Strategic Representation and Realization of Large Distributional Games

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Background

- Consider situations where the payoff to a player depends upon own action and the action-trait distribution of all others.
- A distributional game is a probability measure on the space of players' characteristics - the product of the space of players' traits and the space of players' payoffs.
- A Nash equilibrium distribution (NED) of a distributional game is a probability measure on the product space of players' characteristics and actions such that:
    - its marginal on the space of characteristics is the given game
    - it gives full measure to the characteristics and corresponding best action pairs.
- A strategic game is a mapping from a space of players' names to the space of characteristics.
- A Nash equilibrium of a strategic game is a mapping from the space of players' names to the space of actions, such that each player chooses a best action corresponding to the induced action distribution.

Background, contd.

- General large games (with traits)
  - Strategic form: Khan et. al (2013), Qiao-Yu
  - Distributional form: Khan et. al (2013)
- Conventional large games (all players share some common trait):
  - Strategic form: Schmeidler (1973) (finite action)
  - Distributional form: Mas-Colell (1984)
  - Representation: Rath (1995) (finite action)
- This paper aims to study the relations of two game forms in the general setting.

Large Distributional Games (LDG)

- A: a compact metric set of actions.
- T: a complete separable metric space of traits.
- $\mathcal{M}(T \times A)$: the set of probability measures on $T \times A$ (weak convergence).
- $\mathcal{U}_{(A,T)}$: the the space of real valued continuous functions on $A \times \mathcal{M}(T \times A)$, metrized by supremum norm.

Definition

(a) A *LDG* is a probability measure μ on $T \times \mathcal{U}_{(A,T)}$.

(b) A probability measure τ on $T \times \mathcal{U}_{(A,T)} \times A$ is a *Nash Equilibrium Distribution* (NED) of a *LDG* μ if

(i) $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and

(ii) $\tau(B(\tau)) = 1$ where $B(\tau) = \{(t, u, a) \in T \times \mathcal{U}_{(A,T)} \times A : u(a, \tau_{T \times A}) \geq u(x, \tau_{T \times A})$ for all $x \in A\}$.

▶ Let μ be a *LDG*.

Definition

(c) A NED τ of a game is *symmetric* if there exists a measurable function $h : T \times \mathcal{U}_{(A, \tau)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$, i.e., players with the same characteristics take the same action.

(d) A NED τ of a game can be *symmetrized* if there exists a symmetric NED τ^s of the game such that $B(\tau) = B(\tau^s)$.

(e) Two NEDs τ and τ' of a game μ are *similar* if $\tau_A = \tau'_A$.

Theorem

(a) There exists a NED for any LDG.
(b) There exists a symmetric NED of an atomless LDG if T and A are countable. Furthermore, every NED of such a LDG can be symmetrized.

Large Strategic Games (LSG)

Definition

(a) Given an abstract atomless probability space $(I, \mathcal{I}, \lambda)$, a LSG $\mathcal{G}$ is measurable function from I to $T \times \mathcal{U}_{(A,T)}$.

(b) A *Nash equilibrium* of a LSG $\mathcal{G}$ is a measurable function $f : I \longrightarrow A$ such that such that for λ-almost all $i \in I$,

$$v_i \left(f(i), \lambda \circ (\alpha, f)^{-1} \right) \geq v_i \left(a, \lambda \circ (\alpha, f)^{-1} \right) \text{ for all } a \in A,$$

with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \to T$ abbreviated for $\mathcal{G}_1$, where $\mathcal{G}_k$ is the projection of $\mathcal{G}$ on its kth-coordinate, $k = 1, 2$.

- If A or T is uncountable, a Nash equilibriums need not exist in a LDG when the name space is Lebesgue unit interval.
- A Nash equilibrium of a LSG exists if both A and T are countable (finite or countably infinite), or $(I, \mathcal{I}, \lambda)$ is a saturated probability space. (Qiao-Yu)

Strategic Representation of *LDG*

Definition

Let μ be a *LDG*. A $(I, \mathcal{I}, \lambda)$ *representation* of μ is a *LSG* $\mathcal{G}$ with $(I, \mathcal{I}, \lambda)$ as its name space such that $\mu = \lambda \circ \mathcal{G}^{-1}$.

Let L denote the unit interval, $\mathcal{L}$ its Borel σ-algebra and ℓ the Lebesgue measure on it. $\mathcal{G}$ is a *Lebesgue representation* of μ if $\mathcal{G}$ is a representation of μ with the name space $(L, \mathcal{L}, \ell)$.

Theorem

Let μ be a LDG and $(I, \mathcal{I}, \lambda)$ an arbitrary atomless probability space. Then there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ.

Representation Results

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ, f a measurable mapping from I to A and $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$. Then $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and $\tau_A = \lambda \circ f^{-1}$. Furthermore,
(a) If f is a Nash equilibrium of $\mathcal{G}$ then τ is a NED of μ.
(b) If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$.

The above theorem shows that any Nash equilibrium of a representation induces a NED of the *LDG*. It also shows that if a NED is induced by a strategy profile of the representation, then the strategy profile is a Nash equilibrium of the representation.

What about the converse?

A Partial Converse

Theorem

Given a NED τ of μ and an atomless probability space $(I, \mathcal{I}, \lambda)$, there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ and a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

- ▶ What about a full converse?
- ▶ Namely, in the statement above, given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ, does there exist a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$?
- ▶ In general, the answer is no.

Two Exceptions

Case (1): Representation with Countable Characteristics:

A *LSG* $\mathcal{G}$ has *countable characteristics* if the range of $\mathcal{G}$ is countable. (See Carmona (2008) when the space of characteristics is the space of payoffs.)

Case (2): Saturated Representation:

$(I, \mathcal{I}, \lambda)$ is a saturate probability space.

Theorem

An atomless probability space $(I, \mathcal{I}, \lambda)$ and a NED τ of μ are given. Given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ, if either Case (1) or Case (2) holds, then there is a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

The Similarity Theorem

Theorem

Let A and T be countable. Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ and τ a NED of μ. Then there exists a Nash equilibrium f of $\mathcal{G}$ such that $\tau^* = \lambda \circ (\mathcal{G}, f)^{-1}$ is a NED of μ and τ^* is similar to τ. If in addition, μ is atomless then τ^* can be taken to be symmetric.

- ▶ Example 1 shows that the conclusions of this Theorem cannot be strengthened even with finite actions/one trait.
- ▶ Thus, one cannot go beyond similarity.
- ▶ Counterexamples show that this Theorem cannot be strengthened to the case of uncountable actions/traits.

Representation and Symmetric NEDs

Corollary

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ.
Let τ be a symmetric NED of μ such that τ(graph of h) = 1.
Define $f : I \longrightarrow A$ by $f(i) = h(\mathcal{G}(i))$.
Then $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ and f is a Nash equilibrium of $\mathcal{G}$.

Given a *LDG* $\mathcal{G}$, let $\sigma(\mathcal{G}) = \{\mathcal{G}^{-1}(U) : U \in \mathcal{B}(T \times \mathcal{U}_{(A,T)})\}$, where $\mathcal{B}(T \times \mathcal{U}_{(A,T)})$ is the Borel σ-algebra of $T \times \mathcal{U}_{(A,T)}$. $\sigma(\mathcal{G})$ is the smallest σ-algebra on $\mathcal{I}$ with respect to which $\mathcal{G}$ is measurable.

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ. Then τ is a symmetric NED of μ if and only if $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ for a $\sigma(\mathcal{G})$-measurable Nash equilibrium f of $\mathcal{G}$.

Almost One-to-one Representations

Given any probability space $(I, \mathcal{I}, \lambda)$, a function on I is *almost one-to-one* if it is one-to-one on I except some λ-null set of I.

Theorem

Let $\mathcal{G}$ be a $(L, \mathcal{L}, \ell)$ representation of μ. Assume that $\mathcal{G}$ is almost one-to-one.
(a) If f is a Nash equilibrium of $\mathcal{G}$ then $\tau = \ell \circ (\mathcal{G}, f)^{-1}$ is a symmetric NED of μ.
(b) Let $f : I \longrightarrow A$ be any measurable function and $\tau = \ell \circ (\mathcal{G}, f)^{-1}$. If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$ and τ is symmetric.

▶ If μ is atomless, there exists an almost one-to-one Lebesgue representation.

▶ The result is not true on arbitrary atomless measure spaces.

Examples

To simplify the idea, in each example, we consider a game where all players share a common trait, i.e., the space of characteristics $T \times \mathcal{U}_{(A, T)}$ is now reduced to $\mathcal{U}_A$, the space of real valued continuous functions on $A \times \mathcal{M}(A)$, metrized by supremum norm.

- ▶ Example 1: A *NED* of a *LDG* cannot be induced by a Nash equilibrium of a given strategic Lebesgue representation.
- ▶ Example 2: The *NED* above can be induced by a Nash equilibrium of some other Lebesgue representation.

Example 1

Let the action set be $A = \{a_1, a_2\}$ and the player set be the
Lebesgue interval $(L, \mathcal{L}, \ell)$. Consider a particular function $u \in \mathcal{U}_A$,
defined as follows: $u(a_1, \nu) = 1/2$, $u(a_2, \nu) = 1 - \nu(a_2)$.

Let $\mathcal{G}^1(i) = iu$ for $i \in L$. Define f_1 and f_2 as follows:

$f_1(t) = a_1$ if $t < 1/2$ and $f_1(t) = a_2$ if $t \geq 1/2$.

$f_2(t) = a_2$ if $t < 1/2$ and $f_2(t) = a_1$ if $t \geq 1/2$.

Both f_1 and f_2 are Nash equilibria of $\mathcal{G}^1$.

Let $\tau = \ell \circ (\mathcal{G}^1, f_1)^{-1}$, $\tau' = \ell \circ (\mathcal{G}^1, f_2)^{-1}$ and $\tau^\alpha = \alpha\tau + (1 - \alpha)\tau'$
for $0 < \alpha < 1$.

The LDG μ^1 and τ^α

Consider the *LDG* $\mu^1 = \ell \circ (\mathcal{G}^1)^{-1}$.
For any $\alpha \in (0, 1)$, τ^α is a NED of the *LDG* μ^1.

Example 1, contd.

One can show that

A Negative Result

$\mathcal{G}^1$ is a Lebesgue representation of μ^1. But there is no Nash equilibrium f of $\mathcal{G}^1$ such that $\tau^\alpha = \ell \circ (\mathcal{G}^1, f)^{-1}$, for $0 < \alpha < 1$.

A Similarity Result

However, there exists a Nash equilibrium f' such that (a) $\tau^* = \ell \circ (\mathcal{G}^1, f')^{-1}$ is a NED of μ^1, and (b) τ^α and τ^* are similar.

Example 2

We now show that for any fixed α, the *NED* τ^α of the *LDG* μ^1 in Example 1 indeed can be induced by some Lebesgue representation of the *LDG* and its Nash equilibrium.

In particular, let $\alpha = 1/2$.

Consider the same function u as in Example 1. Define $\mathcal{H} : L \longrightarrow \mathcal{U}_A$ as follows.

$$
\begin{aligned}
\mathcal{H}(i) &= 2iu && \text{if } i < \tfrac{1}{2} \\
&= \mathcal{H}\left(i - \tfrac{1}{2}\right) && \text{if } i \geq \tfrac{1}{2}
\end{aligned}
$$

Since $\mathcal{H}(i) = \mathcal{H}(i - (1/2))$ for each $i \geq 1/2$, $\mathcal{H}$ is not one-to-one.

Example 2, contd.

We can show that

Another Representation of μ^1

$\mathcal{H}$ is a Lebesgue representation of the μ^1 in Example 1.

Moreover, Let $f(i) = a_1$ if $i \in [0, 1/4] \cup (3/4, 1]$ and
$f(i) = a_2$ if $i \in (1/4, 1/2] \cup (1/2, 3/4]$.

Nash equilibrium of $\mathcal{H}$

The NED $\tau^{1/2}$ of μ^1 can be induced by a Nash equilibrium f of $\mathcal{H}$.

Examples on Uncountable Actions/Traits

There are several negative results on the existence of Nash equiilibria in some *LSG* with Lebesgue unit interval as the name space.

- When A is $[-1, 1]$. Examples in RSY or in KRS: *LSG* without Nash equilibrium.
- When T is $[0, 1]$. Example 1 in Qiao-Yu: a *LSG* that has no Nash equilibrium.

Fix any *LSG* $\mathcal{G}$ in those examples. Let $\mu = \lambda \circ \mathcal{G}^{-1}$. There exists a NED in μ.

Further Discussions

- Countably Determined Games
- Realization of NEDs

Countably Determined Games

For simplicity, we consider a game where all players share a common trait.

Denote by $id(r)$ the constant function in $\mathcal{U}_A$ which always assumes value r. Let ψ be the operator on $\mathcal{U}_A$ such that $\psi(u) = u$ if $u = id(0)$ and $u/\parallel u \parallel$ otherwise. ψ is continuous on $\mathcal{U}_{(A,\mathcal{T})} \setminus id(0)$ and is measurable on $\mathcal{U}_A$. Given a game $\mathcal{G}$, consider the game $\bar{\mathcal{G}}$ where $\bar{\mathcal{G}}(i) = \psi(\mathcal{G}(i))$ for all i.

$\mathcal{G}$ is *determined by countable characteristics* if the range of $\bar{\mathcal{G}}$ is countable.

Theorem

Let $\mathcal{G}$ be a game determined by countable characteristics and $\mu = \lambda \circ \mathcal{G}^{-1}$.
(a) $\mathcal{G}$ has a Nash equilibrium f.
(b) If μ is atomless then it has a symmetric NED.

Realization of NEDs

Definition

Given a NED τ of a *LDG* μ, we say that a probability space $(I, \mathcal{I}, \lambda)$ is a *realization* of τ (or, $(I, \mathcal{I}, \lambda)$ *realizes* τ) if every $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ has a Nash equilibrium f such that $\lambda \circ (\mathcal{G}, f)^{-1} = \tau$.

Characterization of NEDs by Realization:

Corollary

Let μ be an atomless LDG and τ a NED of μ.
(a) τ is symmetric if and only if the Lebesuge unit interval is a realization of τ.
(b) If τ is non-symmetric, then an atomless probability space realizes τ if and only if it is saturated.

- Existence of NED and symmetric NED in a *LDG*.
- *LDG* and its Strategic Representation:
- Any Nash equilibrium of a representation of a *LDG* induces a NED of the *LDG*.
- In converse, not all NEDs of a *LDG* can be induced by a Nash equilibrium of a given representation.
    - Two exceptions:
        - Representation with countable characteristics
        - Saturated representation
    - Representation in general: Similarity Theorem
- Characterization of Symmetric NED in a *LDG*
    - $\sigma(\mathcal{G})$-measurable Nash equilibirum
    - Almost one-to-one representation
    - Lebesgue realization