The Nonexistence of Mixed Strategy Nash Equilibria for a Countable Agent Space

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Background and Motivation

- Nash: Every finite-player, finite-action game has an equilibrium in mixed strategies.
- Games with infinitely many players, compact convex action set of each player, payoffs are quasi-concave in own argument. There is a Nash equilibrium. Ma (1969)
- Atomless, countably additive measure space of players:
  - If the set of players is an atomless, countably additive measure space then a game has a pure strategy Nash equilibrium. Schmeidler (1973)
  - DWW (1951) theorem: Every mixed strategy Nash equilibrium can be purified.
- The DWW theorem holds for finitely additive measure spaces.
- **Question:** Does a pure/mixed strategy Nash equilibrium exist in a game over a finitely additive measure space of players?

This Talk

- If the set of players is endowed with a finitely additive measure, then a game may not have a Nash equilibrium (in pure or mixed strategies).
- Main reason: Failure of the upper hemicontinuity of the integral of a correspondence.

Upper Hemicontinuity of the Integral

- Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive measure space and X a metric space.
- Let $F : T \times X \longrightarrow\!\!\!\!\to \mathbb{R}^n$ be a correspondence.
- If $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is upper hemicontinuous then
$$\int_T F(\cdot, x)\, d\mu$$
is upper hemicontinuous (in x).
- This results fails if μ is a finitely additive measure.

Large Games

- Let $E = \{e^1, \dots, e^n\}$ be the set of unit vectors in $\mathbb{R}^n$ and $S = \{s \in \mathbb{R}^n_+ : \sum_{i=1}^n s_i = 1\}$ the unit simplex in $\mathbb{R}^n$.
- Let $\mathcal{U}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive measure space.
- A (non-anonymous large) *game* is a measurable function
$$\mathcal{G} : T \longrightarrow \mathcal{U}.$$
- A $f : T \longrightarrow E$ is a (pure strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t,

$$\mathcal{G}(t)\left(f(t), \int f \, d\mu\right) \geq \mathcal{G}(t)\left(a, \int f \, d\mu\right) \text{ for all } a \in E.$$

Existence of Nash Equilibrium

Theorem (Schmeidler)

Every game has a pure strategy Nash equilibrium.

- Define a correspondence $B : T \times S \longrightarrow E$ by

$$B(t,s) = \{e^i \in E \mid \mathcal{G}(t)(e^i, s) \geq \mathcal{G}(t)(a,s) \text{ for all } a \in E\}.$$

- $B(t,s)$ is nonempty, $B(\cdot,s)$ is measurable and $B(t,\cdot)$ is uhc.
- Let $\Gamma(s) = \int_T B(\cdot,s)\, d\mu$.
  - $\Gamma(s)$ is nonempty for each $s \in S$.
  - $\Gamma(\cdot)$ is uhc (integration preserves uhc).
  - $\Gamma(\cdot)$ is convex valued (by Lyapunov's theorem).
- Γ has a fixed point s^* (by Kakutani's fixed point theorem).
- So, there is $f : T \longrightarrow E$ such that $\int f\, d\mu = s^*$ and for almost all t, $f(t) \in B(t, s^*)$.
- This f is a Nash equilibrium of $\mathcal{G}$. ∎

Finitely Additive Measures

- T is a nonempty set and $\mathcal{T}$ a field of subsets of T.
  (i) $\emptyset, T \in \mathcal{T}$; (ii) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$ and
  (iii) $A, B \in \mathcal{T} \Rightarrow A \setminus B \in \mathcal{T}$.
- μ is a finitely additive probability measure on $\mathcal{T}$ if
  (i) $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(A) \geq 0$ for all $A \in \mathcal{T}$ and
  (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{T}$, $A \cap B = \emptyset$.
- Let $\mathbb{N}$ denote the set of positive integers. Often, we will be concerned with a finitely additive, probability measure on the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$.
- μ is strongly continuous if for every $\epsilon > 0$, there exists a measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.
- If μ is strongly continuous then it is atomless. A countably additive measure μ is strongly continuous iff it is atomless.
- The range of a strongly continuous measure is convex.

Integration on Finitely Additive Measure Spaces

Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability measure space. All functions below are real valued on T.

- f is simple if there exist $\{s_1, \ldots, s_n\}$ and $\{T_1, \ldots, T_n\}$ such that $T_n \in \mathcal{T}$ for all n and $f(t) = s_n$ if $t \in T_n$.
  f is said to be μ-integrable and $\int f \, d\mu = \sum_{i=1}^{n} s_i \mu(T_i)$.
- The outer measure $\mu^* : \mathcal{P}(T) \longrightarrow [0, 1]$ is given by

$$\mu^*(A) = \inf\{\mu(B) : A \subseteq B, \ B \in \mathcal{T}\}.$$

- A sequence of functions $\{f_n\}$ converges hazily to f if

$$\lim_{n \to \infty} \mu^*(\{t \in T : |f_n(t) - f(t)| > \epsilon\}) = 0, \text{ for every } \epsilon > 0.$$

- f is integrable if there exist a sequence of simple functions such that: (i) $\{f_n\}$ converges hazily to f and (ii) $\lim_{m,n \to \infty} \int |f_n - f_m| \, d\mu = 0$. $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$.

A Motivating Example: Lack of UHC

- Let $A = \{0, 1\}$ and $S = [0, 1]$.
  Let μ be a finitely additive probability measure on $\mathcal{P}(\mathbb{N})$ such that the μ-measure of any finite set is zero.
- Define a correspondence $F : \mathbb{N} \times S \longrightarrow\!\!\!\!\rightarrow A$ as:

$$F(t, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Then

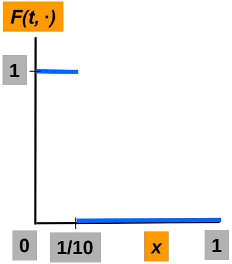
$$\int_{\mathbb{N}} F(\cdot, x)\, d\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

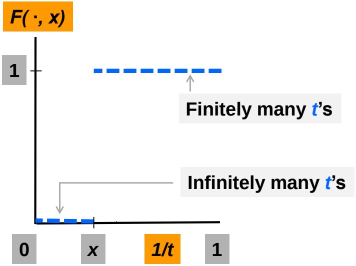
- Clearly, $\int_{\mathbb{N}} F(\cdot, x)\, d\mu$ is not uhc at $x = 0$.
- We have only assumed that the μ-measure of any finite set is zero. In particular, we can take μ to be any strongly continuous measure (such as a density measure).

Graphs of the Correspondence

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

Let $t = 9$.





Example, contd.

- $F : \mathbb{N} \times S \longrightarrow\!\!\!\!\to A$.

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Then

$$\int_{\mathbb{N}} F(\cdot, x)\, d\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

- Clearly, $\int_{\mathbb{N}} F(\cdot, x)\, d\mu$ is not uhc at $x = 0$.
- Let f be a measurable selection. If $x = 0$ then $x < 1/(t+1)$ for all $t \in \mathbb{N}$, which implies that $f(t) = 1$ for all $t \in \mathbb{N}$ and $\int f\, d\mu = 1$.
- If $x > 0$ then $x \leq 1/(t+1)$ for at most finitely many t's. Since the μ-measure of any finite set is zero, $f(t) = 0$ for almost all t and $\int f\, d\mu = 0$.

Games and Nash Equilibria

- Let $A = \{0, 1\}$ and $S = [0, 1]$. Let $\mathcal{U}$ be the set of real valued continuous functions on $A \times S$, endowed with sup norm.
- $\mathbb{N}$ is the set of positive integers. Let μ be a strongly continuous, finitely additive measure on $\mathcal{P}(\mathbb{N})$.
- A *game* is a measurable function $\mathcal{G}$ from $\mathbb{N}$ to $\mathcal{U}$.
- A measurable function f from $\mathbb{N}$ to A is a *Nash equilibrium* of a game $\mathcal{G}$ if

$$\mathcal{G}(t)\left(f(t), \int f \, d\mu\right) \geq \mathcal{G}(t)\left(a, \int f \, d\mu\right)$$

  for all $a \in A$ and for almost all $t \in \mathbb{N}$.
- Note: This notion of Nash equilibrium is in pure strategies.

Nonexistence of Nash Equilibria: Example

- Let $A = \{0, 1\}$ and $S = [0, 1]$. For each $t \in \mathbb{N}$, let the payoff function (on $A \times S$) be

$$u_t(a, x) = \left(x - \frac{1}{t+1}\right)^{a+1}, \ a \in A.$$

Then $t \longrightarrow u_t$ defines a game.

- We will derive the best responses and show that this game has no Nash equilibrium.
- Best responses:

$$\text{argmax}_{a \in A} u_t(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

  - $x = 1/(t+1)$: $u_t(0, x) = u_t(1, x) = 0$.
  - $x < 1/(t+1)$: $u_t(0, x) < 0 < u_t(1, x)$.
  - $x > 1/(t+1)$: $0 < u_t(0, x) < 1$, $u_t(1, x) = [u_t(0, x)]^2$.

Example: contd.

- Best responses:

$$\text{argmax}_{a \in A} u_t(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Suppose that f from $\mathbb{N}$ to A is a Nash equilibrium. Let $x = \int f \, d\mu$.
  - If $x = 0$ then $x < 1/(t+1)$ for all $t \in \mathbb{N}$ which implies that $f(t) = 1$ for all t and $\int f \, d\mu = 1$, a contradiction.
  - If $x > 0$ then $x > 1/(t+1)$ for almost all t (since the measure of a finite set is zero), which implies that $f(t) = 0$ for almost all t and $\int f \, d\mu = 0$, again a contradiction.
- The game does not have a Nash equilibrium.

Nonexistence of Mixed Strategy Nash Equilibria

- We will now consider mixed strategies (formalized as integrals).
- Let $A = S = [0, 1]$. For each $t \in \mathbb{N}$, let the *payoff* be

$$v_t(p, x) = (1 - p)u_t(0, x) + pu_t(1, x).$$

  A $f : \mathbb{N} \longrightarrow A$ is a (mixed strategy) *Nash equilibrium* if

$$v_t\left(f(t), \int f \; d\mu\right) \geq v_t\left(p, \int f \; d\mu\right)$$

  for all $p \in A$ and for almost all $t \in \mathbb{N}$.
- The best responses are as before, i.e., almost all t will choose a pure action, 0 or 1. The preceding arguments show that there is no Nash equilibrium (in mixed strategies).

Nonexistence of Equilibria on General Measure Spaces

- Let T be a nonempty set and $\mathcal{T}$ a field of subsets of T. Let μ be a finitely additive probability measure on $\mathcal{T}$.
- Assume that μ is not countably additive.
  We will show that there is a game on μ which has no pure or mixed strategy Nash equilibrium.

Claim

The following conditions are equivalent.

(i) μ *is countably additive.*

(ii) $\lim_{n\to\infty} \mu(B_n) = \mu(B)$ *whenever* $\{B_n\}$ *is an increasing sequence of sets in* $\mathcal{T}$ *with* $B = \cup_{n=1}^{\infty} B_n \in \mathcal{T}$.

- Let $A = \{0, 1\}$ be the set of actions.
- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that

$$\cup_{n=1}^{\infty} B_n = T \text{ and } \lim_{n \to \infty} \mu(B_n) = c < 1.$$

- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.
- Now we will define the payoffs. Let $x \in [0, 1]$. For each $t \in C_n$, let

$$u_t(a, x) = (x - \ell_n)^{a+1}, \ a \in A \text{ where } \ell_n = c + \frac{1 - c}{n}.$$

- Note that $\ell_1 = 1$, $\ell_n > c$ for each n and $\{\ell_n\}$ is a monotonically decreasing sequence converging to c.

- $u_t(a,x) = (x-\ell_n)^{a+1}$. Best responses:

$$\text{argmax}_{a\in A} u_t(a,x) = \begin{cases} \{0,1\} & \text{if } x = \ell_n \\ 1 & \text{if } x < \ell_n \\ 0 & \text{if } x > \ell_n. \end{cases}$$

- Let $f : T \longrightarrow A$ be a pure strategy Nash equilibrium and $x = \int f \, d\mu$.
  - Suppose that $x \leq c < 1$. Then for all $t \in T$, $f(t) = 1$ which implies that $x = 1$, a contradiction.
  - Now suppose that $x > c$. Then there exists a unique $n_0 \in \mathbb{N}$ such that $\ell_{n_0+1} < x \leq \ell_{n_0}$. If $n \geq n_0 + 1$ and $t \in C_n$ then $f(t) = 0$. So, $x = \int f \, d\mu \leq \sum_{i=1}^{n_0} \mu(C_i) = \mu(B_{n_0}) \leq c$, a contradiction.
- The game does not have a pure strategy Nash equilibrium.
- Similar arguments can be used to show that the game does not have a mixed strategy Nash equilibrium.

Sum up

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space where μ is strongly continuous. Let A be a finite set with at least two elements. Then the following are equivalent.

(i) *Every game $\mathcal{G}$ on T with A as the action space has a pure (mixed) strategy Nash equilibrium.*

(ii) *μ is countably additive.*

- $(ii) \Rightarrow (i)$. If μ is countably additive, then Schmeidler's theorem ensures that every game has a pure/mixed strategy Nash equilibrium.
- $(i) \Rightarrow (ii)$. Suppose that μ is not countably additive. Consider the game in the example. It does not have any pure/mixed strategy Nash equilibrium. ■

Open Questions

1. (Example) Consider a two-player finite-action game of incomplete information. Suppose that the information space of each player has a finitely additive measure. The game may not have a Nash equilibrium.
2. (Theorem) Every game (on a finitely additive measure space of players) has an ϵ-Nash equilibrium.