Nonexistence of Nash Equilibria in Games Over Finitely Additive Measure Spaces

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Background and Motivation

- Games with finite number of players.
- Modeling individual negligibility: Atomless measures, infinitesimals, Loeb spaces, Finitely–additive measures.
  - Standard model: continuum of players with atomless distribution.
  - Countably many agents.
  - Each player has zero mass. Measure of the whole space is 1. So, finitely additive measures.

Literature

- Nash: Every finite-player, finite-action game has an equilibrium in mixed strategies.
- Games with infinitely many players, compact convex action set of each player, payoffs are quasi-concave in own argument. There is a Nash equilibrium. Ma (1969)
- Atomless, countably additive measure space of players:
  - If the set of players is an atomless, countably additive measure space then a game has a pure strategy Nash equilibrium. Schmeidler (1973)
  - DWW (1951) theorem: Every mixed strategy Nash equilibrium can be purified.
- The DWW theorem holds for finitely additive measure spaces.
- **Question:** Does a pure/mixed strategy Nash equilibrium exist in a game over a finitely additive measure space of players?

This Talk

- If the set of players is endowed with a finitely additive measure, then:
  - a game may not have a Nash equilibrium (in pure or mixed strategies).
  - the Nash equilibrium correspondence may not have the closed graph property.
  - a game may not have an ϵ-equilibrium.
  - sufficient conditions for existence of an ϵ-equilibrium.

- Primary reason for nonexistence of equilibrium: The lack of upper hemicontinuity of the integral of a correspondence.

- Let $E = \{e^1, \ldots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{U}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive probability space.
- A (non-anonymous large) *game* is a measurable function $$\mathcal{G} : T \longrightarrow \mathcal{U}.$$
- A *pure strategy profile* (of a game $\mathcal{G}$) is a measurable function $$f : T \longrightarrow E.$$
- A $f : T \longrightarrow E$ is a (pure strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t,

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Existence of Nash Equilibrium

Theorem (Schmeidler)

Every game has a pure strategy Nash equilibrium.

- Define a correspondence $B : T \times S \longrightarrow E$ by

$$B(t,s) = \{e^k \in E : \ \mathcal{G}(t)(e^k, s) \geq \mathcal{G}(t)(a,s) \ \text{ for all } a \in E\}.$$

- $B(t,s)$ is nonempty, $B(\cdot,s)$ is measurable and $B(t,\cdot)$ is uhc.
- Let $\Gamma(s) = \int_T B(\cdot,s)\, \mathrm{d}\mu$.
  - $\Gamma(s)$ is nonempty for each $s \in S$.
  - $\Gamma(\cdot)$ is uhc (integration preserves uhc).
  - $\Gamma(\cdot)$ is convex valued (by Lyapunov's theorem).
- Γ has a fixed point s^* (by Kakutani's fixed point theorem).
- So, there is $f : T \longrightarrow E$ such that $\int_T f \, \mathrm{d}\mu = s^*$ and for almost all t, $f(t) \in B(t, s^*)$.
- This f is a Nash equilibrium of $\mathcal{G}$. ■

Finitely Additive Measures

- T is a nonempty set and $\mathcal{T}$ a field of subsets of T.
  (*i*) $\emptyset, T \in \mathcal{T}$; (*ii*) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$ and
  (*iii*) $A, B \in \mathcal{T} \Rightarrow A \setminus B \in \mathcal{T}$.
- μ is a finitely additive probability measure on $\mathcal{T}$ if
  (*i*) $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(A) \geq 0$ for all $A \in \mathcal{T}$ and
  (*ii*) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{T}$, $A \cap B = \emptyset$.
- Let $\mathbb{N}$ denote the set of positive integers. Often, we will be concerned with a finitely additive, probability measure on the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$.
- μ is strongly continuous if for every $\epsilon > 0$, there exists a measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.
- If μ is strongly continuous then it is atomless. A countably additive measure μ is strongly continuous iff it is atomless.
- The range of a strongly continuous measure is convex.

Games on Finitely Additive Spaces

- Let $E = \{e^1, \dots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{U}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space.
- A *game* is a measurable function
$$\mathcal{G} : T \longrightarrow \mathcal{U}.$$
- A *pure strategy profile* (of a game $\mathcal{G}$) is a measurable function
$$f : T \longrightarrow E.$$
- A $f : T \longrightarrow E$ is a (pure strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t,

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Mixed Strategies

- Pure strategy profile: $f : T \longrightarrow E$.
- Mixed strategy profile: $f : T \longrightarrow S$.
- Given a mixed strategy profile f and $y \in S$, the *payoff to player t* is

$$\mathcal{G}(t)\left(y, \int_T f \, \mathrm{d}\mu\right) = \sum_{k=1}^{L} y_k \mathcal{G}(t)\left(e^k, \int_T f \, \mathrm{d}\mu\right).$$

- A $f : T \longrightarrow E$ is a (mixed strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t,

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T f \, \mathrm{d}\mu\right) \text{ for all } y \in S.$$

Nonexistence of Nash Equilibria: Example

- Let $A = \{0, 1\}$ and $K = [0, 1]$. For each $t \in \mathbb{N}$, let the payoff function (on $A \times K$) be

$$\mathcal{G}(t)(a, x) = \left(x - \frac{1}{t+1}\right)^{a+1}, \ a \in A.$$

- We will derive the best responses and show that this game has no Nash equilibrium.
- Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

  - $x = 1/(t+1)$: $\mathcal{G}(t)(0, x) = \mathcal{G}(t)(1, x) = 0$.
  - $x < 1/(t+1)$: $\mathcal{G}(t)(0, x) < 0 < \mathcal{G}(t)(1, x)$.
  - $x > 1/(t+1)$: $0 < \mathcal{G}(t)(0, x) < 1$, $\mathcal{G}(t)(1, x) = [\mathcal{G}(t)(0, x)]^2$.

Example, contd.

- Best responses:

$$\text{argmax}_{a \in A} u_t(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Suppose that f from $\mathbb{N}$ to K is a Nash equilibrium. Let $x = \int_{\mathbb{N}} f \, d\mu$.
  - If $x = 0$ then $x < 1/(t+1)$ for all $t \in \mathbb{N}$ which implies that $f(t) = 1$ for all t and $\int_{\mathbb{N}} f \, d\mu = 1$, a contradiction.
  - If $x > 0$ then $x > 1/(t+1)$ for almost all t (since the measure of a finite set is zero), which implies that $f(t) = 0$ for almost all t and $\int_{\mathbb{N}} f \, d\mu = 0$, again a contradiction.
- The game does not have a Nash equilibrium in pure or mixed strategies.

Nonexistence of Equilibria on General Measure Spaces

- Let T be a nonempty set and $\mathcal{T}$ a field of subsets of T. Let μ be a finitely additive probability measure on T.
- Assume that μ is not countably additive. We will show that there is a game on μ which has no pure or mixed strategy Nash equilibrium.

Claim

The following conditions are equivalent.

(i) μ *is countably additive.*

(ii) $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ *whenever* $\{A_n\}$ *is an increasing sequence of sets in* $\mathcal{T}$ *with* $A = \cup_{n=1}^{\infty} A_n \in \mathcal{T}$.

The Example

- Let $A = \{0, 1\}$ be the set of actions.
- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that

$$\cup_{n=1}^{\infty} B_n = T \text{ and } \lim_{n\to\infty} \mu(B_n) = c < 1.$$

- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.
- Now we will define the payoffs. Let $x \in [0, 1]$.
For each $t \in C_n$, let

$$\mathcal{G}(t)(a, x) = (x - \ell_n)^{a+1},\ a \in A \text{ where } \ell_n = c + \frac{1 - c}{n}.$$

- Note that $\ell_1 = 1$, $\ell_n > c$ for each n and $\{\ell_n\}$ is a monotonically decreasing sequence converging to c.

The Example, contd.

- $\mathcal{G}(t)(a,x) = (x - \ell_n)^{a+1}$. Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = \ell_n \\ 1 & \text{if } x < \ell_n \\ 0 & \text{if } x > \ell_n. \end{cases}$$

- Let $f : T \longrightarrow [0,1]$ be a mixed strategy Nash equilibrium and $x = \int_T f \, \mathrm{d}\mu$.
  - Suppose that $x \leq c < 1$. Then for all $t \in T$, $f(t) = 1$ which implies that $x = 1$, a contradiction.
  - Now suppose that $x > c$. Then there exists a unique $n_0 \in \mathbb{N}$ such that $\ell_{n_0+1} < x \leq \ell_{n_0}$. If $n \geq n_0 + 1$ and $t \in C_n$ then $f(t) = 0$. So, $x = \int_T f \, \mathrm{d}\mu \leq \sum_{i=1}^{n_0} \mu(C_i) = \mu(B_{n_0}) \leq c$, a contradiction.
- The game does not have a Nash equilibrium in pure or mixed strategies.

A Sequence of Games

- Below we construct a sequence of games, each of which has a pure strategy Nash equilibrium.
- Fix any $n \in \mathbb{N}$, and let

$$\mathcal{G}^n(t) = \begin{cases} \mathcal{G}(t) & \text{if } t \in B_n \\ \bar{u} & \text{if } t \notin B_n, \end{cases}$$

  where $\bar{u}(0, x) = \bar{u}(1, x) = 0$ for any $x \in [0, 1]$.
- Let f^n be a function from T to A such that

$$f^n(t) = 1 \text{ on } B_n \text{ and } f^n(t) = 0 \text{ on } B_n^c.$$

  - Then $\int_T f^n \, \mathrm{d}\mu = \mu(B_n) \leq c < \ell_n$ for every $n \in \mathbb{N}$.
  - It is clear that $f^n(t)$ is a best response to $\int_T f^n \, \mathrm{d}\mu$ for every player t,
- This means f^n is a Nash equilibrium of $\mathcal{G}^n$.

An Implication of Countable Additivity

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space where $\mathcal{T}$ is a σ-algebra and μ is strongly continuous. Then the following are equivalent.

(i) *Every game $\mathcal{G}$ on T with at least two actions has a mixed strategy Nash equilibrium.*

(ii) *μ is countably additive.*

Proof $(ii) \Rightarrow (i)$. Schmeidler (1973).

$(i) \Rightarrow (ii)$. Assume that μ is not countably additive.
Consider the game $\mathcal{G}$ in the example on general measure spaces.
It does not have a pure/mixed strategy Nash equilibrium. ∎

Closed Graph Property

- The Nash equilibrium correspondence assigns the set of Nash equilibria to a game.
- Let $\mathcal{G}^n$, $n \in \mathbb{N}$, and $\mathcal{G}$ be games on $(T, \mathcal{T}, \mu)$. The Nash equilibrium correspondence has the *closed-graph property* if the following holds: if
  - $\{\mathcal{G}^n\}$ converges to $\mathcal{G}$ pointwise,
  - f^n is a Nash equilibrium of $\mathcal{G}^n$ for each n and
  - $\{f^n\}$ converges to f pointwise.

  then f is a Nash equilibrium of $\mathcal{G}$.

Another Implication of Countable Additivity

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space where $\mathcal{T}$ is a σ-algebra and μ is strongly continuous. Then the following are equivalent.

(i) *The Nash equilibrium correspondence of games with at least two actions has the closed graph property.*

(ii) *μ is countably additive.*

Proof $(ii) \Rightarrow (i)$. This is shown in Theorem 2 of Qiao-Yu-Zhang (2015).

$(i) \Rightarrow (ii)$. Assume that μ is not countably additive.
Consider the sequence of games $\{\mathcal{G}^n\}$ and the game $\mathcal{G}$ in the example on general measure spaces.
The closed graph property fails at $\mathcal{G}$. ∎

epsilon-Equilibrium

- Let $\mathcal{G}$ be a game on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$.
- A strategy profile $f : T \longrightarrow S$ is an *ϵ-equilibrium* of $\mathcal{G}$ if
  - there exists $T_\epsilon \in \mathcal{T}$ such that $\mu(T_\epsilon) \leq \epsilon$ and
  - for any $t \in T_\epsilon^c$,

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) - \epsilon$$

for all $a \in E$.

Nonexistence of epsilon-Equilibrium

- The game is on $\mathbb{N}$, with $A = \{0, 1\}$ and $K = [0, 1]$.
- For each player $t \in \mathbb{N}$, the payoff function is

$$\mathcal{G}(t)(0, x) = 0 \qquad \text{and} \qquad \mathcal{G}(t)(1, x) = 1 - 2^t x + 2^{t-1}.$$

- The best responses are:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = (1/2) + 2^{-t} \\ 1 & \text{if } x < (1/2) + 2^{-t} \\ 0 & \text{if } x > (1/2) + 2^{-t}. \end{cases}$$

- This game does not have an ϵ-equilibrium if $0 < \epsilon \leq 1/4$.

Existence of epsilon-Equilibrium

- A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is said to be *tight* if for any $\epsilon > 0$, there exist $\bar{T} \subseteq T$ such that $\mu(\bar{T}) < \epsilon$ and $\mathcal{G}(T \setminus \bar{T})$ is a relatively compact subset of $\mathcal{U}$.

Theorem

Let $\mathcal{G}$ be a game on $(T, \mathcal{T}, \mu)$. If the measure μ is strongly continuous and $\mathcal{G}$ is tight, then $\mathcal{G}$ has a pure strategy ϵ-equilibrium for every $\epsilon > 0$.