The Nonexistence of Mixed Strategy Nash Equilibria for a Countable Agent Space

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Nash: Every finite-player, finite-action game has an equilibrium in mixed strategies.

Games with infinitely many players, compact convex action set of each player, payoffs are quasi-concave in own argument. There is a Nash equilibrium. Ma (1969)

Atomless, countably additive measure space of players:
- If the set of players is an atomless, countably additive measure space then a game has a pure strategy Nash equilibrium. Schmeidler (1973)
- DWW (1951) theorem: Every mixed strategy Nash equilibrium can be purified.

The DWW theorem holds for finitely additive measure spaces.

**Question:** Does a pure/mixed strategy Nash equilibrium exist in a game over a finitely additive measure space of players?
This Talk

- If the set of players is endowed with a finitely additive measure, then a game may not have a Nash equilibrium (in pure or mixed strategies).
- Main reason: Failure of the upper hemicontinuity of the integral of a correspondence.
Let \((T, \mathcal{T}, \mu)\) be an atomless, countably additive measure space and \(X\) a metric space.

Let \(F : T \times X \rightarrow \mathbb{R}^n\) be a correspondence.

If \(F(\cdot, x)\) is measurable and \(F(t, \cdot)\) is upper hemicontinuous then
\[
\int_T F(\cdot, x) \, d\mu
\]
is upper hemicontinuous (in \(x\)).

This result fails if \(\mu\) is a finitely additive measure.
Let $E = \{e^1, \ldots, e^n\}$ be the set of unit vectors in $\mathbb{R}^n$ and $S = \{s \in \mathbb{R}^n_+ : \sum_{i=1}^n s_i = 1\}$ the unit simplex in $\mathbb{R}^n$.

Let $U$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.

Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive measure space.

A (non-anonymous large) game is a measurable function $G : T \rightarrow U$.

A $f : T \rightarrow E$ is a (pure strategy) Nash equilibrium of $G$ if for almost all $t$,

$$G(t) \left( f(t), \int f \, d\mu \right) \geq G(t) \left( a, \int f \, d\mu \right) \text{ for all } a \in E.$$
Definition of Nash Equilibrium

Theorem (Schmeidler)

Every game has a pure strategy Nash equilibrium.

Define a correspondence $B : T \times S \rightarrow E$ by

$$B(t, s) = \{ e^i \in E | G(t)(e^i, s) \geq G(t)(a, s) \text{ for all } a \in E \}.$$

- $B(t, s)$ is nonempty, $B(\cdot, s)$ is measurable and $B(t, \cdot)$ is uhc.
- Let $\Gamma(s) = \int_T B(\cdot, s) \, d\mu$.
  - $\Gamma(s)$ is nonempty for each $s \in S$.
  - $\Gamma(\cdot)$ is uhc (integration preserves uhc).
  - $\Gamma(\cdot)$ is convex valued (by Lyapunov’s theorem).
- $\Gamma$ has a fixed point $s^*$ (by Kakutani’s fixed point theorem).
- So, there is $f : T \rightarrow E$ such that $\int f \, d\mu = s^*$ and for almost all $t$, $f(t) \in B(t, s^*)$.
- This $f$ is a Nash equilibrium of $G$. 

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Nonexistence of Nash Equilibria
Finitely Additive Measures

- $T$ is a nonempty set and $\mathcal{T}$ a field of subsets of $T$.
  - (i) $\emptyset, T \in \mathcal{T}$; (ii) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$ and
  - (iii) $A, B \in \mathcal{T} \Rightarrow A \setminus B \in \mathcal{T}$.

- $\mu$ is a finitely additive probability measure on $\mathcal{T}$ if
  - (i) $\mu(\emptyset) = 0, \mu(T) = 1, \mu(A) \geq 0$ for all $A \in \mathcal{T}$ and
  - (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{T}, A \cap B = \emptyset$.

- Let $\mathbb{N}$ denote the set of positive integers. Often, we will be concerned with a finitely additive, probability measure on the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$.

- $\mu$ is strongly continuous if for every $\epsilon > 0$, there exists a measurable partition $\{F_1, \ldots, F_n\}$ of $T$ such that $\mu(F_i) < \epsilon$ for every $i$.

- If $\mu$ is strongly continuous then it is atomless. A countably additive measure $\mu$ is strongly continuous iff it is atomless.

- The range of a strongly continuous measure is convex.
Let \((T, \mathcal{T}, \mu)\) be a finitely additive probability measure space. All functions below are real valued on \(T\).

- **f is simple** if there exist \(\{s_1, \ldots, s_n\}\) and \(\{T_1, \ldots, T_n\}\) such that \(T_n \in \mathcal{T}\) for all \(n\) and \(f(t) = s_n\) if \(t \in T_n\).
  
  \(f\) is said to be \(\mu\)-integrable and \(\int f \, d\mu = \sum_{i=1}^{n} s_i \mu(T_i)\).

- **The outer measure** \(\mu^* : \mathcal{P}(T) \rightarrow [0, 1]\) is given by

\[
\mu^*(A) = \inf \{ \mu(B) : A \subseteq B, \ B \in \mathcal{T} \}.
\]

- A sequence of functions \(\{f_n\}\) converges hazily to \(f\) if

\[
\lim_{n \to \infty} \mu^*(\{t \in T : |f_n(t) - f(t)| > \epsilon\}) = 0, \quad \text{for every } \epsilon > 0.
\]

- **f is integrable** if there exist a sequence of simple functions such that: (i) \(\{f_n\}\) converges hazily to \(f\) and
  
  (ii) \(\lim_{m,n \to \infty} \int |f_n - f_m| \, d\mu = 0\). \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\).
A Motivating Example: Lack of UHC

Let $A = \{0, 1\}$ and $S = [0, 1]$. Let $\mu$ be a finitely additive probability measure on $\mathcal{P}(\mathbb{N})$ such that the $\mu$-measure of any finite set is zero.

Define a correspondence $F : \mathbb{N} \times S \rightarrow A$ as:

$$F(t, x) = \begin{cases} 
\{0, 1\} & \text{if } x = 1/(t+1) \\
1 & \text{if } x < 1/(t+1) \\
0 & \text{if } x > 1/(t+1)
\end{cases}$$

Then

$$\int_\mathbb{N} F(\cdot, x) \, d\mu = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x > 0.
\end{cases}$$

Clearly, $\int_\mathbb{N} F(\cdot, x) \, d\mu$ is not uhc at $x = 0$.

We have only assumed that the $\mu$-measure of any finite set is zero. In particular, we can take $\mu$ to be any strongly continuous measure (such as a density measure).
Graphs of the Correspondence

\[ F(t, x) = \begin{cases} 
0, 1 & \text{if } x = \frac{1}{t+1} \\
1 & \text{if } x < \frac{1}{t+1} \\
0 & \text{if } x > \frac{1}{t+1}.
\end{cases} \]

Let \( t = 9 \).
Example, contd.

$F : \mathbb{N} \times S \rightarrow A.$

$$F(t, x) = \begin{cases} 
\{0, 1\} & \text{if } x = 1/(t + 1) \\
1 & \text{if } x < 1/(t + 1) \\
0 & \text{if } x > 1/(t + 1). 
\end{cases}$$

Then

$$\int_{\mathbb{N}} F(\cdot, x) \, d\mu = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x > 0. 
\end{cases}$$

Clearly, $\int_{\mathbb{N}} F(\cdot, x) \, d\mu$ is not uhc at $x = 0$.

Let $f$ be a measurable selection. If $x = 0$ then $x < 1/(t + 1)$ for all $t \in \mathbb{N}$, which implies that $f(t) = 1$ for all $t \in \mathbb{N}$ and $\int f \, d\mu = 1$.

If $x > 0$ then $x \leq 1/(t + 1)$ for at most finitely many $t$’s. Since the $\mu$-measure of any finite set is zero, $f(t) = 0$ for almost all $t$ and $\int f \, d\mu = 0$. 
Let $A = \{0, 1\}$ and $S = [0, 1]$. Let $\mathcal{U}$ be the set of real valued continuous functions on $A \times S$, endowed with sup norm.

$\mathbb{N}$ is the set of positive integers. Let $\mu$ be a strongly continuous, finitely additive measure on $\mathcal{P}(\mathbb{N})$.

A game is a measurable function $\mathcal{G}$ from $\mathbb{N}$ to $\mathcal{U}$.

A measurable function $f$ from $\mathbb{N}$ to $A$ is a *Nash equilibrium* of a game $\mathcal{G}$ if

$$\mathcal{G}(t) \left( f(t), \int f \, d\mu \right) \geq \mathcal{G}(t) \left( a, \int f \, d\mu \right)$$

for all $a \in A$ and for almost all $t \in \mathbb{N}$.

Note: This notion of Nash equilibrium is in pure strategies.
Nonexistence of Nash Equilibria: Example

Let $A = \{0, 1\}$ and $S = [0, 1]$. For each $t \in \mathbb{N}$, let the payoff function (on $A \times S$) be

$$u_t(a, x) = \left(x - \frac{1}{t + 1}\right)^{a+1}, \quad a \in A.$$ 

Then $t \to u_t$ defines a game.

We will derive the best responses and show that this game has no Nash equilibrium.

Best responses:

$$\text{argmax}_{a \in A} u_t(a, x) = \begin{cases} 
\{0, 1\} & \text{if } x = 1/(t + 1) \\
1 & \text{if } x < 1/(t + 1) \\
0 & \text{if } x > 1/(t + 1).
\end{cases}$$

- $x = 1/(t + 1)$: $u_t(0, x) = u_t(1, x) = 0$.
- $x < 1/(t + 1)$: $u_t(0, x) < 0 < u_t(1, x)$.
- $x > 1/(t + 1)$: $0 < u_t(0, x) < 1$, $u_t(1, x) = [u_t(0, x)]^2$. 
Example: contd.

- **Best responses:**

\[
\arg\max_{a \in A} u_t(a, x) = \begin{cases} 
\{0, 1\} & \text{if } x = 1/(t + 1) \\
1 & \text{if } x < 1/(t + 1) \\
0 & \text{if } x > 1/(t + 1).
\end{cases}
\]

- **Suppose that** \( f \) **from** \( \mathbb{N} \) **to** \( A \) **is a Nash equilibrium**.

Let \( x = \int f \, d\mu \).

- If \( x = 0 \) then \( x < 1/(t + 1) \) for all \( t \in \mathbb{N} \) which implies that \( f(t) = 1 \) for all \( t \) and \( \int f \, d\mu = 1 \), a contradiction.

- If \( x > 0 \) then \( x > 1/(t + 1) \) for almost all \( t \) (since the measure of a finite set is zero), which implies that \( f(t) = 0 \) for almost all \( t \) and \( \int f \, d\mu = 0 \), again a contradiction.

- **The game does not have a Nash equilibrium.**
We will now consider mixed strategies (formalized as integrals).

Let $A = S = [0, 1]$. For each $t \in \mathbb{N}$, let the payoff be

$$v_t(p, x) = (1 - p)u_t(0, x) + pu_t(1, x).$$

A $f : \mathbb{N} \rightarrow A$ is a (mixed strategy) Nash equilibrium if

$$v_t\left(f(t), \int f \, d\mu\right) \geq v_t\left(p, \int f \, d\mu\right)$$

for all $p \in A$ and for almost all $t \in \mathbb{N}$.

The best responses are as before, i.e., almost all $t$ will choose a pure action, 0 or 1. The preceding arguments show that there is no Nash equilibrium (in mixed strategies).
Let $T$ be a nonempty set and $\mathcal{T}$ a field of subsets of $T$. Let $\mu$ be a finitely additive probability measure on $T$.

Assume that $\mu$ is not countably additive. We will show that there is a game on $\mu$ which has no pure or mixed strategy Nash equilibrium.

**Claim**

The following conditions are equivalent.

(i) $\mu$ is countably additive.

(ii) $\lim_{n \to \infty} \mu(B_n) = \mu(B)$ whenever $\{B_n\}$ is an increasing sequence of sets in $\mathcal{T}$ with $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{T}$. 
Let $A = \{0, 1\}$ be the set of actions.

Since $\mu$ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that

$$\bigcup_{n=1}^{\infty} B_n = T \quad \text{and} \quad \lim_{n \to \infty} \mu(B_n) = c < 1.$$ 

For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.

$\{C_n\}$ is a sequence of pairwise disjoint sets and $\bigcup_{n=1}^{\infty} C_n = T$.

Now we will define the payoffs. Let $x \in [0, 1]$. For each $t \in C_n$, let

$$u_t(a, x) = (x - \ell_n)^{a+1}, \quad a \in A \quad \text{where} \quad \ell_n = c + \frac{1-c}{n}.$$ 

Note that $\ell_1 = 1$, $\ell_n > c$ for each $n$ and $\{\ell_n\}$ is a monotonically decreasing sequence converging to $c$. 
The Example, contd.

- $u_t(a, x) = (x - \ell_n)^{a+1}$. Best responses:

  $$\arg\max_{a \in A} u_t(a, x) = \begin{cases} 
  \{0, 1\} & \text{if } x = \ell_n \\
  1 & \text{if } x < \ell_n \\
  0 & \text{if } x > \ell_n.
  \end{cases}$$

- Let $f : T \rightarrow A$ be a pure strategy Nash equilibrium and $x = \int f \, d\mu$.
  - Suppose that $x \leq c < 1$. Then for all $t \in T$, $f(t) = 1$ which implies that $x = 1$, a contradiction.
  - Now suppose that $x > c$. Then there exists a unique $n_0 \in \mathbb{N}$ such that $\ell_{n_0+1} < x \leq \ell_{n_0}$. If $n \geq n_0 + 1$ and $t \in C_n$ then $f(t) = 0$. So, $x = \int f \, d\mu \leq \sum_{i=1}^{n_0} \mu(C_i) = \mu(B_{n_0}) \leq c$, a contradiction.

- The game does not have a pure strategy Nash equilibrium.

- Similar arguments can be used to show that the game does not have a mixed strategy Nash equilibrium.
Theorem

Let \((T, \mathcal{T}, \mu)\) be a finitely additive measure space where \(\mu\) is strongly continuous. Let \(A\) be a finite set with at least two elements. Then the following are equivalent.

(i) Every game \(G\) on \(T\) with \(A\) as the action space has a pure (mixed) strategy Nash equilibrium.

(ii) \(\mu\) is countably additive.

\(\triangleright\) (ii) \(\Rightarrow\) (i). If \(\mu\) is countably additive, then Schmeidler’s theorem ensures that every game has a pure/mixed strategy Nash equilibrium.

\(\triangleright\) (i) \(\Rightarrow\) (ii). Suppose that \(\mu\) is not countably additive. Consider the game in the example. It does not have any pure/mixed strategy Nash equilibrium.
Open Questions

1. (Example) Consider a two-player finite-action game of incomplete information. Suppose that the information space of each player has a finitely additive measure. The game may not have a Nash equilibrium.

2. (Theorem) Every game (on a finitely additive measure space of players) has an $\epsilon$-Nash equilibrium.