Individualized Representation and Realization of Large Distributionalized Games

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Background

- Consider situations where the payoff to a player depends upon own action and the trait-action distribution of all others.
- A large distributionalized game (LDG) is a probability measure on the space of players' characteristics–the product of the space of players' traits and the space of players' payoffs.
- A Nash equilibrium distribution (NED) of an LDG is a probability measure on the product space of players' characteristics and actions such that:
  - its marginal on the space of characteristics is the given game,
  - it gives full measure to the characteristics and corresponding best action pairs.
- A large individualized game (LIG) is a mapping from a space of players' names to the space of characteristics.
- A Nash equilibrium of an LIG is a mapping from the space of players' names to the space of actions, such that each player chooses a best action corresponding to the induced trait-action distribution.

Background, contd.

- General large games (with traits):
  - Individualized form: Khan et al. (2013), Qiao-Yu (2013)
  - Distributionalized form: Khan et al. (2013).
- Conventional large games (all players share a common trait):
  - Individualized form: Schmeidler (1973) (finite action)
  - Distributionalized form: Mas-Colell (1984)
  - Representation: Rath (1995) (finite action).
- This paper examines the relationships among equilibria of the two game forms (distributionalized and individualized) in the general setting.

Large Distributionalized Games (LDG)

- A: a compact metric space of actions.
- T: a complete separable metric space of traits.
- $\mathcal{M}(T \times A)$: the set of probability measures on $T \times A$ (weak convergence).
- $\mathcal{U}_{(A,T)}$: the space of real valued continuous functions on $A \times \mathcal{M}(T \times A)$, metrized by supremum norm.

Definition

(a) An *LDG* is a probability measure μ on $T \times \mathcal{U}_{(A,T)}$.

(b) A probability measure τ on $T \times \mathcal{U}_{(A,T)} \times A$ is a *Nash Equilibrium Distribution* (NED) of an LDG μ if

(i) $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and

(ii) $\tau(B(\tau)) = 1$ where $B(\tau) = \{(t, u, a) \in T \times \mathcal{U}_{(A,T)} \times A :$
$u(a, \tau_{T \times A}) \geq u(x, \tau_{T \times A})$ for all $x \in A\}$.

NEDs of LDGs

- Let μ be an LDG.

Definition

(c) A NED τ of a game is *symmetric* if there exists a measurable function $h : T \times \mathcal{U}_{(A,T)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$, i.e., players with the same characteristics take the same action.

(d) A NED τ of a game can be *symmetrized* if there exists a symmetric NED τ^s of the game such that $B(\tau) = B(\tau^s)$.

(e) Two NEDs τ and τ' of a game μ are *similar* if $\tau_{T\times A} = \tau'_{T\times A}$.

Theorem

(a) Every LDG has a NED.
(b)There exists a symmetric NED of an atomless LDG if T and A are countable. Furthermore, every NED of such an LDG can be symmetrized.

Large Individualized Games (*LIG*)

Definition

(a) Given an abstract atomless probability space $(I, \mathcal{I}, \lambda)$, an *LIG* $\mathcal{G}$ is measurable function from I to $T \times \mathcal{U}_{(A,T)}$.

(b) A *Nash equilibrium* of an LIG $\mathcal{G}$ is a measurable function $f : I \longrightarrow A$ such that such that for λ-almost all $i \in I$,

$$v_i \left(f(i), \lambda \circ (\alpha, f)^{-1} \right) \geq v_i \left(a, \lambda \circ (\alpha, f)^{-1} \right) \text{ for all } a \in A,$$

with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \to T$ abbreviated for $\mathcal{G}_1$, where $\mathcal{G}_k$ is the projection of $\mathcal{G}$ on its k^{th}-coordinate, $k = 1, 2$.

- If A or T is uncountable, a Nash equilibriums need not exist in an LIG when the name space is the Lebesgue unit interval.
- A Nash equilibrium of an LIG exists if both A and T are countable (finite or countably infinite), or
  $(I, \mathcal{I}, \lambda)$ is a saturated probability space. (Qiao-Yu)

Individualized Representation of LDGs

Definition

Let μ be an LDG. A $(I, \mathcal{I}, \lambda)$ *representation* of μ is an LIG $\mathcal{G}$ with $(I, \mathcal{I}, \lambda)$ as its name space such that $\mu = \lambda \circ \mathcal{G}^{-1}$.

Let L denote the unit interval, $\mathcal{B}[0,1]$ its Borel σ-algebra and ℓ the Lebesgue measure on it. $\mathcal{G}$ is a *Lebesgue representation* of μ if $\mathcal{G}$ is a representation of μ with the name space $(L, \mathcal{B}[0,1], \ell)$.

Theorem

Let μ be an LDG and $(I, \mathcal{I}, \lambda)$ an arbitrary atomless probability space. Then there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ.

Representation Results

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ, f a measurable mapping from I to A and $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$. Then $\tau_{T \times \mathcal{U}_{(A,T)}} = \mu$ and $\tau_A = \lambda \circ f^{-1}$.

Furthermore,

(a) If f is a Nash equilibrium of $\mathcal{G}$ then τ is a NED of μ.

(b) If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$.

The above theorem shows that any Nash equilibrium of a representation induces a NED of the LDG.

It also shows that if a NED is induced by a strategy profile of the representation, then the strategy profile is a Nash equilibrium of the representation.

What about the converse?

A Partial Converse

Theorem

Given a NED τ of μ and an atomless probability space $(I, \mathcal{I}, \lambda)$, there is a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ and a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

- What about a full converse?
- Namely, in the statement above, given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ, does there exist a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$?
- In general, the answer is no.

Two Exceptions

Case (1): Representation with Countable Characteristics:

An LIG $\mathcal{G}$ has *countable characteristics* if the range of $\mathcal{G}$ is countable. (See Carmona (2008) when the space of characteristics is the space of payoffs.)

Case (2): Saturated Representation:

$(I, \mathcal{I}, \lambda)$ is a saturated probability space.

Theorem

Let an atomless probability space $(I, \mathcal{I}, \lambda)$ and a NED τ of μ be given. Given a $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ,
if either Case (1) or Case (2) holds,
then there is a Nash equilibrium f of $\mathcal{G}$ such that $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$.

The Similarity Theorem

Theorem

Let A and T be countable.
Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ and τ a NED of μ.
Then there exists a Nash equilibrium f of $\mathcal{G}$ such that $\tau^* = \lambda \circ (\mathcal{G}, f)^{-1}$ is a NED of μ and τ^* is similar to τ.
If in addition, μ is atomless then τ^* can be taken to be symmetric.

- Example 1 shows that the conclusions of this Theorem cannot be strengthened even with finite actions/one trait.
- Thus, one cannot go beyond similarity.
- Counterexamples show that this Theorem cannot be strengthened to the case of uncountable actions/traits.

Representation and Symmetric NEDs

Corollary

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ.
Let τ be a symmetric NED of μ such that τ(graph of h) = 1.
Define $f : I \longrightarrow A$ by $f(i) = h(\mathcal{G}(i))$.
Then $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ and f is a Nash equilibrium of $\mathcal{G}$.

Given an LDG $\mathcal{G}$, let $\sigma(\mathcal{G}) = \{\mathcal{G}^{-1}(U) : U \in \mathcal{B}(T \times \mathcal{U}_{(A,T)})\}$, where $\mathcal{B}(T \times \mathcal{U}_{(A,T)})$ is the Borel σ-algebra of $T \times \mathcal{U}_{(A,T)}$.
$\sigma(\mathcal{G})$ is the smallest σ-algebra on $\mathcal{I}$ wrt. which $\mathcal{G}$ is measurable.

Theorem

Let $\mathcal{G}$ be a $(I, \mathcal{I}, \lambda)$ representation of μ. Then τ is a symmetric NED of μ if and only if $\tau = \lambda \circ (\mathcal{G}, f)^{-1}$ for a $\sigma(\mathcal{G})$-measurable Nash equilibrium f of $\mathcal{G}$.

Almost One-to-one Representations

Given any probability space $(I, \mathcal{I}, \lambda)$, a function on I is *almost one-to-one* if it is one-to-one on I except some λ-null set of I.

Theorem

Let $\mathcal{G}$ be a $(L, \mathcal{B}[0,1], \ell)$ representation of μ. Assume that $\mathcal{G}$ is almost one-to-one.
(a) If f is a Nash equilibrium of $\mathcal{G}$ then $\tau = \ell \circ (\mathcal{G}, f)^{-1}$ is a symmetric NED of μ.
(b) Let $f : I \longrightarrow A$ be any measurable function and $\tau = \ell \circ (\mathcal{G}, f)^{-1}$. If τ is a NED of μ then f is a Nash equilibrium of $\mathcal{G}$ and τ is symmetric.

- If μ is atomless, there exists an almost one-to-one Lebesgue representation.
- The result is not true on arbitrary atomless measure spaces.

Examples

To simplify the idea, in each example, we consider a game where all players share a common trait, i.e., the space of characteristics $T \times \mathcal{U}_{(A,T)}$ is now reduced to $\mathcal{U}_A$, the space of real valued continuous functions on $A \times \mathcal{M}(A)$, metrized by supremum norm.

- Example 1: A NED of an LDG cannot be induced by a Nash equilibrium of a given strategic Lebesgue representation.
- Example 2: The NED above can be induced by a Nash equilibrium of some other Lebesgue representation.

Example 1

Let the action set be $A = \{a_1, a_2\}$ and the player set be the Lebesgue interval $(L, \mathcal{B}[0,1], \ell)$. Consider a particular function $u \in \mathcal{U}_A$, defined as follows: $u(a_1, \nu) = 1/2$, $u(a_2, \nu) = 1 - \nu(a_2)$.

Let $\mathcal{G}^1(i) = iu$ for $i \in L$.
Define f_1 and f_2 as follows:

$$f_1(i) = a_1 \text{ if } i < 1/2 \text{ and } f_1(i) = a_2 \text{ if } i \geq 1/2.$$

$$f_2(i) = a_2 \text{ if } i < 1/2 \text{ and } f_2(i) = a_1 \text{ if } i \geq 1/2.$$

Both f_1 and f_2 are Nash equilibria of $\mathcal{G}^1$.

Let $\tau = \ell \circ (\mathcal{G}^1, f_1)^{-1}$, $\tau' = \ell \circ (\mathcal{G}^1, f_2)^{-1}$ and $\tau^\alpha = \alpha\tau + (1-\alpha)\tau'$ for $0 < \alpha < 1$.

The LDG μ^1 and τ^α

Consider the LDG $\mu^1 = \ell \circ (\mathcal{G}^1)^{-1}$.
For any $\alpha \in (0,1)$, τ^α is a NED of the LDG μ^1.

Example 1, contd.

One can show that

A Negative Result

$\mathcal{G}^1$ is a Lebesgue representation of μ^1. But there is no Nash equilibrium f of $\mathcal{G}^1$ such that $\tau^\alpha = \ell \circ (\mathcal{G}^1, f)^{-1}$, for $0 < \alpha < 1$.

A Similarity Result

However, there exists a Nash equilibrium f' such that:
(*a*) $\tau^* = \ell \circ \left(\mathcal{G}^1, f'\right)^{-1}$ is a NED of μ^1, and (*b*) τ^α and τ^* are similar.

Example 2

For any fixed α, the NED τ^α of the LDG μ^1 in Example 1 can be induced by some Lebesgue representation of the LDG and its Nash equilibrium.

In particular, let $\alpha = 1/2$.

Consider the same function u as in Example 1. Define $\mathcal{H} : L \longrightarrow \mathcal{U}_A$ as follows.

$$\begin{aligned} \mathcal{H}(i) &= 2iu & \text{if } i < \tfrac{1}{2} \\ &= \mathcal{H}\left(i - \tfrac{1}{2}\right) & \text{if } i \geq \tfrac{1}{2} \end{aligned}$$

Since $\mathcal{H}(i) = \mathcal{H}(i - (1/2))$ for each $i \geq 1/2$, $\mathcal{H}$ is not one-to-one.

Example 2, contd.

We can show that

Another Representation of μ^1

$\mathcal{H}$ is a Lebesgue representation of the μ^1 in Example 1.

Moreover, Let $f(i) = a_1$ if $i \in [0, 1/4] \cup (3/4, 1]$ and
$f(i) = a_2$ if $i \in (1/4, 1/2] \cup (1/2, 3/4]$.

Nash equilibrium of $\mathcal{H}$

The NED $\tau^{1/2}$ of μ^1 can be induced by a Nash equilibrium f of $\mathcal{H}$.

Further Discussions

- Countably Determined Games
- Realization of NEDs

Countably Determined Games

Consider games with a common trait for all the players.

An LIG $\mathcal{G}$ is *determined* by another LIG $\bar{\mathcal{G}}$ if it has the same set of Nash equilibria as $\bar{\mathcal{G}}$ and $\mathcal{G}_1 = \bar{\mathcal{G}}_1$. An LIG $\mathcal{G}$ is *determined by countable characteristics* if it is determined by an LIG with countable characteristics.

If an LIG has countable characteristics then it is an LIG determined by countable characteristics but not the converse.

An LDG μ is *determined* by $\bar{\mu}$ if any representation of it can be determined by some representation of $\bar{\mu}$. An LDG μ is *determined by countable characteristics* if it is determined by a purely atomic LDG.

Proposition

(a) There exists a NE in any LIG $\mathcal{G}$ that is determined by countable characteristics.
(b) If μ is an atomless LDG determined by countable characteristics, then μ has a symmetric NED and every NED of μ can be symmetrized.

Countably Determined Games

Consider games with a common trait for all the players.

Denote by $id(r)$ the constant function in $\mathcal{U}_A$ with value r.
Let ψ be the operator on $\mathcal{U}_A$ such that

$$\psi(u) = u \text{ if } u = id(0) \text{ and } u/ \parallel u \parallel \text{ otherwise.}$$

ψ is continuous on $\mathcal{U}_A \setminus id(0)$ and is measurable on $\mathcal{U}_A$.

Given a game $\mathcal{G}$, consider the game $\bar{\mathcal{G}}$ where $\bar{\mathcal{G}}(i) = \psi(\mathcal{G}(i))$ for all i.
$\mathcal{G}$ is *determined by countable characteristics* if the range of $\bar{\mathcal{G}}$ is countable.

Theorem

Let $\mathcal{G}$ be a game determined by countable characteristics and $\mu = \lambda \circ \mathcal{G}^{-1}$.
(a) $\mathcal{G}$ has a Nash equilibrium f.
(b) If μ is atomless then it has a symmetric NED.
(c) The similarity theorem (above) holds.

Realization of NEDs

Definition

Given a NED τ of an LDG μ, we say that a probability space $(I, \mathcal{I}, \lambda)$ is a *realization* of τ (or, $(I, \mathcal{I}, \lambda)$ *realizes* τ) if every $(I, \mathcal{I}, \lambda)$ representation $\mathcal{G}$ of μ has a Nash equilibrium f such that $\lambda \circ (\mathcal{G}, f)^{-1} = \tau$.

Characterization of NEDs by Realization:

Theorem

Let μ be an atomless LDG and τ a NED of μ.
(a) τ is symmetric if and only if the Lebesuge unit interval is a realization of τ.
(b) If τ is non-symmetric, then an atomless probability space realizes τ if and only if it is saturated.

Conclusions

- Existence of NED and symmetric NED in an LDG.
- LDG and its Individualized Representation.
- Any Nash equilibrium of a representation of an LDG induces a NED of the LDG.
- Converse: not all NEDs of an LDG can be induced by a Nash equilibrium of a given representation.
  - Two exceptions:
    - Representation with countable characteristics,
    - Saturated representation.
  - Representation in general: Similarity Theorem.
- Characterization of Symmetric NED in an LDG:
  - $\sigma(\mathcal{G})$-measurable Nash equilibirum
  - Almost one-to-one Lebesgue representation
- Countably determined games
- Realization: symmetric and non-symmetric case