Strategic Representation and Realization of Large Distributional Games

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Consider situations where the payoff to a player depends upon own action and the trait-action distribution of all others.

A distributional game is a probability measure on the space of players’ characteristics—the product of the space of players’ traits and the space of players’ payoffs.

A Nash equilibrium distribution (NED) of a distributional game is a probability measure on the product space of players’ characteristics and actions such that:

- its marginal on the space of characteristics is the given game
- it gives full measure to the characteristics and corresponding best action pairs.

A strategic game is a mapping from a space of players’ names to the space of characteristics.

A Nash equilibrium of a strategic game is a mapping from the space of players’ names to the space of actions, such that each player chooses a best action corresponding to the induced trait-action distribution.
General large games (with traits)
- Strategic form: Khan et al. (2013), Qiao-Yu (2013)
- Distributional form: Khan et al. (2013)

Conventional large games (all players share some common trait):
- Strategic form: Schmeidler (1973) (finite action)
- Distributional form: Mas-Colell (1984)
- Representation: Rath (1995) (finite action)

This paper examines the relationships among equilibria of the two game forms (distributional and strategic) in the general setting.
Large Distributional Games (LDG)

- \(A\): a compact metric set of actions.
- \(T\): a complete separable metric space of traits.
- \(\mathcal{M}(T \times A)\): the set of probability measures on \(T \times A\) (weak convergence).
- \(\mathcal{U}_{(A,T)}\): the space of real valued continuous functions on \(A \times \mathcal{M}(T \times A)\), metrized by supremum norm.

Definition

(a) A **LDG** is a probability measure \(\mu\) on \(T \times \mathcal{U}_{(A,T)}\).

(b) A probability measure \(\tau\) on \(T \times \mathcal{U}_{(A,T)} \times A\) is a **Nash Equilibrium Distribution** (NED) of a LDG \(\mu\) if

\[
(i) \quad \tau_{T \times \mathcal{U}_{(A,T)}} = \mu \quad \text{and} \\
(ii) \quad \tau(B(\tau)) = 1 \quad \text{where} \quad B(\tau) = \{(t, u, a) \in T \times \mathcal{U}_{(A,T)} \times A : u(a, \tau_{T \times A}) \geq u(x, \tau_{T \times A}) \quad \text{for all} \quad x \in A\}.
\]
Let $\mu$ be a LDG.

**Definition**

(c) A NED $\tau$ of a game is *symmetric* if there exists a measurable function $h : T \times \mathcal{U}_{(A,T)} \to A$ such that $\tau(\text{graph of } h) = 1$, i.e., players with the same characteristics take the same action.

(d) A NED $\tau$ of a game can be *symmetrized* if there exists a symmetric NED $\tau^s$ of the game such that $B(\tau) = B(\tau^s)$.

(e) Two NEDs $\tau$ and $\tau'$ of a game $\mu$ are *similar* if $\tau_A = \tau'_A$.

**Theorem**

(a) There exists a NED for any LDG.

(b) There exists a symmetric NED of an atomless LDG if $T$ and $A$ are countable. Furthermore, every NED of such a LDG can be symmetrized.
Large Strategic Games (LSG)

Definition

(a) Given an abstract atomless probability space \((I, \mathcal{I}, \lambda)\), a \textit{LSG} \(G\) is measurable function from \(I\) to \(T \times U_{(A, T)}\).

(b) A \textit{Nash equilibrium} of a \textit{LSG} \(G\) is a measurable function \(f : I \rightarrow A\) such that such that for \(\lambda\)-almost all \(i \in I\),

\[
v_i \left( f(i), \lambda \circ (\alpha, f)^{-1} \right) \geq v_i \left( a, \lambda \circ (\alpha, f)^{-1} \right) \quad \text{for all } a \in A,
\]

with \(v_i\) abbreviated for \(G_2(i)\), and \(\alpha : I \rightarrow T\) abbreviated for \(G_1\), where \(G_k\) is the projection of \(G\) on its \(k^{th}\)-coordinate, \(k = 1, 2\).

- If \(A\) or \(T\) is uncountable, a Nash equilibriums need not exist in a \textit{LDG} when the name space is Lebesgue unit interval.

- A Nash equilibrium of a \textit{LSG} exists if both \(A\) and \(T\) are countable (finite or countably infinite), or \((I, \mathcal{I}, \lambda)\) is a saturated probability space. (Qiao-Yu)
**Definition**

Let $\mu$ be a LDG. A $(I, \mathcal{I}, \lambda)$ representation of $\mu$ is a LSG $\mathcal{G}$ with $(I, \mathcal{I}, \lambda)$ as its name space such that $\mu = \lambda \circ \mathcal{G}^{-1}$.

Let $L$ denote the unit interval, $\mathcal{L}$ its Borel $\sigma$-algebra and $\ell$ the Lebesgue measure on it. $\mathcal{G}$ is a Lebesgue representation of $\mu$ if $\mathcal{G}$ is a representation of $\mu$ with the name space $(L, \mathcal{L}, \ell)$.

**Theorem**

Let $\mu$ be a LDG and $(I, \mathcal{I}, \lambda)$ an arbitrary atomless probability space. Then there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of $\mu$. 

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of $\mu$, $f$ a measurable mapping from $I$ to $A$ and $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$. Then $\tau_{T \times \mathcal{U}(A, T)} = \mu$ and $\tau_{A} = \lambda \circ f^{-1}$. Furthermore,

(a) If $f$ is a Nash equilibrium of $\mathcal{G}$ then $\tau$ is a NED of $\mu$.
(b) If $\tau$ is a NED of $\mu$ then $f$ is a Nash equilibrium of the representation.

The above theorem shows that any Nash equilibrium of a representation induces a NED of the LDG.

It also shows that if a NED is induced by a strategy profile of the representation, then the strategy profile is a Nash equilibrium of the representation.

What about the converse?
A Partial Converse

Theorem

Given a NED $\tau$ of $\mu$ and an atomless probability space $(I, I, \lambda)$, there is a $(I, I, \lambda)$ representation $G$ of $\mu$ and a Nash equilibrium $f$ of $G$ such that $\tau = \lambda \circ (G, f)^{-1}$.

What about a full converse?

Namely, in the statement above, given a $(I, I, \lambda)$ representation $G$ of $\mu$, does there exist a Nash equilibrium $f$ of $G$ such that $\tau = \lambda \circ (G, f)^{-1}$?

In general, the answer is no.
Case (1): Representation with Countable Characteristics:

A LSG $\mathcal{G}$ has *countable characteristics* if the range of $\mathcal{G}$ is countable. (See Carmona (2008) when the space of characteristics is the space of payoffs.)

Case (2): Saturated Representation:

$(I, \mathcal{I}, \lambda)$ is a saturate probability space.

**Theorem**

An atomless probability space $(I, \mathcal{I}, \lambda)$ and a NED $\tau$ of $\mu$ are given. Given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of $\mu$,

if either Case (1) or Case (2) holds,

then there is a Nash equilibrium $f$ of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$. 
The Similarity Theorem

Theorem

Let $A$ and $T$ be countable. Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of $\mu$ and $\tau$ a NED of $\mu$. Then there exists a Nash equilibrium $f$ of $\mathcal{G}$ such that $\tau^* = \lambda \circ (\mathcal{G}, f)^{-1}$ is a NED of $\mu$ and $\tau^*$ is similar to $\tau$. If in addition, $\mu$ is atomless then $\tau^*$ can be taken to be symmetric.

- Example 1 shows that the conclusions of this Theorem cannot be strengthened even with finite actions/one trait.
- Thus, one cannot go beyond similarity.
- Counterexamples show that this Theorem cannot be strengthened to the case of uncountable actions/traits.
Corollary

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of $\mu$.
Let $\tau$ be a symmetric NED of $\mu$ such that $\tau(\text{graph of } h) = 1$.
Define $f : I \rightarrow A$ by $f(i) = h(\mathcal{G}(i))$.
Then $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ and $f$ is a Nash equilibrium of $\mathcal{G}$.

Given a LDG $\mathcal{G}$, let $\sigma(\mathcal{G}) = \{\mathcal{G}^{-1}(U) : U \in \mathcal{B}(T \times \mathcal{U}_{(A,T)})\}$, where $\mathcal{B}(T \times \mathcal{U}_{(A,T)})$ is the Borel $\sigma$-algebra of $T \times \mathcal{U}_{(A,T)}$. $\sigma(\mathcal{G})$ is the smallest $\sigma$-algebra on $\mathcal{I}$ with respect to which $\mathcal{G}$ is measurable.

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of $\mu$. Then $\tau$ is a symmetric NED of $\mu$ if and only if $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ for a $\sigma(\mathcal{G})$-measurable Nash equilibrium $f$ of $\mathcal{G}$.
Given any probability space \((I, \mathcal{I}, \lambda)\), a function on \(I\) is *almost one-to-one* if it is one-to-one on \(I\) except some \(\lambda\)-null set of \(I\).

**Theorem**

Let \(G\) be a \((L, \mathcal{L}, \ell)\) representation of \(\mu\). Assume that \(G\) is almost one-to-one.

(a) If \(f\) is a Nash equilibrium of \(G\) then \(\tau = \ell \circ (G, f)^{-1}\) is a symmetric NED of \(\mu\).

(b) Let \(f : I \rightarrow A\) be any measurable function and \(\tau = \ell \circ (G, f)^{-1}\). If \(\tau\) is a NED of \(\mu\) then \(f\) is a Nash equilibrium of \(G\) and \(\tau\) is symmetric.

- If \(\mu\) is atomless, there exists an almost one-to-one Lebesgue representation.
- The result is not true on arbitrary atomless measure spaces.
Examples

To simplify the idea, in each example, we consider a game where all players share a common trait, i.e., the space of characteristics $T \times \mathcal{U}_{(A,T)}$ is now reduced to $\mathcal{U}_A$, the space of real valued continuous functions on $A \times \mathcal{M}(A)$, metrized by supremum norm.

- **Example 1:** A NED of a LDG cannot be induced by a Nash equilibrium of a given strategic Lebesgue representation.

- **Example 2:** The NED above can be induced by a Nash equilibrium of some other Lebesgue representation.
Example 1

Let the action set be $A = \{a_1, a_2\}$ and the player set be the Lebesgue interval $(L, \mathcal{L}, \ell)$. Consider a particular function $u \in \mathcal{U}_A$, defined as follows: $u(a_1, \nu) = 1/2$, $u(a_2, \nu) = 1 - \nu(a_2)$.

Let $G_1(i) = iu$ for $i \in L$. Define $f_1$ and $f_2$ as follows:

$$f_1(t) = a_1 \text{ if } t < 1/2 \text{ and } f_1(t) = a_2 \text{ if } t \geq 1/2.$$  

$$f_2(t) = a_2 \text{ if } t < 1/2 \text{ and } f_2(t) = a_1 \text{ if } t \geq 1/2.$$  

Both $f_1$ and $f_2$ are Nash equilibria of $G_1$.

Let $\tau = \ell \circ (G_1, f_1)^{-1}$, $\tau' = \ell \circ (G_1, f_2)^{-1}$ and $\tau^\alpha = \alpha \tau + (1 - \alpha)\tau'$ for $0 < \alpha < 1$.

The LDG $\mu^1$ and $\tau^\alpha$

Consider the LDG $\mu^1 = \ell \circ (G_1)^{-1}$. For any $\alpha \in (0, 1)$, $\tau^\alpha$ is a NED of the LDG $\mu^1$. 
Example 1, contd.

One can show that

<table>
<thead>
<tr>
<th>A Negative Result</th>
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<tbody>
<tr>
<td>$\mathcal{G}^1$ is a Lebesgue representation of $\mu^1$. But there is no Nash equilibrium $f$ of $\mathcal{G}^1$ such that $\tau^\alpha = \ell \circ (\mathcal{G}^1, f)^{-1}$, for $0 &lt; \alpha &lt; 1$.</td>
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<thead>
<tr>
<th>A Similarity Result</th>
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<td>However, there exists a Nash equilibrium $f'$ such that (a) $\tau^* = \ell \circ (\mathcal{G}^1, f')^{-1}$ is a NED of $\mu^1$, and (b) $\tau^\alpha$ and $\tau^*$ are similar.</td>
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Example 2

We now show that for any fixed $\alpha$, the $NED$ $\tau^\alpha$ of the $LDG$ $\mu^1$ in Example 1 indeed can be induced by some Lebesgue representation of the $LDG$ and its Nash equilibrium.

In particular, let $\alpha = 1/2$.

Consider the same function $u$ as in Example 1. Define $\mathcal{H} : L \rightarrow \mathcal{U}_A$ as follows.

$$\mathcal{H}(i) = \begin{cases} 2iu & \text{if } i < \frac{1}{2} \\ \mathcal{H}(i - \frac{1}{2}) & \text{if } i \geq \frac{1}{2} \end{cases}$$

Since $\mathcal{H}(i) = \mathcal{H}(i - (1/2))$ for each $i \geq 1/2$, $\mathcal{H}$ is not one-to-one.
Example 2, contd.

We can show that

Another Representation of $\mu^1$

$\mathcal{H}$ is a Lebesgue representation of the $\mu^1$ in Example 1.

Moreover, Let $f(i) = a_1$ if $i \in [0, 1/4] \cup (3/4, 1]$ and $f(i) = a_2$ if $i \in (1/4, 1/2] \cup (1/2, 3/4]$.

Nash equilibrium of $\mathcal{H}$

The NED $\tau^{1/2}$ of $\mu^1$ can be induced by a Nash equilibrium $f$ of $\mathcal{H}$. 
Negative results on the existence of Nash equilibria in some \( LSG \) with Lebesgue unit interval as the name space.

- When \( A \) is \([-1, 1]\). Examples in RSY or in KRS: \( LSG \) without Nash equilibrium.
- When \( T \) is \([0, 1]\). Example 1 in Qiao-Yu: a \( LSG \) that has no Nash equilibrium.

Fix any \( LSG \) \( \mathcal{G} \) in those examples. Let \( \mu = \lambda \circ \mathcal{G}^{-1} \). There exists a NED of \( \mu \).
Further Discussions

- Countably Determined Games
- Realization of NEDs
Consider games with a common trait for all the players.

Denote by \( id(r) \) the constant function in \( \mathcal{U}_A \) which always assumes value \( r \). Let \( \psi \) be the operator on \( \mathcal{U}_A \) such that \( \psi(u) = u \) if \( u = id(0) \) and \( u / \| u \| \) otherwise. \( \psi \) is continuous on \( \mathcal{U}_A \setminus id(0) \) and is measurable on \( \mathcal{U}_A \). Given a game \( G \), consider the game \( \overline{G} \) where \( \overline{G}(i) = \psi(G(i)) \) for all \( i \).

\( G \) is determined by countable characteristics if the range of \( \overline{G} \) is countable.

Theorem

Let \( G \) be a game determined by countable characteristics and \( \mu = \lambda \circ G^{-1} \).
(a) \( G \) has a Nash equilibrium \( f \).
(b) If \( \mu \) is atomless then it has a symmetric NED.
(c) The similarity theorem (above) holds.
Realization of NEDs

**Definition**

Given a NED $\tau$ of a LDG $\mu$, we say that a probability space $(I, I, \lambda)$ is a realization of $\tau$ (or, $(I, I, \lambda)$ realizes $\tau$) if every $(I, I, \lambda)$ representation $G$ of $\mu$ has a Nash equilibrium $f$ such that $\lambda \circ (G, f)^{-1} = \tau$.

**Characterization of NEDs by Realization:**

**Corollary**

Let $\mu$ be an atomless LDG and $\tau$ a NED of $\mu$.

(a) $\tau$ is symmetric if and only if the Lebesgue unit interval is a realization of $\tau$.

(b) If $\tau$ is non-symmetric, then an atomless probability space realizes $\tau$ if and only if it is saturated.
Conclusions

- Existence of NED and symmetric NED in a LDG.
- *LDG* and its Strategic Representation:
  - Any Nash equilibrium of a representation of a *LDG* induces a NED of the *LDG*.
  - Converse: not all NEDs of a *LDG* can be induced by a Nash equilibrium of a given representation.
    - Two exceptions:
      - Representation with countable characteristics
      - Saturated representation
    - Representation in general: Similarity Theorem
- Characterization of Symmetric NED in a *LDG*
  - $\sigma(G)$-measurable Nash equilibrium
  - Almost one-to-one Lebesgue representation
- Countably determined games
- Realization: symmetric and non-symmetric case