Strategic Representation and Realization of Large Distributional Games

M. Ali Khan
Johns Hopkins University

Kali P. Rath
University of Notre Dame

Haomiao Yu
Ryerson University

Yongchao Zhang
Shanghai University of Finance and Economics

August 04, 2013

Background

- Consider situations where the payoff to a player depends upon own action and the trait-action distribution of all others.
- A distributional game is a probability measure on the space of players' characteristics–the product of the space of players' traits and the space of players' payoffs.
- A Nash equilibrium distribution (NED) of a distributional game is a probability measure on the product space of players' characteristics and actions such that:
  - its marginal on the space of characteristics is the given game
  - it gives full measure to the characteristics and corresponding best action pairs.
- A strategic game is a mapping from a space of players' names to the space of characteristics.
- A Nash equilibrium of a strategic game is a mapping from the space of players' names to the space of actions, such that each player chooses a best action corresponding to the induced trait-action distribution.

Background, contd.

- General large games (with traits)
  - Strategic form: Khan et al. (2013), Qiao-Yu (2013)
  - Distributional form: Khan et al. (2013)
- Conventional large games (all players share some common trait):
  - Strategic form: Schmeidler (1973) (finite action)
  - Distributional form: Mas-Colell (1984)
  - Representation: Rath (1995) (finite action)
- This paper examines the relationships among equilibria of the two game forms (distributional and strategic) in the general setting.

Large Distributional Games (LDG)

- A: a compact metric set of actions.
- T: a complete separable metric space of traits.
- $\mathcal{M}(T \times A)$: the set of probability measures on $T \times A$ (weak convergence).
- $\mathcal{U}_{(A,T)}$: the space of real valued continuous functions on $A \times \mathcal{M}(T \times A)$, metrized by supremum norm.

Definition

(a) A *LDG* is a probability measure μ on $T \times \mathcal{U}_{(A,T)}$.

(b) A probability measure τ on $T \times \mathcal{U}_{(A,T)} \times A$ is a *Nash Equilibrium Distribution* (NED) of a LDG μ if

(i) $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and

(ii) $\tau(B(\tau)) = 1$ where $B(\tau) = \{(t, u, a) \in T \times \mathcal{U}_{(A,T)} \times A : u(a, \tau_{T \times A}) \geq u(x, \tau_{T \times A}) \text{ for all } x \in A\}$.

NEDs of *LDG*

- Let μ be a *LDG*.

Definition

(*c*) A NED τ of a game is *symmetric* if there exists a measurable function $h : T \times \mathcal{U}_{(A,T)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$, i.e., players with the same characteristics take the same action.

(*d*) A NED τ of a game can be *symmetrized* if there exists a symmetric NED τ^s of the game such that $B(\tau) = B(\tau^s)$.

(*e*) Two NEDs τ and τ' of a game μ are *similar* if $\tau_A = \tau'_A$.

Theorem

(a) There exists a NED for any LDG.
(b)There exists a symmetric NED of an atomless LDG if T and A are countable. Furthermore, every NED of such a LDG can be symmetrized.

Large Strategic Games (*LSG*)

Definition

(a) Given an abstract atomless probability space $(I, \mathcal{I}, \lambda)$, a *LSG* $\mathcal{G}$ is measurable function from I to $T \times \mathcal{U}_{(A,T)}$.

(b) A *Nash equilibrium* of a *LSG* $\mathcal{G}$ is a measurable function $f : I \longrightarrow A$ such that such that for λ-almost all $i \in I$,

$$v_i\left(f(i), \lambda \circ (\alpha, f)^{-1}\right) \geq v_i\left(a, \lambda \circ (\alpha, f)^{-1}\right) \text{ for all } a \in A,$$

with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \to T$ abbreviated for $\mathcal{G}_1$, where $\mathcal{G}_k$ is the projection of $\mathcal{G}$ on its k^{th}-coordinate, $k = 1, 2$.

- If A or T is uncountable, a Nash equilibriums need not exist in a *LDG* when the name space is Lebesgue unit interval.
- A Nash equilibrium of a *LSG* exists if both A and T are countable (finite or countably infinite), or $(I, \mathcal{I}, \lambda)$ is a saturated probability space. (Qiao-Yu)

Strategic Representation of *LDG*

Definition

Let μ be a *LDG*. A $(I, \mathcal{I}, \lambda)$ *representation* of μ is a *LSG* $\mathcal{G}$ with $(I, \mathcal{I}, \lambda)$ as its name space such that $\mu = \lambda \circ \mathcal{G}^{-1}$.

Let L denote the unit interval, $\mathcal{L}$ its Borel σ-algebra and ℓ the Lebesgue measure on it. $\mathcal{G}$ is a *Lebesgue representation* of μ if $\mathcal{G}$ is a representation of μ with the name space $(L, \mathcal{L}, \ell)$.

Theorem

Let μ be a LDG and $(I, \mathcal{I}, \lambda)$ an arbitrary atomless probability space. Then there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ.

Representation Results

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ, f a measurable mapping from I to A and $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$. Then $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and $\tau_A = \lambda \circ f^{-1}$. Furthermore,
(a) If f is a Nash equilibrium of $\mathcal{G}$ then τ is a NED of μ.
(b) If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$.

The above theorem shows that any Nash equilibrium of a representation induces a NED of the *LDG*.

It also shows that if a NED is induced by a strategy profile of the representation, then the strategy profile is a Nash equilibrium of the representation.

What about the converse?

A Partial Converse

Theorem

Given a NED τ of μ and an atomless probability space $(I, \mathcal{I}, \lambda)$, there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ and a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

- What about a full converse?
- Namely, in the statement above, given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ, does there exist a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$?
- In general, the answer is no.

Two Exceptions

Case (1): Representation with Countable Characteristics:

A *LSG* $\mathcal{G}$ has *countable characteristics* if the range of $\mathcal{G}$ is countable. (See Carmona (2008) when the space of characteristics is the space of payoffs.)

Case (2): Saturated Representation:

$(I, \mathcal{I}, \lambda)$ is a saturate probability space.

Theorem

An atomless probability space $(I, \mathcal{I}, \lambda)$ and a NED τ of μ are given. Given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ,

if either Case (1) or Case (2) holds,

then there is a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

The Similarity Theorem

Theorem

Let A and T be countable. Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ and τ a NED of μ. Then there exists a Nash equilibrium f of $\mathcal{G}$ such that $\tau^* = \lambda \circ (\mathcal{G}, f)^{-1}$ is a NED of μ and τ^* is similar to τ. If in addition, μ is atomless then τ^* can be taken to be symmetric.

- Example 1 shows that the conclusions of this Theorem cannot be strengthened even with finite actions/one trait.
- Thus, one cannot go beyond similarity.
- Counterexamples show that this Theorem cannot be strengthened to the case of uncountable actions/traits.

Representation and Symmetric NEDs

Corollary

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ.
Let τ be a symmetric NED of μ such that τ(graph of h) = 1.
Define $f : I \longrightarrow A$ by $f(i) = h(\mathcal{G}(i))$.
Then $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ and f is a Nash equilibrium of $\mathcal{G}$.

Given a *LDG* $\mathcal{G}$, let $\sigma(\mathcal{G}) = \{\mathcal{G}^{-1}(U) : U \in \mathcal{B}(T \times \mathcal{U}_{(A,T)})\}$, where $\mathcal{B}(T \times \mathcal{U}_{(A,T)})$ is the Borel σ-algebra of $T \times \mathcal{U}_{(A,T)}$. $\sigma(\mathcal{G})$ is the smallest σ-algebra on $\mathcal{I}$ with respect to which $\mathcal{G}$ is measurable.

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ. Then τ is a symmetric NED of μ if and only if $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ for a $\sigma(\mathcal{G})$-measurable Nash equilibrium f of $\mathcal{G}$.

Almost One-to-one Representations

Given any probability space $(I, \mathcal{I}, \lambda)$, a function on I is *almost one-to-one* if it is one-to-one on I except some λ-null set of I.

Theorem

Let $\mathcal{G}$ be a $(L, \mathcal{L}, \ell)$ representation of μ. Assume that $\mathcal{G}$ is almost one-to-one.
(a) If f is a Nash equilibrium of $\mathcal{G}$ then $\tau = \ell \circ (\mathcal{G}, f)^{-1}$ is a symmetric NED of μ.
(b) Let $f : I \longrightarrow A$ be any measurable function and $\tau = \ell \circ (\mathcal{G}, f)^{-1}$. If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$ and τ is symmetric.

- If μ is atomless, there exists an almost one-to-one Lebesgue representation.
- The result is not true on arbitrary atomless measure spaces.

Examples

To simplify the idea, in each example, we consider a game where all players share a common trait, i.e., the space of characteristics $T \times \mathcal{U}_{(A,T)}$ is now reduced to $\mathcal{U}_A$, the space of real valued continuous functions on $A \times \mathcal{M}(A)$, metrized by supremum norm.

- Example 1: A *NED* of a *LDG* cannot be induced by a Nash equilibrium of a given strategic Lebesgue representation.
- Example 2: The *NED* above can be induced by a Nash equilibrium of some other Lebesgue representation.

Example 1

Let the action set be $A = \{a_1, a_2\}$ and the player set be the Lebesgue interval $(L, \mathcal{L}, \ell)$. Consider a particular function $u \in \mathcal{U}_A$, defined as follows: $u(a_1, \nu) = 1/2$, $u(a_2, \nu) = 1 - \nu(a_2)$.

Let $\mathcal{G}^1(i) = iu$ for $i \in L$. Define f_1 and f_2 as follows:

$$f_1(t) = a_1 \text{ if } t < 1/2 \text{ and } f_1(t) = a_2 \text{ if } t \geq 1/2.$$

$$f_2(t) = a_2 \text{ if } t < 1/2 \text{ and } f_2(t) = a_1 \text{ if } t \geq 1/2.$$

Both f_1 and f_2 are Nash equilibria of $\mathcal{G}^1$.

Let $\tau = \ell \circ (\mathcal{G}^1, f_1)^{-1}$, $\tau' = \ell \circ (\mathcal{G}^1, f_2)^{-1}$ and $\tau^\alpha = \alpha\tau + (1-\alpha)\tau'$ for $0 < \alpha < 1$.

The LDG μ^1 and τ^α

Consider the *LDG* $\mu^1 = \ell \circ (\mathcal{G}^1)^{-1}$.
For any $\alpha \in (0,1)$, τ^α is a NED of the *LDG* μ^1.

Example 1, contd.

One can show that

A Negative Result

$\mathcal{G}^1$ is a Lebesgue representation of μ^1. But there is no Nash equilibrium f of $\mathcal{G}^1$ such that $\tau^\alpha = \ell \circ (\mathcal{G}^1, f)^{-1}$, for $0 < \alpha < 1$.

A Similarity Result

However, there exists a Nash equilibrium f' such that (a) $\tau^* = \ell \circ \left(\mathcal{G}^1, f'\right)^{-1}$ is a NED of μ^1, and (b) τ^α and τ^* are similar.

Example 2

We now show that for any fixed α, the *NED* τ^α of the *LDG* μ^1 in Example 1 indeed can be induced by some Lebesgue representation of the *LDG* and its Nash equilibrium.

In particular, let $\alpha = 1/2$.

Consider the same function u as in Example 1. Define $\mathcal{H} : L \longrightarrow \mathcal{U}_A$ as follows.

$$\begin{aligned} \mathcal{H}(i) &= 2iu & \text{if } i < \tfrac{1}{2} \\ &= \mathcal{H}\left(i - \tfrac{1}{2}\right) & \text{if } i \geq \tfrac{1}{2} \end{aligned}$$

Since $\mathcal{H}(i) = \mathcal{H}(i - (1/2))$ for each $i \geq 1/2$, $\mathcal{H}$ is not one-to-one.

Example 2, contd.

We can show that

Another Representation of μ^1

$\mathcal{H}$ is a Lebesgue representation of the μ^1 in Example 1.

Moreover, Let $f(i) = a_1$ if $i \in [0, 1/4] \cup (3/4, 1]$ and
$f(i) = a_2$ if $i \in (1/4, 1/2] \cup (1/2, 3/4]$.

Nash equilibrium of $\mathcal{H}$

The NED $\tau^{1/2}$ of μ^1 can be induced by a Nash equilibrium f of $\mathcal{H}$.

Examples on Uncountable Actions/Traits

Negative results on the existence of Nash equiilibria in some *LSG* with Lebesgue unit interval as the name space.

- When A is $[-1, 1]$. Examples in RSY or in KRS:
  LSG without Nash equilibrium.
- When T is $[0, 1]$. Example 1 in Qiao-Yu:
  a *LSG* that has no Nash equilibrium.

Fix any *LSG* $\mathcal{G}$ in those examples. Let $\mu = \lambda \circ \mathcal{G}^{-1}$.
There exists a NED of μ.

Further Discussions

- Countably Determined Games
- Realization of NEDs

Countably Determined Games

Consider games with a common trait for all the players.

Denote by $id(r)$ the constant function in $\mathcal{U}_A$ which always assumes value r. Let ψ be the operator on $\mathcal{U}_A$ such that $\psi(u) = u$ if $u = id(0)$ and $u / \parallel u \parallel$ otherwise. ψ is continuous on $\mathcal{U}_A \setminus id(0)$ and is measurable on $\mathcal{U}_A$. Given a game $\mathcal{G}$, consider the game $\bar{\mathcal{G}}$ where $\bar{\mathcal{G}}(i) = \psi(\mathcal{G}(i))$ for all i.

$\mathcal{G}$ is *determined by countable characteristics* if the range of $\bar{\mathcal{G}}$ is countable.

Theorem

Let $\mathcal{G}$ be a game determined by countable characteristics and $\mu = \lambda \circ \mathcal{G}^{-1}$.
(a) $\mathcal{G}$ has a Nash equilibrium f.
(b) If μ is atomless then it has a symmetric NED.
(c) The similarity theorem (above) holds.

Realization of NEDs

Definition

Given a NED τ of a *LDG* μ, we say that a probability space $(I, \mathcal{I}, \lambda)$ is a *realization* of τ (or, $(I, \mathcal{I}, \lambda)$ *realizes* τ) if every $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ has a Nash equilibrium f such that $\lambda \circ (\mathcal{G}, f)^{-1} = \tau$.

Characterization of NEDs by Realization:

Corollary

Let μ be an atomless LDG and τ a NED of μ.
(a) τ is symmetric if and only if the Lebesuge unit interval is a realization of τ.
(b) If τ is non-symmetric, then an atomless probability space realizes τ if and only if it is saturated.

Conclusions

- Existence of NED and symmetric NED in a *LDG*.
- *LDG* and its Strategic Representation:
- Any Nash equilibrium of a representation of a *LDG* induces a NED of the *LDG*.
- Converse: not all NEDs of a *LDG* can be induced by a Nash equilibrium of a given representation.
  - Two exceptions:
    - Representation with countable characteristics
    - Saturated representation
  - Representation in general: Similarity Theorem
- Characterization of Symmetric NED in a *LDG*
  - $\sigma(\mathcal{G})$-measurable Nash equilibirum
  - Almost one-to-one Lebesgue representation
- Countably determined games
- Realization: symmetric and non-symmetric case