General Equilibrium Theory on a Finitely-Additive Measure Space of Agents: A Viable Option?

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Background and Motivation

- General equilibrium theory: Arrow-Debreu model, Continuum model with an atomless measure, Finitely–additive economy.
- Modeling individual negligibility: Atomless measures, infinitesimals, Loeb spaces, Finitely–additive measures.
  - Standard model: continuum of players with atomless distribution.
  - Countably many agents.
  - Each player has zero mass. Measure of the whole space is 1. So, finitely additive measures.
- Literature: Weiss, Armstrong-Richter, Basile.

This Talk

- If the set of agents is endowed with a finitely additive measure, then
  - An economy may not have a competitive equilibrium.
  - The (competitive) equilibrium allocation correspondence may not have the closed graph property.
  - Sufficient conditions for existence of an ϵ-competitive equilibrium.

- Primary reason for nonexistence of equilibrium: The lack of upper hemicontinuity of the integral of a correspondence.

Upper Hemicontinuity of the Integral

- Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive measure space and X a metric space.
- Let $F : T \times X \longrightarrow\rightarrow \mathbb{R}^n$ be a correspondence.
- If $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is upper hemicontinuous then

$$\int_T F(\cdot, x)\, \mathrm{d}\mu$$

  is upper hemicontinuous (in x).

- This results fails if μ is a finitely additive measure.

Finitely Additive Measures

- T is a nonempty set and $\mathcal{T}$ a field of subsets of T.
  (*i*) $\emptyset,\ T \in \mathcal{T}$; (*ii*) $A,\ B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$ and
  (*iii*) $A,\ B \in \mathcal{T} \Rightarrow A \setminus B \in \mathcal{T}$.
- μ is a finitely additive probability measure on $\mathcal{T}$ if
  (*i*) $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(A) \geq 0$ for all $A \in \mathcal{T}$ and
  (*ii*) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in \mathcal{T}$, $A \cap B = \emptyset$.
- Let $\mathbb{N}$ denote the set of positive integers. Often, we will be concerned with a finitely additive, probability measure on the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$.
- μ is strongly continuous if for every $\epsilon > 0$, there exists a measurable partition $\{F_1, \ldots,\ F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.
- If μ is strongly continuous then it is atomless. A countably additive measure μ is strongly continuous iff it is atomless.
- The range of a strongly continuous measure is convex.

A Motivating Example: Lack of UHC

- Let $A = \{0, 1\}$ and $S = [0, 1]$.
  Let μ be a finitely additive probability measure on $\mathcal{P}(\mathbb{N})$ such that the μ-measure of any finite set is zero.
- Define a correspondence $F : \mathbb{N} \times S \longrightarrow\!\!\!\to A$ as:

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Then

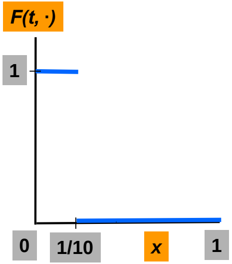
$$\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

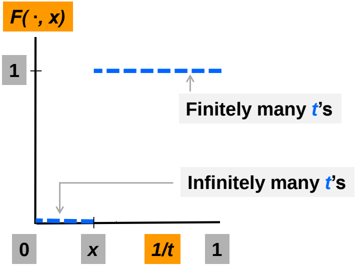
- Clearly, $\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu$ is not uhc at $x = 0$.
- We have only assumed that the μ-measure of any finite set is zero. In particular, we can take μ to be any strongly continuous measure (such as a density measure).

Graphs of the Correspondence

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

Let $t = 9$.





Example, contd.

- $F : \mathbb{N} \times S \longrightarrow\!\!\!\!\to A$.

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/(t+1) \\ 1 & \text{if } x < 1/(t+1) \\ 0 & \text{if } x > 1/(t+1). \end{cases}$$

- Then

$$\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

- Clearly, $\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu$ is not uhc at $x = 0$.
- Let f be a measurable selection of $F(\cdot, x)$.
  - If $x = 0$ then $x < 1/(t+1)$ for all $t \in \mathbb{N}$, which implies that $f(t) = 1$ for all $t \in \mathbb{N}$ and $\int f\, \mathrm{d}\mu = 1$.
  - If $x > 0$ then $x > 1/(t+1)$ for almost all t, i.e., $f(t) = 0$ for almost all t and $\int f\, \mathrm{d}\mu = 0$.

Commodities and Preferences

- There are L goods and the commodity space is $\mathbb{R}_+^L$.
- A consumer has a complete, transitive, continuous and monotone (i.e., $x \gg y \Rightarrow x \succ y$) preference relation defined over $\mathbb{R}_+^L$.
- Let $\mathcal{U}$ denote the class of real valued, continuous utility functions on $\mathbb{R}_+^L$ which represents these preferences, endowed with the compact open topology.
- A preference relation is strongly monotone if $x \geq y$ and $x \neq y$ imply that $x \succ y$.

Economies and Competitive Equilibria

- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space.
- An *economy* is a measurable mapping $\mathcal{E} = (u, \omega) : T \longrightarrow \mathcal{U} \times \mathbb{R}_+^L$ such that ω is integrable and $\int_T \omega \, \mathrm{d}\mu \gg 0$.
- An *allocation* of $\mathcal{E}$ is a measurable mapping f from T to $\mathbb{R}_+^L$ such that $\int_T f \, \mathrm{d}\mu \leq \int_T \omega \, \mathrm{d}\mu$.
- Given a price vector $p \in \mathbb{R}_+^L$, the *budget set* of consumer t is $B_t(p) = \{x \in \mathbb{R}_+^L : p \cdot x \leq p \cdot \omega_t\}$.
- A *competitive equilibrium* of $\mathcal{E}$ is a pair (p, f), where $p \in \mathbb{R}_+^L \setminus \{0\}$, f is an allocation and μ-a.e.;
  (*a*) $f_t \in B_t(p)$ and (*b*) $u_t(f_t) \geq u_t(x)$ for all $x \in B_t(p)$.
- An allocation f of $\mathcal{E}$ is a *competitive allocation* if for some p, (p, f) is a competitive equilibrium.

Nonexistence of a CE: An Example on Integers

- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$.
- Let $\{a_n\}$ be a decreasing sequence of positive numbers in $(0, 1)$ with $\lim_{n\to\infty} a_n = 0$.
- For $t \in \mathbb{N}$, define the utility function

$$u_t(x_1, x_2) = \sqrt{a_t x_1} + \sqrt{x_2}.$$

  The underlying preferences are continuous, strictly concave, strongly monotone and homothetic.
- The endowment assignment ω is arbitrary.
- We will show that there is no competitive equilibrium.
- Let $p_1 > 0$ and $p_2 > 0$. The unique solution of the agent's utility maximization problem subject to budget constraint is

$$D_{t1} = \frac{a_t p_2}{p_1} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_t p_2}, \quad D_{t2} = \frac{p_1}{p_2} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_t p_2}.$$

Example on Integers, contd. (I)

- Assume that $p_1, p_2 \in \mathbb{R}^2_+$ is a pair of competitive equilibrium prices. We must have
  $p_1 > 0$ and $p_2 > 0$ since u_t is strongly monotone for each t.
- For any positive integer m,

$$
\begin{aligned}
0 &\le \int_{\mathbb{N}} D_{t1}\,\mathrm{d}\mu = \int_{t\le m} D_{t1}\,\mathrm{d}\mu + \int_{t>m} D_{t1}\,\mathrm{d}\mu \\
&= \int_{t>m} \frac{a_t p_2}{p_1} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_t p_2}\,\mathrm{d}\mu \\
&\le \frac{a_m p_2}{p_1} \int_{t>m} \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1}\,\mathrm{d}\mu \\
&= \frac{a_m p_2}{p_1} \int_{\mathbb{N}} \left(\omega_{t1} + \frac{p_2}{p_1}\omega_{t2}\right)\mathrm{d}\mu = \frac{a_m p_2}{p_1}\left(\bar{\omega}_1 + \frac{p_2}{p_1}\bar{\omega}_2\right).
\end{aligned}
$$

Since $\lim_{m\to\infty} a_m = 0$, $\int_{\mathbb{N}} D_{t1}\,\mathrm{d}\mu = 0 \neq \bar{\omega}_1$.

Example on Integers, contd. (II)

$$\int_{\mathbb{N}} D_{t2}\,\mathrm{d}\mu = \int_{\mathbb{N}} \frac{p_1}{p_2} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_t p_2}\,\mathrm{d}\mu$$
$$\leq \frac{p_1}{p_2} \int_{\mathbb{N}} \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1}\,\mathrm{d}\mu = \frac{p_1}{p_2}\left(\bar{\omega}_1 + \frac{p_2}{p_1}\bar{\omega}_2\right).$$

In addition, for each $m \in \mathbb{N}$, we have

$$\int_{\mathbb{N}} D_{t2}\,\mathrm{d}\mu = \int_{t\leq m} D_{t2}\,\mathrm{d}\mu + \int_{t>m} D_{t2}\,\mathrm{d}\mu$$
$$= \int_{t>m} \frac{p_1}{p_2} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_t p_2}\,\mathrm{d}\mu$$
$$\geq \int_{t>m} \frac{p_1}{p_2} \times \frac{p_1\omega_{t1} + p_2\omega_{t2}}{p_1 + a_m p_2}\,\mathrm{d}\mu$$
$$= \frac{p_1}{p_2} \times \frac{1}{p_1 + a_m p_2} \int_{t>m} (p_1\omega_{t1} + p_2\omega_{t2})\,\mathrm{d}\mu$$

Example on Integers, contd. (III)

$$\begin{aligned}\int_{\mathbb{N}} D_{t2}\,\mathrm{d}\mu &\geq \frac{p_1}{p_2} \times \frac{1}{p_1 + a_m p_2} \int_{t>m} (p_1\omega_{t1} + p_2\omega_{t2})\,\mathrm{d}\mu \\ &= \frac{p_1}{p_2} \times \frac{1}{p_1 + a_m p_2} \int_{\mathbb{N}} (p_1\omega_{t1} + p_2\omega_{t2})\,\mathrm{d}\mu \\ &= \frac{p_1}{p_2} \times \frac{1}{p_1 + a_m p_2} (p_1\bar{\omega}_1 + p_2\bar{\omega}_2).\end{aligned}$$

By letting $m \to \infty$, we obtain that

$$\int_{\mathbb{N}} D_{t2}\,\mathrm{d}\mu \geq \frac{p_1}{p_2} \times \frac{1}{p_1}(p_1\bar{\omega}_1 + p_2\bar{\omega}_2) = \frac{p_1}{p_2}\left(\bar{\omega}_1 + \frac{p_2}{p_1}\bar{\omega}_2\right).$$

Therefore,

$$\int_{\mathbb{N}} D_{t2}\,\mathrm{d}\mu = \frac{p_1}{p_2}\bar{\omega}_1 + \bar{\omega}_2 > \bar{\omega}_2.$$

Hence the market for each good cannot be cleared.

Example on General Measure Spaces (Main Ideas)

- We will consider a general finitely additive measure space and consider a sequence of economies.
  - Each element of the sequence of economies has a competitive equilibrium.
  - However, the limit economy *does not have* a competitive equilibrium.
- Fact: Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space. Then the following are equivalent.
  (i) μ is not countably additive.
  (ii) There is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$.

The Economies

- Let $(T, \mathcal{T}, \mu)$ be given.
- There is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$.
- Let $A_1 = B_1$ and $A_n = B_n \setminus B_{n-1}$ for $n \geq 2$. Then $\{A_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} A_n = T$.
- Define $\mathcal{E} = (u, \omega)$ as follows. If $t \in A_n$ then

$$u_t(x_1, x_2) = \frac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2}\right).$$

- Define $\mathcal{E}^n = (u^n, \omega^n)$ as follows. If $t \in A_m$ then

$$\begin{aligned} u_t^n(x_1, x_2) &= \frac{m+1}{m} x_1^{\frac{m}{m+1}} + x_2, && m \leq n \\ &= x_1 + x_2, && m > n \\ \omega_t^n &= \left(\frac{c+1}{2}, \frac{c+1}{2}\right). \end{aligned}$$

Equilibrium of $\mathcal{E}^n$

$$
\begin{aligned}
u_t^n(x_1, x_2) &= \frac{m+1}{m} x_1^{\frac{m}{m+1}} + x_2, \qquad m \le n \\
&= x_1 + x_2, \qquad m > n \\
\omega_t^n &= \left(\frac{c+1}{2}, \frac{c+1}{2}\right)
\end{aligned}
$$

- Let $p_1^n = p_2^n = 1$.
- For $t \in A_m$ and $m \le n$,

  max $u_t^n(x_1, x_2)$ subject to $x_1 + x_2 \le c + 1$ gives

$$D_{t1}^n = 1, \qquad D_{t2}^n = c.$$

- For $t \in A_m$ and $m > n$, let

$$D_{t1}^n = \frac{\frac{c+1}{2} - \mu(B_n)}{1 - \mu(B_n)}, \qquad D_{t2}^n = c + 1 - D_{t1}^n.$$

Equilibrium of $\mathcal{E}^n$, contd.

Demands:

$$D^n_{t1} = 1 \qquad D^n_{t2} = c = c + 1 - D^n_{t1}$$

$$D^n_{t1} = \frac{\frac{c+1}{2} - \mu(B_n)}{1 - \mu(B_n)} \qquad D^n_{t2} = c + 1 - D^n_{t1}.$$

$$\begin{aligned}
\int_T D^n_{t1}\,\mathrm{d}\mu &= \int_{B_n} D^n_{t1}\,\mathrm{d}\mu + \int_{T\setminus B_n} D^n_{t1}\,\mathrm{d}\mu \\
&= \mu(B_n) + \frac{\frac{c+1}{2} - \mu(B_n)}{1 - \mu(B_n)}(1 - \mu(B_n)) \\
&= \frac{c+1}{2} \\
\int_T D^n_{t2}\,\mathrm{d}\mu &= \int_T (c + 1 - D^n_{t1})\,\mathrm{d}\mu = \frac{c+1}{2}.
\end{aligned}$$

The Economy $\mathcal{E}$

- Preferences and endowments: Let $t \in A_n$.

$$u_t(x_1, x_2) = \frac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right).$$

- Assume that $p_1, p_2 \in \mathbb{R}^2_+$ is a pair of competitive equilibrium prices. We must have
  $p_1 > 0$ and $p_2 > 0$ since u_t is strongly monotone for each t.
- Let $p_2 = 1$ and $p_1 > 0$. Then

$$D_{t1} = \min \left\{ \frac{1}{p_1^{n+1}}, \frac{c+1}{2}(1 + \frac{1}{p_1}) \right\}, \quad D_{t2} = \frac{c+1}{2}(1+p_1) - p_1 D_{t1}.$$

- To show that there is *no* competitive equilibrium, we will consider two cases: (i) $1 \geq p_1$ and (ii) $1 < p_1$.

Nonexistence of Equilibrium (Case 1)

Case 1: $1 \geq p_1$.

$$D_{t1} = \min\left\{\frac{1}{p_1^{n+1}}, \frac{c+1}{2}(1+\frac{1}{p_1})\right\} \geq \min\left\{1, \frac{c+1}{2}(1+\frac{1}{p_1})\right\}$$
$$= \frac{c+1}{2} + \min\left\{1 - \frac{c+1}{2}, \frac{c+1}{2p_1}\right\}.$$

Let $\epsilon = \min\left\{1 - \frac{c+1}{2}, \frac{c+1}{2p_1}\right\} > 0$.

Then $D_{t1} \geq \frac{c+1}{2} + \epsilon$ for any $t \in T$. Therefore,

$$\frac{c+1}{2} = \int_T D_{t1}\, \mathrm{d}\mu \geq \frac{c+1}{2} + \epsilon,$$

a contradiction.

Nonexistence of Equilibrium (Case 2)

Case 2: $1 < p_1$. Note that $D_{t1} \leq \frac{1}{p_1^{n+1}}$ for any $t \in A_n$.

For any positive integer m,

$$\begin{aligned}
\frac{c+1}{2} &= \int_T D_{t1}\, \mathrm{d}\mu = \int_{B_m} D_{t1}\, \mathrm{d}\mu + \int_{T \setminus B_m} D_{t1}\, \mathrm{d}\mu \\
&\leq \int_{B_m} 1\, \mathrm{d}\mu + \int_{T \setminus B_m} \frac{1}{p_1^{m+2}}\, \mathrm{d}\mu \\
&= \mu(B_m) + \frac{1}{p_1^{m+2}} \mu(T \setminus B_m).
\end{aligned}$$

Let m go to infinity. Then

$$\frac{c+1}{2} \leq c, \qquad \text{a contradiction.}$$

Thus, $\mathcal{E}$ does not have a competitive equilibrium.

An Implication of Countable Additivity

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space where $\mathcal{T}$ is a σ-algebra and μ is strongly continuous. Then the following are equivalent.

(i) *Every economy $\mathcal{E}$ on T with strongly monotone preferences has a competitive equilibrium.*

(ii) *μ is countably additive.*

Proof *(ii)* $\Rightarrow$ *(i)*. Aumann (1966).

(i) $\Rightarrow$ *(ii)*. Assume that μ is not countably additive.
Consider the economy $\mathcal{E}$ in the Example on integers. It does not have a competitive equilibrium. ■

Closed Graph Property

- Let $\{\mathcal{E}^n\}$ and $\mathcal{E}$ be economies on $(T, \mathcal{T}, \mu)$ and (p^n, f^n) a competitive equilibrium of $\mathcal{E}^n$.
- Suppose that
  - $\{\mathcal{E}^n\}$ converges to $\mathcal{E}$ pointwise,
  - $\{\int_T \omega^n \, \mathrm{d}\mu\} \to \int_T \omega \, \mathrm{d}\mu$ and
  - $\{f^n\}$ converges to f pointwise.
- $\mathcal{E}$ has the *closed graph property* if f is a competitive allocation of $\mathcal{E}$.

Another Implication of Countable Additivity

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space where $\mathcal{T}$ is a σ-algebra and μ is strongly continuous. Then the following are equivalent.

(i) *Every economy $\mathcal{E}$ on T with strongly monotone preferences which is the limit of a sequence of economies with strongly monotone preferences has the closed graph property.*

(ii) *μ is countably additive.*

Proof *(ii)* $\Rightarrow$ *(i)*. Follows from Theorem B of Kannai (1970).

(i) $\Rightarrow$ *(ii)*. Assume that μ is not countably additive. Consider the Example on general measure spaces. There, each of the economies $\mathcal{E}^n$ has a competitive equilibrium but the limit economy $\mathcal{E}$ does not. Thus, $\mathcal{E}$ does not have the closed graph property. ∎

epsilon-Competitive Equilibrium

- Let $\mathcal{E}$ be economy on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$.
- (p, f) is an *ϵ-competitive equilibrium* of $\mathcal{E}$ if
  - $p \in \mathbb{R}^L_+ \setminus \{0\}$,
  - f is an allocation,
  - for almost all t, $f_t \in B_t(p)$ and
  - there exists $T_\epsilon \in \mathcal{T}$ such that:

    (a) $\mu(T_\epsilon) \leq \epsilon$ and

    (b) for almost all $t \in T_\epsilon^c$,

    $$u_t(f_t) \geq u_t(y) - \epsilon \text{ for any } y \in B_t(p).$$

Existence of epsilon-Competitive Equilibrium

- An economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exist $T_1 \subseteq T$ such that $\mu(T_1) > 1 - \epsilon$ and $\mathcal{E}(T_1)$ is relatively compact.

Theorem

Let $\mathcal{E}$ be an economy on $(T, \mathcal{T}, \mu)$. If μ is strongly continuous and $\mathcal{E}$ is tight then $\mathcal{E}$ has an ϵ-competitive equilibrium (p, f) for any $\epsilon > 0$.