strategy profile and mixed strategy profile
    - In equilibrium:
      every mixed strategy *induces* a behavior strategy, and
      every behavior strategy can be *lifted* to a mixed strategy
    - A mixed strategy profile is a mixed strategy Nash equilibrium if and only if it has the ex-post Nash property
    - The key tools are rich Fubini extension and the exact law of large numbers (ELLN)

Equivalences

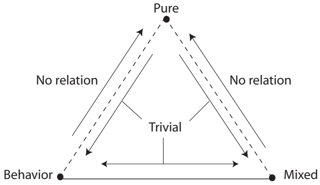


Figure 1a: Classification of Strategy Profiles in Equilibrium in Finite Games

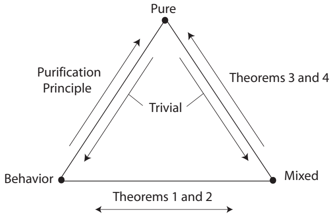


Figure 1b: Classification of Strategy Profiles in Equilibrium in Infinite Games

Strategy

- von Neumann (1928): "*draws* depending on chance" and the "free decisions of individual players"
- Non-Cooperative Games: players make decisions independently
- A n-player game with complete information (Nash 1950, 1951): pure vs. mixed strategies
  - $I = \{1, \cdots, n\}$, $A = \prod_{i=1}^{n} A_i$ where A_i (a finite set) is the set of pure strategies for player $i \in I$
  - Pure Strategy Profile: $a \in A$
  - Mixed Strategy Profile: $\sigma \in \prod_{i=1}^{n} \mathcal{M}(A_i)$, where $\mathcal{M}(A_i)$ denotes the set of probability distributions on A_i
  - In general, a pure strategy Nash equilibrium (PSNE) may not exist, a mixed strategy Nash equilibrium (MSNE) exists in a finite game.

Mixed Strategies

Aumann (1964)

Mathematically, the random device–the set of sides of the coin or of points on the edge of a roulette wheel–constitutes a probability measure space, sometimes called a *sample space*; a mixed strategy is a function from this sample space to the set of all pure strategies. In other words *what we have here is precisely a random variable whose values are pure strategies*.

...The correct condition on a mixed strategy is that this *corresponding* function be measurable. Thus we *define a mixed strategy to be a measurable function from* $\Omega \times X$ *into* Y....

Elements of a Large Game

- *Space of players* $(I, \mathcal{I}, \lambda)$: a non-atomic probability space; e.g., the Lebesgue interval [0, 1]
- *Action set* A: a compact metric space
- *Space of characteristics* $\mathcal{U}_A$: space of real-valued continuous functions on $A \times \mathcal{M}(A)$ endowed with its sup-norm topology where $\mathcal{M}(A)$ is the set of all probability measures on A endowed with the weak topology
- A *large game* $\mathcal{G}^0$ is a measurable function from I to $\mathcal{U}_A$
- A *pure strategy profile* of $\mathcal{G}^0$ is a measurable function from I to A

Pure Strategy Nash Equilibrium (PSNE)

Definition

A *pure strategy Nash equilibrium (PSNE)* of a game $\mathcal{G}^0$ is a pure strategy profile such that for λ-almost all $i \in I$,

$$u_i(f(i), \lambda f^{-1}) \geq u_i(a, \lambda f^{-1})$$

and for all $a \in A$, where $u_i \equiv \mathcal{G}^0(i)$.

PSNE in Non-atomic games

- Measure theoretic frameworks:
  - Large games (with a continuum of players)
  - Finite games with diffused incomplete information
- Action space: finite
  - the Dvoretzky-Wald-Wolfowitz (1951) purification principle guarantees a pure-strategy Nash equilibrium (PSNE) exists
  - Schmeidler (1973), Khan et al. (2006)
  - Radner-Rosenthal (1982), Milgrom-Weber (1985)
- Action space: countable
  - The existence of a PSNE can be obtained through the BV marriage lemma
  - Khan-Rath (2009) extension of DWW
- Action space: uncountable?!

PSNE in Non-atomic games, contd.

- When the action space is uncountable,
  - A PSNE may not exist!
    - Counterexamples (with the Lebesgue unit interval): Rath-Sun-Yamashige (1995), Khan-Rath-Sun (1997)
  - Positive results
    - Loeb space: Khan-Sun (1996, 1999)
    - Saturated space: Keisler-Sun (2009), Khan et al. (2013)
    - Keisler-Sun (2009): Saturated space is not only a sufficient, but also a necessity condition for all games (with uncountable actions) to have PSNE!

Mixed Strategy Nash Equilibrium

Mixed strategy (profile) is given up in non-atomic games. Why?

Osborne (2008)

One interpretation of a mixed strategy Nash equilibrium is that each player conditions her action on the realization of a random variable, where the random variable observed by each player is independent of the random variable observed by every other player.

As players make decisions independently, these random variables of players shall be mutually independent.

- In a finite game, there is no problem in defining a mixed strategy profile
- But in a non-atomic game, this will involve some measurability problems

How to define mixed strategies?

- How to define a "meaningful" mixed strategy profile in a large game?
- Recall again, in non-cooperative game theory, players are not coordinating their randomizations. Thus, it is to ask how to define a decision-theoretic foundation based on the explicit description of the players' randomizations and independence?
- One now needs to work with a process with a continuum of independent random variables to define mixed strategy profile
- There is a problem!

The "Problem"

- Consider any two probability spaces $(I, \mathcal{I}, \lambda)$ (space of players) and $(\Omega, \mathcal{F}, P)$ (sample space)
- $\mathcal{I} \otimes \mathcal{F}$: the usual product σ-algebra (including all the null subsets) generated by $\{S \times T : S \in \mathcal{I}, T \in \mathcal{F}\}$,
- $\lambda \otimes P$ as the product probability measure on $\mathcal{I} \otimes \mathcal{F}$.
- Given any mapping F from $I \times \Omega$ to a complete separable metric space X, let F_i denote the marginal mapping $F(i, \cdot)$ on Ω, and F_ω the marginal mapping $F(\cdot, \omega)$ on I.

Definition

A process F is said to be *essentially pairwise independent* if for λ-almost all $i \in I$, F_i and $F_{i'}$ are independent for λ-almost all $i' \in I$.

Essential pairwise independence is more general than the usual pairwise and mutual independence.

The "Problem," contd.

Constant Random Variables: Sun (2006)

Let f be a function from $I \times \Omega$ to a complete separable metric space X. If f is jointly measurable on the product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$, and if f is essentially pairwise independent, then, for λ-almost all $i \in I$, f_i is a constant random variable.

And so, with this result, we have a precise and clear articulation of the "problem": non-compatibility problem of measurability and independence.

- Related literature: Doob (1937), Judd (1985), Feldman and Gilles (1985), Sun (1996, 1998, 2006)

Some other subtle measurability issues:

- (1) The Lebesgue unit interval is not suitable for modeling a continuum of players acting independently.
  - It has too few measurable sets in the sense that countably many measurable subsets of players determine the Lebesgue σ-algebra
- (2) The usual continuum product sample space (by Kolomogorov construction) is inadequate for the study of a continuum of independent players
  - Events in the usual continuum product sample space depend only on the actions of countably many players, and therefore such a sample probability space is inadequate for the study of a continuum of independent players, especially when one needs to consider the aggregate behavior of all the players

Saturated Spaces

A probability space is said to be countably-generated if its σ-algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not countably-generated.

Definition (Saturated Space)

A probability space $(I, \mathcal{I}, \lambda)$ is saturated if it is nowhere countably-generated, in the sense that, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not countably-generated, where $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.

Solution to the Problem

Definition: Fubini Extension

A probability space $(I \times \Omega, \mathcal{W}, Q)$ is said to be a Fubini extension of the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function F on $(I \times \Omega, \mathcal{W})$,

(i) F_i is P-integrable on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and F_ω is λ-integrable on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;

(ii) $\int_\Omega F_i \,\mathrm{d}P$ and $\int_I F_\omega \,\mathrm{d}\lambda$ are integrable on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ respectively, in addition,
$\int_{I \times \Omega} F \,\mathrm{d}Q = \int_I \left(\int_\Omega F_i \,\mathrm{d}P \right) \mathrm{d}\lambda = \int_\Omega \left(\int_I F_\omega \,\mathrm{d}\lambda \right) \mathrm{d}P.$

To reflect the fact that $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Solution to the Problem, contd.

Definition: Rich Fubini Extension

A Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is said to be rich if there is a $\mathcal{I} \boxtimes \mathcal{F}$-measurable process G from $I \times \Omega$ to the interval $[0, 1]$, such that G is essentially pairwise independent, and G_i induces the uniform distribution on $[0, 1]$ for λ-almost all $i \in I$.

We say that such a rich Fubini extension is based on $(I, \mathcal{I}, \lambda)$, and the process G witnesses the richness of the Fubini extension.

Theorem

The probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if there is a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ based on it.

Sun (2006), Podczeck (2010), Wang-Zhang (2012)

Mixed Strategy Nash Equilibrium (MSNE)

Definitions

A *mixed strategy profile* of a large game $\mathcal{G}^0$ is a $\mathcal{I} \boxtimes \mathcal{F}$-measurable function $g : I \times \Omega \longrightarrow A$ where the process g is essentially pairwise independent.

A *MSNE* of $\mathcal{G}^0$ is a mixed strategy profile g^*, such that for λ-almost all $i \in I$,

$$\int_\Omega u_i(g_i^*(\omega), \lambda g_\omega^{*-1}) dP \geq \int_\Omega u_i(\eta(\omega), \lambda g_\omega^{*-1}) dP \qquad (1)$$

for all random variables $\eta : \Omega \longrightarrow A$.

Behavior Strategies

Another method of randomization:

"However, instead of mixing pure strategies, a player could specify a probability distribution over the *alternatives* in each information set and thus plan his action in any given play. We will call the aggregate of such distributions a *behavior strategy*. –Kuhn (1950)

"A player chooses a probability distribution on the alternatives in each of his information sets, thus randomizing on the occasion of a choice as he knows it." –Kuhn (1953)

Instead of using random variables to incorporate all the independence in a strategy profile explicitly, one can work with a behavior strategy profile and consider a distribution over alternatives locally for any given player of a given type.

Behavior Strategy Nash Equilibrium (BSNE)

Definitions

A *behavior strategy profile* of a game $\mathcal{G}^0$ is a measurable function from I to $\mathcal{M}(A)$.

A *BSNE* of $\mathcal{G}^0$ is a behavior strategy profile $h^* : I \longrightarrow \mathcal{M}(A)$, such that for λ-almost all $i \in I$,

$$\int_A u_i \left(a, \int_I h^*(j) d\lambda \right) dh_i^* \geq \int_A u_i \left(a, \int_I h^*(j) d\lambda \right) d\nu \qquad (2)$$

for all $\nu \in \mathcal{M}(A)$.

Interconnections: MSNE and BSNE

- In a finite game with complete information, a BSNE is trivially equivalent to a MSNE.
- It is natural to ask if the equivalence holds for large games.
- The following propositions are needed:

ELLN (Sun 2006)

Assume that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension. If F is an essentially pairwise independent and $\mathcal{I} \boxtimes \mathcal{F}$-measurable process, then the sample distribution λF_{ω}^{-1} is the same as the distribution $(\lambda \boxtimes P)\, F^{-1}$ for P-almost all $\omega \in \Omega$.

Equivalence of BSNE and MSNE

Consolidation of Distributions

Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension, X a complete separable metric space, and f a measurable mapping from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. Then there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F : I \times \Omega \to X$ such that the process F is essentially pairwise independent and $f(i)$ is the induced distribution by F_i, for λ-almost all $i \in I$.

Theorem 1

The following equivalence holds for a large game $\mathcal{G}^0$:
(i) Every MSNE induces a BSNE;
(ii) Every BSNE can be lifted to a MSNE.

Sketch of the Proof of Theorem 1

- Suppose g^* is a MSNE of a game $\mathcal{G}^0$. Let $h^*(i) = Pg_i^{*-1}$ for all $i \in I$.
  - By the ELLN, $\int_I h^*(i) d\lambda = \int_I Pg_i^{*-1} d\lambda = \lambda g_\omega^{*-1}$.
  - For any random variable $\eta : \Omega \longrightarrow A$, $P\eta^{-1} \in \mathcal{M}(A)$. At the same time, for any $\nu \in \mathcal{M}(A)$, there exists a random variable $\eta : \Omega \longrightarrow A$ such that $\nu = P\eta^{-1}$
  - By the ELLN and the change of variable theorem, h^* where $h^*(i) = Pg_i^{*-1}$ for all $i \in I$ is a BSNE of $\mathcal{G}^0$.
- Suppose h^* is a BSNE of a game $\mathcal{G}^0$.
  - There is a $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g^* from $I \times \Omega$ to A such that g^* is an essentially pairwise independent process, and the distribution Pg_i^{*-1} is the given distribution $h^*(i)$ for λ-almost all $i \in I$.
  - By the ELLN, one can show such a process g^* is a MSNE.

Ex-Post Nash Property

It would also be of interest to ask once uncertainty is resolved, if a player has incentive to depart *ex-post* from her optimal strategy taken in the *ex-ante* game when she finds herself in the realized *ex-post* game.

Definition

A mixed strategy profile g^* of a game is said to have the *ex-post Nash property* if for P-almost all $\omega \in \Omega$, g^*_ω is a PSNE for the same game with the empirical action distribution λg^{*-1}_ω.

Theorem 2

A mixed strategy profile of $\mathcal{G}^0$ is a MSNE if and only if it has the ex-post Nash property.

Sketch of the Proof of Theorem 2

- Suppose g^* is a Nash equilibrium in mixed strategies.
  - By the ELLN, first observe for any mixed strategy profile g, $\lambda g_\omega^{-1}(\cdot) = \int_I P g_i^{-1}(\cdot) d\lambda$.
  - Let $\xi = \int_I P g_i^{*-1}(\cdot) d\lambda$. By the ELLN, for λ-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i^*(\omega), \xi\right) dP \geq \int_\Omega u_i\left(\eta(\omega), \xi\right) dP \text{ for all r.v. } \eta : \Omega \longrightarrow A.$$

  - i.e., for λ-almost all $i \in I$, for P-almost all $\omega \in \Omega$,

$$u_i(g_i^*(\omega), \xi) = \max_{a \in A} u_i(a, \xi).$$

  - By the Fubini property, we have, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

$$u_i\left(g_\omega^*(i), \xi\right) = \max_{a \in A} u_i(a, \xi).$$

Sketch of the Proof, contd.

- $\cdots$
  - By the ELLN again, hence, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

    $$u_i\left(g^*_\omega(i), \lambda g^{*-1}_\omega\right) = \max_{a \in A} u_i\left(a, \lambda g^{*-1}_\omega\right).$$

  - g^* has expost Nash property.
- Now, suppose that a mixed strategy profile g has ex-post Nash property:
  - i.e., for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

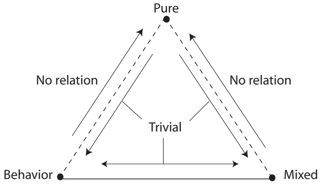
    $$u_i\left(g_\omega(i), \lambda g^{-1}_\omega\right) = \max_{a \in A} u_i(a, \lambda g^{-1}_\omega).$$

  - By the ELLN and the Fubini property, $\cdots$, one can show that for λ-almost all $i \in I$,

    $$\int_\Omega u_i\left(g_i(\omega), \lambda g^{-1}_\omega\right) dP \geq \int_\Omega u_i\left(\eta(\omega), \lambda g^{-1}_\omega\right) dP, \forall \eta : \Omega \longrightarrow A.$$

  - Thus, this verifies that g is a MSNE of $\mathcal{G}^0$.

Back to the Figure



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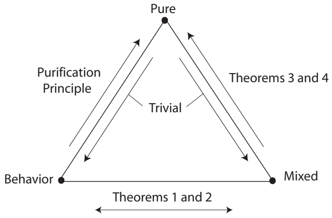

Figure 1b: Classification of Strategy Profiles in Equilibrium in Infinite Games